Toward a Unified Description of Bound and Unbound States

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The aim of this talk is to show that the complex scaling method provides us with a unified description of bound states, resonant states and continuum scattering states.

1. **Framework of the complex scaling method**

   Complex Scaled Resolution of Identity

   Complex Scaled Green's Function

2. **Coulomb breakup reactions**

3. **Complex scaled Lippmann-Schwinger equation**

1. Resolution of Identity in the Complex Scaling Method

Completeness Relation (Resolution of Identity)

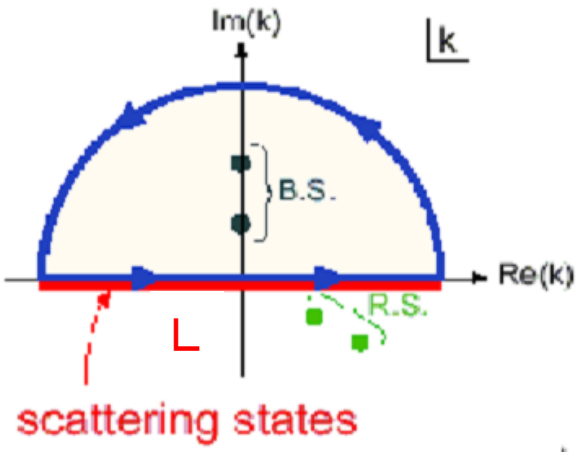
R.G. Newton, J. Math. Phys. **1** (1960), 319

$$1 = \sum_{n=b} |u_n\rangle\langle\tilde{u}_n| + \frac{1}{\pi}\int_R dk\, |\psi_k\rangle\langle\tilde{\psi}_k|$$

Bound states

Continuum states

- **Resonant states**
- **non-resonant continuum states**



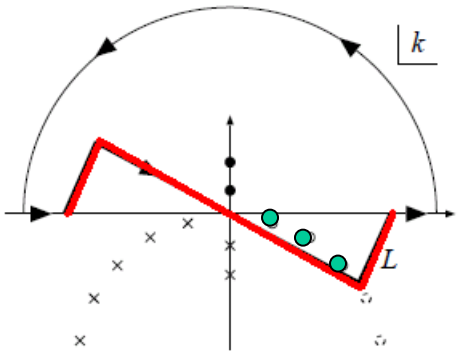
Among the continuum states, resonant states are considered as an extension of bound states because they result from interactions.

Separation of resonant states from continuum states

$$1=\sum_{n=b}|u_n\rangle\langle\tilde{u}_n|+\boxed{\sum_{n=r}^{N_r(L)}|u_r\rangle\langle\tilde{u}_r|}+\boxed{\frac{1}{\pi}\int_L dk\,|\psi_k\rangle\langle\tilde{\psi}_k|}$$

Resonant states

Deformed continuum states



Deformation of the contour

T. Berggren, Nucl. Phys. A 109, 265 (1968)

Matrix elements of resonant states

$$\langle\tilde{u}_1|\hat{O}|u_2\rangle=\lim_{\alpha\to 0}\int_R dr\, e^{-\alpha r^2}\tilde{u}_1^*\hat{O}u_2$$

Convergence Factor Method

Ya.B. Zel'dovich, Sov. Phys. JETP 12, 542 (1961).

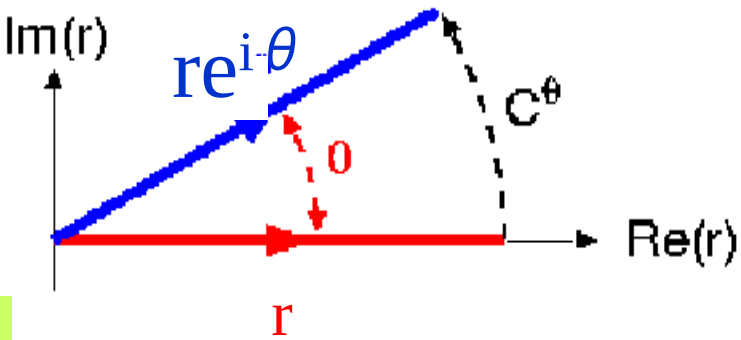
N. Hokkyo, Prog. Theor. Phys. 33, 1116 (1965).

Complex scaling method

J. Aguilar and J.M.Combes, J. Math. Phys. **22**, 269 (1971)

E.Balslev and J.M.Combes, J. Math. Phys. **22**, 280 (1971)

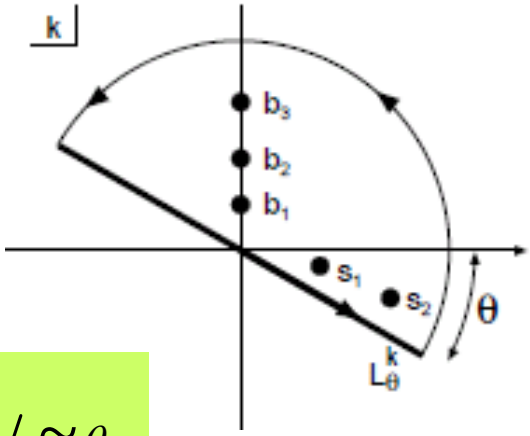
coordinate: $r \rightarrow re^{i\theta}$



$$\langle \tilde{u}_1 | \hat{O} | u_2 \rangle = \lim_{\alpha \to 0} \int_R dr\, e^{-\alpha r^2}\, \tilde{u}_1^* \hat{O} u_2$$
$$= \int_{R^\theta} d(re^{i\theta})\, \tilde{u}_1^*(re^{i\theta}) \hat{O}(\theta) u_2(re^{i\theta})$$

B. Gyarmati and T. Vertse, Nucl. Phys. **A160**, 523 (1971).

momentum: $k \rightarrow ke^{-i\theta}$



$$1 = \sum_{n=b} |u_n^\theta\rangle\langle \tilde{u}_n^\theta| + \sum_{n=r}^{N_r^\theta} |u_n^\theta\rangle\langle \tilde{u}_n^\theta| + \frac{1}{\pi}\int_{L_\theta^k} dk\, |\psi_k^\theta\rangle\langle \tilde{\psi}_k^\theta|$$

Resonant states **Rotated Continuum states**

Complex Scaling Method

physical picture of the complex scaling method

Resonance state

The resonance wave function behaves asymptotically as

$$\varphi(r) \underset{r\to\infty}{\longrightarrow} e^{ikr} \quad (r \to \infty).$$

When the resonance energy is expressed as

$$E = E_r - i\frac{\Gamma}{2}, \qquad \left(\theta_r = \frac{1}{2}\tan^{-1}\left(\frac{\Gamma}{2E_r}\right)\right),$$

the corresponding momentum is

$$k = \kappa - i\gamma = \sqrt{2E} = \sqrt{2\,|\,E\,|}e^{-i\theta_r}$$

$$= |\,k_r\,|\,e^{-i\theta_r},$$

and the asymptotic resonance wave function

$$\varphi(r) \underset{r\to\infty}{\longrightarrow} e^{ikr} = e^{i|k|e^{(-i\theta_r)}r}$$

$$= e^{i|k|r\cos\theta_r} \cdot \underline{e^{|k|r\sin\theta_r}}.$$

Diverge!

This asymptotic divergence of the resonance wave function causes difficulties in the resonance calculations.

In the method of complex scaling, a radial coordinate r is transformed as

$$U(\theta); \quad r \to re^{i\theta}, \quad p \to pe^{-i\theta}.$$

Then the asymptotic form of the resonance wave function becomes

$$\varphi(r) \underset{r\to\infty}{\longrightarrow} e^{ikr} = e^{i|k|e^{-i\theta_r} \underline{re^{i\theta}}}$$

$$= e^{i|k|re^{i(\theta-\theta_r)}}$$

$$= e^{i|k|r\cos(\theta-\theta_r)} \cdot \underline{e^{-|k|r\sin(\theta-\theta_r)}}$$

Converge!

It is now apparent that when $\pi/2>(\theta-\theta_r)>0$ the wave function converges asymptotically. This result leads to the conclusion that the resonance parameters (E_r, Γ) can be obtained as an eigenvalue of a bound-state type wave function.

This is an important reason why we use the complex scaling method.

Eigenvalue Problem of the Complex Scaled Hamiltonian

- Complex scaling transformation

$$U(\theta) f(r) = e^{i3\theta/2} f(re^{i\theta})$$

- Complex Scaled Schoedinger Equation

$$H_\theta \chi_\theta = E_\theta \chi_\theta$$

$$H_\theta = U(\theta) H U^{-1}(\theta), \qquad H = T + V$$

$$\chi_\theta = U(\theta)\chi$$

ABC Theorem

J.Aguilar and J. M. Combes; Commun. Math. Phys. 22 (1971), 269.

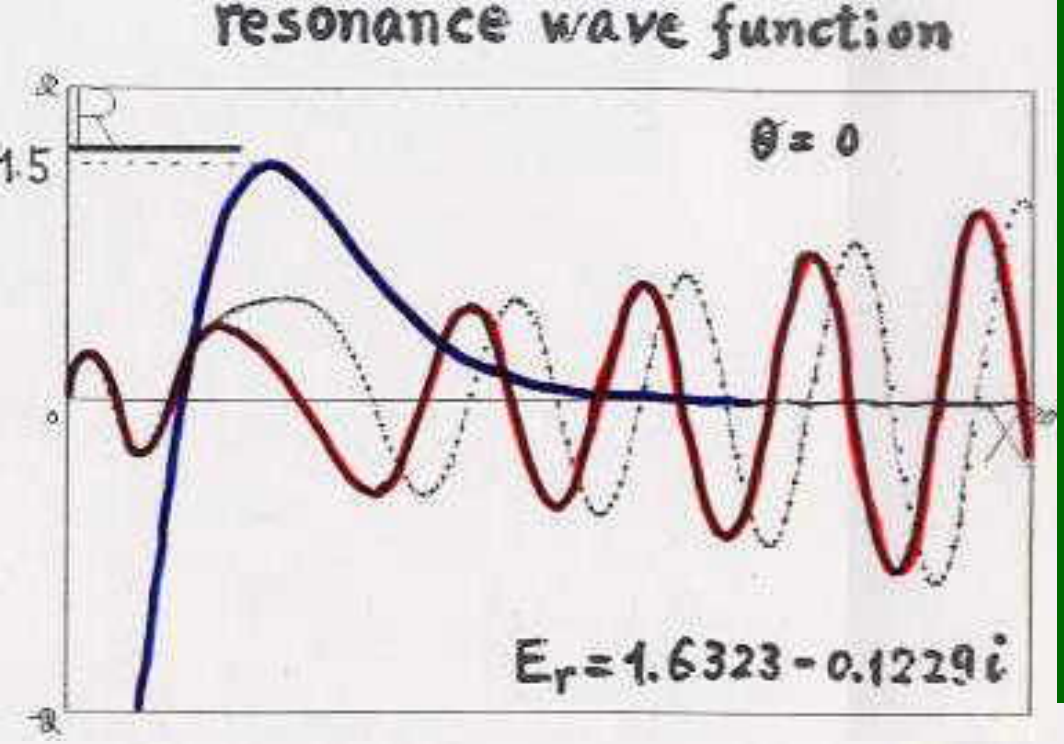
E. Balslev and J.M. Combes; Commun. Math. Phys. 22(1971), 280.

i) χ_θ is an L^2-class function:

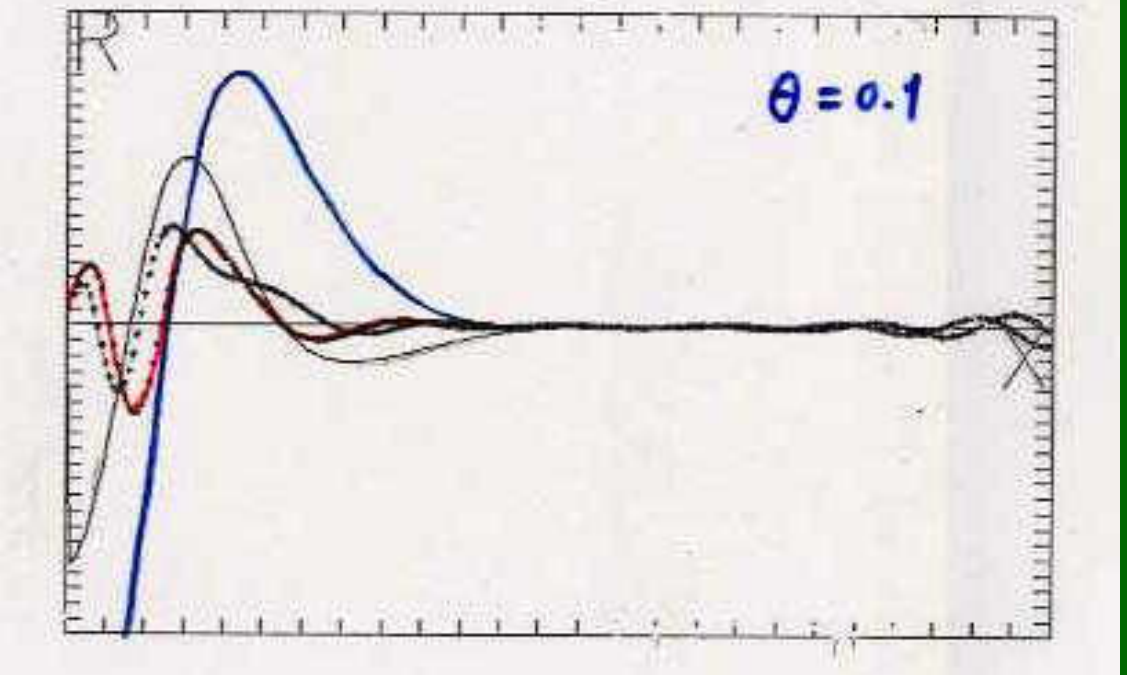
$$\chi_\theta = \sum_i c_i(\theta) u_i, \qquad \| u_i \| < \infty$$

ii) E_θ is independent on θ ($\theta \geq \frac{1}{2} \arg(E^{res})$)

$$E^{res} = E_r - i\Gamma / 2$$



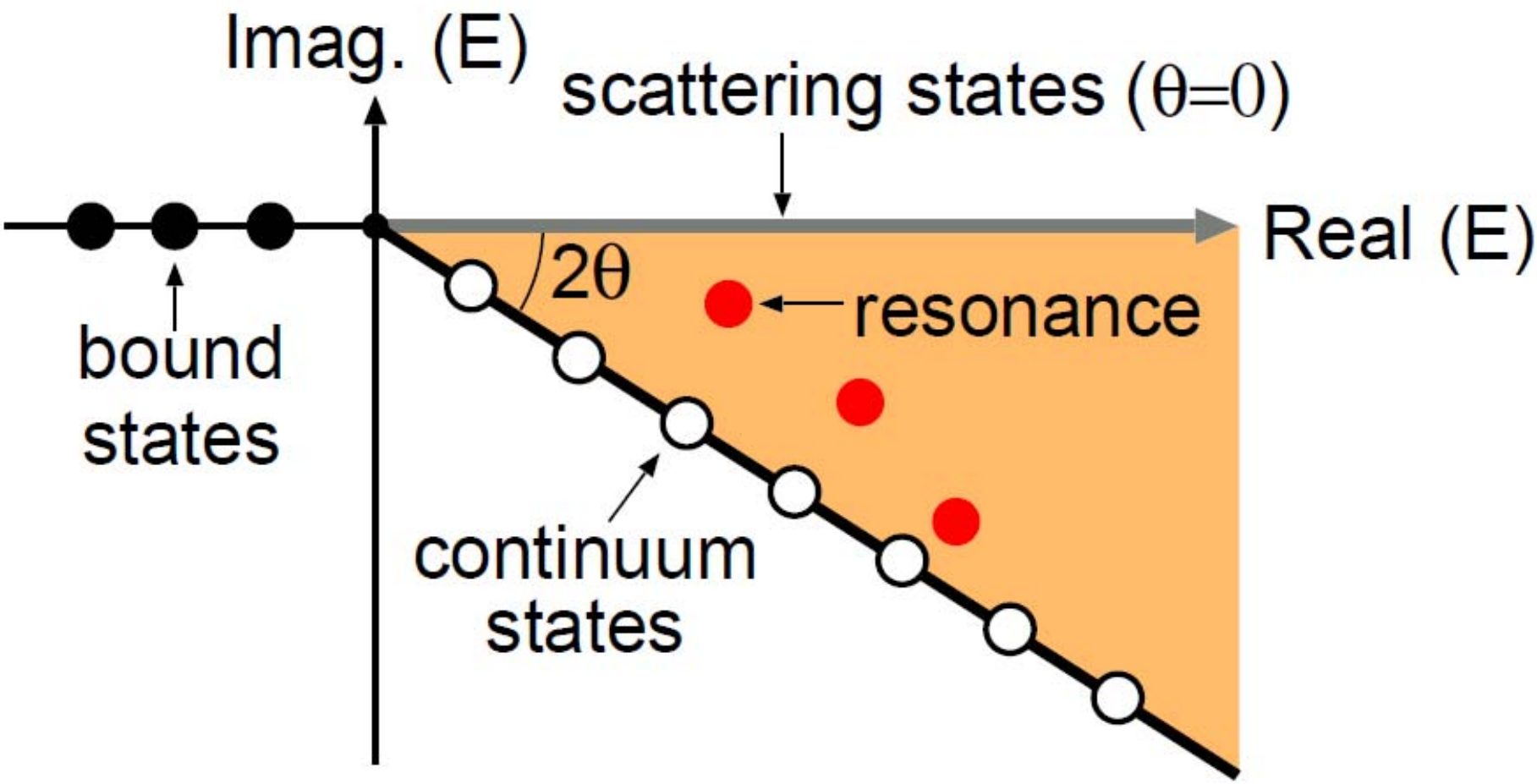
Exponentially diverge!



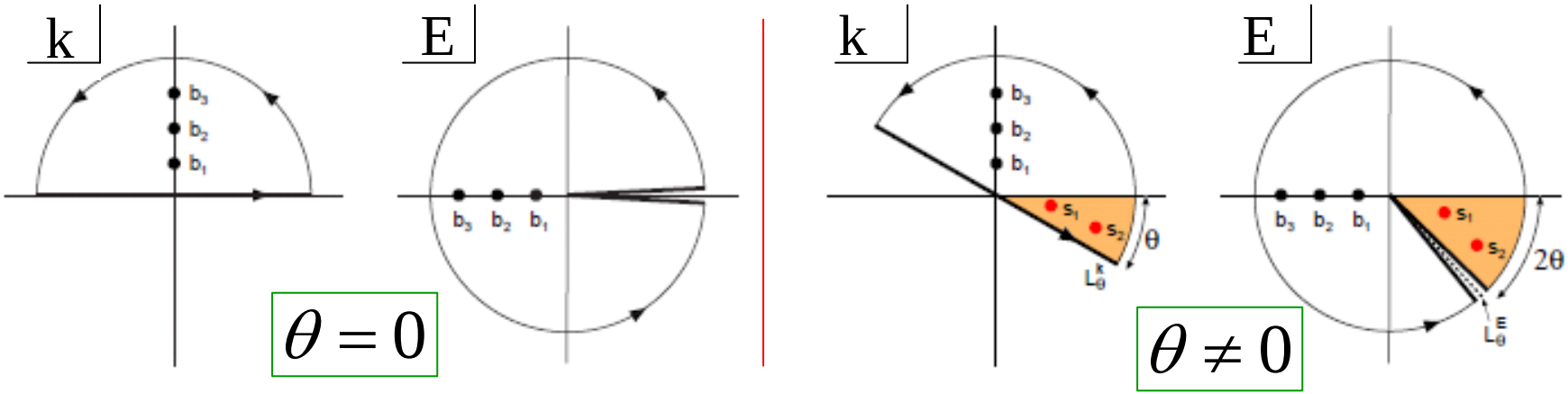
Rapidly Converge!

Resonant States; Eigenvalues of $H(\theta)$ with a L^2 basis set

(L^2 basis set ; Gaussian basis functions)



Resolution of Identity in Complex Scaling Method

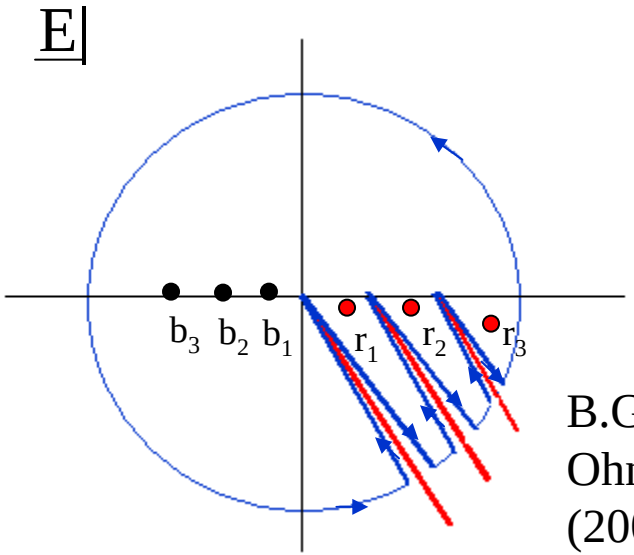


Single Channel system

B.Giraud and K.Kato, Ann.of Phys. 308 (2003), 115.

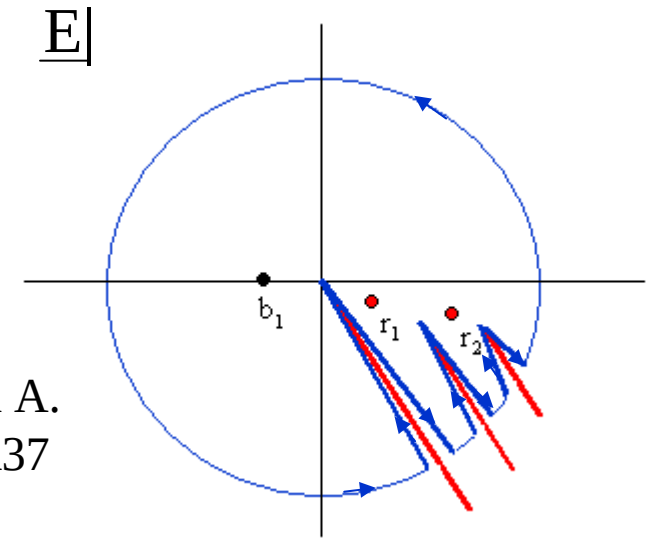
$$1 = \sum_{n=b} |u_n^\theta\rangle\langle\tilde{u}_n^\theta| + \sum_{n=r}^{N_r^\theta} |u_n^\theta\rangle\langle\tilde{u}_n^\theta| + \frac{1}{\pi}\int_{L_\theta^k} dk\, |\psi_k^\theta\rangle\langle\tilde{\psi}_k^\theta|$$

Resonant states **Rotated Continuum states**



$\theta \neq 0$

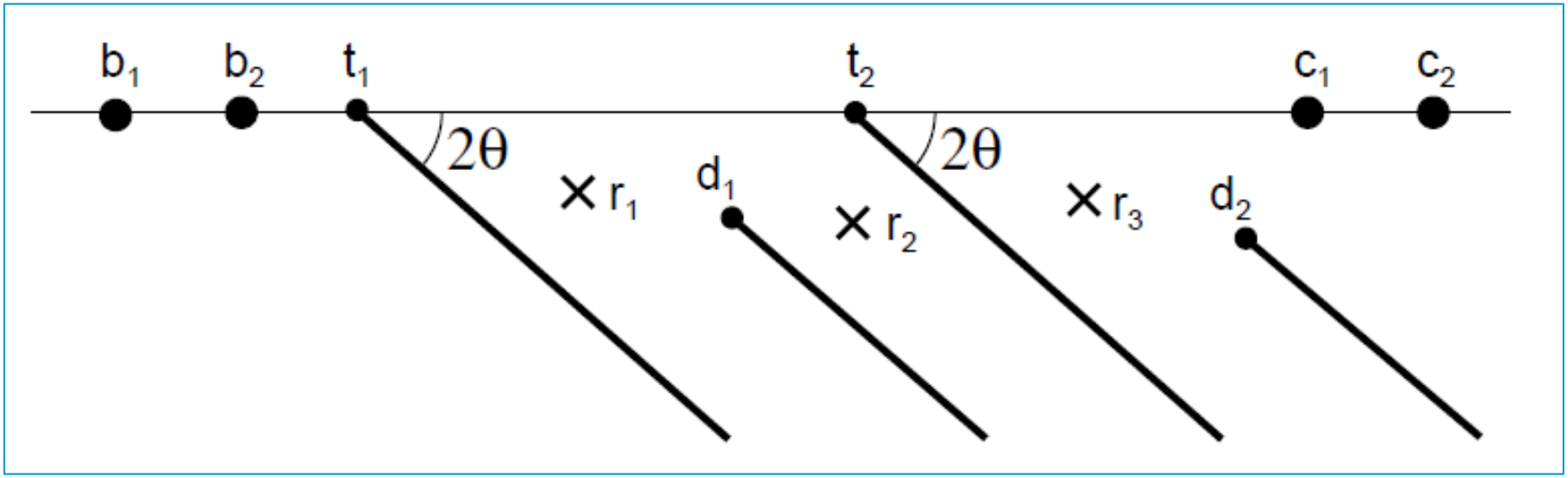
B.Giraud, K.Kato and A. Ohnishi, J. of Phys. A37 (2004),11575

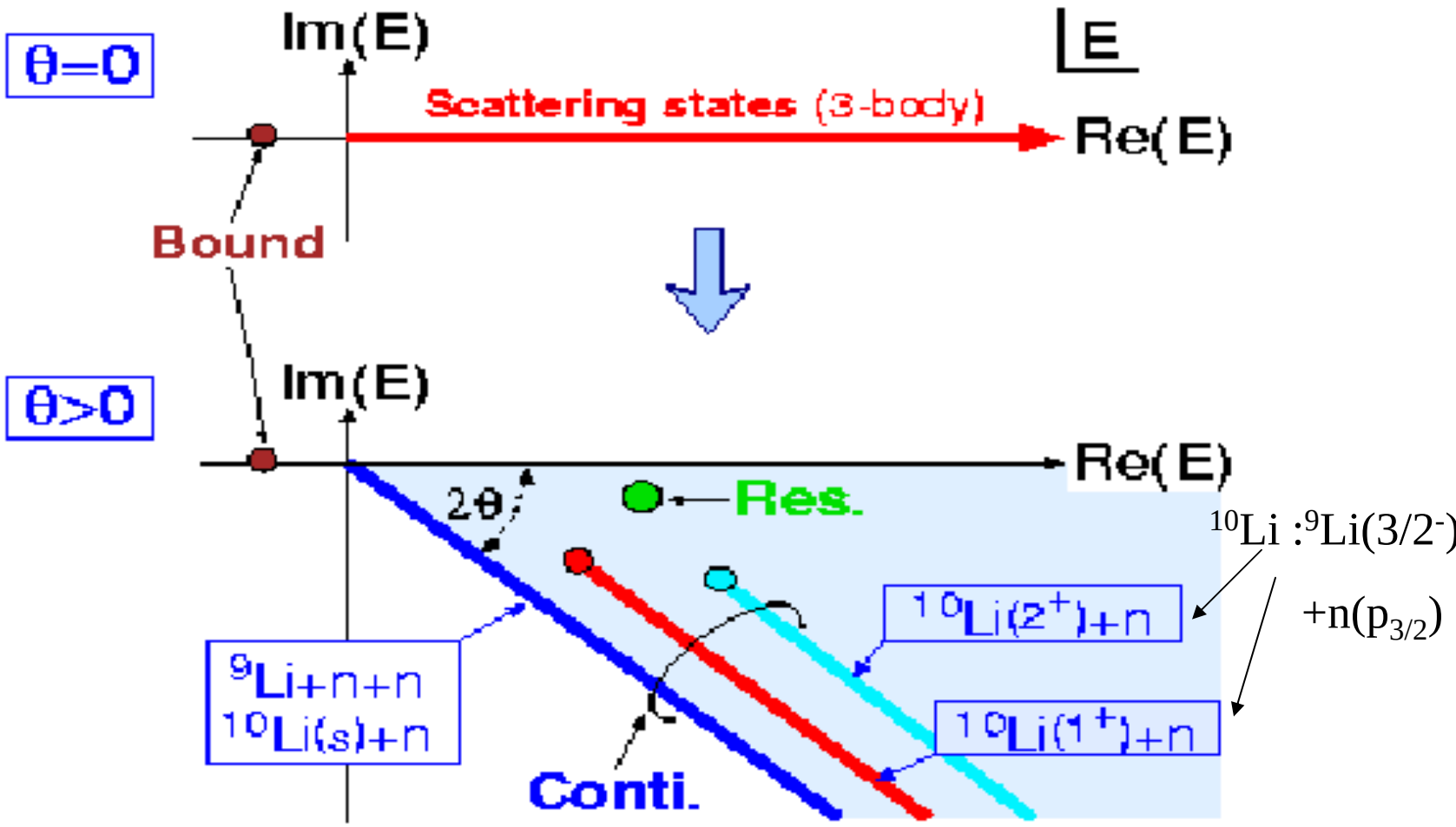


Coupled Channel system

Three-body system

Bany-body system





T. Myo, A. Ohnishi and K. Kato, Prog. Theor. Phys. **99** (1998), 801.

- Extended completeness relation in CSM

$$1=\sum_{n=b}|u_n^\theta\rangle\langle\tilde{u}_n^\theta|+\sum_{n=r}^{N_r^\theta}|u_n^\theta\rangle\langle\tilde{u}_n^\theta|+\frac{1}{\pi}\int_{L_\theta^k}dk\,|\psi_k^\theta\rangle\langle\tilde{\psi}_k^\theta|+\frac{1}{\pi}\int_{L_\theta^{k'}}dk'|\psi_{k'}^\theta\rangle\langle\tilde{\psi}_{k'}^\theta|+\frac{1}{\pi}\int_{L_\theta^{k''}}dk''|\psi_{k''}^\theta\rangle\langle\tilde{\psi}_{k''}^\theta|$$

Resonances ^{9}Li+n+n ^{10}Li(1$^+$)+n ^{10}Li(2$^+$)+n

Complex Scaled Green's Functions

Green's operator

$$G^{(+)} = \frac{1}{E - H + i\varepsilon}$$

Complex scaled Green's operator

$$G_\theta^{(+)} = \frac{1}{E - H(\theta)}$$

Resolution of Identity

$$1 = \sum_{n=b} |u_n^\theta\rangle\langle\tilde{u}_n^\theta| + \sum_{n=r}^{N_r^\theta} |u_n^\theta\rangle\langle\tilde{u}_n^\theta| + \frac{1}{\pi}\int_{L_\theta^k} dk\, |\psi_k^\theta\rangle\langle\tilde{\psi}_k^\theta|$$

Complex Scaled Green's function

$$\mathcal{G}^\theta(E,\boldsymbol{\xi},\boldsymbol{\xi}') = \left\langle \boldsymbol{\xi} \left| \frac{1}{E - H(\theta)} \right| \boldsymbol{\xi}' \right\rangle$$

$$= \sum_B \frac{\phi_B(\boldsymbol{\xi})\tilde{\phi}_B^*(\boldsymbol{\xi}')}{E - E_B} + \sum_R \frac{\phi_R(\boldsymbol{\xi})\tilde{\phi}_R^*(\boldsymbol{\xi}')}{E - E_R} + \sum_C \frac{\phi_C(\boldsymbol{\xi})\tilde{\phi}_C^*(\boldsymbol{\xi}')}{E - E_C}$$

2. Strength Functions and Coulomb Breakup Reaction

- Strength function

$$\mathcal{S}_\lambda(E) = \sum_\nu \langle \tilde{\Phi}_I | \hat{O}^\dagger_\lambda | \nu \rangle \langle \tilde{\nu} | \hat{O}_\lambda | \Phi_I \rangle \, \delta(E - E_\nu) = -\frac{1}{\pi} \mathrm{Im}\,[R(E)]$$

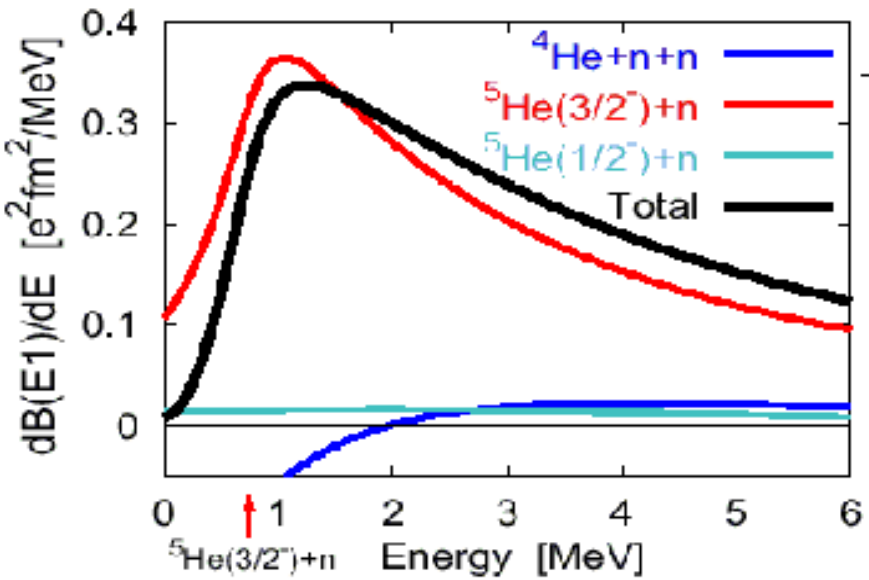
- Response function and Green's function

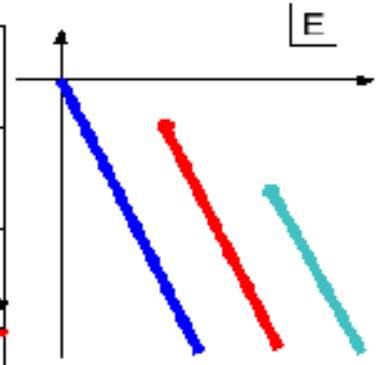
$$R(E) = \int d\boldsymbol{\xi}\, d\boldsymbol{\xi}'\, \tilde{\Phi}^*_I(\boldsymbol{\xi})\, \hat{O}^\dagger_\lambda\, \underline{\mathcal{G}(E, \boldsymbol{\xi}, \boldsymbol{\xi}')}\, \hat{O}_\lambda\, \Phi_I(\boldsymbol{\xi}')$$

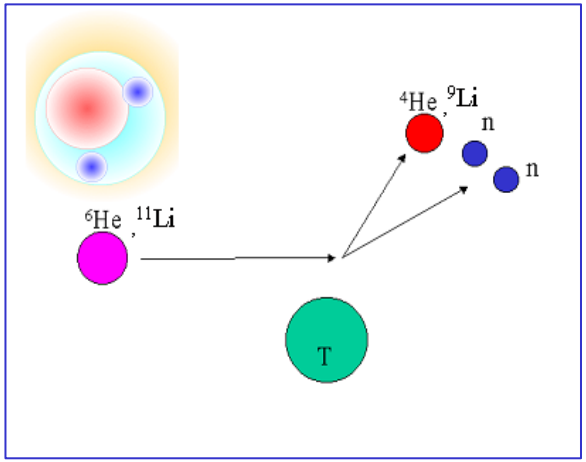
$$\mathcal{G}^\theta(E, \boldsymbol{\xi}, \boldsymbol{\xi}') = \left\langle \boldsymbol{\xi} \left| \frac{\mathbf{1}}{E - H(\theta)} \right| \boldsymbol{\xi}' \right\rangle$$

$$= \sum_B \frac{\phi_B(\boldsymbol{\xi}) \tilde{\phi}^*_B(\boldsymbol{\xi}')}{E - E_B} + \sum_R \frac{\phi_R(\boldsymbol{\xi}) \tilde{\phi}^*_R(\boldsymbol{\xi}')}{E - E_R} + \sum_C \frac{\phi_C(\boldsymbol{\xi}) \tilde{\phi}^*_C(\boldsymbol{\xi}')}{E - E_C}$$

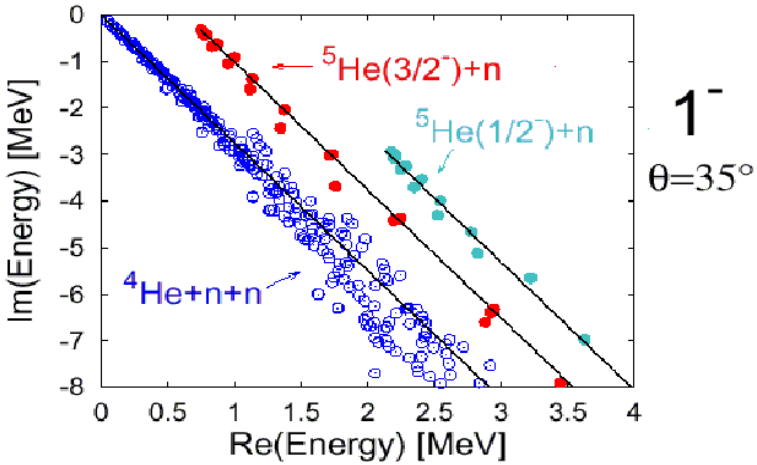
- E1 transition ($0^+ \rightarrow 1^-$)

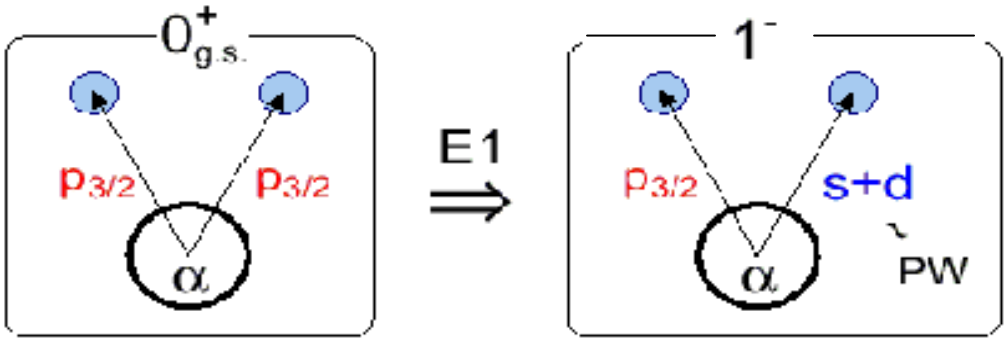






- 1^-, energies of ^{6}He with CSM





- ^{6}He $\Rightarrow$ ^{5}He(3/2⁻)+n $\Rightarrow$ ^{4}He—n—n
- threshold effect of ^{5}He+n

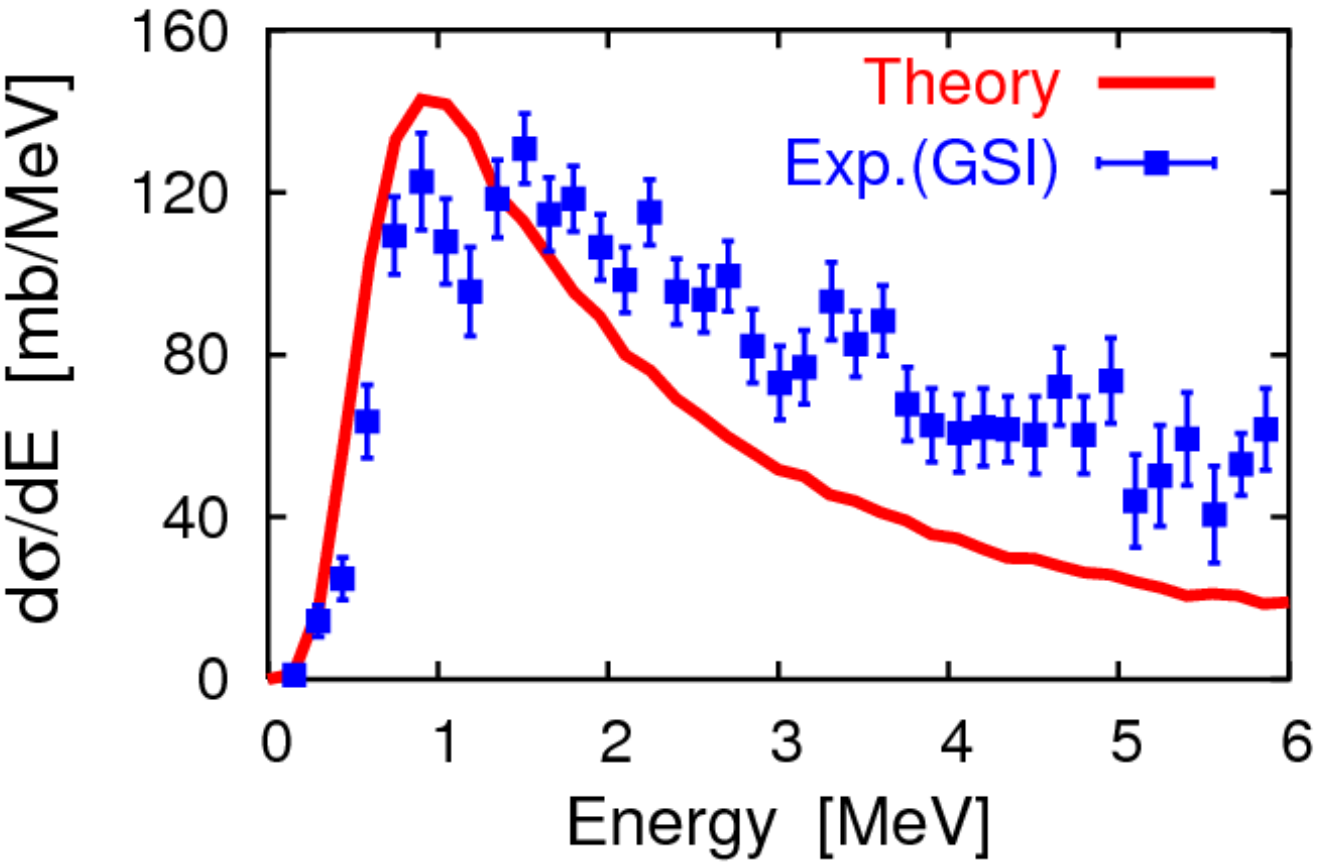
  $\Rightarrow$ Low energy enhancement

T. Myo, K. Kato, S. Aoyama and K. Ikeda, PRC63(2001), 054313

Coulomb breakup strength of ^{6}He

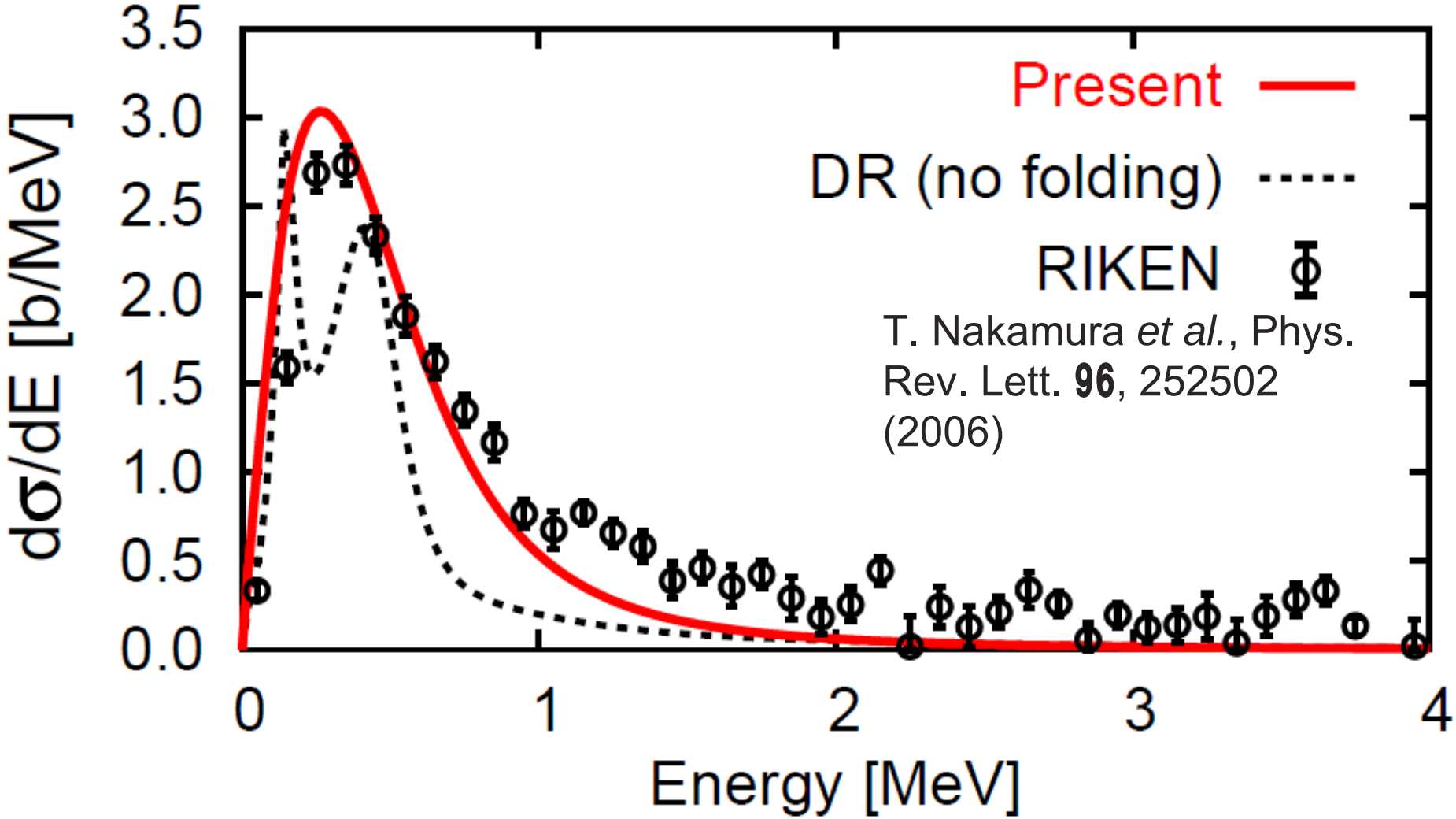
$$\frac{d\sigma_{E\lambda}}{dE} \propto N_{E\lambda}(E) \cdot \frac{dB(E\lambda, E)}{dE} \qquad N_{E\lambda}(E): \text{virtual photon number}$$

^{6}He : 240MeV/A, Pb Target (T. Aumann et.al, PRC59(1999)1252)



T.Myo, K. Kato,
S. Aoyama and K. Ikeda
PRC63(2001)054313.

Coulomb breakup cross section of ^{11}Li



3. Complex scaled Lippmann-Schwinger equation

$$(H_0 + V)\Psi = E\Psi$$

$H_0 = T + V_C$

V; Short Range Interaction

$$H_0 \Psi_0 = E \Psi_0$$

（Ψ_0; regular at origin）

Solutions of Lippmann-Schwinger Equation

$$\Psi = \Psi_0 + \frac{1}{E - H} V \Psi_0$$

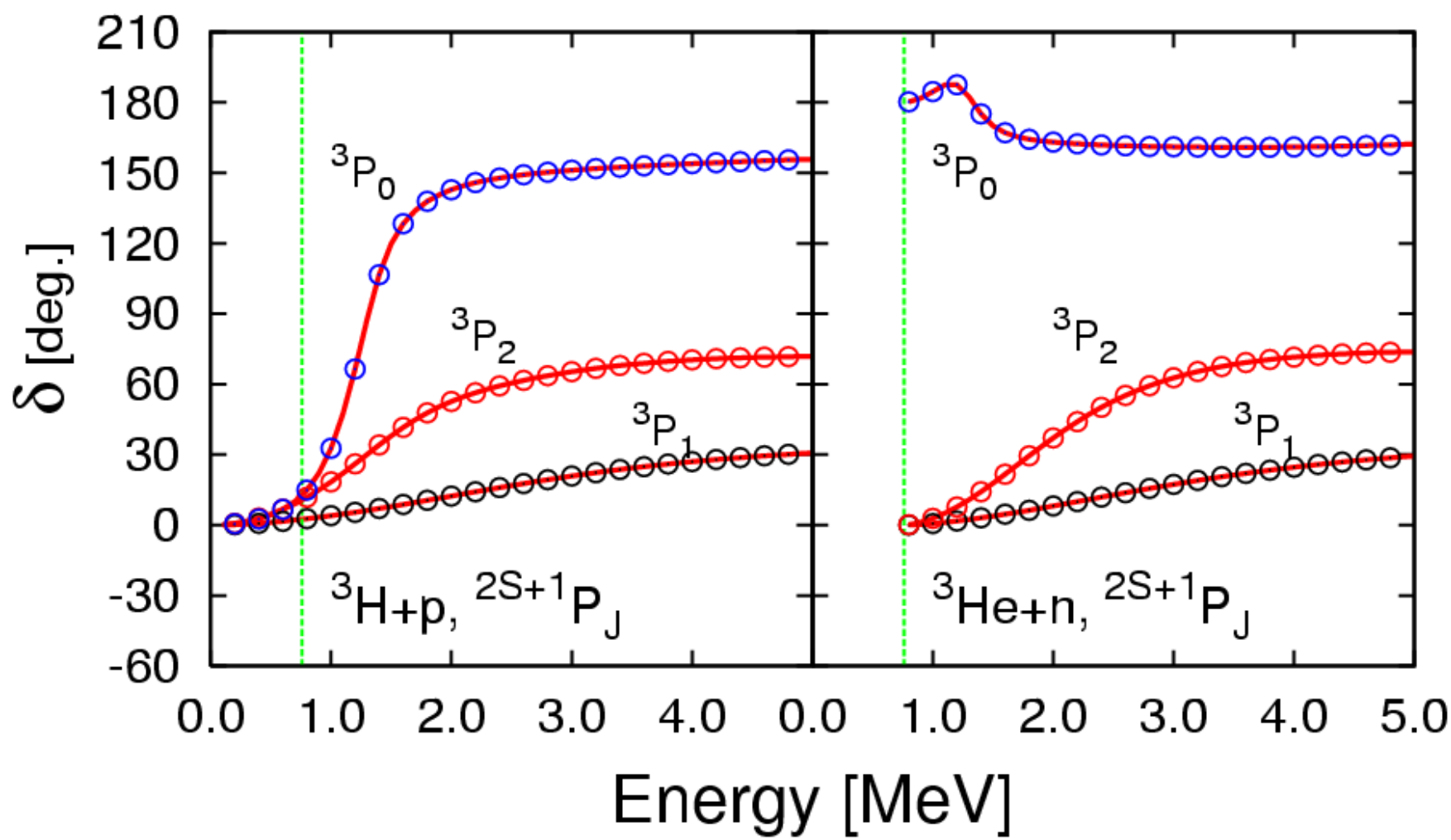
Outgoing waves

Complex Scaling

$$\Psi^{(+)} = \Psi_0 + \frac{1}{E - H(\theta)} V \Psi_0$$

A. Kruppa, R. Suzuki and K. Kato, phys. Rev.C75 (2007), 044602

^{4}He: (^{3}He+p)+(^{3}He+n) Coupled-Channel Model



- Lines : Runge-Kutta method
- Circles : CSM+Base

Complex-scaled Lippmann-Schwinger Eq.

- CSLM solution

$$|\Psi^+(k,K)\rangle = |k,K\rangle + \sum_i U^{-1}\boxed{|\phi_i^\theta\rangle \frac{1}{E-E_i^\theta}\langle\phi_i^\theta|}U\hat{V}|k,K\rangle$$

- B(E1) Strength

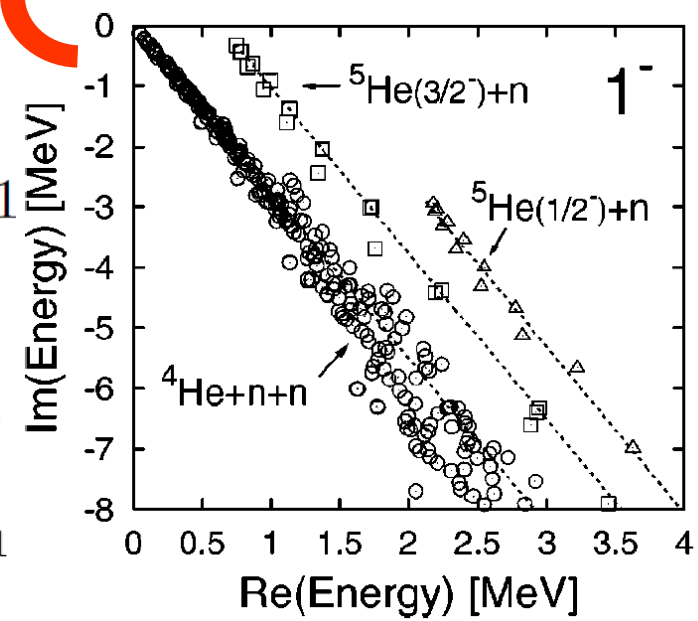
$$\frac{d^6B(E1)}{d\boldsymbol{k}d\boldsymbol{K}} = |\langle\Psi\ \ (\boldsymbol{k},\boldsymbol{K})|\hat{O}\ \ (E1$$

$$\langle\Phi_{\mathrm{g.s.}}|\hat{O}(E1)|\Psi^+(k,K)\rangle = \langle\Phi_{\mathrm{g.s.}}|\hat{O}(E1)|k.K\rangle$$

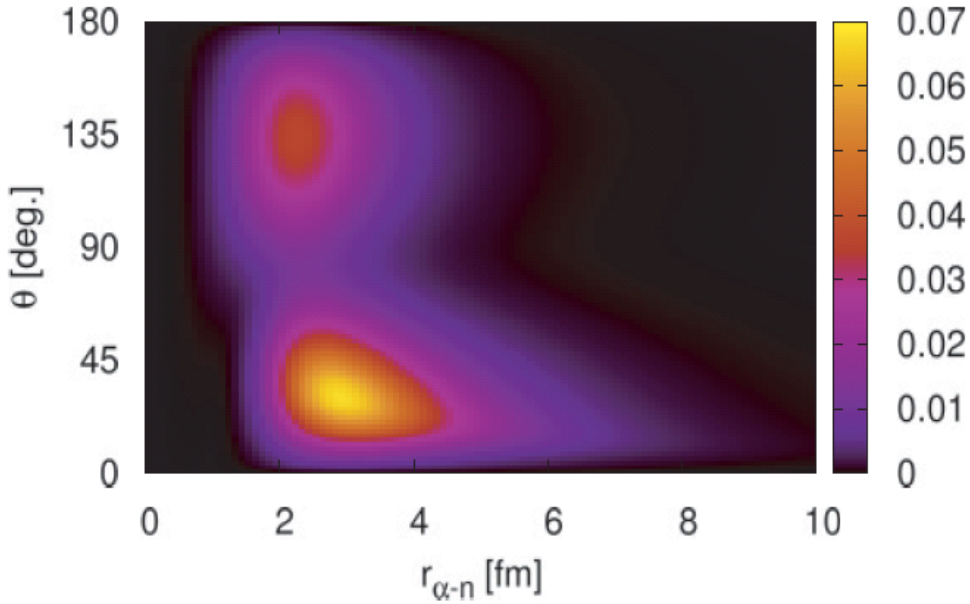
Direct breakup

$$+\sum_i\langle\Phi_{\mathrm{g.s.}}|\hat{O}(E1)U^{-1}$$

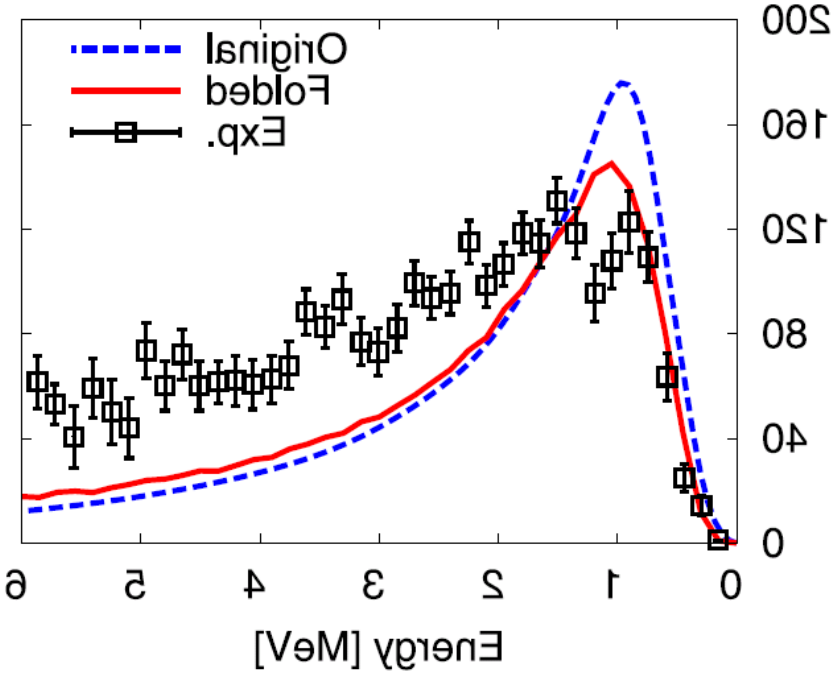
Final state interaction (FSI)

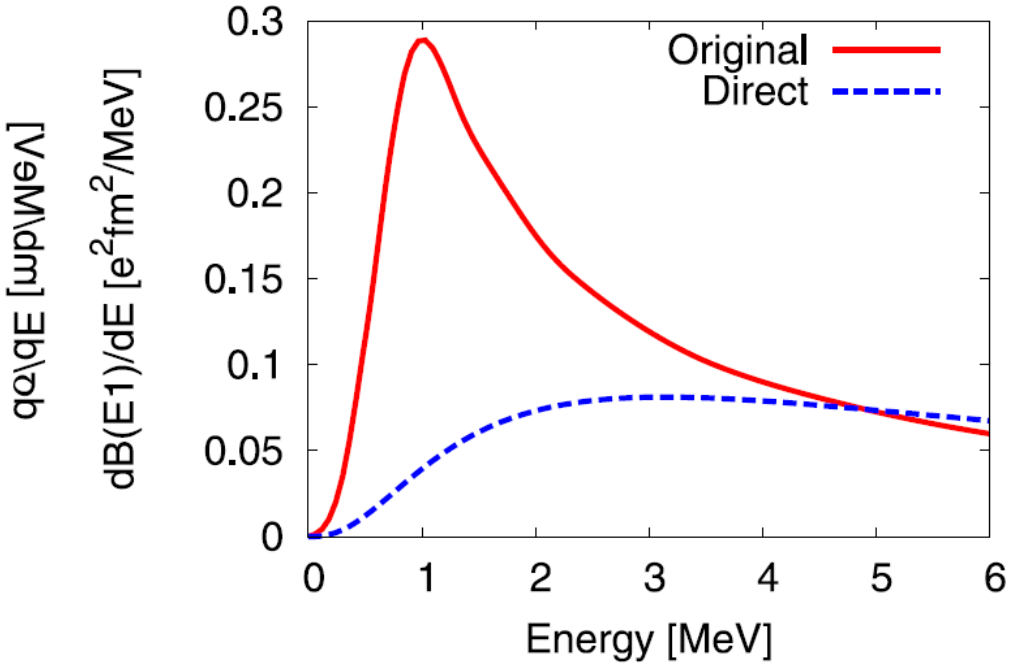


Two-neutron distribution of ^{6}He



(T. Aumann et.al, PRC59(1999)1252)





Summary and conclusion

- **The resolution of identity in the complex scaling method is presented to treat the resonant states in the same way as bound states.**
- **The complex scaling method is shown to describe not only resonant states but also continuum states on the rotated branch cuts.**
- **We presented several applications of the extended resolution of identity in the complex scaling method; strength functions of the Coulomb break reactions and three-body scattering states.**

Collaborators

Y. Kikuchi, K. Yamamoto, A. Wano, T. Myo, M. Takashina, C. Kurokawa, R. Suzuki, K. Arai, H. Masui, S. Aoyama, K. Ikeda, A. Kruppa. B. Giraud