中南大學
CENTRAL SOUTH UNIVERSITY

Investigation of the $D_s^+ \to \pi^+\pi^-K^+$ and $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay

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Background and Motivations

- Experiments：

① $D_s^+(D^+) \to K^+K^-\pi^+$:

P. L. Frabetti et al. [E687], Phys. Lett. B 351, 591-600 (1995).

R. E. Mitchell et al. [CLEO], Phys. Rev. D 79, 072008 (2009).

M. Ablikim et al. [BESIII], Phys. Rev. D 104, 012016 (2021).

② $D_s^+ \to \pi^+\pi^0\eta$:

M. Ablikim et al. [BESIII], Phys. Rev. Lett. 123, 112001 (2019)

③ $D_s^+ \to \pi^+\pi^-\pi^+$:

B. Aubert et al. [BaBar], Phys. Rev. D 79, 032003 (2009).

M. Ablikim et al. [BESIII], Phys. Rev. D 106, 112006 (2022).

④ $D_s^+ \to K_s^0K_s^0\pi^+$:

M. Ablikim et al. [BESIII], Phys. Rev. D 105, L051103 (2022).

⑤ $D_s^+ \to K_s^0K^+\pi^0$:

M. Ablikim et al. [BESIII], Phys. Rev. Lett. 129, 18 (2022).

- Theories：

① $D_s^+ \to K^+K^-\pi^+$:

J. Y. Wang et al. Phys. Lett. B 821, 136617 (2021).

Z. Y. Wang et al. Phys. Rev. D 105, 016025 (2022).

R. Escribano et al. arXiv:2302.03312 [hep-ph].

② $D_s^+ \to \pi^+\pi^0\eta$:

R. Molina et al. Phys. Lett. B 803, 135279 (2020).

③ $D_s^+ \to \pi^+\pi^-\pi^+$:

J. M. Dias et al. Phys. Rev. D 94, 096002 (2016).

N. N. Achasov et al. Phys. Rev. D 107, 056009 (2023).

④ $D_s^+ \to K_s^0K_s^0\pi^+$:

L. R. Dai et al. Eur. Phys. J. C 82, 225 (2022).

⑤ $D_s^+ \to K_s^0K^+\pi^0$:

X. Zhu et al. Phys. Rev. D 107, 034001 (2023).

Background and Motivations.

- $D_s^+ \to \pi^+\pi^-K^+$:

J.M. Link et al. [FOCUS Collaboration], Phys. Lett. B 601, 10-19 (2004).

Decay channel	Fit fraction (%)	Phase ϕ_j (degrees)	Amplitude coefficient
$\rho(770)K^+$	$38.83 \pm 5.31 \pm 2.61$	0 (fixed)	1 (fixed)
$K^*(892)\pi^+$	$21.64 \pm 3.21 \pm 1.14$	$161.7 \pm 8.6 \pm 2.2$	$0.747 \pm 0.080 \pm 0.031$
NR	$15.88 \pm 4.92 \pm 1.53$	$43.1 \pm 10.4 \pm 4.4$	$0.640 \pm 0.118 \pm 0.026$
$K^*(1410)\pi^+$	$18.82 \pm 4.03 \pm 1.22$	$-34.8 \pm 12.1 \pm 4.3$	$0.696 \pm 0.097 \pm 0.025$
$K_0^*(1430)\pi^+$	$7.65 \pm 5.0 \pm 1.70$	$59.3 \pm 19.5 \pm 13.2$	$0.444 \pm 0.141 \pm 0.060$
$\rho(1450)K^+$	$10.62 \pm 3.51 \pm 1.04$	$-151.7 \pm 11.1 \pm 4.4$	$0.523 \pm 0.091 \pm 0.020$
C.L. = 5.5%	$\chi^2 = 38.5$	d.o.f. = 43 (#bins) - 17 (#free parameters)	

$$\frac{\Gamma(D_s^+ \to K^+\pi^+\pi^-)}{\Gamma(D_s^+ \to K^+K^-\pi^+)} = 0.127 \pm 0.007 \pm 0.014\,.$$

Medina Ablikim et al. [BESIII Collaboration], JHEP 08, 196 (2022).

Amplitude	Phase ϕ_n (rad)	FF(%)	Statistical significance(σ)
$D_s^+ \to K^+\rho^0$	0.0 (fixed)	$32.5 \pm 3.1 \pm 3.6$	>10
$D_s^+ \to K^+\rho(1450)^0$	$2.72 \pm 0.14 \pm 0.24$	$12.7 \pm 3.2 \pm 2.7$	>10
$D_s^+ \to K^+f_0(500)$	$0.98 \pm 0.17 \pm 0.19$	$7.0 \pm 2.2 \pm 4.0$	6.8
$D_s^+ \to K^+f_0(980)$	$5.02 \pm 0.15 \pm 0.15$	$4.4 \pm 1.3 \pm 1.1$	6.9
$D_s^+ \to K^+f_0(1370)$	$6.03 \pm 0.14 \pm 0.26$	$19.9 \pm 3.1 \pm 2.9$	>10
$D_s^+ \to K^*(892)^0\pi^+$	$3.03 \pm 0.09 \pm 0.04$	$30.3 \pm 1.9 \pm 1.8$	>10
$D_s^+ \to K^*(1410)^0\pi^+$	$5.62 \pm 0.14 \pm 0.09$	$4.7 \pm 2.2 \pm 2.1$	5.2
$D_s^+ \to K_0^*(1430)^0\pi^+$	$1.89 \pm 0.19 \pm 0.18$	$18.9 \pm 2.5 \pm 2.4$	8.6

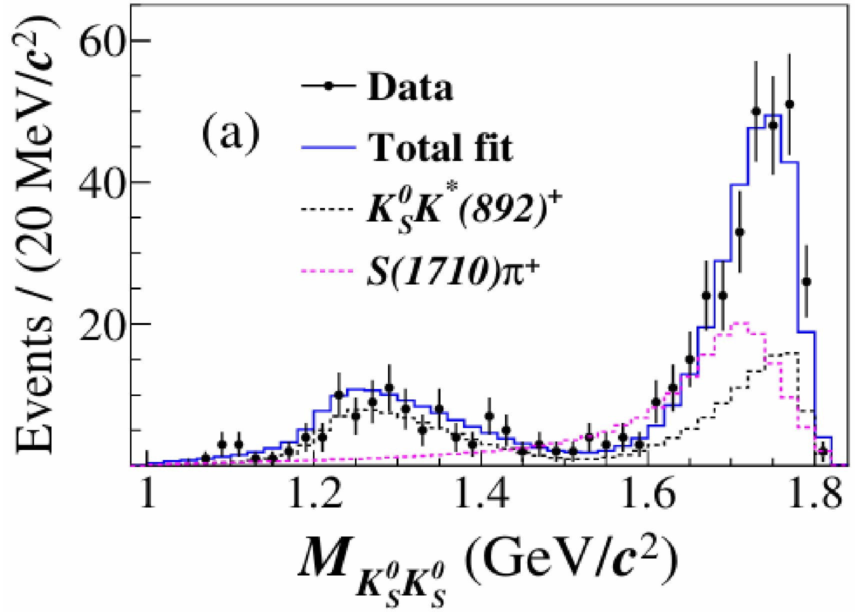
Intermediate process	BF(10^{-3})	PDG(10^{-3})
$D_s^+ \to K^+\rho^0$	$1.99 \pm 0.20 \pm 0.22$	2.5 ± 0.4
$D_s^+ \to K^+\rho(1450)^0$	$0.78 \pm 0.20 \pm 0.17$	0.69 ± 0.64
$D_s^+ \to K^*(892)^0\pi^+$	$1.85 \pm 0.13 \pm 0.11$	1.41 ± 0.24
$D_s^+ \to K^*(1410)^0\pi^+$	$0.29 \pm 0.13 \pm 0.13$	1.23 ± 0.28
$D_s^+ \to K_0^*(1430)^0\pi^+$	$1.15 \pm 0.16 \pm 0.15$	0.50 ± 0.35
$D_s^+ \to K^+f_0(500)$	$0.43 \pm 0.14 \pm 0.24$	-
$D_s^+ \to K^+f_0(980)$	$0.27 \pm 0.08 \pm 0.07$	-
$D_s^+ \to K^+f_0(1370)$	$1.22 \pm 0.19 \pm 0.18$	-
$D_s^+ \to (K^+\pi^+\pi^-)_{NR}$	-	1.03 ± 0.34

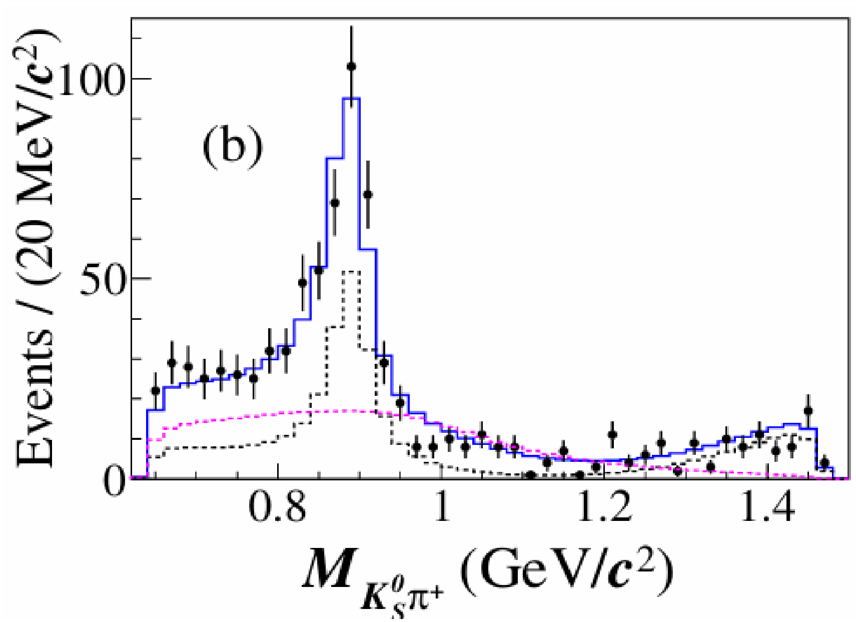
$$\mathcal{B}(D_s^+ \to K^+\pi^+\pi^-) = (6.11 \pm 0.18_{\text{stat.}} \pm 0.11_{\text{syst.}}) \times 10^{-3}$$

Background and Motivations.

- $D_s^+ \to K_s^0 K_s^0 \pi^+$:

M. Ablikim et al. [BESIII], Phys. Rev. D 105, L051103 (2022).





Amplitude	BF (10^{-3})
$D_s^+ \to K_S^0 K^*(892)^+ \to K_S^0 K_S^0 \pi^+$	$3.0 \pm 0.3 \pm 0.1$
$D_s^+ \to S(1710)\pi^+ \to K_S^0 K_S^0 \pi^+$	$3.1 \pm 0.3 \pm 0.1$

$$M_{S(1710)} = (1.723 \pm 0.011_{\text{stat}} \pm 0.002_{syst})GeV/c^2$$

$$\Gamma_{S(1710)} = (0.140 \pm 0.014_{\text{stat}} \pm 0.004_{syst})GeV/c^2$$

Formalism

Formalism

- The processes of three-body decay:

Feynman diagrams
- quark level

→ **hadronize**
- hadron level

→ **S-wave**
- propagators, two-body scattering amplitudes(Bethe-Salpeter equation)

→ **other resonances**
- relativistic amplitude

→ **differential width distribution**
- fitting experimental data

→ **branching fractions**

Propagators.

- The diagonal matrix G is two intermediate meson propagators:

$$G_{ii}(s) = i\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m_1^2+i\varepsilon}\frac{1}{(p_1+p_2-q)^2-m_2^2+i\varepsilon}.$$

- The integral is logarithmically divergent, there are two methods to solve this problem:

✓ the three-momentum cut off:

$$G_{ii}(s) = \int_0^{q_{\max}}\frac{q^2dq}{(2\pi)^2}\frac{\omega_1+\omega_2}{\omega_1\omega_2\left[s-(\omega_1+\omega_2)^2+i\varepsilon\right]}$$

$$\omega_i = \sqrt{\left(\vec{q}^{\,2}+m_i^2\right)} \qquad s=\left(p_1+p_2\right)^2$$

✓ the dimensional regularization method:

$$\begin{aligned}G_{ii}(s) = {} & \frac{1}{16\pi^2}\{a_\mu + \ln\frac{m_1^2}{\mu^2} + \frac{m_2^2-m_1^2+s}{2s}\ln\frac{m_2^2}{m_1^2} \\ & +\frac{q_{cm}(s)}{\sqrt{s}}[\ln\left(s-\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right)+\ln\left(s+\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right) \\ & -\ln\left(-s-\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right)-\ln\left(-s+\left(m_2^2-m_1^2\right)+2q_{cm}(s)\sqrt{s}\right)]\}\end{aligned}$$

- The value of the subtraction constant :

✓ a relationship between two regularization method :

$$a_\mu = 16\pi^2[G^{CO}(s_{thr}, q_{max}) - G^{DR}(s_{thr}, \mu)],$$

G. Montaña et al., Phys. Rev. D 107, 054014 (2023).

✓ a calculation which adopted by other references :

$$a_{PP'}(\mu) = -2\log\left(1+\sqrt{1+\frac{m_1^2}{\mu^2}}\right)+\cdots,$$

J. A. Oller et al., Phys. Lett. B 500, 263-272 (2001).

Propagators.

- The value of the parameter for pseudoscalar-pseudoscalar interaction:

➢ $\mu = 0.6$ GeV

$$a_{\pi^+K^-} = -1.57,\ a_{\pi^0\bar{K}^0} = -1.57,\ a_{\eta\bar{K}^0} = -1.66$$

$$a_{\pi^+\pi^-} = -1.30,\ a_{\pi^0\pi^0} = -1.29,\ a_{K^+K^-} = -1.63,\ a_{K^0\bar{K}^0} = -1.63,\ a_{\eta\eta} = -1.68$$

➢ $\mu = 0.6$ GeV

$$a_{\pi^+K^-} = -1.66,\ a_{\pi^0\bar{K}^0} = -1.66,\ a_{\eta\bar{K}^0} = -1.71$$

$$a_{\pi^+\pi^-} = -1.41,\ a_{\pi^0\pi^0} = -1.41,\ a_{K^+K^-} = -1.66,\ a_{K^0\bar{K}^0} = -1.66,\ a_{\eta\eta} = -1.71$$

Gloria Montaña, Angels Ramos, Laura Tolos, Juan M. Torres-Rincon, Arxiv: 2211.01896 (2022).

M. Y. Duan, J. Y. Wang, G. Y. Wang, E. Wang, and D. M. Li, Eur. Phys. J. C 80, 1041 (2020).

Wang, Zhong-Yu, Yi, Jing-Yu, Sun, Zhi-Feng and Xiao, C. W, Phys Rev D.105.016025 (2021).

Two-body scattering amplitudes.

- T is the two-body scattering amplitudes, it can be evaluated by the coupled channel Bethe-Salpeter equation of ChUA:

$$T = [1 - VG]^{-1} V,$$

- The interaction potentials of each coupled channel for PP→PP processes:

➢ I = 0: $\pi^+\pi^-$, $\pi^0\pi^0$, K^+K^-, $K^0\overline{K}^0$, $\eta\eta$

$$V_{11} = -\frac{1}{2f^2}s, \quad V_{12} = -\frac{1}{\sqrt{2}f^2}(s - m_\pi^2), \quad V_{13} = -\frac{1}{4f^2}s,$$

$$V_{14} = -\frac{1}{4f^2}s, \quad V_{15} = -\frac{1}{3\sqrt{2}f^2}m_\pi^2, \quad V_{22} = -\frac{1}{2f^2}m_\pi^2,$$

$$V_{23} = -\frac{1}{4\sqrt{2}f^2}s, \quad V_{24} = -\frac{1}{4\sqrt{2}f^2}s, \quad V_{25} = -\frac{1}{6f^2}m_\pi^2,$$

$$V_{33} = -\frac{1}{2f^2}s, \quad V_{34} = -\frac{1}{4f^2}s,$$

$$V_{35} = -\frac{1}{12\sqrt{2}f^2}(9s - 6m_\eta^2 - 2m_\pi^2), \quad V_{44} = -\frac{1}{2f^2}s,$$

$$V_{45} = -\frac{1}{12\sqrt{2}f^2}(9s - 6m_\eta^2 - 2m_\pi^2),$$

$$V_{55} = -\frac{1}{18f^2}(16m_K^2 - 7m_\pi^2),$$

➢ I = 1/2 : $K^+\pi^-$, $K^0\pi^0$, $K^0\eta$

$$V_{11} = \frac{-1}{6f^2}\left(\frac{3}{2}s - \frac{3}{2s}\left(m_\pi^2 - m_K^2\right)^2\right)$$

$$V_{12} = \frac{1}{2\sqrt{2}f^2}\left(\frac{3}{2}s - m_\pi^2 - m_K^2 - \frac{\left(m_\pi^2 - m_K^2\right)^2}{2s}\right),$$

$$V_{13} = \frac{1}{2\sqrt{6}f^2}\left(\frac{3}{2}s - \frac{7}{6}m_\pi^2 - \frac{1}{2}m_\eta^2 - \frac{1}{3}m_K^2 + \frac{3}{2s}\left(m_\pi^2 - m_K^2\right)\left(m_\eta^2 - m_K^2\right)\right),$$

$$V_{22} = \frac{-1}{4f^2}\left(-\frac{s}{2} + m_\pi^2 + m_K^2 - \frac{\left(m_\pi^2 - m_K^2\right)^2}{2s}\right)$$

$$V_{23} = -\frac{1}{4\sqrt{3}f^2}\left(\frac{3}{2}s - \frac{7}{6}m_\pi^2 - \frac{1}{2}m_\eta^2 - \frac{1}{3}m_K^2 + \frac{3}{2s}\left(m_\pi^2 - m_K^2\right)\left(m_\eta^2 - m_K^2\right)\right)$$

$$V_{33} = -\frac{1}{4f^2}\left(-\frac{3}{2}s - \frac{2}{3}m_\pi^2 + m_\eta^2 + 3m_K^2 - \frac{3}{2s}\left(m_\eta^2 - m_K^2\right)^2\right)$$

J. A. Oller and E. Oset, Nucl. Phys. A 620, 438-456 (1997).

Two-body scattering amplitudes.

- The interaction potentials for VV→VV processes(Tree-level transition amplitudes of the four-vector-contact diagrams and of the t(u)-channel vector-exchange diagrams):

➢ I = 0: $K^*\bar{K}^*$, $\rho\rho$, $\omega\omega$, $\omega\phi$, $\phi\phi$

	$K^*\bar{K}^*$	$\rho\rho$	$\omega\omega$	$\omega\phi$	$\phi\phi$
$K^*\bar{K}^*$	$6g^2$	$2\sqrt{3}g^2$	$-2g^2$	$4g^2$	$-4g^2$
$\rho\rho$		$8g^2$	0	0	0
$\omega\omega$			0	0	0
$\omega\phi$				0	0
$\phi\phi$					0

➢ I = 1 : $K^*\bar{K}^*$, $\rho\rho$, $\rho\omega$, $\rho\phi$

	$K^*\bar{K}^*$	$\rho\rho$	$\rho\omega$	$\rho\phi$
$K^*\bar{K}^*$	$2g^2$	0	$-2\sqrt{2}g^2$	$4g^2$
$\rho\rho$		0	0	0
$\rho\omega$			0	0
$\rho\phi$				0

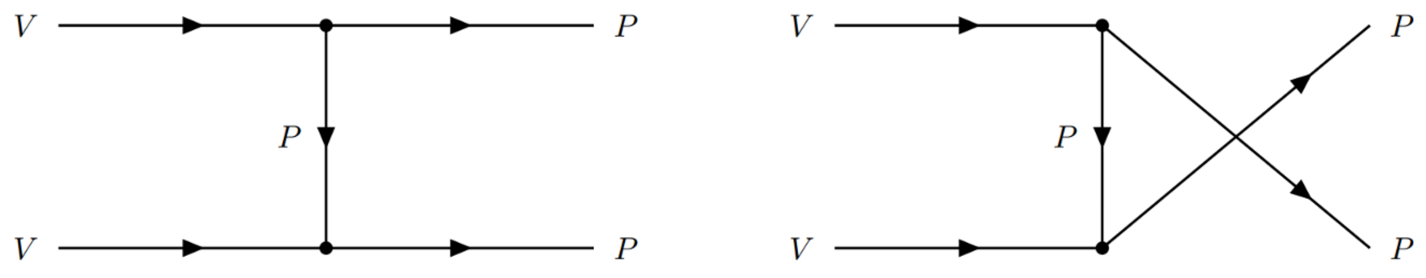
	$K^*\bar{K}^*$	$\rho\rho$	$\omega\omega$	$\omega\phi$	$\phi\phi$
$K^*\bar{K}^*$	$\frac{g^2\left(M_\rho^2M_\phi^2+\left(2M_\rho^2+3M_\phi^2\right)M_\omega^2\right)\left(4M_{K^*}^2-3s\right)}{4M_\rho^2M_\phi^2M_\omega^2}$	$\frac{\sqrt{3}g^2\left(2M_\rho^2+2M_{K^*}^2-3s\right)}{2M_{K^*}^2}$	$-\frac{g^2\left(2M_\omega^2+2M_{K^*}^2-3s\right)}{2M_{K^*}^2}$	$\frac{g^2\left(M_\phi^2+M_\omega^2+2M_{K^*}^2-3s\right)}{M_{K^*}^2}$	$\frac{g^2\left(-2M_\phi^2-2M_{K^*}^2+3s\right)}{M_{K^*}^2}$
$\rho\rho$		$2g^2\left(4-\frac{3s}{M_\rho^2}\right)$	0	0	0
$\omega\omega$			0	0	0
$\omega\phi$				0	0
$\phi\phi$					0

	$K^*\bar{K}^*$	$\rho\rho$	$\rho\omega$	$\rho\phi$
$K^*\bar{K}^*$	$\frac{g^2\left(M_\rho^2M_\phi^2-\left(M_\phi^2-2M_\rho^2\right)M_\omega^2\right)\left(4M_{K^*}^2-3s\right)}{4M_\rho^2M_\phi^2M_\omega^2}$	0	$-\frac{g^2\left(M_\rho^2+M_\omega^2+2M_{K^*}^2-3s\right)}{\sqrt{2}M_{K^*}^2}$	$\frac{g^2\left(M_\rho^2+M_\phi^2+2M_{K^*}^2-3s\right)}{M_{K^*}^2}$
$\rho\rho$		0	0	0
$\rho\omega$			0	0
$\rho\phi$				0

L. S. Geng and E. Oset, Phys. Rev. D 79, 074009 (2009).

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- The interaction potentials of each coupled channel for VV→PP processes(The t(u)-channel pseudoscalar-exchange diagrams):



(a) The t-channel　　(b) The u- channel

$$\mathcal{L}_{VPP} = -ig \left\langle V_\mu \left[P, \partial^\mu P\right]\right\rangle$$

$$g = M_V/(2f_\pi) \qquad M_V = 0.84566\,\text{GeV}$$

$$f_\pi = 0.093$$

$$V_{K^{*+}K^{*-}\to K^0\bar{K}^0} = -\frac{4}{t-m_\pi^2} g^2 \epsilon_{1\mu} k_3^\mu \epsilon_{2\nu} k_4^\nu,$$

$$V_{K^{*0}\bar{K}^{*0}\to K^0\bar{K}^0} = -2\left(\frac{3}{t-m_\eta^2} + \frac{1}{t-m_\pi^2}\right) g^2 \epsilon_{1\mu} k_3^\mu \epsilon_{2\nu} k_4^\nu,$$

$$V_{\phi\phi\to K^0\bar{K}^0} = -4g^2\left(\frac{1}{t-m_K^2}\epsilon_{1\mu} k_3^\mu \epsilon_{2\nu} k_4^\nu + \frac{1}{u-m_K^2}\epsilon_{1\mu} k_4^\mu \epsilon_{2\nu} k_3^\nu\right),$$

$$V_{\omega\phi\to K^0\bar{K}^0} = 2\sqrt{2}g^2\left(\frac{1}{t-m_K^2}\epsilon_{1\mu} k_3^\mu \epsilon_{2\nu} k_4^\nu + \frac{1}{u-m_K^2}\epsilon_{1\mu} k_4^\mu \epsilon_{2\nu} k_3^\nu\right),$$

$$V_{\rho\phi\to K^0\bar{K}^0} = -2\sqrt{2}\left(\frac{1}{t-m_K^2}g^2\epsilon_{1\mu} k_3^\mu \epsilon_{2\nu} k_4^\nu + \frac{1}{u-m_K^2}g^2\epsilon_{1\mu} k_4^\mu \epsilon_{2\nu} k_3^\nu\right)$$

- The form factor for each VPP vertex of the exchanged pseudoscalar meson:

$$F = \frac{\Lambda^2 - m_{ex}^2}{\Lambda^2 - q^2},$$

Z. L. Wang and B. S. Zou, Eur. Phys. J. C 82, 509 (2022).
M. Bando et al., Phys. Rept. 164, 217-314 (1988)

The decay of $D_s^+ \to \pi^+\pi^-K^+$

$D_s^+ \to \pi^+\pi^- K^+$

- The external and internal W-emission mechanism:

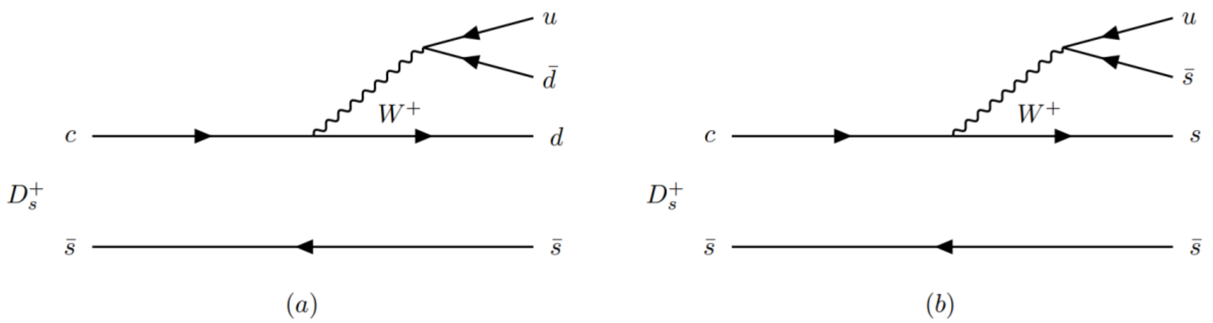


FIG. 1: W-external emission mechanism for the $D_s^+ \to K^+\pi^+\pi^-$ decay.

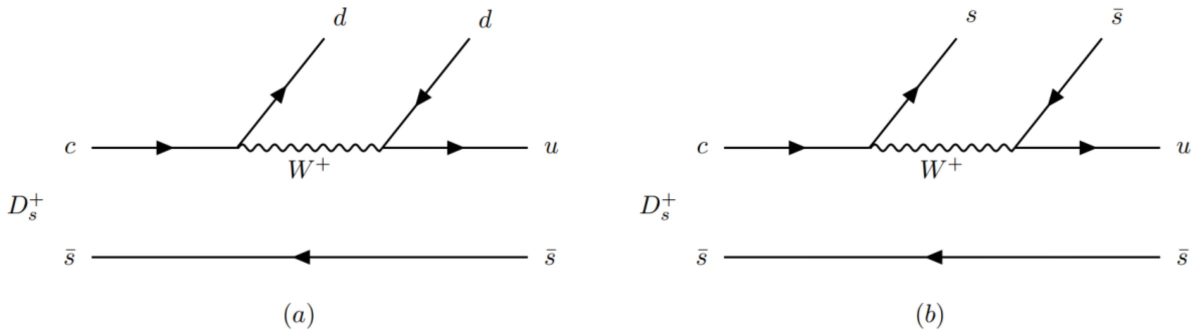


FIG. 2: W-internal emission mechanism for the $D_s^+ \to K^+\pi^+\pi^-$ decay.

$$\begin{aligned}H^{(1a)} =& V_P V_{cd} V_{ud} \left\{ \left(u\bar{d} \to \pi^+\right) \left[d\bar{s} \to d\bar{s} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right. \\ & \left. + \left(d\bar{s} \to K^0\right) \left[u\bar{d} \to u\bar{d} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right\} \\ =& V_P V_{cd} V_{ud} \left\{ \left(u\bar{d} \to \pi^+\right) \left[M_{23} \to (M \cdot M)_{23}\right] \right. \\ & \left. + \left(d\bar{s} \to K^0\right) \left[M_{12} \to (M \cdot M)_{12}\right] \right\},\end{aligned}$$

$$\begin{aligned}H^{(1b)} =& V'_P V_{cs} V_{us} \left\{ \left(u\bar{s} \to K^+\right) \left[\bar{s}\bar{s} \to \bar{s}\bar{s} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right. \\ & \left. + \left(\bar{s}\bar{s} \to -\frac{2}{\sqrt{6}}\eta\right) \left[u\bar{s} \to u\bar{s} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right\} \\ =& V'_P V_{cs} V_{us} \left\{ \left(u\bar{s} \to K^+\right) \left[M_{33} \to (M \cdot M)_{33}\right] \right. \\ & \left. + \left(\bar{s}\bar{s} \to -\frac{2}{\sqrt{6}}\eta\right) \left[M_{13} \to (M \cdot M)_{13}\right] \right\}.\end{aligned}$$

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}|(u\bar{u} - d\bar{d})\rangle, \quad |\eta\rangle = \frac{1}{\sqrt{6}}|(u\bar{u} + d\bar{d} - 2s\bar{s})\rangle.$$

$$\begin{aligned}H^{(2a)} =& \beta V_P V_{cd} V_{ud} \left\{ \left(d\bar{d} \to -\frac{1}{\sqrt{2}}\pi^0\right) \left[u\bar{s} \to u\bar{s} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right. \\ & + \left(d\bar{d} \to \frac{1}{\sqrt{6}}\eta\right) \left[u\bar{s} \to u\bar{s} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \\ & \left. + \left(u\bar{s} \to K^+\right) \left[d\bar{d} \to d\bar{d} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right\} \\ =& \beta V_P V_{cd} V_{ud} \left\{ \left(d\bar{d} \to -\frac{1}{\sqrt{2}}\pi^0\right) \left[M_{13} \to (M \cdot M)_{13}\right] \right. \\ & \left. + \left(d\bar{d} \to \frac{1}{\sqrt{6}}\eta\right) \left[M_{13} \to (M \cdot M)_{13}\right] + \left(u\bar{s} \to K^+\right) \left[M_{22} \to (M \cdot M)_{22}\right] \right\},\end{aligned}$$

$$\begin{aligned}H^{(2b)} =& \beta V'_P V_{cs} V_{us} \left\{ \left(s\bar{s} \to -\frac{2}{\sqrt{6}}\eta\right) \left[u\bar{s} \to u\bar{s} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right. \\ & \left. + \left(u\bar{s} \to K^+\right) \left[s\bar{s} \to s\bar{s} \cdot \left(u\bar{u} + d\bar{d} + s\bar{s}\right)\right] \right\} \\ =& \beta V'_P V_{cs} V_{us} \left\{ \left(s\bar{s} \to -\frac{2}{\sqrt{6}}\eta\right) \left[M_{13} \to (M \cdot M)_{13}\right] \right. \\ & \left. + \left(u\bar{s} \to K^+\right) \left[M_{33} \to (M \cdot M)_{33}\right] \right\},\end{aligned}$$

$D_s^+ \to \pi^+\pi^-K^+$

- The SU(3) matrix M in quark and hadron level:

$$M = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}.$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix},$$

- The hadonizations at the hadron level:

$$(M\cdot M)_{12} = \frac{2}{\sqrt{6}}\pi^+\eta + K^+\bar{K}^0,$$

$$(M\cdot M)_{13} = \frac{1}{\sqrt{2}}\pi^0K^+ + \pi^+K^0 - \frac{1}{\sqrt{6}}\eta K^+,$$

$$(M\cdot M)_{22} = \pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 + \frac{1}{6}\eta\eta - \frac{1}{\sqrt{3}}\pi^0\eta + K^0\bar{K}^0,$$

$$(M\cdot M)_{23} = \pi^-K^+ - \frac{1}{\sqrt{2}}\pi^0K^0 - \frac{1}{\sqrt{6}}K^0\eta,$$

$$(M\cdot M)_{33} = K^+K^- + K^0\bar{K}^0 + \frac{2}{3}\eta\eta.$$

- The contributions for different mechanism:

$$H^{(1a)} = V_P V_{cd} V_{ud}\left(\pi^+\pi^-K^+ - \frac{1}{\sqrt{2}}\pi^+\pi^0K^0 + \frac{1}{\sqrt{6}}\pi^+K^0\eta + K^+K^0\bar{K}^0\right),$$

$$H^{(1b)} = V_P' V_{cs} V_{us}\left(K^+K^+K^- + K^+K^0\bar{K}^0 + K^+\eta\eta - \frac{1}{\sqrt{3}}\pi^0K^+\eta - \frac{2}{\sqrt{6}}\pi^+K^0\eta\right),$$

$$H^{(2a)} = \beta \times V_P V_{cd} V_{ud}\left(\pi^+\pi^-K^+ + K^+K^0\bar{K}^0 - \frac{1}{\sqrt{2}}\pi^+\pi^0K^0 + \frac{1}{\sqrt{6}}\eta\pi^+K^0\right),$$

$$H^{(2b)} = \beta \times V_P' V_{cs} V_{us}\left(K^+K^+K^- + K^+K^0\bar{K}^0 + \eta\eta K^+ - \frac{1}{\sqrt{3}}\eta\pi^0K^+ - \frac{2}{\sqrt{6}}\eta\pi^+K^0\right).$$

- The relationship of the CKM matrix elements :

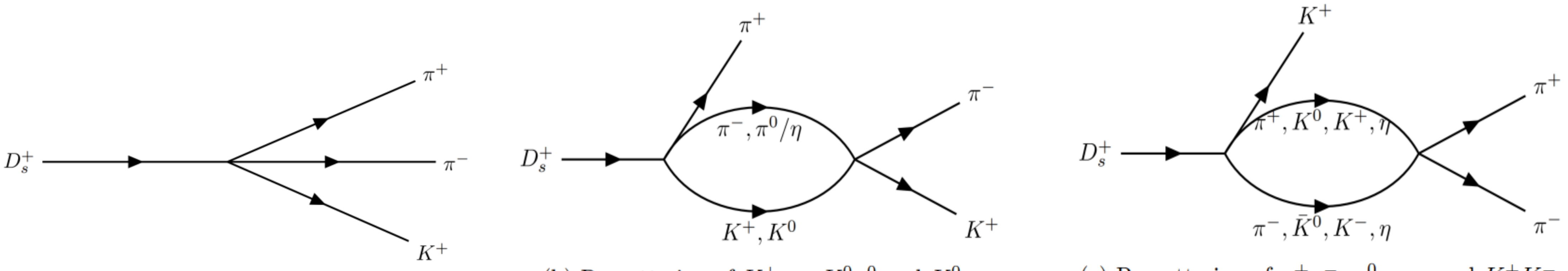
$$V_{cd}V_{ud} = -V_{us}V_{cs}$$

$D_s^+ \to \pi^+\pi^- K^+$

- The total contributions for the decay $D_s^+ \to K^+\pi^+\pi^-$:

$$
\begin{aligned}
H =& H^{(a)} + H^{(b)} + H^{(2a)} + H^{(2b)} \\
=& V_{cd}V_{ud}(1+\beta)\Big[V_P\big(\pi^+\pi^-K^+ - \frac{1}{\sqrt{2}}\pi^+\pi^0K^0 + \frac{1}{\sqrt{6}}\eta\pi^+K^0 + K^+K^0\bar{K}^0\big) \\
&+ V_P'\big(-K^+K^+K^- - \eta\eta K^+ + \frac{2}{\sqrt{6}}\eta\pi^+K^0 - K^+K^0\bar{K}^0 + \frac{1}{\sqrt{3}}\eta\pi^0K^+\big)\Big] \\
=& C_1\big(\pi^+\pi^-K^+ - \frac{1}{\sqrt{2}}\pi^+\pi^0K^0 + \frac{1}{\sqrt{6}}\eta\pi^+K^0 + K^+K^0\bar{K}^0\big) \\
&- C_2\big(K^+K^+K^- + \eta\eta K^+ - \frac{2}{\sqrt{6}}\eta\pi^+K^0 + K^+K^0\bar{K}^0\big).
\end{aligned}
$$

- Tree-level production and final state interactions via rescattering mechanism:



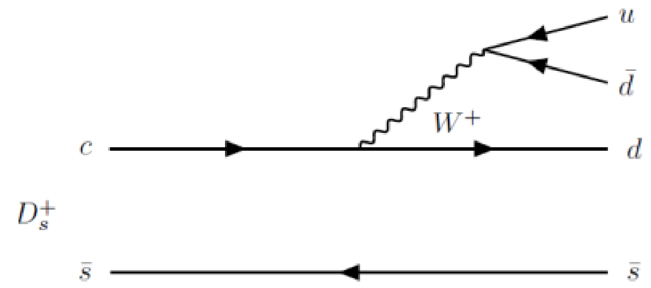
(a) Tree-level production. (b) Rescattering of $K^+\pi^-$, $K^0\pi^0$ and $K^0\eta$. (c) Rescattering of $\pi^+\pi^-$, $\pi^0\eta$, $\eta\eta$ and K^+K^-.

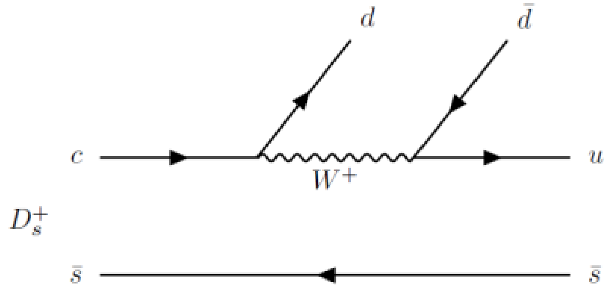
$D_s^+ \to \pi^+\pi^- K^+$

- The amplitudes for the decay $D_s^+ \to K^+\pi^+\pi^-$ in the S-wave:

$$
\begin{aligned}
t(s_{12}, s_{23}) = & C_1 \Big[1 + G_{\pi^- K^+}(s_{23}) T_{\pi^- K^+ \to \pi^- K^+}(s_{23}) + G_{\pi^+\pi^-}(s_{12}) T_{\pi^+\pi^- \to \pi^+\pi^-}(s_{12}) \\
& - \frac{1}{\sqrt{2}} G_{\pi^0 K^0}(s_{23}) T_{\pi^0 K^0 \to \pi^- K^+}(s_{23}) + \frac{1}{\sqrt{6}} G_{\eta K^0}(s_{23}) T_{\eta K^0 \to \pi^- K^+}(s_{23}) \\
& + G_{K^0 \bar{K}^0}(s_{12}) T_{K^0 \bar{K}^0 \to \pi^+\pi^-}(s_{12}) \Big] - C_2 \Big[G_{K^+K^-}(s_{12}) T_{K^+K^- \to \pi^+\pi^-}(s_{12}) \\
& + G_{\eta\eta}(s_{12}) T_{\eta\eta \to \pi^+\pi^-}(s_{12}) - \frac{2}{\sqrt{6}} G_{\eta K^0}(s_{23}) T_{\eta K^0 \to \pi^- K^+}(s_{23}) \\
& + G_{K^0 \bar{K}^0}(s_{12}) T_{K^0 \bar{K}^0 \to \pi^+\pi^-}(s_{12}) \Big]
\end{aligned}
$$

- The contribution of other intermediate states:





$$M_{K^*(892)}(s_{12}, s_{23}) = \frac{D_{K^*(892)} e^{i\alpha_{K^*(892)}}}{s_{23} - m^2_{K^*(892)} + i m_{K^*(892)} \Gamma_{K^*(892)}} \left[(m_K^2 - m_\pi^2) \frac{m^2_{D_s^+} - m_\pi^2}{m^2_{K^*(892)}} - s_{13} + s_{12} \right],$$

$$M_{K^*(1430)}(s_{12}, s_{23}) = \frac{D_{K^*(1430)} e^{i\alpha_{K^*(1430)}}}{s_{23} - m^2_{K^*(1430)} + i m_{K^*(1430)} \Gamma_{K^*(1430)}} \left[(s_{23} - m_K^2 - m_\pi^2) \cdot (s_{13} + s_{12} - m_K^2 - m_\pi^2) \right],$$

$$M_\rho(s_{12}, s_{23}) = \frac{D_\rho e^{i\alpha_\rho}}{s_{12} - m_\rho^2 + i m_\rho \Gamma_\rho} (s_{23} - s_{13}),$$

$$M_{f_0(1370)}(s_{12}, s_{23}) = \frac{D_{f_0(1370)} e^{i\alpha_{f_0(1370)}}}{s_{12} - m^2_{f_0(1370)} + i m_{f_0(1370)} \Gamma_{f_0(1370)}} \left[(s_{12} - 2m_\pi^2) \cdot (s_{13} + s_{23} - 2m_\pi^2) \right],$$

$$s_{12} + s_{23} + s_{13} = m^2_{D_s^+} + m_K^2 + m_\pi^2 + m_\pi^2,$$

$D_s^+ \to \pi^+\pi^- K^+$

- The double differential width distribution of three-body decay:

$$\frac{d^2\Gamma}{ds_{12}ds_{23}} = \frac{1}{(2\pi)^3}\frac{1}{32m_{D_s^+}^3}\left(\left|t(s_{12},s_{23}) + M_{K^*(892)} + M_{K^*(1430)} + M_{f_0(1370)} + M_{\rho} + M_{\rho(1450)}\right|^2\right)$$

- The limits of integral variable for the invariant masses are higher than 1.2 GeV, we need to smoothly extrapolate $G(s)T(s)$ above the energy cut $\sqrt{s} \geq \sqrt{s_{cut}} = 1.1$ GeV :

$$G(s)T(s) = G(s_{\text{cut}})T(s_{\text{cut}})e^{-\alpha(\sqrt{s}-\sqrt{s_{\text{cut}}})}, \quad \text{for } \sqrt{s} > \sqrt{s_{\text{cut}}}$$

- The parameters need to be fitted:

S-wave: C_1, C_2, α

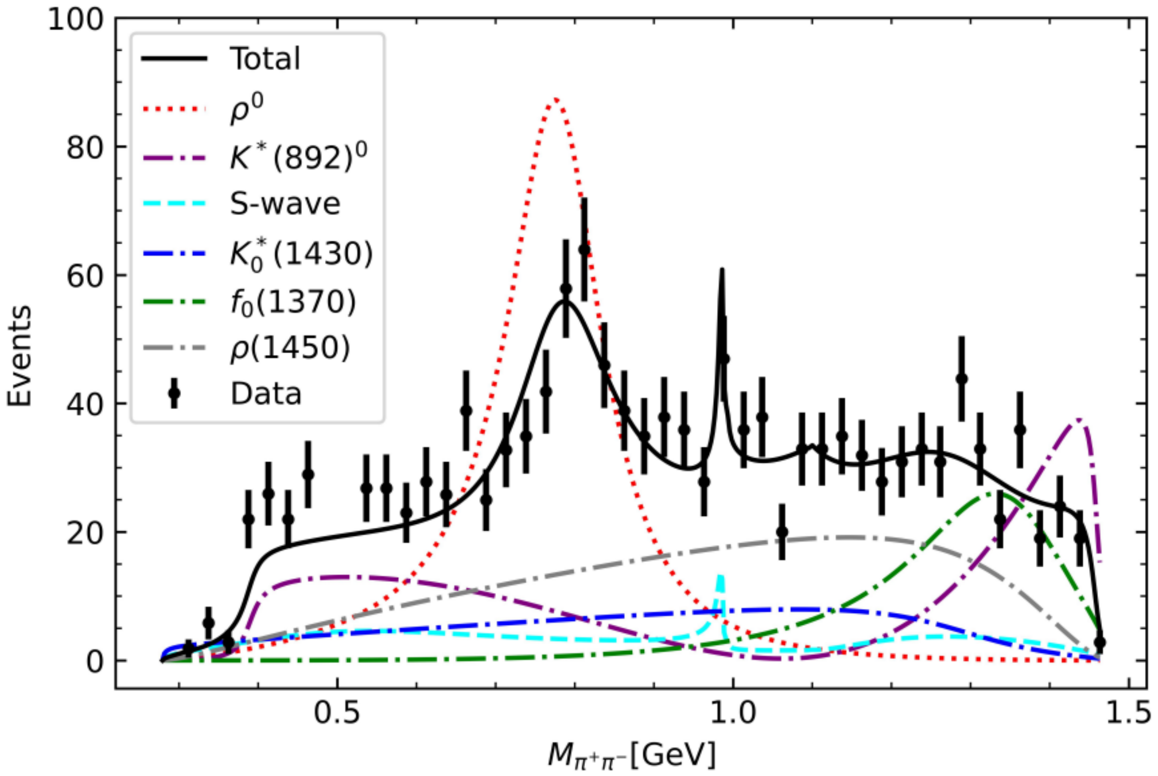
other resonances: $D_\rho, \alpha_\rho, D_{K^*(892)}, \alpha_{K^*(892)}, D_{K^*(1430)}, \alpha_{K^*(1430)}, D_{f_0(1370)}, \alpha_{f_0(1370)}, D_{\rho(1450)}, \alpha_{\rho(1450)},$

V. R. Debastiani, W. H. Liang, J. J. Xie, and E. Oset, Phys. Lett. B 766, 59 (2017).

$D_s^+ \to \pi^+\pi^-K^+$

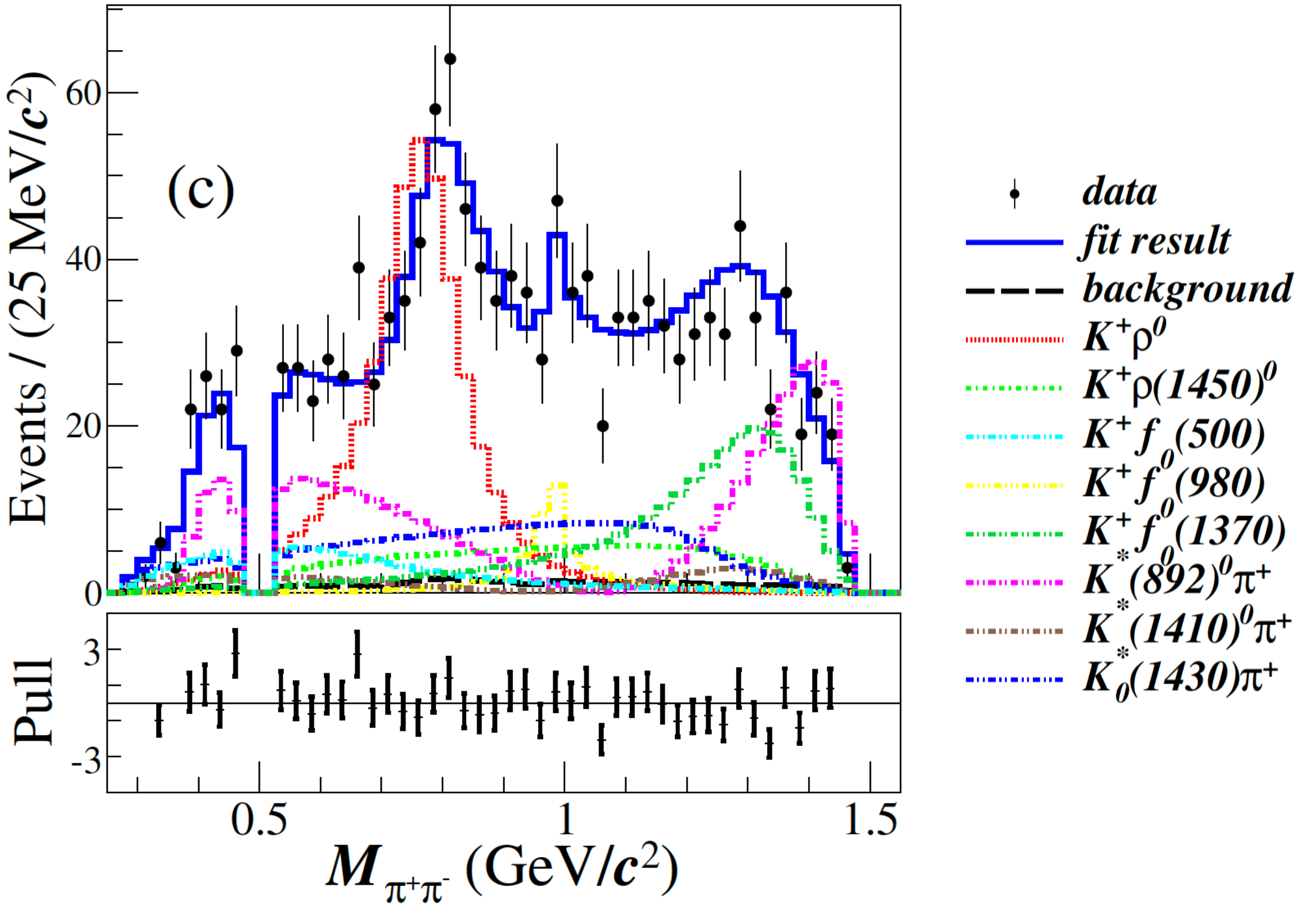
- $\boldsymbol{\pi^+\pi^-}$ $\quad \boldsymbol{\chi^2/dof = 183.37/128 = 1.43}$

Our Results:



Medina Ablikim et al. [BESIII Collaboration], JHEP 08, 196 (2022).

BESIII Experiment:



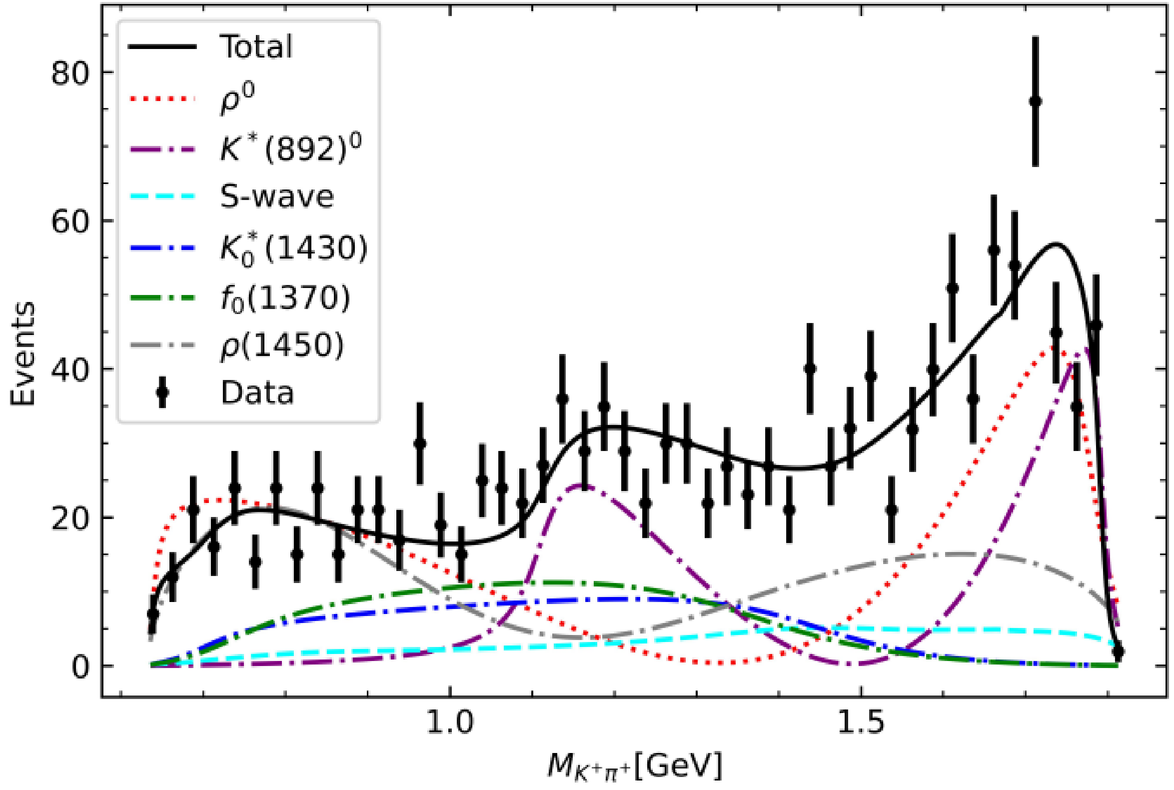
Parameters	C_1	C_2	α	D_ρ	α_ρ	$D_{K^*(892)}$	$\alpha_{K^*(892)}$	$D_{K^*(1430)}$	$\alpha_{K^*(1430)}$	$D_{f_0(1370)}$	$\alpha_{f_0(1370)}$	$D_{\rho(1450)}$	$\alpha_{\rho(1450)}$	$\chi^2/dof.$
Fit	263.74	-63.08	12.34	80.77	0.18	62.99	3.54	-62.50	1.21	-60.24	3.07	-456.70	0.93	1.43

$D_s^+ \to \pi^+\pi^- K^+$

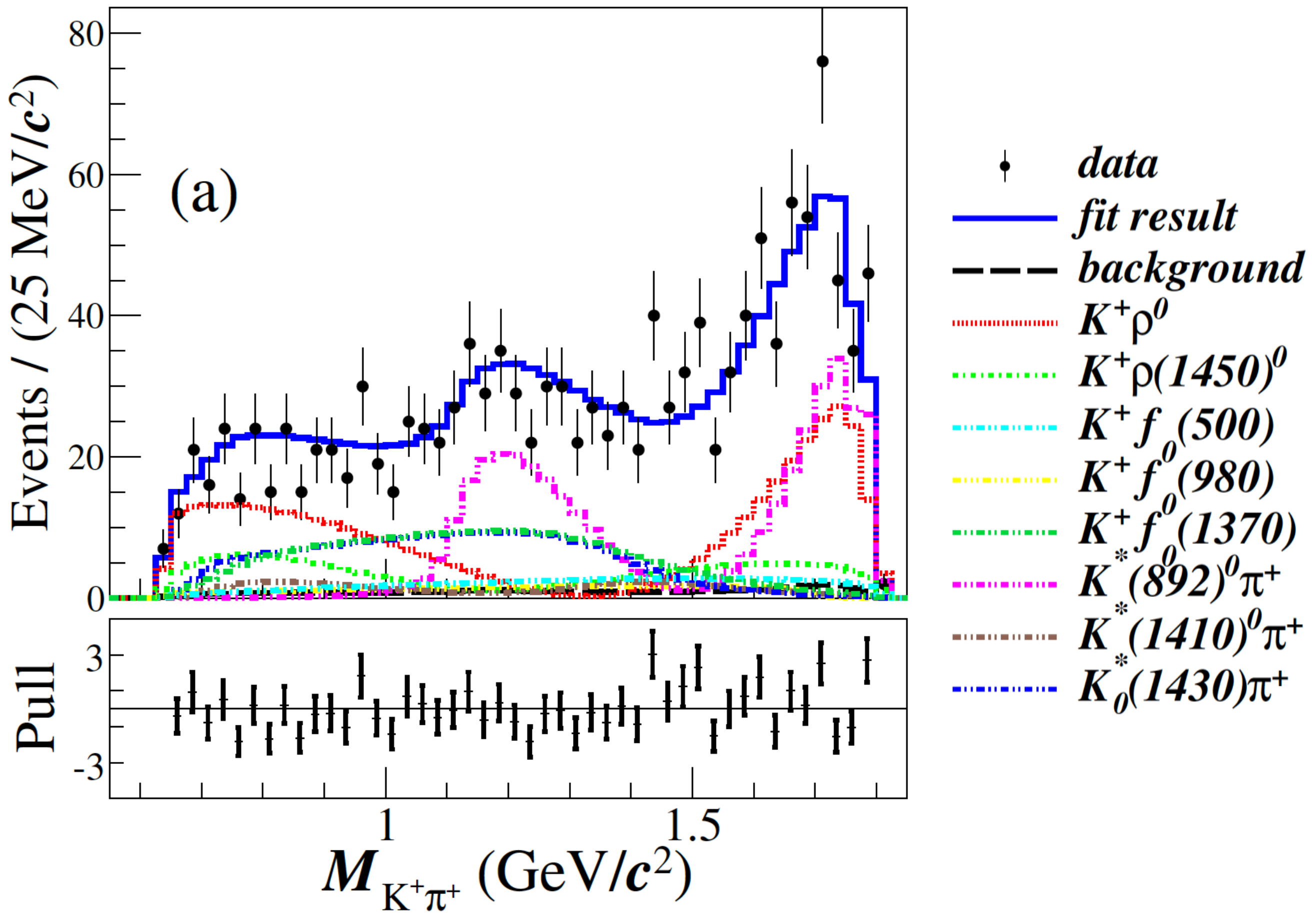
- $K^+\pi^+$ $\chi^2/dof = 183.37/128 = 1.43$

Medina Ablikim et al. [BESIII Collaboration], JHEP 08, 196 (2022).

Our Results:



BESIII Experiment:



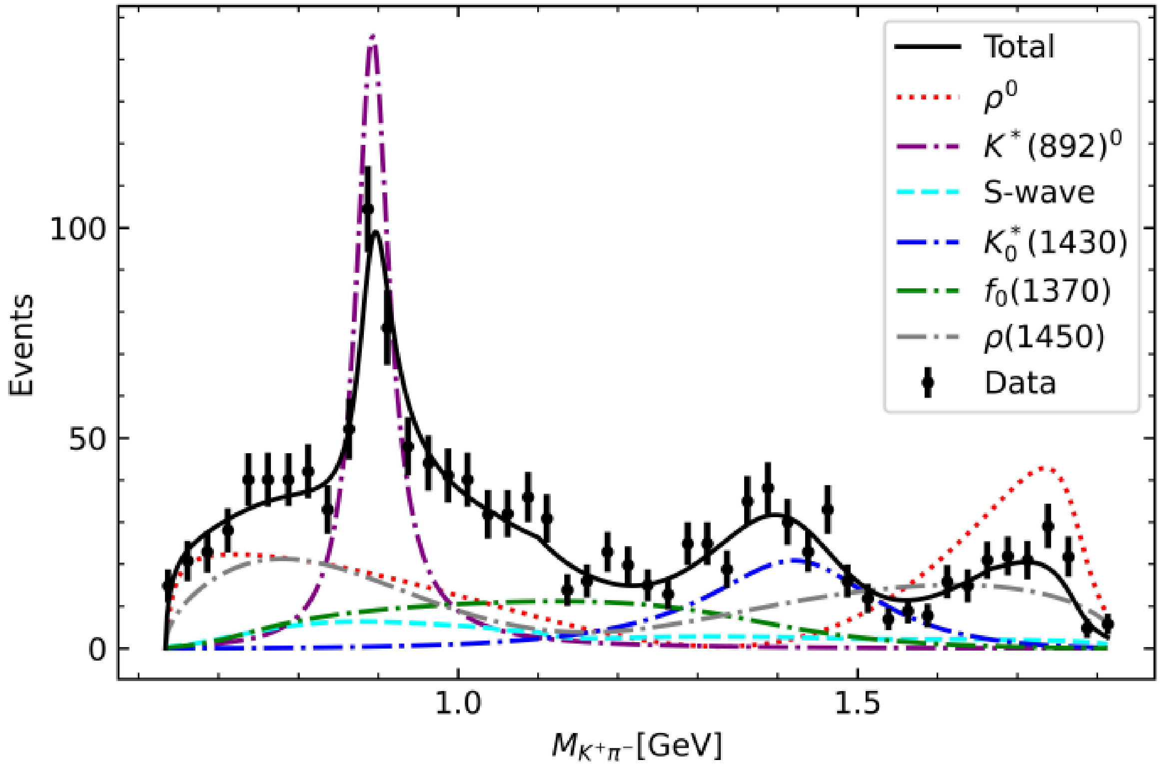
Parameters	C_1	C_2	α	D_ρ	α_ρ	$D_{K^*(892)}$	$\alpha_{K^*(892)}$	$D_{K^*(1430)}$	$\alpha_{K^*(1430)}$	$D_{f_0(1370)}$	$\alpha_{f_0(1370)}$	$D_{\rho(1450)}$	$\alpha_{\rho(1450)}$	$\chi^2/dof.$
Fit	263.74	-63.08	12.34	80.77	0.18	62.99	3.54	-62.50	1.21	-60.24	3.07	-456.70	0.93	1.43

$D_s^+ \to \pi^+\pi^-K^+$

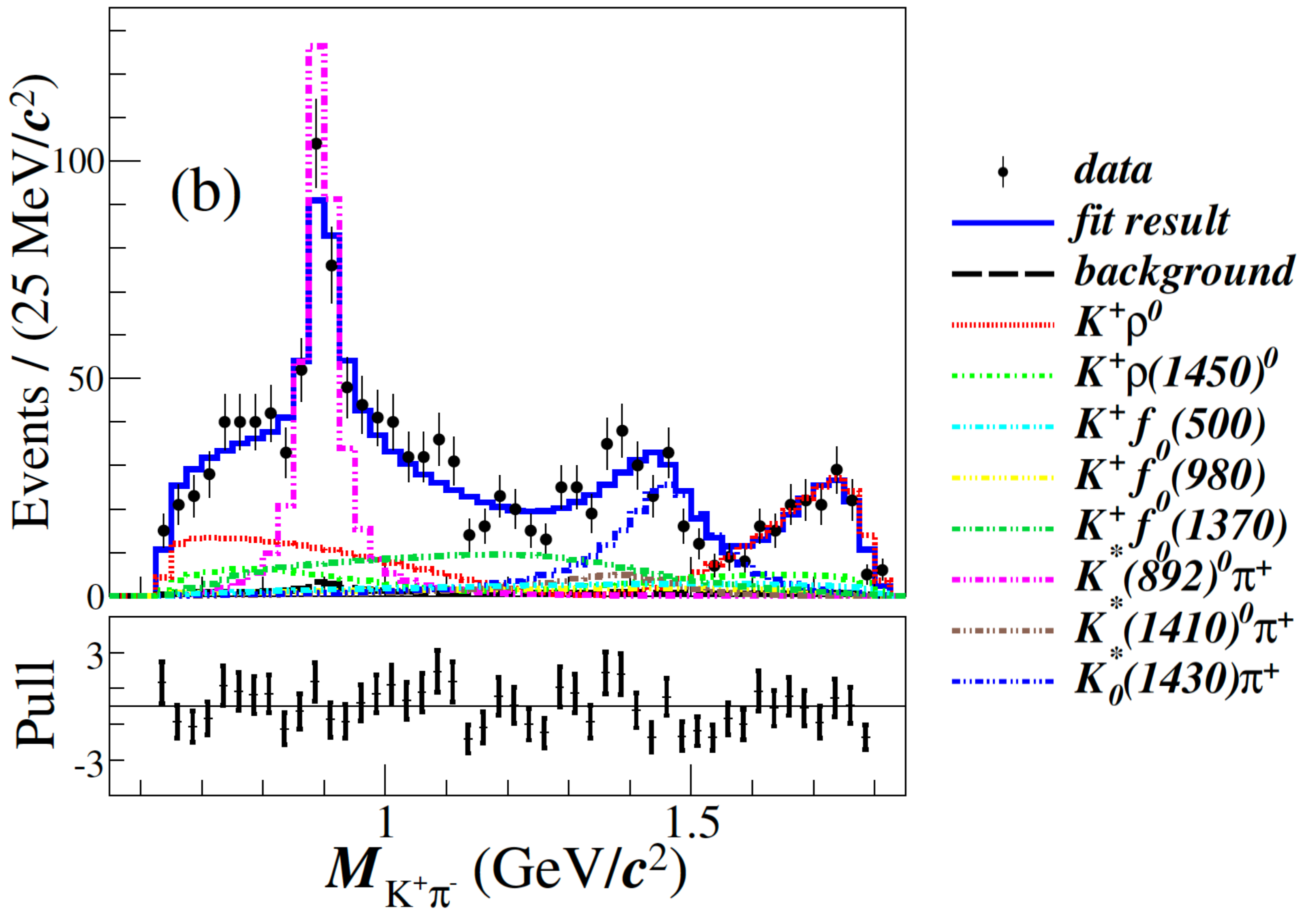
Medina Ablikim et al. [BESIII Collaboration], JHEP 08, 196 (2022).

- $K^+\pi^-$ $\quad \chi^2/dof = 183.37/128 = 1.43$

Our Results:



BESIII Experiment:



Parameters	C_1	C_2	α	D_ρ	α_ρ	$D_{K^*(892)}$	$\alpha_{K^*(892)}$	$D_{K^*(1430)}$	$\alpha_{K^*(1430)}$	$D_{f_0(1370)}$	$\alpha_{f_0(1370)}$	$D_{\rho(1450)}$	$\alpha_{\rho(1450)}$	$\chi^2/dof.$
Fit	263.74	-63.08	12.34	80.77	0.18	62.99	3.54	-62.50	1.21	-60.24	3.07	-456.70	0.93	1.43

Branching fractions

- The ratios of the branching fractions between different resonances :

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ f_0(500) \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 0.20^{+0.02}_{-0.02},$$

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ f_0(980) \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 0.06^{+0.02}_{-0.02},$$

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ \rho \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 1.59^{+0.02}_{-0.03},$$

$$\frac{\mathcal{B}\left[D_S^+ \to f_0(1370) K^+ \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 0.58^{+0.06}_{-0.11},$$

$$\frac{\mathcal{B}\left[D_S^+ \to K^+ \rho(1450) \to K^+\pi^+\pi^-\right]}{\mathcal{B}\left[D_S^+ \to K^*(892)^0 \pi^+ \to K^+\pi^+\pi^-\right]} = 1.28^{+0.02}_{-0.05},$$

- The branching ratios for intermediate :

$$B(D_s^+ \to K^*(892)\pi^+, K^*(892) \to K^+\pi^-) = (1.85 \pm 0.13 \pm 0.11) \times 10^{-3}$$

Decay process	Ours (10^{-3})	BESIII (10^{-3})	PDG (10^{-3})
$D_s^+ \to K^+ f_0(500)$	$0.38 \pm 0.03^{+0.03}_{-0.03}$	$0.43 \pm 0.14 \pm 0.24$	-
$D_s^+ \to K^+ f_0(980)$	$0.11 \pm 0.01^{+0.04}_{-0.04}$	$0.27 \pm 0.08 \pm 0.07$	-
$D_s^+ \to K^+ \rho^0$	$2.94 \pm 0.27^{+0.03}_{-0.05}$	$1.99 \pm 0.20 \pm 0.22$	2.5 ± 0.4
$D_s^+ \to K^+ f_0(1370)$	$1.07 \pm 0.10^{+0.11}_{-0.20}$	$1.22 \pm 0.19 \pm 0.18$	-
$D_s^+ \to K_0^*(1430)^0 \pi^+$	$1.06 \pm 0.10^{+0.01}_{-0.02}$	$1.15 \pm 0.16 \pm 0.15$	0.50 ± 0.35
$D_s^+ \to K^+ \rho(1450)^0$	$2.38 \pm 0.22^{+0.04}_{-0.09}$	$0.78 \pm 0.20 \pm 0.17$	0.69 ± 0.64

The decay of $D_s^+ \to K_S^0 K_S^0 \pi^+$

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- The external and internal W-emission mechanism:

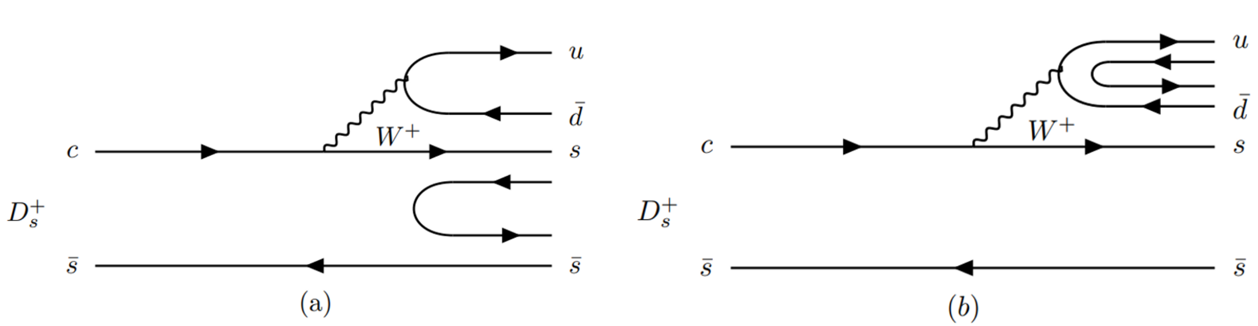


FIG. 1: W-external emission mechanism for the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay.

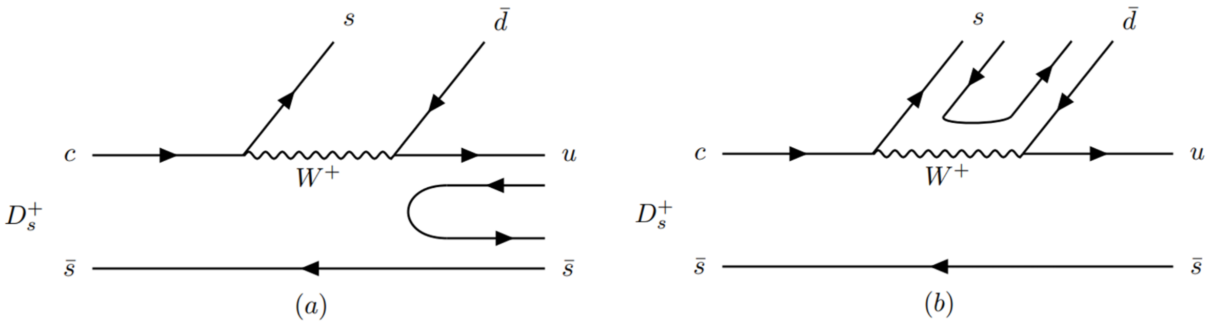


FIG. 2: W-external emission mechanism for the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay.

- The SU(3) matrix M in hadron level:

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \quad V = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{2}}\omega & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix},$$

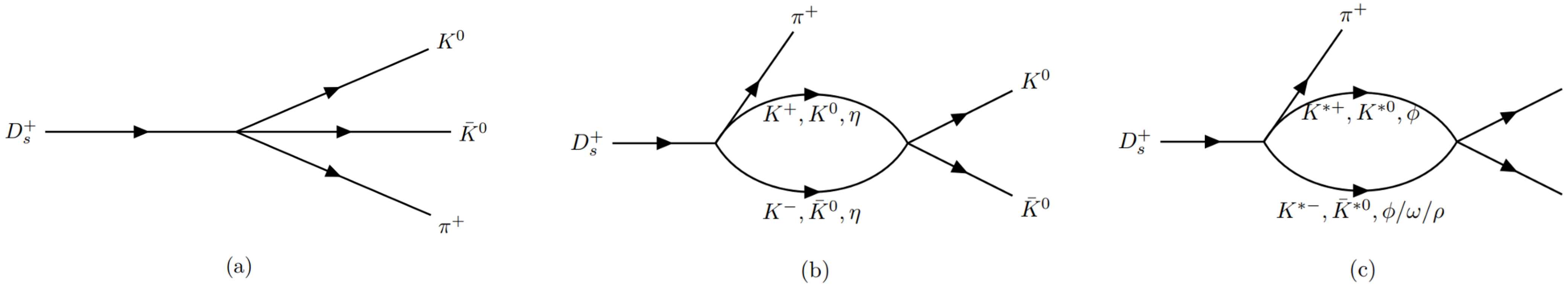
- The total contributions for the decay $D_s^+ \to K_s^0 K_s^0 \pi^+$:

$$|H\rangle = |H^{(1a)}\rangle + |H^{(1b)}\rangle + |H^{(2a)}\rangle + |H^{(2b)}\rangle$$

$$= C_1 \pi^+ K^+ K^- + C_2 \pi^+ K^0 \bar{K}^0 + \frac{2}{3} C_3 \pi^+ \eta\eta + C_4 \pi^+ K^{*+} K^{*-} + C_5 \pi^+ K^{*0} \bar{K}^{*0} + C_6 \pi^+ \phi\phi + \frac{1}{\sqrt{2}} C_7 \pi^+ \omega\phi + \frac{1}{\sqrt{2}} C_8 \pi^+ \rho^0 \phi,$$

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- Tree-level production and final state interactions via rescattering mechanism:



- The amplitudes for the decay $D_s^+ \to K_s^0 K_s^0 \pi^+$ in the S-wave:

$$
\begin{aligned}
t(M_{12})|_{K^0\bar{K}^0\pi^+} =& C_1 G_{K^+K^-}(M_{12}) T_{K^+K^-\to K^0\bar{K}^0}(M_{12}) + C_2 + C_2 G_{K^0\bar{K}^0}(M_{12}) T_{K^0\bar{K}^0\to K^0\bar{K}^0}(M_{12}) \\
&+ \frac{2}{3} C_3 G_{\eta\eta}(M_{12}) T_{\eta\eta\to K^0\bar{K}^0}(M_{12}) + C_4 G_{K^{*+}K^{*-}}(M_{12}) T_{K^{*+}K^{*-}\to K^0\bar{K}^0}(M_{12}) \\
&+ C_5 G_{K^{*0}\bar{K}^{*0}}(M_{12}) T_{K^{*0}\bar{K}^{*0}\to K^0\bar{K}^0}(M_{12}) + C_6 G_{\phi\phi}(M_{12}) T_{\phi\phi\to K^0\bar{K}^0}(M_{12}) \\
&+ \frac{1}{\sqrt{2}} C_7 G_{\omega\phi}(M_{12}) T_{\omega\phi\to K^0\bar{K}^0}(M_{12}) + \frac{1}{\sqrt{2}} C_8 G_{\rho^0\phi}(M_{12}) T_{\rho^0\phi\to K^0\bar{K}^0}(M_{12}),
\end{aligned}
$$

$$
|K_S^0\rangle = \frac{1}{\sqrt{2}} \left(|K^0\rangle - |\bar{K}^0\rangle\right)
$$

$$
\begin{aligned}
t(M_{12})|_{K_S^0K_S^0\pi^+} =& -\frac{1}{2} C_1 G_{K^+K^-}(M_{12}) T_{K^+K^-\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_2 - \frac{1}{2} C_2 G_{K^0\bar{K}^0}(M_{12}) T_{K^0\bar{K}^0\to K^0\bar{K}^0}(M_{12}) \\
&- \frac{1}{3} C_3 G_{\eta\eta}(M_{12}) T_{\eta\eta\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_4 G_{K^{*+}K^{*-}}(M_{12}) T_{K^{*+}K^{*-}\to K^0\bar{K}^0}(M_{12}) \\
&- \frac{1}{2} C_5 G_{K^{*0}\bar{K}^{*0}}(M_{12}) T_{K^{*0}\bar{K}^{*0}\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_6 G_{\phi\phi}(M_{12}) T_{\phi\phi\to K^0\bar{K}^0}(M_{12}) \\
&- \frac{1}{2\sqrt{2}} C_7 G_{\omega\phi}(M_{12}) T_{\omega\phi\to K^0\bar{K}^0}(M_{12}) - \frac{1}{2\sqrt{2}} C_8 G_{\rho^0\phi}(M_{12}) T_{\rho^0\phi\to K^0\bar{K}^0}(M_{12}),
\end{aligned}
$$

L. R. Dai et al. Eur. Phys. J. C 82, 225 (2022).

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- The diagonal matrix G is two intermediate meson propagators(dimensional regularization method):

$$G_{ii}(s) = \frac{1}{16\pi^2}\{a_\mu + \ln\frac{m_1^2}{\mu^2} + \frac{m_2^2 - m_1^2 + s}{2s}\ln\frac{m_2^2}{m_1^2} + \frac{q_{cm}(s)}{\sqrt{s}}[\ln\left(s - \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right) + \ln\left(s + \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right) - \ln\left(-s - \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right) - \ln\left(-s + \left(m_2^2 - m_1^2\right) + 2q_{cm}(s)\sqrt{s}\right)]\}$$

- The value of the subtraction constant : J. A. Oller and U.-G. Meißner, Phys. Lett. B 500, 263-272 (2001)

$$a_{PP'}(\mu) = -2\log\left(1 + \sqrt{1 + \frac{m_1^2}{\mu^2}}\right) + \cdots,$$

✓ the pseudoscalar-pseudoscalar interaction:

$\mu = 0.6$ GeV

✓ the vector-vector meson interaction:

$\mu = 1.0$ GeV

✓ In our formalism: the pseudoscalar-vector interactions

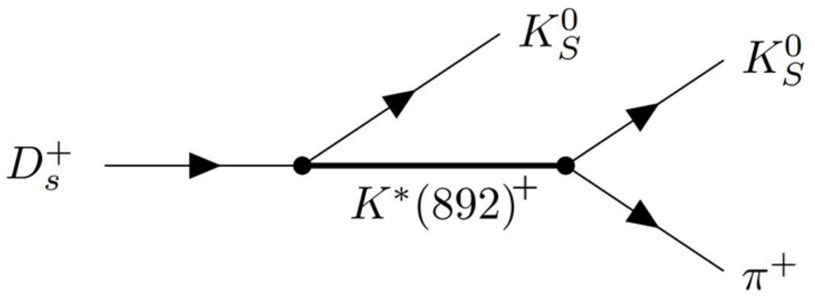
μ: a free parameter

J. A. Oller and E. Oset, Nucl. Phys. A 620, 438-456 (1997) .

L. S. Geng and E. Oset, Phys. Rev. D 79, 074009 (2009).

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- The contribution of the vector resonance generated in the P-wave:



$$t_{K^*(892)^+}(M_{12}, M_{23}) = \frac{\mathcal{D} e^{i\alpha_{K^*(892)^+}}}{M_{23}^2 - M_{K^*(892)^+}^2 + iM_{K^*(892)^+}\Gamma_{K^*(892)^+}} \times \left[\frac{(m_{D_s^+}^2 - m_{K_s^0}^2)(m_{K_s^0}^2 - m_{\pi^+}^2)}{M_{K^*(892)^+}^2} - M_{12}^2 + M_{13}^2 \right],$$

$$M_{12}^2 + M_{13}^2 + M_{23}^2 = m_{D_s^+}^2 + m_{K_S^0}^2 + m_{K_S^0}^2 + m_{\pi^+}^2.$$

- The double differential width distribution of three-body decay:

$$\frac{d^2\Gamma}{dM_{12}dM_{23}} = \frac{1}{(2\pi)^3} \frac{M_{12}M_{23}}{8m_{D_s^+}^3} \frac{1}{2} \mid \mathcal{M} \mid^2, \qquad \mathcal{M} = t(M_{12})|_{K_S^0 K_S^0 \pi^+} + t_{K^*(892)^+}(M_{12}, M_{23}) + (1 \leftrightarrow 2),$$

- The parameters need to be fitted:

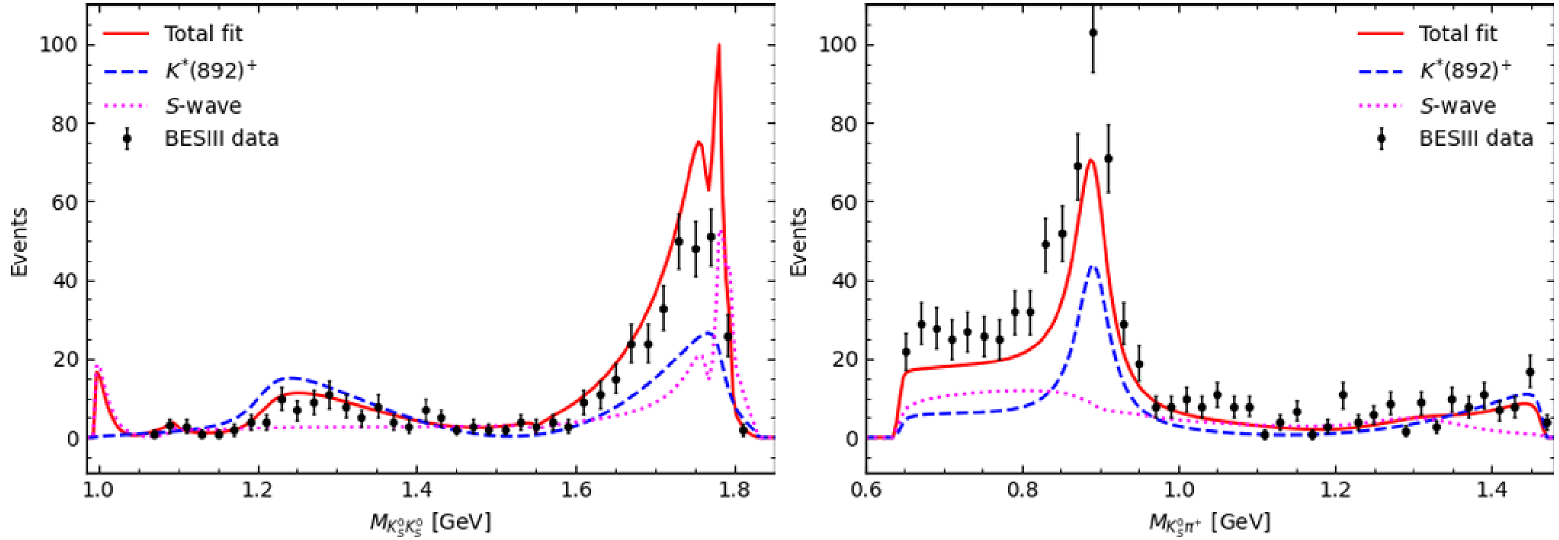
S-wave: $\mu, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$

P-wave: $D_{K^*(892)}, \alpha_{K^*(892)},$

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- Fitting results: (Combined fit)

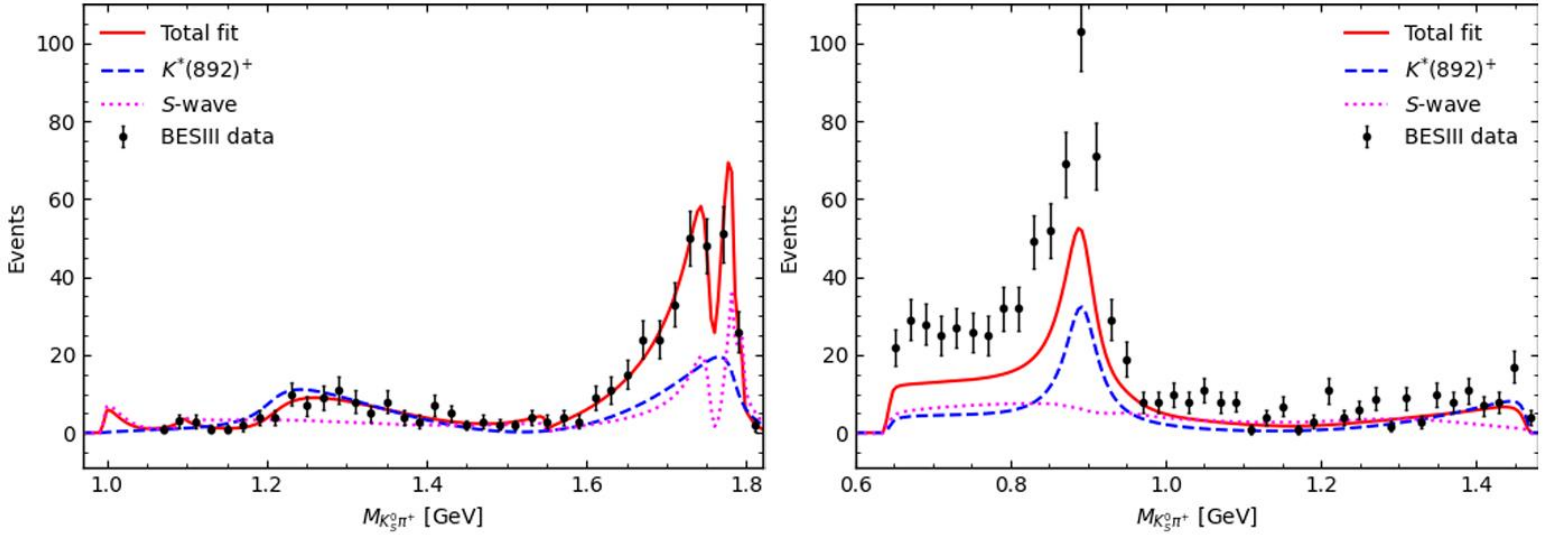
Parameters	μ	C_1	C_2	C_3	C_4	C_5	C_6	C_7	$C8$	D	$\alpha_{K^*(892)^+}$	$\chi^2/dof.$
Fit	0.648 GeV	8640.90	2980.71	-1902.86	56906.35	-13433.15	-58284.22	102835.76	202807.71	54.80	0.0024	2.55



	This work	Ref. [64]	Ref. [96]	Ref. [43]	Ref. [62]	Ref. [44]
Parameters	$\mu = 0.648$	$\mu = 0.716$	$q_{max} = 0.931$	$\mu = 1.0$	$q_{max} = 1.0$	$q_{max} = 1.0$
$a_0(980)$	$1.0598 + 0.024i$	$1.0419 + 0.0345i$	$1.0029 + 0.0567i$	...	...	...
$f_0(980)$	$0.9912 + 0.003i$	...	$0.9912 + 0.0135i$	...	...	...
$a_0(1710)$	$1.7981 + 0.0018i$	$1.7936 + 0.0094i$	...	$1.780 - 0.066i$	$1.72 - 0.010i$	$1.76 \pm 0.03i$
$f_0(1710)$	$1.7676 + 0.0093i$	...	...	$1.726 - 0.014i$	...	...

$D_s^+ \to K_s^0 K_s^0 \pi^+$

- Fitting results: (Fit only for $K_S^0 K_S^0$ spectrum)



- The ratios of the branching fractions between different resonances :

$$\frac{\mathcal{B}(D_s^+ \to S(980)\pi^+, S(980) \to K_S^0 K_S^0)}{\mathcal{B}(D_s^+ \to K_s^0 K^*(892)^+, K^*(892)^+ \to K_S^0 \pi^+)} = 0.122^{+0.032}_{-0.023}, \quad \frac{\mathcal{B}(D_s^+ \to S(1710)\pi^+, S(1710) \to K_S^0 K_S^0)}{\mathcal{B}(D_s^+ \to K_s^0 K^*(892)^+, K^*(892)^+ \to K_S^0 \pi^+)} = 0.552^{+0.460}_{-0.297},$$

- The branching ratios for intermediate :

$$\mathcal{B}(D_s^+ \to S(980)\pi^+, S(980) \to K_S^0 K_S^0) = (0.36 \pm 0.04^{+0.10}_{-0.06}) \times 10^{-3},$$
$$\mathcal{B}(D_s^+ \to S(1710)\pi^+, S(1710) \to K_S^0 K_S^0) = (1.66 \pm 0.17^{+1.38}_{-0.89}) \times 10^{-3},$$

$$B\left(D_S^+ \to K^*(892) K_S^0 \to K_S^0 K_S^0 \pi^+\right) = (3.0 \pm 0.3 \pm 0.1) \times 10^{-3};$$
$$B\left(D_S^+ \to S(1710)\pi^+ \to K_S^0 K_S^0 \pi^+\right) = (3.1 \pm 0.3 \pm 0.1) \times 10^{-3}.$$

Conclusions

Conclusions

- Based on the measurements for the decay $D_s^+ \to \pi^+\pi^-K^+$ and $D_s^+ \to K_S^0 K_S^0 \pi^+$, we adopt the chiral unitary approach to investigate these processes theoretically via considering the contributions of the W-external and –internal emission mechanisms. Besides, the contributions of the other intermediate resonances are also take into account.

- $D_s^+ \to \pi^+\pi^-K^+$: we take a combined fit to reproduce the $\pi^+\pi^-$, $K^+\pi^-$ and $K^+\pi^+$ invariant mass distributions by considering the coherent effects between the S and P waves, and then calculate the branching fractions of the dominant decay channels, the results are almost in good agreement with the experimental measurements and PDG within the uncertainties .

- $D_s^+ \to K_S^0 K_S^0 \pi^+$: the fitted results show that the enhancement around 1.7 GeV in $K_S^0 K_S^0$ mass spectrum is overlapped with two visible peaks, indicating the mixing signal originated from the resonances $a_0(1710)$ and $f_0(1710)$ due to their different poles (masses).

Thank you!