Roy-Steiner equations for πN

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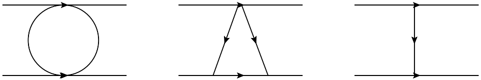
Outline

Contents

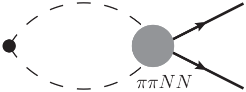
Motivation: Why πN scattering?

- **Low energies:** test chiral dynamics in the baryon sector
  $\Rightarrow$ low-energy theorems e.g. for the scattering lengths
- **Higher energies:** resonances, baryon spectrum
- **Input for NN scattering:** LECs c_i, πNN coupling



- **Crossed channel $\pi\pi \to \bar{N}N$:** nucleon form factors

$\Rightarrow$ probe the structure of the nucleon
$\Rightarrow$ $\sigma_{\pi N}$ term



Motivation: Why Roy-Steiner equations?

Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** $\Rightarrow$ the actual parametrization of the data is irrelevant once it is used in the integral.
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with high precision:
  - $\pi\pi$-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - πK-scattering: [Büttiker et al. (2004)]
  - $\gamma\gamma \to \pi\pi$ scattering: [Hoferichter et al. (2011)]

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Warm up: Roy-equations for $\pi\pi$

- $\pi\pi \to \pi\pi \Rightarrow$ fully crossing symmetric in Mandelstam variables s, t, and $u = 4M_\pi - s - t$
- Start from twice-subtracted **fixed-t** DRs of the generic form

$$T^I(s,t) = c(t) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2}\left[\frac{s^2}{(s'-s)} + \frac{u^2}{(s'-u)}\right] \mathrm{Im}T^I(s',t)$$

- Subtraction functions c(t) are determined via crossing symmetry functions of the I=0,2 scattering lengths: a_0^0 and a_0^2
- PW-expansion of these DRs yields the **Roy-equations** [Roy (1971)]

$$t_J^I(s) = ST_J^I(s) + \sum_{J'=0}^{\infty}(2J'+1)\sum_{I'=0,1,2}\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s',s)\, t_{J'}^{I'}(s')$$

- $K_{JJ'}^{II'}(s',s) \equiv$ kernels $\Rightarrow$ analytically known

Solving Roy-equations: flow information

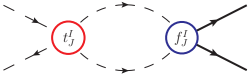
- **Roy-equations** rigorously valid for a finite energy range
  $\Rightarrow$ introduce a matching point s_m
- only partial waves with $J \leq J_{\max}$ are solved
- assume isospin limit
- **Input**
  - High-energy region: $\mathrm{Im}\, t_{IJ}(s)$ for $s \geq s_m$ and for all J
  - Higher partial waves: $\mathrm{Im}\, t_{IJ}(s)$ for $J > J_{\max}$ and for all s
- **Output**
  - Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\max}$ and $s_{\mathrm{th}} \leq s \leq s_m$
  - Constraints on subtraction constants

Roy-Steiner equations for πN: difficulties

Key difficulties compared to $\pi\pi$ Roy-equations

- Crossing: coupling between $\pi N \to \pi N$ (s-channel) and $\pi\pi \to \bar{N}N$ (t -channel)
  $\Rightarrow$ hyperbolic dispersion relations [Hite, Steiner 1973], [Büttiker et al. 2004]
- Unitarity in t-channel, e.g. in single-channel approximation

$$\mathrm{Im} f_\pm^J(t) = \sigma_t^\pi f_\pm^J(t) t_J^I(t)^*$$



$\Rightarrow$ **Watson's theorem**: phase of $f_\pm^J(t)$ equals δ_{IJ} [Watson 1954]

solve with Muskhelishvili-Omnès techniques [Muskhelishvili 1953, Omnès 1958]

$\Rightarrow$ Omnès function: $\Omega_J^I(t) = \exp\left\{ \frac{t}{\pi} \int_{t_{th}}^{tm} dt' \frac{\delta_J^I(t)}{t'(t'-t)} \right\}$

- Large pseudo-physical region in t -channel
  $\Rightarrow$ $\bar{K}K$ intermediate states for s-wave in the region of the $f_0(980)$

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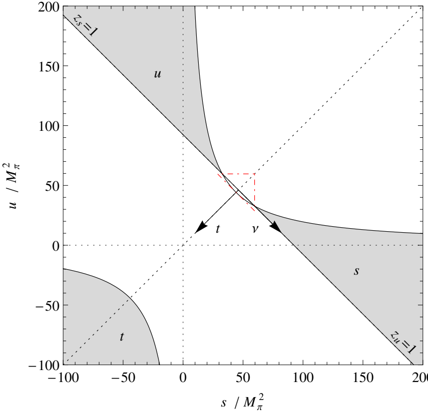
πN-scattering basics

$$\pi^a(q) + N(p) \to \pi^b(q') + N(p')$$

- **Isospin Structure**:
  $T^{ba} = \delta^{ba} T^+ + \epsilon^{ab} T^-$
- **Lorentz Structure**: $I \in \{+, -\}$
  $T^I = \bar{u}(p') \left(A^I + \frac{\not{q} + \not{q}'}{2} B^I \right) u(p)$
  $D^I = A^I + \nu B^I, \quad \nu = \frac{s-u}{4m}$
- **Isospin basis**: $I_s \in \{1/2, 3/2\}$
  $\{T^+, T^-\} \Leftrightarrow \{T^{1/2}, T^{3/2}\}$
- PW projection:
  s-channel pw: $f^I_{l\pm}$
  t-channel pw: $f^J_\pm$

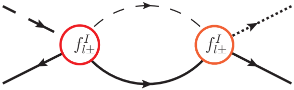
Bose symmetry $\Rightarrow$ even/odd $J \Leftrightarrow I = +/-$



πN-scattering basics: Unitarity relations

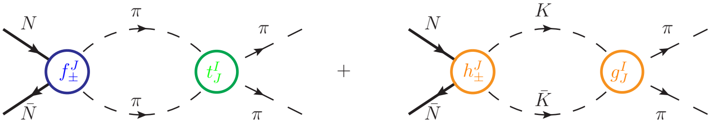
- **s-channel** unitarity relations $(I_s \in \{1/2, 3/2\})$:

$$\mathrm{Im}\, f_{l\pm}^{I_s}(W) = q \left| f_{l\pm}^{I_s}(W) \right|^2 \theta(W - W_+) + \frac{1 - \left(\eta_{l\pm}^{I_s}(W)\right)^2}{4q} \theta(W - W_{\mathrm{inel}})$$



- **t-channel** unitarity relations: 2-body intermediate states: $\pi\pi + \bar{K}K + \cdots$

$$\mathrm{Im}\, f_{\pm}^{J}(t) = \sigma_t^{\pi} \left(t_J^{I_t}(t)\right)^* f_{\pm}^{J}(t)\, \theta(t - t_\pi) + 2 c_J \sqrt{2}\, k_t^{2J} \sigma_t^{K} \left(g_J^{I_t}(t)\right)^* h_{\pm}^{J}(t)\, \theta(t - t_K)$$



- Only linear in $f_{\pm}^{J}(t) \Rightarrow$ less restrictive

Roy-Steiner equations for πN: HDR's

- **Hyperbolic DRs**: $(s-a)(u-a) = b = (s'-a)(u'-a)$ with $a, b \in \mathbb{R}$

$$A^+(s,t;a) = \frac{1}{\pi}\int_{s_+}^{\infty} \mathrm{d}s' \left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right] \mathrm{Im}\, A^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \,\frac{\mathrm{Im}\, A^+(s',t')}{t'-t}$$

$$B^+(s,t;a) = N^+(s,t) + \frac{1}{\pi}\int_{s_+}^{\infty} \mathrm{d}s' \left[\frac{1}{s'-s} - \frac{1}{s'-u}\right] \mathrm{Im}\, B^+(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \,\frac{\nu}{\nu'}\,\frac{\mathrm{Im}\, B^+(s',t')}{t'-t}$$

$$N^+(s,t) = g^2\left(\frac{1}{m^2-s} - \frac{1}{m^2-u}\right)$$

similar for A^-, B^- and N^- [Hite/Steiner (1973)]

- Why **HDR**?
  - Combine all physical regions $\Rightarrow$ crucial for t-channel projection
  - Evade double-spectral regions $\Rightarrow$ the PW decompositions converge
  - Range of convergence can be maximized by tuning the free hyperbola parameter a
  - No kinematical cuts, manageable kernel functions

Roy-Steiner equations for πN: derivation

- Recipe to derive **Roy-Steiner** equations:
  - Expand imaginary parts in terms of s- and t-channel partial waves
  - Project onto s- and t-channel partial waves
  - Combine the resulting equations using s- and t-channel PW unitarity relations
- Similar structure to $\pi\pi$ **Roy equations**
- **Validity**: assuming Mandelstam analyticity
  - s-channel $\Rightarrow$ optimal for $a = -23.2 M_\pi^2$

$$s \in \left[s_+ = (m + M_\pi)^2, 97.30\, M_\pi^2\right] \quad \Leftrightarrow \quad W \in [W_+ = 1.08 \text{ GeV}, 1.38 \text{ GeV}]$$

  - t-channel $\Rightarrow$ optimal for $a = -2.71 M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2, 205.45\, M_\pi^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}]\ .$$

Roy-Steiner equations for πN: subtractions

- **Subtractions** are necessary to ensure the convergence of DR integrals
  $\Rightarrow$ asymptotic behavior
- Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
- Parametrize high-energy information in (a priori unknown) subtraction constants
  $\Rightarrow$ matching to ChPT
- Subthreshold expansion around $\nu = t = 0$

$$\bar{A}^+(\nu, t) = \sum_{m,n=0}^{\infty} a^+_{mn} \nu^{2m} t^n \qquad \bar{A}^-(\nu, t) = \sum_{m,n=0}^{\infty} a^-_{mn} \nu^{2m+1} t^n$$

where

$$\bar{A}^+(s, t) = A^+(s, t) - \frac{g^2}{m} \qquad \bar{A}^-(s, t) = A^-(s, t) \ ,$$

similar expansion for $B^+(s, t)$ and $B^-(s, t)$

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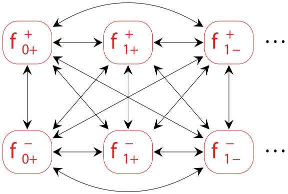
Solving Roy-Steiner equations for πN: Recoupling schemes

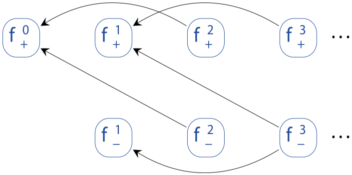
- s-channel subproblem:
  - Kernels are diagonal for $I \in \{+,-\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\}$ $\Rightarrow$ all partial-waves are interrelated
  - Once the t-channel PWs are known

    $\Rightarrow$ Structure similar to $\pi\pi$ Roy-equations



- t-channel subproblem:
  - Only higher PWs couple to lower ones
  - Only PWs with even or odd J are coupled
  - No contribution from f_+^J to f_-^{J+1}

    $\Rightarrow$ Leads to Muskhelishvili-Omnès problem



Solving t-channel equations

- Elastic-channel approximation: generic form of the integral equation

$$f(t) = \Delta(t) + (a + bt)(t - 4m^2) + \frac{t^2(t-4m^2)}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\mathrm{Im}\, f(t')}{t'(t'^2 - 4m^2)(t'-t)}$$

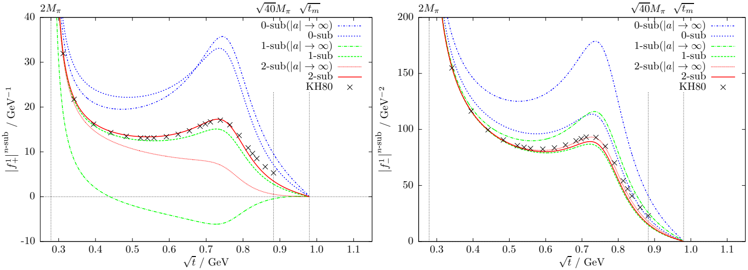
- $\Delta(t)$: Born terms, s-channel integrals, higher t -channel partial waves
  $\Rightarrow$ left-hand cut
- Introduce subtractions at $\nu = t = 0 \Rightarrow$ subthreshold parameters a, b
- Solution in terms of Omnès function:

$$f(t) = \Delta(t) + (t - 4m^2)\Omega(t)(1 - t\dot{\Omega}(0))a + t(t-4m^2)\Omega(t)b$$
$$- \Omega(t)\frac{t^2(t-4m^2)}{\pi}\left\{ \int_{4M_\pi^2}^{t_m} dt' \frac{\Delta(t')\mathrm{Im}\,\Omega(t')^{-1}}{t'^2(t'-4m^2)(t'-t)} + \int_{4M_\pi^2}^{\infty} dt' \frac{\Omega(t')^{-1}\mathrm{Im}\, f(t')}{t'^{(}t'-4m^2)(t'-t)} \right\}$$

$$\Omega(t) = \exp\left\{ \frac{t}{\pi} \int_{t_\pi}^{t_m} \frac{dt'}{t'} \frac{\delta(t')}{t'-t} \right\}$$

Solving t-channel: P-wave results

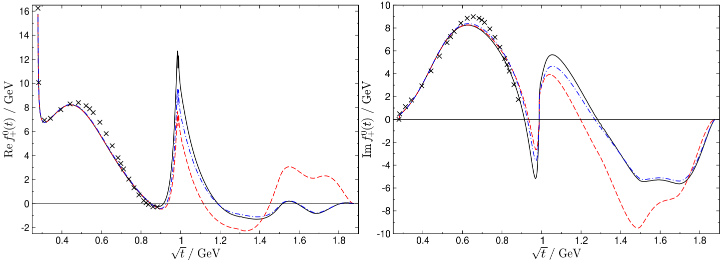
- Important for electromagnetic nucleon form factors
- elastic channel approximation $\Rightarrow \sqrt{t_m} = 0.98$ GeV and $\mathrm{Im} f^J_\pm(t) = 0$ above
- First step: check **consistency** with KH80 [Höhler 1983]
- Input needed:
  - $\pi\pi$ phase shifts: [Caprini, Colangelo, Leutwyler, (in preparation)]
  - πN phase shifts: SAID [Arndt et al. 2008], KH80
  - πN at high energies: Regge model [Huang et al. 2010]
  - πN parameters: KH80



MO solutions in general consistent with KH80 results

Solving t-channel: S-wave results

- Important for the $\sigma_{\pi N}$ term
- $\bar{K}K$ channel important $\Rightarrow$ two-channel Muskhelishvili-Omnès problem
- also needed:
  - $K\bar{K}$ s-wave partial waves: [Büttiker. (2004)]
  - KN s-wave pw: SAID [Arndt et al. 2008], KH80
  - Hyperon couplings from [Jülich model 1989]
  - KN subthreshold parameters neglected



MO solutions in general consistent with KH80 results

Solving s-channel: General form

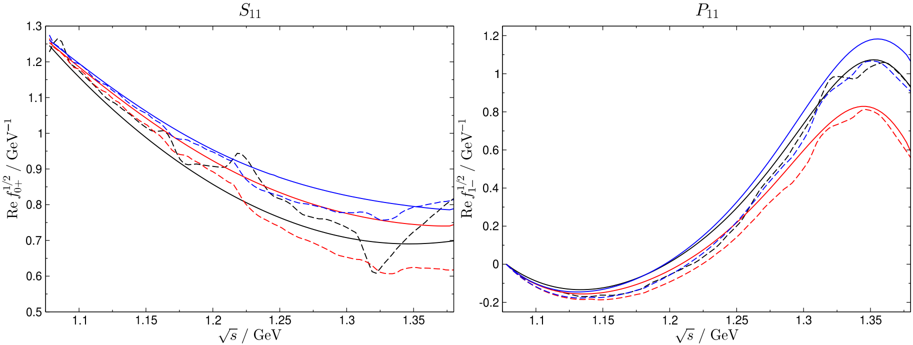
- General form of the s-channel integral equation

$$f^I_{l+}(W) = \Delta^I_{l+}(W) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \left\{ K^I_{ll'}(W,W')\,\mathrm{Im}\, f^I_{l'+}(W') + K^I_{ll'}(W,-W')\,\mathrm{Im}\, f^I_{(l'+1)-}(W') \right\}$$

  $\Rightarrow$ form of $\pi\pi$ Roy-Equations
- $\Delta^I_{l+}(W) \equiv$ t-channel contribution and pole term
- valid up to $W_m = 1.38$ GeV
- **Input**:
  - RS t-channel solutions for S and P waves
  - s-channel partial waves for $J > 1$ [SAID analysis]
  - s-channel partial waves for $W_m < W < 2.5$ GeV [SAID analysis]
  - high energy contribution for $W > 2.5$ GeV: Regge model [Huang et al. 2010]
- **Output**:
  - Self-consistent solution for S and P waves for $s_{\text{th}} \leq s \leq s_m$
  - Constraints on subtraction constants $\Rightarrow$ subthreshold parameters

Solving s-channel: consistency with KH80

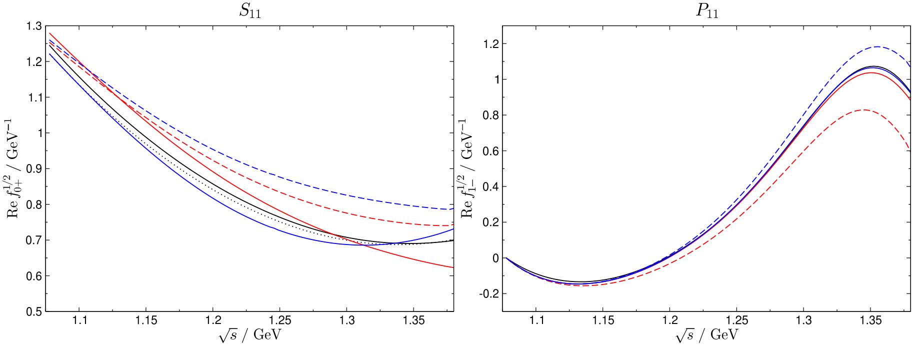
- Consistency with KH80
  - parametrize SAID S and P waves up to W<Wm
    Imposing a continuous and differentiable matching point
  - Compare between the input (LHS) and the output (RHS)



$\Rightarrow$ important discrepancies

Solving s-channel: fitting subthreshold parameters

- Next step $\Rightarrow$ fit only subthreshold parameters, but keep phase shifts fixed
- Minimize the χ-like function: $\chi^2 = \sum_{l,I_s,\pm} \sum_{j=1}^{N} \left(\mathrm{Re} f_{l\pm}^{I_s}(W_j) - F[f_{l\pm}^{I_s}](W_j)\right)^2$
  $\Rightarrow F[f_{l\pm}^{I_s}](W_j) \equiv$ right hand side of RS-equations
- subthreshold parameters change in less than 10%



- Still important differences, especially for the S-waves
  $\Rightarrow$ the pw parametrizations have to be included in the fit $\Rightarrow$ In progress

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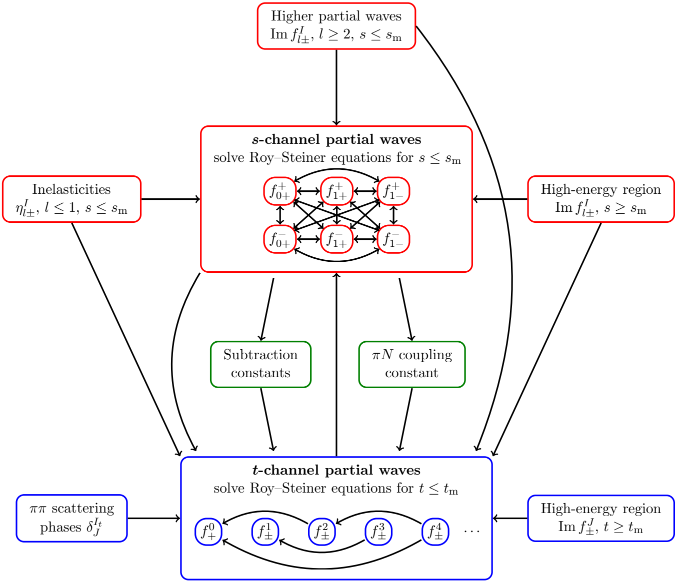
Summary & Outlook

- **What has been done**:
  - Derived a closed system of Roy-Steiner equations (PWHDRs) for πN scattering
  - Constructed unitarity relations including $\bar{K}K$ intermediate states for the t-channel PWs
  - Optimized the range of convergence by tuning a for s- and t-channel each
  - Implemented subtractions at several orders
  - Solved the t-channel MO problem for a single- and two-channel approximation
    $\Rightarrow$ t-channel RS/MO machinery works
  - Numerical solution of the s-channel subproblem fitting the subthreshold parameters
- **What needs to be done**:
  - Self-consistent, iterative solution of the full RS system
    $\Rightarrow$ lowest PWs & low-energy parameters
  - Possible improvements: higher PWs, more inelastic input, ...

Spare slides

Roy-Steiner equations for πN: flow of information



πN-scattering basics: partial waves

- **s-channel** projection:

$$f^I_{l\pm}(W) = \frac{1}{16\pi W}\Big\{(E+m)\big[A^I_l(s) + (W-m)B^I_l(s)\big] + (E-m)\big[-A^I_{l\pm1}(s) + (W+m)B^I_{l\pm1}(s)\big]\Big\}$$

$$X^I_l(s) = \int_{-1}^{1} \mathrm{d}z_s\, P_l(z_s) X^I(s,t)\Big|_{t=t(s,z_s)=-2q^2(1-z_s)} \quad \text{for } X\in\{A,B\} \text{ and } W=\sqrt{s}$$

- **McDowell symmetry**: $f^I_{l+}(W) = -f^I_{(l+1)-}(-W) \qquad \forall\, l \geq 0$
- **t-channel** projection:

$$f^J_+(t) = -\frac{1}{4\pi}\int_0^1 \mathrm{d}z_t\, P_J(z_t)\Big\{\frac{p_t^2}{(p_t q_t)^J} A^I(s,t)\Big|_{s=s(t,z_t)} - \frac{m}{(p_t q_t)^{J-1}} z_t B^I(s,t)\Big|_{s=s(t,z_t)}\Big\} \qquad \forall\, J\geq 0$$

$$f^J_-(t) = \frac{1}{4\pi}\frac{\sqrt{J(J+1)}}{2J+1}\frac{1}{(p_t q_t)^{J-1}}\int_0^1 \mathrm{d}z_t \big[P_{J-1}(z_t) - P_{J+1}(z_t)\big] B^I(s,t)\Big|_{s=s(t,z_t)} \qquad \forall\, J\geq 1$$

- **Bose symmetry** $\Rightarrow$ even/odd $J \Leftrightarrow I = +/-$

Roy-Steiner equations for πN: s-channel

s-channel RS equations

$$f^I_{l+}(W) = N^I_{l+}(W) + \frac{1}{\pi}\int_{W_+}^{\infty} dW' \sum_{l'=0}^{\infty} \Big\{ K^I_{ll'}(W,W')\,\mathrm{Im}\, f^I_{l'+}(W') + K^I_{ll'}(W,-W')\,\mathrm{Im}\, f^I_{(l'+1)-}(W') \Big\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} dt' \sum_J \Big\{ G_{lJ}(W,t')\,\mathrm{Im}\, f^J_+(t') + H_{lJ}(W,t')\,\mathrm{Im}\, f^J_-(t') \Big\}$$

$$= -f^I_{(l+1)-}(-W) \qquad \forall\, l \geq 0\,, \quad \text{[Hite/Steiner (1973)]}$$

- $K^I_{ll'}(W,W')$, $G_{lJ}(W,t')$ and $H_{lJ}(W,t')$-Kernels: analytically known,
  e.g. $K^I_{ll'}(W,W') = \frac{\delta_{ll'}}{W'-W} + \dots \quad \forall\, l,l' \geq 0$,
- **Validity**: assuming Mandelstam analyticity
  $\Rightarrow$ optimal for $a = -23.2 M_\pi^2$

$$s \in \big[s_+ = (m+M_\pi)^2, 97.30\, M_\pi^2\big] \quad \Leftrightarrow \quad W \in [W_+ = 1.08\text{ GeV}, 1.38\text{ GeV}]$$

Roy-Steiner equations for πN: t-channel

t-channel RS equations

$$f_+^J(t) = \tilde{N}_+^J(t) + \frac{1}{\pi}\int_{W_+}^{\infty} \mathrm{d}W' \sum_{l=0}^{\infty} \Big\{ \tilde{G}_{Jl}(t,W')\,\mathrm{Im}\, f_{l+}^I(W') + \tilde{G}_{Jl}(t,-W')\,\mathrm{Im}\, f_{(l+1)-}^I(W') \Big\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \sum_{J'} \Big\{ \tilde{K}_{JJ'}^1(t,t')\,\mathrm{Im}\, f_+^{J'}(t') + \tilde{K}_{JJ'}^2(t,t')\,\mathrm{Im}\, f_-^{J'}(t') \Big\} \qquad \forall\, J \geq 0\ ,$$

$$f_-^J(t) = \tilde{N}_-^J(t) + \frac{1}{\pi}\int_{W_+}^{\infty} \mathrm{d}W' \sum_{l=0}^{\infty} \Big\{ \tilde{H}_{Jl}(t,W')\,\mathrm{Im}\, f_{l+}^I(W') + \tilde{H}_{Jl}(t,-W')\,\mathrm{Im}\, f_{(l+1)-}^I(W') \Big\}$$

$$+ \frac{1}{\pi}\int_{t_\pi}^{\infty} \mathrm{d}t' \sum_{J'} \tilde{K}_{JJ'}^3(t,t')\,\mathrm{Im}\, f_-^{J'}(t') \qquad \forall\, J \geq 1\ ,$$

- **Validity**: assuming Mandelstam analyticity
  $\Rightarrow$ optimal for $a = -2.71 M_\pi^2$

$$t \in [t_\pi = 4M_\pi^2, 205.45\, M_\pi^2] \quad \Leftrightarrow \quad \sqrt{t} \in [\sqrt{t_\pi} = 0.28 \text{ GeV}, 2.00 \text{ GeV}]\ .$$

RS-eqs for πN: Range of convergence

- Subthreshold expansion around $\nu = t = 0$

$$A^+(\nu,t) = \frac{g^2}{m} + d^+_{00} + d^+_{01} t + a^+_{10}\nu^2 + \mathcal{O}\big(\nu^2 t, t^2\big)$$

$$A^-(\nu,t) = \nu a^-_{00} + a^-_{01}\nu t + a^-_{10}\nu^3 + \mathcal{O}\big(\nu^5, \nu t^2, \nu^3 t\big)$$

$$B^+(\nu,t) = g^2 \frac{4m\nu}{(m^2-s_0)^2} + \nu b^+_{00} + \mathcal{O}\big(\nu^3, \nu t\big) \ ,$$

$$B^-(\nu,t) = g^2\left[\frac{2}{m^2-s_0} - \frac{t}{(m^2-s_0)^2}\right] - \frac{g^2}{2m^2} + b^-_{00} + b^-_{01} t + +b^-_{10}\nu^2 + \mathcal{O}\big(\nu^2, \nu^2 t, t^2\big)$$

- pseudovector Born terms: $D^I = A^I + \nu B^I$

$$\bar{D}^+ = d^+_{00} + d^+_{01} t + d^+_{10}\nu^2$$

$$d^+_{mn} = a^+_{mn} + b^+_{m-1,n} \ , \qquad d^-_{mn} = a^-_{mn} + b^-_{mn} \ .$$

- Sum rules for subthreshold parameters:

$$d^+_{00} = -\frac{g^2}{m} + \frac{1}{\pi}\int_{s_+}^{\infty} \mathrm{d}s'\, h_0(s')\,\big[\mathrm{Im}\, A^+(s', z'_s)\big]_{(0,0)} + \frac{1}{\pi}\int_{t_\pi}^{\infty} \frac{\mathrm{d}t'}{t'}\,\big[\mathrm{Im}\, A^+(t', z'_t)\big]_{(0,0)}$$

$$h_0(s') = \frac{2}{s'-s_0} - \frac{1}{s'-a}$$

RS-eqs for πN: Range of convergence

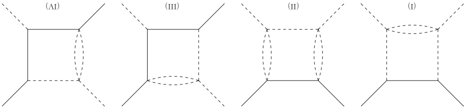
- Assumption: Mandelstam analyticity [Mandelstam (1958,1959)]

  $\Rightarrow$ T(s,t) can be written in terms double spectral densities: ρ_{st}, ρ_{su}, ρ_{ut}

  $$T(s,t) = \frac{1}{\pi^2}\iint ds'du' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\iint dt'du' \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\iint ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

  integration ranges defined by the support of the double spectral densities ρ

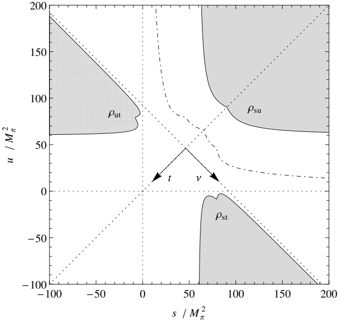
- Boundaries of ρ are given lowest lying intermediate states



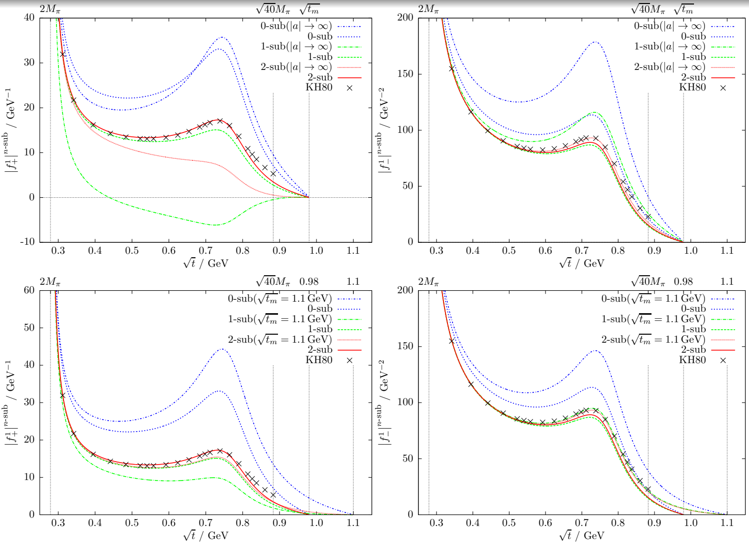
- They limit the range of validity of the HDRS:
  - Pw expansion converge

    $\Rightarrow z = \cos\theta \in$ Lehman ellipses [Lehmann (1958)]
  - the hyperbolae $(s-a)(u-a) = b$ does not enter any double spectral region

    $\Rightarrow$ for a value of a, constraints on b yield ranges in s & t



Solving t-channel: P-wave results



MO solutions in general consistent with KH80 results

Solving t-channel equations: S-waves

- Generic coupled-channel integral equation

$$\mathbf{f}(t) = \boldsymbol{\Delta}(t) + \frac{1}{\pi}\int_{t_\pi}^{t_m} dt' \frac{T^*(t')\Sigma(t')\mathbf{f}(t')}{t'-t} + \frac{1}{\pi}\int_{t_m}^{\infty} dt' \frac{\operatorname{Im}\mathbf{f}(t')}{t'-t}$$

- Formal solution as in the single-channel case (now with Omnès matrix $\Omega(t)$)
  $\Rightarrow$ Two-channel Muskhelishvili-Omnès problem

$$\mathbf{f}(t) = \begin{pmatrix} f_+^0(t) \\ h_+^0(t) \end{pmatrix} \qquad \operatorname{Im}\Omega(t) = (T(t))^*\Sigma(t)\Omega(t)$$

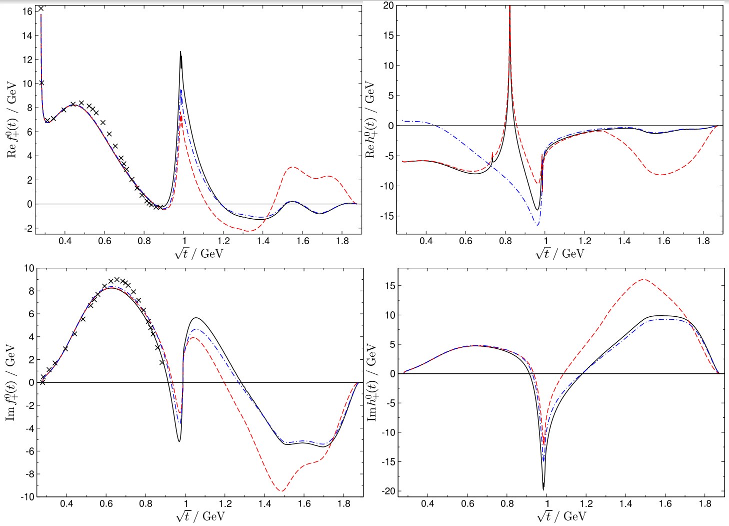
- Two linearly independent solutions Ω_1, Ω_2 [Muskhelishvili 1953]
- In general no analytical solution for the Omnès matrix but for its determinant [Moussallam 2000]

$$\det\Omega(t) = \exp\left\{\frac{t}{\pi}\int_{t_\pi}^{t_m} dt' \frac{\psi(t')}{t'(t'-t)}\right\}.$$

Solving t-channel S-wave equations: input

- Input needed:
  - $\pi\pi$ s-wave partial waves: [Caprini, Colangelo, Leutwyler, (in preparation)]
  - $K\bar{K}$ s-wave partial waves: [Büttiker. (2004)]
  - πN and KN s-wave pw: SAID [Arndt et al. 2008], KH80
  - πN at high energies: Regge model [Huang et al. 2010]
  - πN parameters: KH80
  - Hyperon couplings from [Jülich model 1989]
  - KN subthreshold parameters neglected
- Two-channel approximation beaks down at $\sqrt{t_0} = 1.3$ GeV $\Rightarrow 4\pi$ channel
- From t_0 to $t = 2$ GeV, different approximations considered

Solving t-channel: S-wave results



MO solutions in general consistent with KH80 results

Solving s-channel: threshold parameters

- Precise data for pionic atoms [Gotta et al. 2005, 2010]
  $\Rightarrow$ Impose as a **constraint** scattering lengths from a combined analysis of pionic hydrogen and deuterium [Baru et al. 2011]

$$a_{0+}^{1/2} = (170.5 \pm 2.0)10^{-3}M_\pi^{-1} \quad a_{0+}^{3/2} = (-86.5 \pm 1.8)10^{-3}M_\pi^{-1}$$

$$\mathrm{Re}\, f_{l\pm}^I(s) = \mathbf{q}^{2l}\left(a_{l\pm}^I + b_{l\pm}^I \mathbf{q}^2 + \cdots\right)$$

- s-channel scattering lengths from RS sum rules

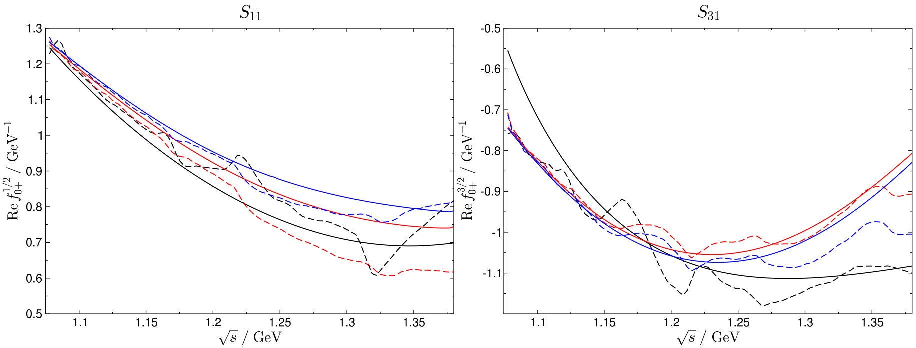
		1-sub	2-sub	3-sub	KH80
S11	$a_{0+}^{1/2}[10^{-3}M_\pi^{-1}]$	**-40.9**	146.0	175.0	173±3
S31	$a_{0+}^{3/2}[10^{-3}M_\pi^{-1}]$	**-72.3**	-132.4	-103.5	-101±4

$\Rightarrow$ three subtractions needed

Solving s-channel: consistency with KH80

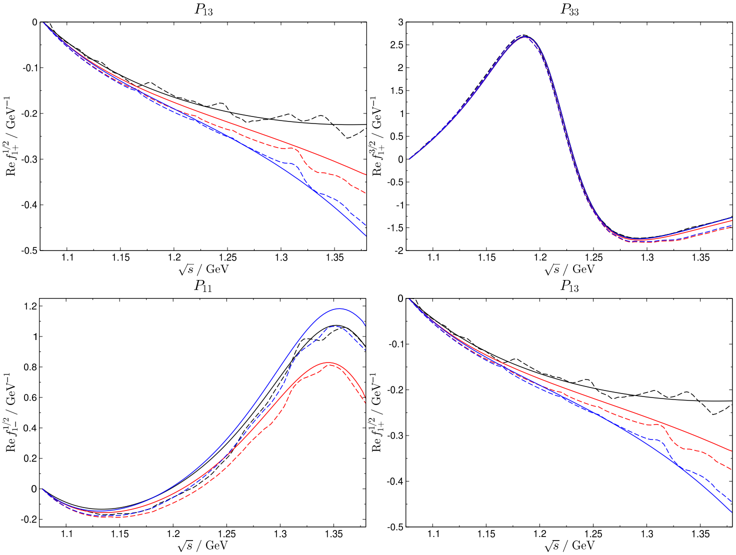
- Consistency with KH80
  - parametrize SAID S and P waves up to W<Wm
    Imposing a continuous and differentiable matching point
  - Compare between the input (LHS) and the output (RHS)

S-WAVES



$\Rightarrow$ important discrepancies

Solving s-channel: consistency with KH80. P-waves



Solving s-channel: fitting subthreshold parameters

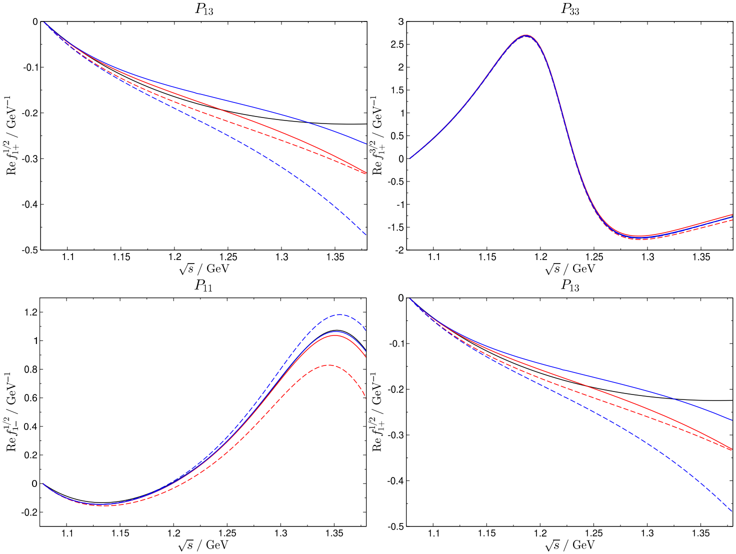
- Next step $\Rightarrow$ fit only subthreshold parameters, but keep phase shifts fixed
- Minimize the χ-like function:

$$\chi^2 = \sum_{l,I_s,\pm} \sum_{j=1}^{N} \left(\text{Re} f_{l\pm}^{I_s}(W_j) - F[f_{l\pm}^{I_s}](W_j) \right)^2$$

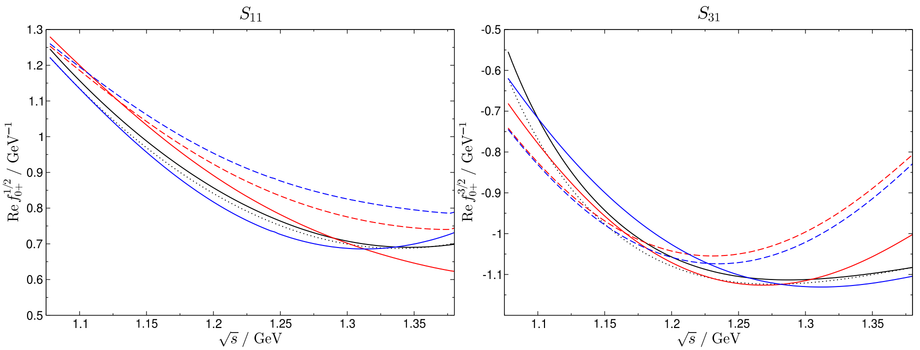
$\Rightarrow F[f_{l\pm}^{I_s}](W_j) \equiv$ right hand side of RS-equations

- small change in the subthreshold parameters

Solving s-channel: fitting subthreshold parameters. P-waves



Solving s-channel: fitting subthreshold parameters. S-waves



- Still important differences, especially for the S-waves
  $\Rightarrow$ the pw parametrizations have to be included in the fit
  In progress