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# Solution of Loop Integrals in Quantum Field Theory using Modern Summation Methods

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# Outline of talk

1. Symbolic summation techniques

2. Application to massive 3-loop integrals

# A warm up example

GIVEN
$$F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\times$$
$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}\right.
+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\left.\vphantom{\frac{\Gamma}{\Gamma}}\right)}_{f(n,k,j)}.$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

# A warm up example

GIVEN $F(n) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)} \times$

$$\times \underbrace{\left( \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right)}_{f(n,k,j)}.$$

FIND the first coefficients of the $\epsilon$-expansion

$$F(n) = F_0(n) + \varepsilon F_1(n) + \dots$$

Arose in the context of
I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

# A warm up example

GIVEN $$F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\left(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}\right.
+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}}_{f(n,k,j)}\left.\vphantom{\frac{1}{1}}\right).$$

Step 1: Compute the first coefficients of the $\epsilon$-expansion

$$f(n,k,j)=\qquad f_0(n,k,j)+\qquad \varepsilon f_1(n,k,j)+\dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

# A warm up example

$$\text{GIVEN } F(n)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\times$$

$$\times\underbrace{\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)} + \frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg)}_{f(n,k,j)}.$$

Step 2: Simplify the sums in

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f(n,k,j)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

# Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i}$$

# Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

# Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

# Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(n,k,j) = g(a+1)-g(0)$$

# Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j)=g(j+1)-g(j)}$$

Summing the telescoping equation over $j$ from $0$ to $a$ gives

$$\sum_{j=0}^{a} f(n,k,j) = g(a+1)-g(0)$$

$$=\underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\,\to\,\infty}$$

# Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(n,k,j) = \frac{1}{n!}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# Simplify

$$\sum_{k=0}^{\infty}\underbrace{\sum_{j=0}^{\infty}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

# Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{f(n,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

**no solution** ☹

# Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

# Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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for all $0 \leq k \leq n$ and all $n \geq 0$.

Sigma *computes*: $c_0(n) = -n$, $c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k) + (-n-1)S_1(n) - kS_1(k+n) - 2}{(k+n+1)(n+1)^2}$$

# Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k)\Big]}$$

# Zeilberger's creative telescoping paradigm

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$$A(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

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# Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\sum_{k=1}^{a} f(n,k)+c_1(n)\sum_{k=1}^{a} f(n+1,k)}$$

# Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

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$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\, \mathsf{A}(n) + c_1(n)\, \mathsf{A}(n+1)}$$

# Zeilberger's creative telescoping paradigm

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$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n), c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over $k$ from $1$ to $a$ gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \quad || \quad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \{c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} | c \in \mathbb{R}\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

## Summation package Sigma

(based on difference field algorithms/theory
see, e.g., Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

# Simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

# GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+$$

## Sigma produces

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}.$$

# GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

## Sigma produces

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)=\frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}.$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)=\frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!}.$$

# GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-e\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$
$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+$$

## Sigma produces

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_2(n,k,j)=\frac{1}{96n(n+1)}\Big(S_1(n)^4+(12\zeta_2+54S_2(n))S_1(n)^2$$
$$+104S_3(n)S_1(n)-48S_{2,1}(n)S_1(n)+51S_2(n)^2+36\zeta_2S_2(n)$$
$$+126S_4(n)-48S_{3,1}(n)-96S_{1,1,2}(n)\Big)$$

# GIVEN

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}$$

$$+\frac{\Gamma(k+1)}{\Gamma(k+2+n)}\frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg).$$

$$=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_0(n,k,j)+\varepsilon\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\varepsilon^3\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_1(n,k,j)+\dots$$

# Sigma produces

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}f_3(n,k,j)=\frac{1}{960n(n+1)}\Big(S_1(n)^5+(20\zeta_2+130S_2(n))S_1(n)^3+$$

$$(40\zeta_3+380S_3(n))S_1(n)^2+\left(135S_2(n)^2+60\zeta_2S_2(n)+510S_4(n)\right)S_1(n)$$

$$-240S_{3,1}(n)S_1(n)-240S_{1,1,2}(n)S_1(n)+160\zeta_2S_3(n)+S_2(n)(120\zeta_3$$

$$+380S_3(n))+624S_5(n)+\left(-120S_1(n)^2-120S_2(n)\right)S_{2,1}(n)$$

$$-240S_{4,1}(n)-240S_{1,1,3}(n)+240S_{2,2,1}(n)\Big)$$

# Toolbox 1: Symbolic summation algorithms

# 1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$S(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in $k$;
$n$: extra parameter

FIND a recurrence for $S(n)$

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FIND a recurrence for $S(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, in preparation)

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(Abramov/Bronstein/Petkovšek/CS, in preparation)

3. Find a "closed form"

S(n)=combined solutions in terms of indefinite nested sums.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{k=1}^{A} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)};$$

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $\mathbf{mySum} = \sum_{k=1}^{A} \frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)};$

## Compute a recurrence

In[3]:= **rec = GenerateRecurrence[mySum, n][[1]]**

Out[3]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[4]:= **rec = LimitRec[rec, SUM[n], {n}, A]**

Out[4]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S_1(n)+1}{(n+1)^3}$

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In[4]:= **rec = LimitRec[rec, SUM[n], {n}, A]**

Out[4]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S_1(n)+1}{(n+1)^3}$

## Solve a recurrence

In[5]:= **recSol = SolveRecurrence[rec, SUM[n]]**

Out[5]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S_1(n)^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}\}\}$

In[1]:= << Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

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In[4]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, A]}$

Out[4]= $-n\mathsf{SUM}[n]+(1+n)(2+n)\mathsf{SUM}[n+1] == \frac{(n+1)S_1(n)+1}{(n+1)^3}$

## Solve a recurrence

In[5]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[5]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S_1(n)^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}\}\}$

## Combine the solutions

In[6]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}, n, 2]}$

Out[6]= $\frac{S_1(n)^2+\sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}$

# Iterative application from inside to outside transforms

definite sums

$$\downarrow$$

indefinite sums

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$\|$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\|$$

$$\boxed{\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+s-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{n}\frac{\mathsf{sign}(a)^i}{i^{|a|}}$, $a\in\mathbb{Z}\setminus\{0\}$.

# Toolbox 2: Special function algorithms

# Computer algebra and special functions:

**Harmonic sums** (Vermaseren, Remmiddi, Blümlein; Hoffman, Broadhurst,...)

$$S_{2,1}(n) = \boxed{\sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j}}$$

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**Integral representation:**

$$= \int_0^1 \frac{x^n - 1}{1 - x} \left( \int_0^x \frac{\int_0^y \frac{1}{1-z} dz}{y} dy - \zeta(2) \right) dx, \qquad \zeta(z) := \sum_{i=1}^{\infty} 1/i^z$$

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**Asymptotic expansion:**

$$= \left(\frac{1}{30n^5} - \frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n}\right)\ln(n)$$
$$-\frac{1}{100n^5} - \frac{1}{6n^4} + \frac{13}{36n^3} - \frac{1}{4n^2} - \frac{1}{n} + 2\zeta(3) + O\left(\frac{\ln(n)}{n^6}\right).$$

**limit computations** | **numerical evaluation**

# Computer algebra and special functions:

## Generalization to cyclotomic harmonic sums

$$\boxed{\sum_{k=1}^{n}\frac{(-1)^k}{2k+1}} =$$

**Integral representation:**

$$=-(-1)^n\int_0^1\frac{x^{2n}}{x^2+1}dx+\frac{(-1)^n}{2n+1}-1+\frac{\pi}{4},$$

**Asymptotic expansion:**

$$=(-1)^n\left(-\frac{3}{64n^5}-\frac{1}{16n^4}+\frac{3}{16n^3}-\frac{1}{4n^2}+\frac{1}{4n}\right)+\frac{\pi}{4}-1+O\left(\frac{1}{n^6}\right)$$

**limit computations** | **numerical evaluation**

(J. Ablinger, J. Blümlein, CS; J. Math. Phys. 2011 [arXiv:1105.6063 [math-ph]])

$$\sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j^2}}{(2i+1)^2} = \text{asymptotic expansion?}$$

In[1]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[2]:= **SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]**

$$\sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j^2}}{(2i+1)^2} = \text{asymptotic expansion?}$$

In[1]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[2]:= **SExpansion[S[{{2, 1, 2}, {1, 0, 2}}, n], n, 10]**

Out[2]=

$$\left(-\frac{16\ln 2^2}{3}+\frac{3}{128n^{10}}-\frac{367}{5760n^9}+\frac{7}{96n^8}-\frac{221}{2016n^7}+\frac{5}{24n^6}-\frac{127}{360n^5}+\frac{1}{2n^4}-\frac{11}{18n^3}+\frac{2}{3n^2}-\frac{2}{3n}-\frac{1936}{15}\right)\frac{1}{4}(\pi-4)^2+\left(-\frac{32\ln 2^2}{3}+\frac{3}{64n^{10}}-\frac{367}{2880n^9}+\frac{7}{48n^8}-\frac{221}{1008n^7}+\frac{5}{12n^6}-\frac{127}{180n^5}+\frac{1}{n^4}-\frac{11}{9n^3}+\frac{4}{3n^2}-\frac{4}{3n}-\frac{3872}{45}\right)\frac{1}{4}(\pi-4)-\frac{968}{45}\frac{1}{4}(\pi-4)^4-\frac{3872}{45}\frac{1}{4}(\pi-4)^3+8\text{li4half}+\frac{\ln 2^4}{3}-\frac{16\ln 2^2}{3}+7\ln 2\ \text{z3}+\frac{125891}{1075200n^{10}}-\frac{10259}{80640n^9}+\frac{92257}{645120n^8}-\frac{5507}{20160n^7}+\frac{2837}{5760n^6}-\frac{509}{720n^5}+\frac{161}{192n^4}-\frac{31}{36n^3}+\frac{19}{24n^2}-\frac{2}{3n}-\frac{968}{45}$$

# More involved massive 3-loop diagrams

Emergence of new nested sums :

$$\sum_{i=1}^{n}\binom{2i}{i}(-2)^i\sum_{j=1}^{i}\frac{1}{j\binom{2j}{j}}S_{1,2}\left(\tfrac{1}{2},-1;j\right)$$

# More involved massive 3-loop diagrams

Emergence of new nested sums :

$$\sum_{i=1}^{n} \binom{2i}{i} (-2)^i \sum_{j=1}^{i} \frac{1}{j\binom{2j}{j}} S_{1,2}\left(\tfrac{1}{2}, -1; j\right)$$

$$= \int_0^1 dx \frac{x^n - 1}{x - 1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{w_{17},-1,0}(x) - 2\mathrm{H}^*_{w_{18},-1,0}(x)\right]$$

$$+ \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^n - 1}{x + 1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{12}(x) - 2\mathrm{H}^*_{13}(x)\right] + c_3 \int_0^1 dx \frac{(-8x)^n - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1-x}}\ ,$$

with the constant

$$c_3 = \sum_{j=0}^{\infty} S_{1,2}\left(\tfrac{1}{2}, -1; j\right) \frac{(j!)^2}{j(2j)!\pi}$$

and the letters

$$w_{12} = \frac{1}{\sqrt{x(8-x)}}, \quad w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}}, \qquad w_{17} = \frac{1}{\sqrt{x(8+x)}}, \quad w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}}.$$

(J. Ablinger, J. Bümlein, J. Raab, C. Schneider 2013.)

For more details see: Johannes Blümlein's talk (Saturday, 16:10)

# The full machinery:

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum[**

$$\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\left(\frac{\Gamma(k+1)\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(k+2+n)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}+\frac{\Gamma(k+1)\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(k+2+n)\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\right)$$

$$\{\{j,0,\infty\},\{k,0,\infty\}\},\{n\},\{1\},\mathrm{ExpandIn}\to\{\varepsilon,0,2\}]$$

# The full machinery:

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EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= **EvaluateMultiSum[**

$$\frac{e^{-\varepsilon\gamma}}{\Gamma(\varepsilon+1)}\Bigg(\frac{\Gamma(k+1)\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+n)}{\Gamma(k+2+n)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+n)\Gamma(k+j+2)}+\frac{\Gamma(k+1)\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+n)}{\Gamma(k+2+n)\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+n)\Gamma(k+j+2+\frac{\varepsilon}{2})}\Bigg)$$

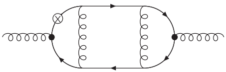
$$\{\{j,0,\infty\},\{k,0,\infty\}\},\{n\},\{1\},\mathrm{ExpandIn}\to\{\varepsilon,0,2\}]$$

Out[4]=

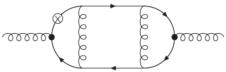
$$\Bigg\{\frac{S_1(n)^2+3S_2(n)}{2n(n+1)!},\\ \frac{-S_1(n)^3-3S_2(n)S_1(n)-8S_3(n)}{6n(n+1)!},\\ \frac{1}{96n(n+1)}\Big(S_1(n)^4+(12\zeta_2+54S_2(n))S_1(n)^2+104S_3(n)S_1(n)\\ -48S_{2,1}(n)S_1(n)+51S_2(n)^2+36\zeta_2S_2(n)+126S_4(n)-48S_{3,1}(n)-96S_{1,1,2}(n)\Big)\Bigg\}$$

# Example 1: 3–Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH), F. Wissbrock (DESY)



(massive 3–loop ladder graph with operator insertion)

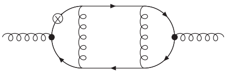


J. Blümlein
A. Hasselhuhn

$$= \frac{C_3}{(n+1)(n+2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=2}^{n+2} \binom{n+2}{l} \sum_{j=2}^{l} \binom{l}{j} \Bigg\{$$

$$\sum_{k=1}^{j} \binom{j}{k} \sum_{r=0}^{l-k} \binom{l-k}{r} (-1)^{l+j+k+r} \frac{B\left(k,m+1+\frac{\varepsilon}{2}\right)\Gamma(k+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(k+r+\frac{\varepsilon}{2})} \frac{B\left(r+l-1,n+1+\frac{\varepsilon}{2}\right)}{(n+3-j)} \frac{B\left(k+m-\frac{\varepsilon}{2},r+1+n-\right.}{(k+r+1+m+n-\varepsilon}$$

$$+ \sum_{r=0}^{l-j} \binom{l-j}{r} (-1)^{l+j+r} \frac{B\left(r+l-1,n+1+\frac{\varepsilon}{2}\right)\Gamma(j+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(j+r+\frac{\varepsilon}{2})} \frac{B\left(j,m+1+\frac{\varepsilon}{2}\right)B\left(j+m-\frac{\varepsilon}{2},r+1+n-\frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(n+3-j)} \Bigg\}$$



J. Blümlein
A. Hasselhuhn

$$= \frac{C_3}{(n+1)(n+2)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=2}^{n+2}\binom{n+2}{l}\sum_{j=2}^{l}\binom{l}{j}\Bigg\{$$

$$\sum_{k=1}^{j}\binom{j}{k}\sum_{r=0}^{l-k}\binom{l-k}{r}(-1)^{l+j+k+r}\frac{B\left(k,m+1+\frac{\varepsilon}{2}\right)\Gamma(k+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(k+r+\frac{\varepsilon}{2})}\frac{B\left(r+l-1,n+1+\frac{\varepsilon}{2}\right)}{(n+3-j)}\frac{B\left(k+m-\frac{\varepsilon}{2},r+1+n-\right.}{(k+r+1+m+n-\varepsilon}$$

$$+\sum_{r=0}^{l-j}\binom{l-j}{r}(-1)^{l+j+r}\frac{B\left(r+l-1,n+1+\frac{\varepsilon}{2}\right)\Gamma(j+r+m+n+\frac{\varepsilon}{2})}{\Gamma(m+1)\Gamma(n+1)\Gamma(j+r+\frac{\varepsilon}{2})}\frac{B\left(j,m+1+\frac{\varepsilon}{2}\right)B\left(j+m-\frac{\varepsilon}{2},r+1+n-\frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(n+3-j)}\Bigg\}$$

$\|$ EvaluateMultiSums

$$\frac{C_3}{(n+1)(n+2)(n+3)}\Bigg\{\frac{1}{6}S_1^3(n)+\frac{n^2+12n+16}{2(n+1)(n+2)}S_1(n)^2+\frac{4(2n+3)}{(n+1)^2(n+2)}S_1(n)$$

$$+2\left[-2^{n+3}+3-(-1)^n\right]\zeta_3+\left[\frac{3n^2+40n+56}{2(n+1)(n+2)}-\frac{1}{2}S_1(n)\right]S_2(n)$$

$$-(-1)^nS_{-3}(n)+\frac{8(2n+3)}{(n+1)^3(n+2)}-\frac{3n+17}{3}S_3(n)-2(-1)^nS_{-2,1}(n)-(n+3)S_{2,1}(n)$$

$$+2^{n+4}S_{1,2}\left(\frac{1}{2},1;n\right)+2^{n+3}\boxed{S_{1,1,1}\left(\frac{1}{2},1,1;n\right)}\Bigg\}+O(\varepsilon)$$

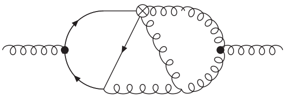
$$\boxed{S_{1,1,1}\left(\frac{1}{2},1,1;n\right)} = \sum_{i=1}^{n} \frac{\left(\frac{1}{2}\right)^i \displaystyle\sum_{j=1}^{i} \frac{\displaystyle\sum_{k=1}^{j} \frac{1}{k}}{j}}{i}$$

$$\boxed{S_{1,1,1}\left(\frac{1}{2},1,1;n\right)} = \sum_{i=1}^{n} \frac{\left(\frac{1}{2}\right)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i}$$

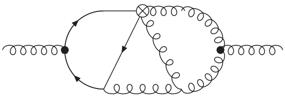
|| asymptotic expansion (HarmonicSums package)

$$2^{-n}\left(+\frac{541}{n^6}-\frac{75}{n^5}+\frac{13}{n^4}-\frac{3}{n^3}+\frac{1}{n^2}-\frac{1}{2n}\right)(\ln(n)+\gamma)^2$$
$$+2^{-n-3}\left(-\frac{114686}{5n^6}+\frac{44099}{15n^5}-\frac{1372}{3n^4}+\frac{266}{3n^3}-\frac{20}{n^2}\right)(\ln(n)+\gamma)$$
$$+2^{-n}\left(+\frac{541}{n^6}-\frac{75}{n^5}+\frac{13}{n^4}-\frac{3}{n^3}+\frac{1}{n^2}-\frac{1}{2n}\right)\zeta(2)+\frac{3\zeta(3)}{4}$$
$$+2^{-n-9}\left(\frac{69280576}{45n^6}-\frac{1582096}{9n^5}+\frac{69184}{3n^4}-\frac{3264}{n^3}+\frac{256}{n^2}\right)+O\left(\frac{1}{2^n n^7}\right)$$

(J. Ablinger, J. Blümlein, CS; r arXiv:1302.0378 [math-ph])



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

# Simplify

$$\|$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-\big(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s)\big)$$

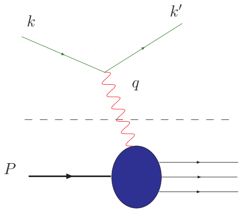
$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+ \Big(-\frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+ \big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$- \frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

# Example 2: Heavy Flavor Wilson Coefficients

# Unpolarized Deep–Inelastic Scattering (DIS):




$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$ Bjorken–x

$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu}L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.

$$F_i(x,Q^2) = \sum_k C_{i,k}\left(\frac{Q^2}{\mu^2},x\right) \otimes f_k\left(\frac{\mu^2}{\mu_0^2},x\right)$$

$$C_{i,k}\left(\frac{Q^2}{\mu^2},x\right) = C_{i,k}^{\text{light}}\left(\frac{Q^2}{\mu^2},x\right) + C_{i,k}^{\text{heavy}}\left(\frac{Q^2}{\mu^2},x\right)$$

# Heavy Flavor Wilson Coefficients

Present team:

J. Ablinger[a], J. Blümlein[b], A. De Freitas[b] A. Hasselhuhn[a,b], A. von Manteuffel[c], C. Raab[b], C. Schneider[a], M. Round[a], F. Wißbrock[b]

[a] RISC, J. Kepler University, Linz, Austria
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There are eight massive Wilson coefficients in the unpolarized case.

- The OMEs $A_{qq,Q}^{\mathsf{PS}}, A_{qg,Q}$ were calculated.

  J. Ablinger, J. Blümlein, S. Klein, C. Schneider, F. Wissbrock, 2010.

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production by extension of `QGRAF` (P. Nogueira)
apply color algebra by `Color` (T.v. Ritbergen, A.N. Schellekens, J.A.M. Vermaseren)

↓

diagrams with local operator insertions
for the respective Wilson coefficient

production by extension of `QGRAF` (P. Nogueira)
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↓

**diagrams with local operator insertions for the respective Wilson coefficient**

extension of `MATAD` (M. Steinhauser)

↓

**series of moments**

production by extension of `QGRAF` (P. Nogueira)
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↓

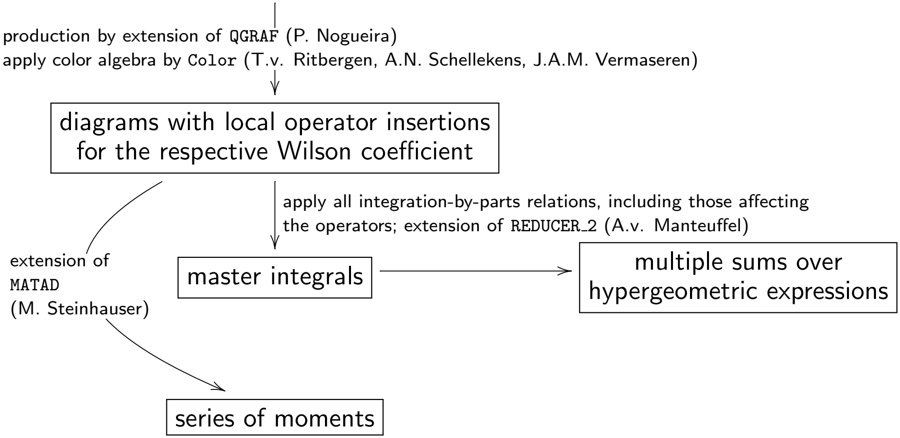
**diagrams with local operator insertions for the respective Wilson coefficient**

↓ apply all integration-by-parts relations, including those affecting the operators; extension of `REDUCER_2` (A.v. Manteuffel)

**master integrals**

extension of `MATAD` (M. Steinhauser) →

**series of moments**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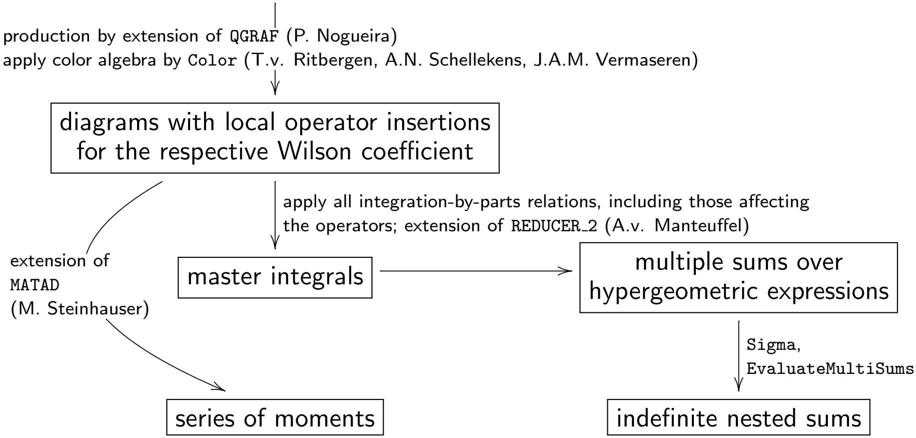
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Sigma, EvaluateMultiSums
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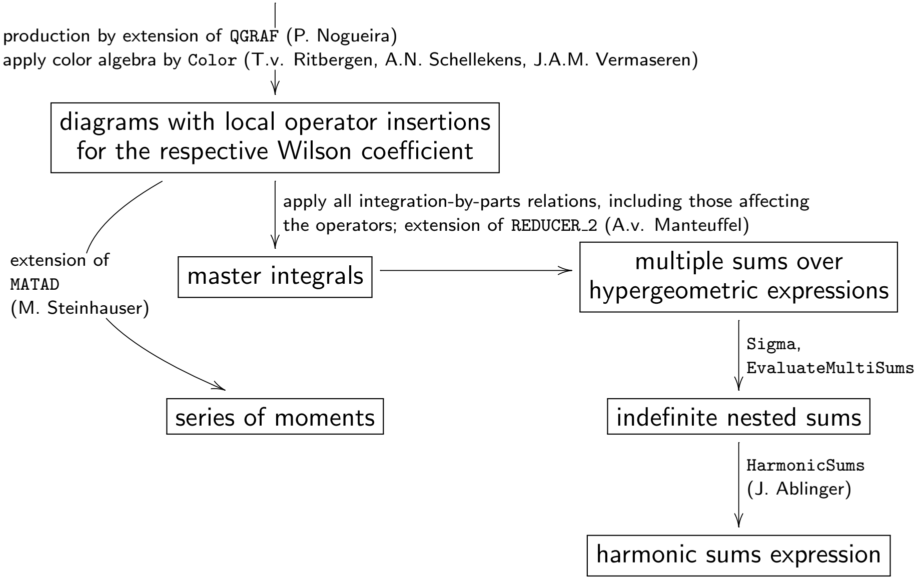
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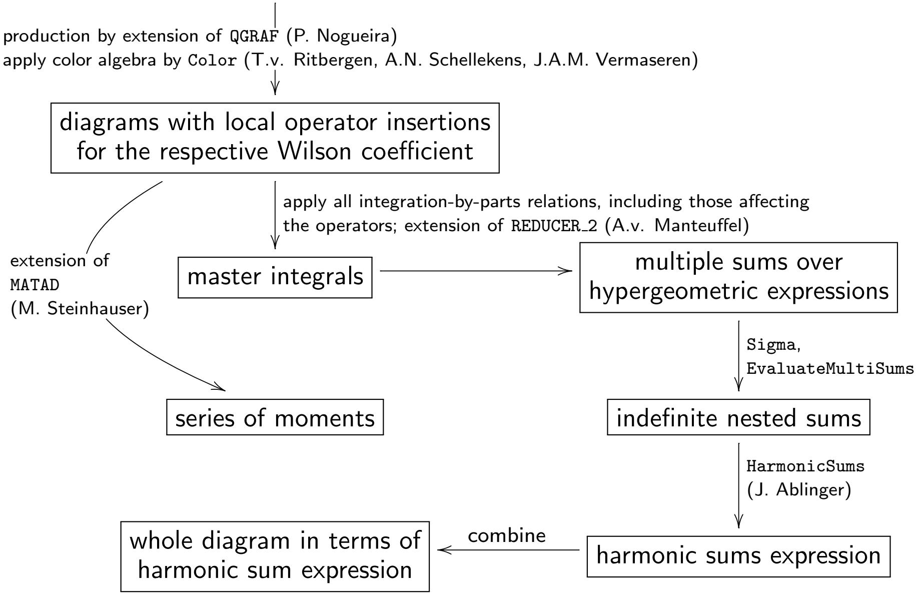
HarmonicSums (J. Ablinger)
harmonic sums expression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whole diagram in terms of harmonic sum expression
combine
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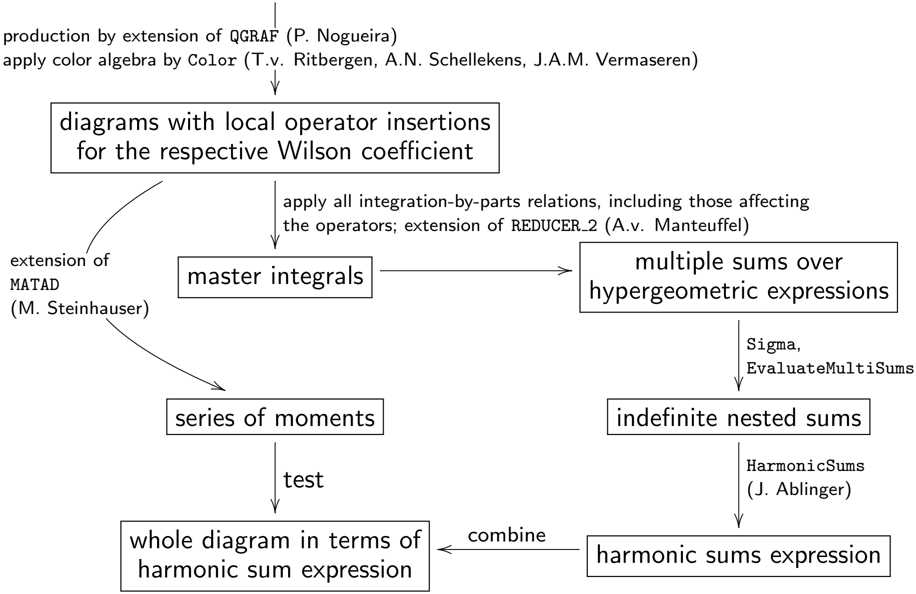
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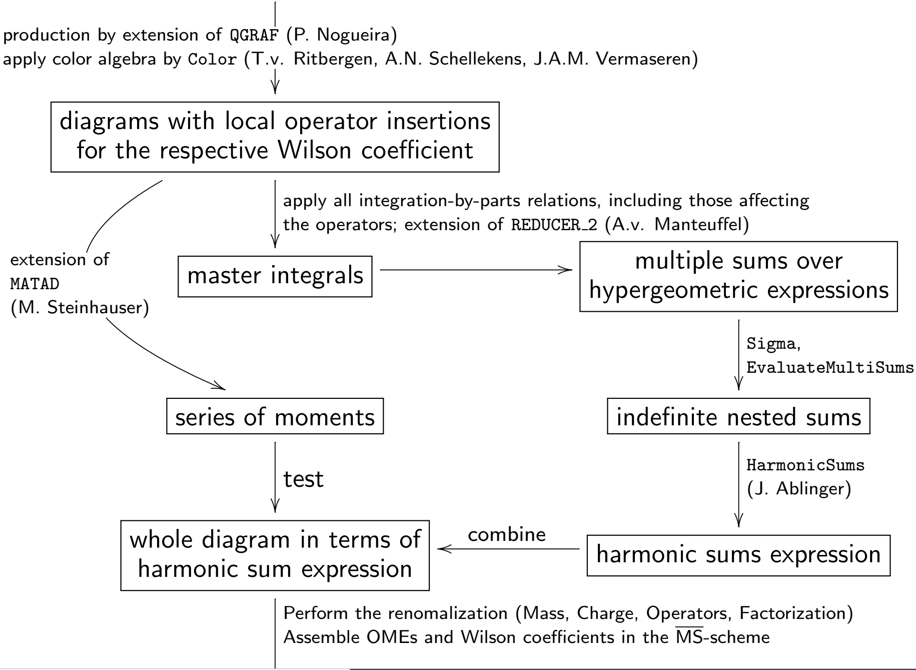
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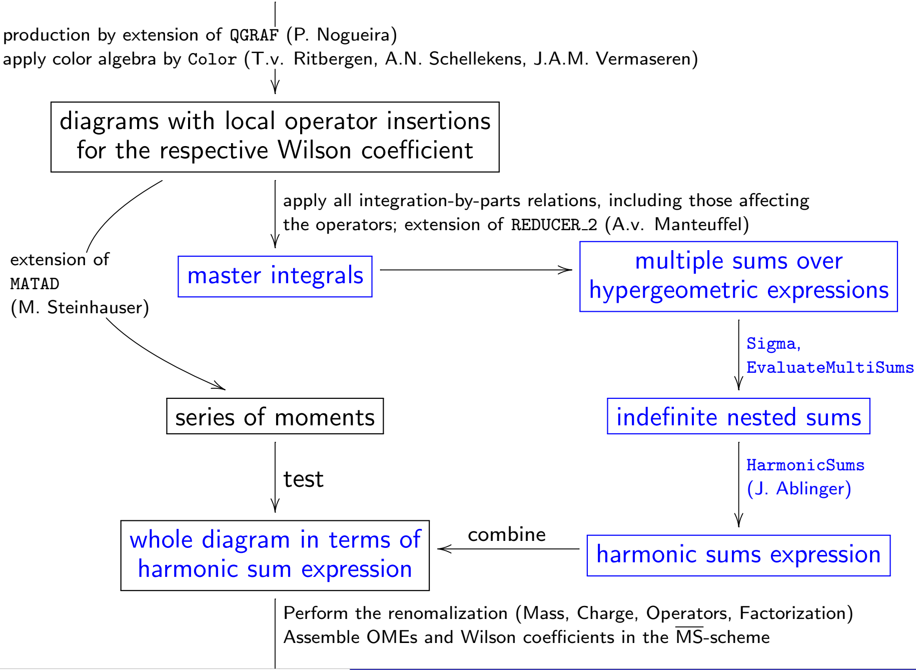
whole diagram in terms of harmonic sum expression
combine
harmonic sums expression
Perform the renomalization (Mass, Charge, Operators, Factorization)
Assemble OMEs and Wilson coefficients in the $\overline{\text{MS}}$-scheme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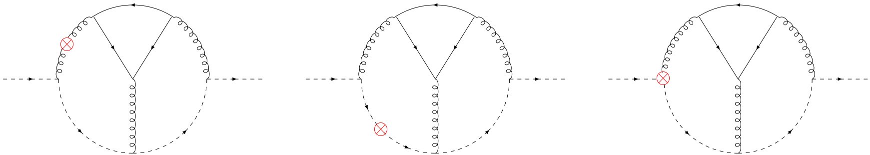
# Calculation of Benz-Diagrams

$D_1(n)$ $D_2(n)$ $D_3(n)$



$A_{gq,Q}$ $A_{qq,Q}^{\mathrm{NS}}$ $A_{qq,Q}^{\mathrm{NS}}$

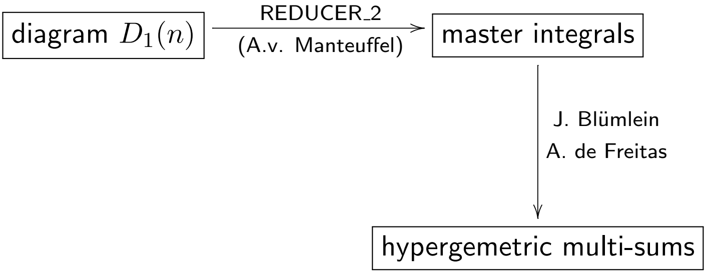
Only one scalar diagram needs to be calculated to obtain the two others :

$$D_2(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k D_1(k), \qquad \text{conjugation}$$

$$D_3(n) = \sum_{k=0}^{n} D_2(k)$$

[Use summation techniques.]

# Symbolic summation



$$\sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^{i+1}e^{-\frac{3\varepsilon\gamma}{2}}\times$$

$$\times\ \frac{n!\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(i-\frac{\varepsilon}{2}\right)\Gamma(-\varepsilon+i+j+1)\Gamma\left(k-\frac{3\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}+j+k+1\right)\Gamma(n+2)\Gamma(-\varepsilon-j+n+1)}{i!j!(k-i)!(n-j)!\Gamma\left(-\frac{\varepsilon}{2}+i+1\right)\Gamma\left(-\frac{3\varepsilon}{2}+i+j+2\right)\Gamma\left(\frac{\varepsilon}{2}+n+2\right)\Gamma\left(-\frac{\varepsilon}{2}+k+n+2\right)}$$

$$\sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^{i+1}e^{-\frac{3\varepsilon\gamma}{2}}\times$$

$$\times\ \frac{n!\Gamma\left(1-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(i-\frac{\varepsilon}{2}\right)\Gamma(-\varepsilon+i+j+1)\Gamma\left(k-\frac{3\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}+j+k+1\right)\Gamma(n+2)\Gamma(-\varepsilon-j+n+1)}{i!j!(k-i)!(n-j)!\Gamma\left(-\frac{\varepsilon}{2}+i+1\right)\Gamma\left(-\frac{3\varepsilon}{2}+i+j+2\right)\Gamma\left(\frac{\varepsilon}{2}+n+2\right)\Gamma\left(-\frac{\varepsilon}{2}+k+n+2\right)}$$

||`Sigma, EvaluateMultiSums, HarmonicSums`

$$-\frac{8}{3}\Big[\frac{S_1(n)}{n+1}+\frac{1}{(n+1)^2}\Big]\varepsilon^{-3}$$

$$+\Big[\frac{4S_1(n)^2}{3(n+1)}+\frac{8S_1(n)}{3(n+1)^2}-\frac{4S_2(n)}{3(n+1)}\Big]\varepsilon^{-2}$$

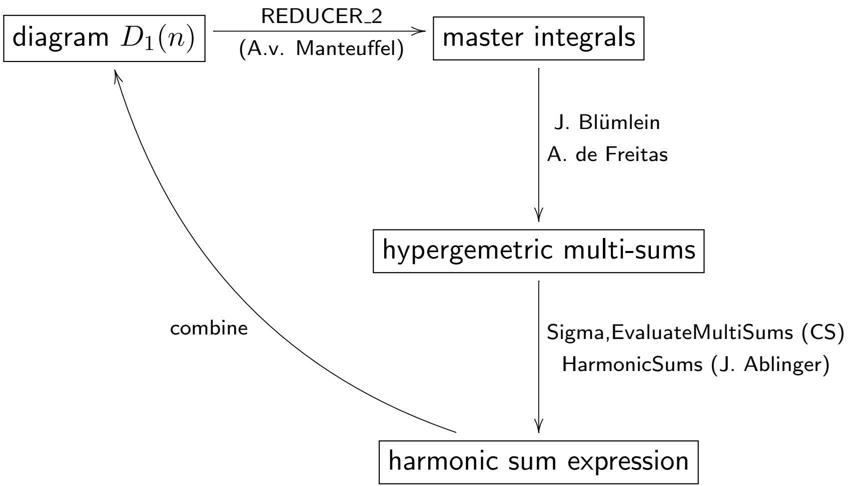
$$+\Big[\frac{4S_1(n)^3}{9(n+1)}-\frac{4S_{2,1}(n)}{3(n+1)}-\frac{4S_1(n)^2}{3(n+1)^2}-\frac{8S_1(n)}{3(n+1)^3}-\frac{8S_3(n)}{9(n+1)}-\frac{8}{3(n+1)^4}-\zeta_2\frac{(n+1)S_1(n)+1}{(n+1)^2}\Big]\varepsilon^{-1}$$

$$+\Big[S_1(n)\Big(\frac{2S_{2,1}(n)}{n+1}+\frac{2S_3(n)}{9(n+1)}+\frac{8}{3(n+1)^4}\Big)+\frac{2S_{2,1}(n)}{(n+1)^2}+\frac{2S_{3,1}(n)}{3(n+1)}-\frac{8S_{2,1,1}(n)}{3(n+1)}$$

$$+\zeta_2\Big(\frac{S_1(n)^2}{2(n+1)}+\frac{S_1(n)}{(n+1)^2}-\frac{S_2(n)}{2(n+1)}\Big)+\zeta_3\Big(-\frac{5S_1(n)}{3(n+1)}-\frac{5}{3(n+1)^2}\Big)$$

$$+\frac{S_1(n)^4}{9(n+1)}+\frac{4S_1(n)^3}{9(n+1)^2}+\frac{4S_1(n)^2}{3(n+1)^3}-\frac{4S_2(n)}{3(n+1)^3}+\frac{2S_3(n)}{9(n+1)^2}-\frac{S_4(n)}{3(n+1)}\Big]\varepsilon^0+O(\varepsilon)$$

# Symbolic summation

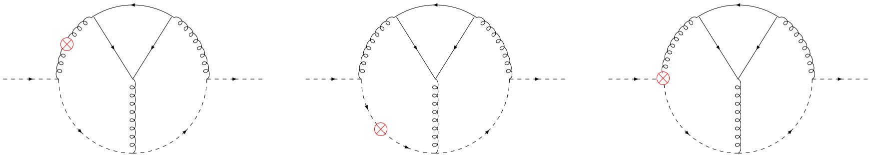


# Constant term of

$$D_1(n) = \left[\frac{S_2(n)}{96(n+1)} - \frac{52n^2+32n+7}{432(n+1)^3}\right] S_1(n) - \zeta_2 \left[\frac{3n+4}{32(n+1)^2} - \frac{S_1(n)}{32(n+1)}\right]$$
$$- \frac{7S_1^3(n)}{288(n+1)} + \frac{(10n+3)S_1(n)^2}{96(n+1)^2} + \frac{(10n+11)S_2(n)}{96(n+1)^2} + \frac{5S_3(n)}{144(n+1)}$$
$$- \frac{204n^3+592n^2+527n+130}{432(n+1)^4}$$

# Calculation of Benz-Diagrams

$D_1(n)$ $\quad$ $D_2(n)$ $\quad$ $D_3(n)$



$A_{gq,Q}$ $\quad$ $A_{qq,Q}^{\mathrm{NS}}$ $\quad$ $A_{qq,Q}^{\mathrm{NS}}$

Only one scalar diagram needs to be calculated to obtain the two others :

$$D_2(n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k D_1(k), \qquad \text{conjugation}$$

$$D_3(n) = \sum_{k=0}^{n} D_2(k)$$

[Use summation techniques.]

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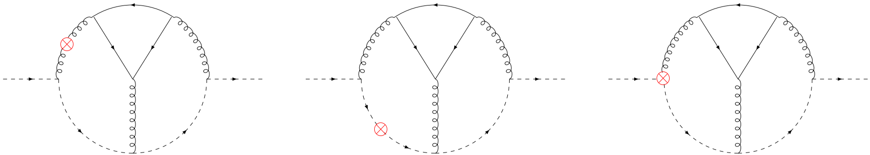
$$D_2(n) = \frac{\left(8n^2+13n+9\right)S_1(n)}{48(n+1)^3} - \frac{S_1(n)^2}{32(n+1)} + \frac{(1-7n)S_2(n)}{96(n+1)^2} - \frac{(3n+4)\zeta_2}{32(n+1)^2}$$
$$- \frac{204n^3+592n^2+527n+130}{432(n+1)^4}$$

$$D_3(n) = -\frac{32n^2+127n+23}{864(n+1)^2} S_2(n) - \left[\frac{68n^3+180n^2+165n+41}{144(n+1)^3} - \frac{7}{96} S_2(n)\right] S_1(n)$$
$$\frac{1}{24} S_2(n)^2 - \frac{1}{32}\zeta_2 \left(3S_1(n) + S_2(n) - \frac{3n+4}{(n+1)^2}\right) - \frac{1}{96} S_1(n)^3 + \frac{(8n+5)}{96(n+1)} S_1(n)^2$$
$$+ \frac{7}{48} S_3(n) - \frac{5}{48} S_4(n) + \frac{1}{24} S_{2,1}(n) + \frac{1}{12} S_{3,1}(n) - \frac{204n^3+592n^2+527n+130}{432(n+1)^4}$$

# Calculation of Benz-Diagrams

$D_1(n)$ $D_2(n)$ $D_3(n)$



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- The (ongoing) development of computer algebra algorithms for special functions is crucial.
- Massive Wilson coefficients for DIS will be calculated with the present techniques very soon.
- Multi-leg calculations (e.g., for two loop diagrams) are in preperation (cooperation with J. Blümlein, J. Gluza, T. Riemann).