Chiral Kinetic Theory and Berry Phase

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S.Pu, J.H.Gao, Q.Wang, PRD 83, 094017(2011)
J.H.Gao, Z.T.Liang, S.Pu, Q.Wang, X.N. Wang, PRL 109, 232301(2012)
J.W. Chen, S.Pu, Q.Wang, X.N. Wang, PRL 110, 262301(2013)

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Outline

- **Core question:**

how to describe quantum phenomena like anomaly in semi-classical way for charged massless fermion (chiral) in gauge field?

- **Quantum Kinetic Theory:**

(1) Chiral magnetic and Vortical effect (CME/CVE)

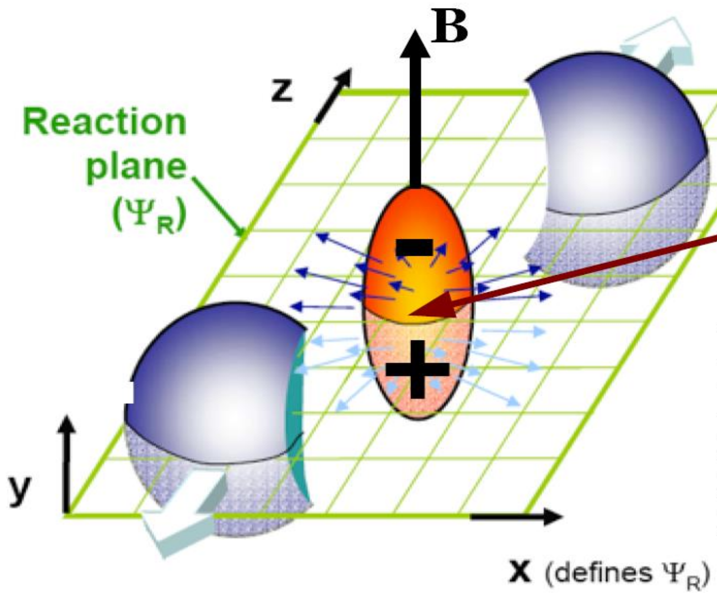
(2) Berry phase and monopole structure

(3) Covariant Chiral Kinetic Equation (CCKE) in quanum kinetic theory

Ultra-high Magnetic field in HIC

$\langle Q \rangle = 0$

$\langle Q^2 \rangle \neq 0$



Fluctuating EDM of QGP
P- and CP-odd effect
Kharzeev ('06),
Kharzeev and Zhitnitsky ('07)
Kharzeev, Mclerran and Warringa ('08)

$$eB \approx 10^3 - 10^4\ \mathrm{MeV}^2 \sim 10^{18}\ \mathrm{Gauss}$$

Chirality and Helicity

- Chiraltiy $\psi_L = \frac{1}{2}(1-\gamma^5)\psi, \quad \psi_R = \frac{1}{2}(1+\gamma^5)\psi$

- Helicity $h = \boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|}$

- In the chiral limit (massless quark) with $m_f = 0$

Helicity	RH chirality	LH chirality
Particle	+1	-1
Anti-particle	-1	+1

Axial Anomaly and Winding number

- All gauge field configurations are classified by the topological winding numbers ($\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} F^a_{\rho\sigma}$)

$$Q_w = \frac{g^2}{32\pi^2}\int d^4x F_{\alpha\beta}\tilde{F}^{\alpha\beta}$$
$$= N_{CS}(\mathrm{t}=\infty) - N_{CS}(t=-\infty)$$

- Axial anomaly

$$j^5_\mu = \sum_f \langle\bar{\psi}_f\gamma_\mu\gamma_5\psi_f\rangle_A$$

Average over gluon field configuration

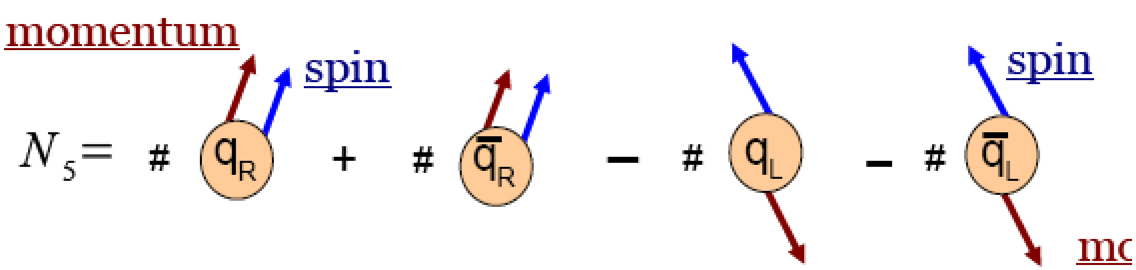
$$\partial_\sigma j^\sigma_5 = -\frac{N_f g^2}{16\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a$$

Axial Anomaly and Winding number

- Chiral charge number at chiral limit:

$$N_5 = N_R - N_L = (n_R - \bar{n}_R) - (n_L - \bar{n}_L)$$
$$= (n_R + \bar{n}_L) - (n_L + \bar{n}_R)$$
$$= n(h = +1) - n(h = -1)$$

R,L denote chirality



R,L denote helicity

- Assuming $N_R(t = 0) = N_L(t = 0)$, then we have

$$N_5(t = \infty) = -2N_f Q_w = -2N_f \Delta N_{CS}$$

Chiral Magnetic Effect

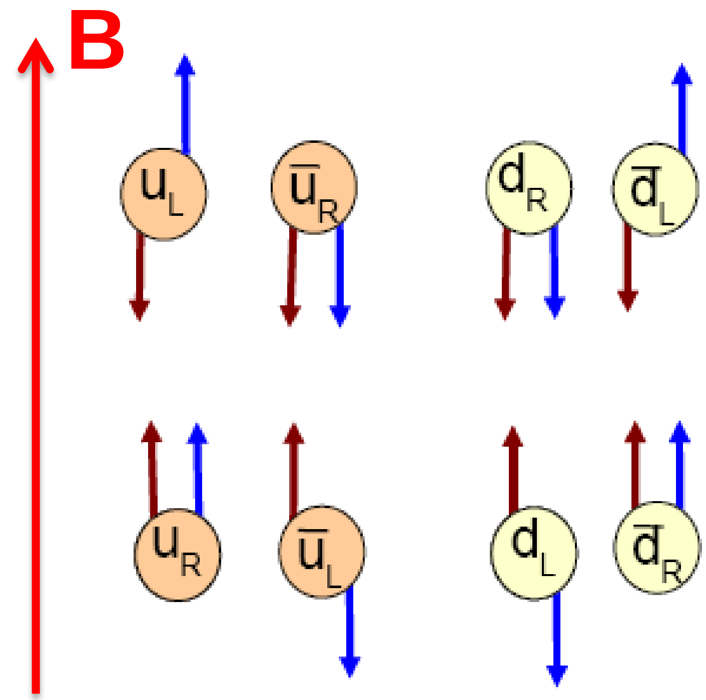
- Magnetic field aligns spin depending on electric charge; The momenta of quarks and antiquarks align along the magnetic field.
- Quarks with RH-helicity move opposite to those with LH-helicity
- Momentum-down:

  $d_R + \bar{u}_R \ (Q_e = -)$

  $u_L + \bar{d}_L \ (Q_e = +)$
- Momentum-up:

  $u_R + \bar{d}_R \ (Q_e = +)$

  $d_L + \bar{u}_L \ (Q_e = -)$



What is Berry phase

- **Let us consider a quantum system with two sets of parameters, rapidly changing parameter $\vec{r}$ and slowly changing one $\vec{R}(t)$. The Schroedinger equation is**

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H(\mathbf{R}(t))\left|\psi(t)\right\rangle \qquad (1)$$

- **The energy eigen-states satisfy at a moment when $\vec{R}(t)$ changes slowly with time**

$$H(\mathbf{R}(t))\left|n,\mathbf{R}(t)\right\rangle = E_n(\mathbf{R}(t))\left|n,\mathbf{R}(t)\right\rangle$$

$$\langle n,\mathbf{R}(t)|n',\mathbf{R}(t)\rangle = \delta_{n,n'}$$

What is Berry phase

- **We assume the initial state is in an eigen-state**

$$|\psi(t)\rangle = a_n(t)e^{-i\int_0^t dt' E_n(t')}\,|n,\mathbf{R}(t)\rangle \qquad (2)$$

- **Substitute the above into Schroedinger Eq. (1),**

$$\begin{aligned} 0 &= e^{-i\int_0^t dt' E_n(t')}\frac{da_n(t)}{dt}\,|n,\mathbf{R}(t)\rangle \\ &\quad + e^{-i\int_0^t dt' E_n(t')}a_n(t)\frac{d}{dt}\,|n,\mathbf{R}(t)\rangle \end{aligned} \qquad (3)$$

- **Then we have**

$$\frac{d}{dt}a_n(t) = -a_n(t)\,\underbrace{\langle n,\mathbf{R}(t)|\,\frac{d}{dt}\,|n,\mathbf{R}(t)\rangle}_{i\tilde{\gamma}_n(t)}$$

What is Berry phase

- **We can verify that** $i\tilde{\gamma}_n(t) = \langle n, \mathbf{R}(t)| \frac{d}{dt} |n, \mathbf{R}(t)\rangle$ **is imaginary**

$$\begin{aligned} \langle n, \mathbf{R}(t)|n, \mathbf{R}(t)\rangle &= 1 \\ \langle n, \mathbf{R}(t)| \frac{d}{dt} |n, \mathbf{R}(t)\rangle + [\frac{d}{dt} \langle n, \mathbf{R}(t)|]\, |n, \mathbf{R}(t)\rangle &= 0 \\ \langle n, \mathbf{R}(t)| \frac{d}{dt} |n, \mathbf{R}(t)\rangle + \langle n, \mathbf{R}(t)| \frac{d}{dt} |n, \mathbf{R}(t)\rangle^* &= 0 \end{aligned}$$

- **We can define a phase factor**

$$\frac{d}{dt} a_n(t) = -i\tilde{\gamma}_n(t) a_n(t) \rightarrow \begin{aligned} a_n(t) &= \exp[-i\gamma_n(t)] a_n(0) \\ \boxed{\gamma_n(t)} &= \int_0^t dt' \tilde{\gamma}_n(t') \end{aligned}$$

Berry phase

What is Berry phase

- **The solution to the Schroedinger equation becomes**

$$|\psi(t)\rangle = \sum_n a_n(0) \exp[\underbrace{-i\gamma_n(t)}_{\text{Berry phase}} - i\int_0^t dt' E_n(\mathbf{R}(t'))] \, |n, \mathbf{R}(t)\rangle$$

- **If we consider a loop in parameter space** $\mathbf{R}(t_f) = \mathbf{R}(t_0)$

$$\begin{aligned} \gamma_n(\mathrm{C}) &= -i\int_{t_0}^{t_f} dt \frac{d\mathbf{R}}{dt} \cdot \langle n, \mathbf{R}(t)| \nabla_{\mathbf{R}} |n, \mathbf{R}(t)\rangle \\ &= -i\oint_C d\mathbf{R} \cdot \langle n, \mathbf{R}(t)| \nabla_{\mathbf{R}} |n, \mathbf{R}(t)\rangle \end{aligned}$$

- **This induces a Berry connection in parameter space (R-space)**

$$\mathbf{A}(\mathbf{R}) = -i\langle n, \mathbf{R}| \nabla_{\mathbf{R}} |n, \mathbf{R}\rangle \qquad \text{Berry connection}$$

$$\gamma_n(\mathrm{C}) = \oint_C d\mathbf{R} \cdot \mathbf{A}(\mathbf{R}) = \int_{\text{Area}} d\boldsymbol{\sigma} \cdot \boxed{(\nabla \times \mathbf{A})} \longrightarrow \text{Berry curvature}$$

Berry phase of chiral fermion

- **Now we consider chiral fermion with Hamiltonian $\boldsymbol{H = \sigma \cdot \vec{p}}$,**

$$
\begin{aligned}
H u_{\mathbf{p}}(e) &= e|\mathbf{p}| u_{\mathbf{p}}(e) \\
\sigma \cdot \hat{\mathrm{p}} &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}
\end{aligned}
$$

- **The positive and negative helicity states**

$$
u_{\mathbf{p}}(\uparrow) = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad u_{\mathbf{p}}(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}
$$

Berry phase of chiral fermion

- **The Berry connection can be evaluated as,**

$$
\begin{aligned}
\mathbf{a}(\mathbf{p}) &= -iu_{\mathbf{p}}^{\dagger}(\uparrow)\nabla_{\mathbf{p}}u_{\mathbf{p}}(\uparrow) \\
&= -i(e^{i\phi}\cos\frac{\theta}{2},\sin\frac{\theta}{2})\left(\mathbf{e}_{\theta}\frac{1}{|\mathbf{p}|}\frac{\partial}{\partial\theta}+\mathbf{e}_{\phi}\frac{1}{|\mathbf{p}|\sin\theta}\frac{\partial}{\partial\phi}\right)\begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \\
&= -\mathbf{e}_{\phi}\frac{1}{|\mathbf{p}|\sin\theta}\cos^2\frac{\theta}{2} = \boxed{-\mathbf{e}_{\phi}\frac{1}{2|\mathbf{p}|}\frac{\cos(\theta/2)}{\sin(\theta/2)}}
\end{aligned}
$$

- **where in spherical coordinates,**

$$
\nabla_{\mathbf{p}} = \mathbf{e}_{p}\frac{\partial}{\partial|\mathbf{p}|}+\mathbf{e}_{\theta}\frac{1}{|\mathbf{p}|}\frac{\partial}{\partial\theta}+\mathbf{e}_{\phi}\frac{1}{|\mathbf{p}|\sin\theta}\frac{\partial}{\partial\phi}
$$

Berry phase of chiral fermion

- **The Berry curvature for chiral fermion is**

$$\boldsymbol{\Omega} = \nabla_{\mathbf{p}} \times \mathbf{a}(\mathbf{p}) \;=\; -\frac{1}{|\mathbf{p}|^2 \sin\theta} \frac{\partial}{\partial\theta} (\cos^2\frac{\theta}{2}) \hat{\mathbf{p}} = \frac{\mathbf{p}}{2|\mathbf{p}|^3}$$

- **This is a monopole, for $\vec{p} \neq 0$, the divergence is vanishing because**

$$\begin{aligned} \nabla_{\mathbf{p}} \cdot [\nabla_{\mathbf{p}} \times \mathbf{a}(\mathbf{p})] \;&=\; 0 \\ \nabla_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{|\mathbf{p}|^3}\right) \;&=\; \frac{3}{|\mathbf{p}|^3} - \frac{3\mathbf{p}}{|\mathbf{p}|^4} \cdot \nabla_{\mathbf{p}} |\mathbf{p}| = 0 \end{aligned}$$

Berry phase of chiral fermion

- **But it is non-vanishing when calculating the total flux on a sphere**

$$\int d^3p \nabla_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{|\mathbf{p}|^3} \right) = \oint d\boldsymbol{\sigma} \cdot \frac{\mathbf{p}}{|\mathbf{p}|^3} = 4\pi$$

- **Therefore we obtain the monopole form of the Berry curvature**

$$\nabla_{\mathbf{p}} \cdot \mathbf{\Omega} = \nabla_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{2|\mathbf{p}|^3} \right) = 2\pi\delta^{(3)}(\mathbf{p})$$

Analogy to magnetic field

- Berry curvature

$$\mathbf{\Omega}(\mathbf{R})$$

- Berry connection

$$\mathbf{a}(\mathbf{R})$$

- Geometric phase

$$\oint_C d\mathbf{R}\cdot\mathbf{a}(\mathbf{R}) = \int\int d\boldsymbol{\sigma}\cdot\mathbf{\Omega}(\mathbf{R})$$

- Chern-Simons number

$$\oiint d\boldsymbol{\sigma}\cdot\mathbf{\Omega}(\mathbf{R}) \quad = \quad \text{integer}$$

- Magnetic field

$$\mathbf{B}(\mathbf{r})$$

- Vector potential

$$\mathbf{A}(\mathbf{r})$$

- Ahanonrov-Bohm phase

$$\oint_C d\mathbf{r}\cdot\mathbf{A}(\mathbf{r}) \quad = \quad \int\int d\mathbf{S}\cdot\mathbf{B}(\mathbf{r})$$

- Dirac monopole

$$\oiint d\mathbf{S}\cdot\mathbf{B}(\mathbf{r}) \quad = \quad \text{integer}\times\frac{h}{e}$$

Phase space description of charged fermion

- **A charged fermion in EM field, treat (x,p) in equal footing,**

$$S(\boldsymbol{x},\boldsymbol{p}) = \int dt[(\boldsymbol{p}+\boldsymbol{A}(\boldsymbol{x}))\cdot\dot{\boldsymbol{x}} - \phi(\boldsymbol{x}) - \epsilon(\boldsymbol{p})]$$

- **EOM can be derived from Euler-Lagrange equation**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{p}}} - \frac{\partial L}{\partial \boldsymbol{p}} = 0 \to \dot{\boldsymbol{x}} = \frac{\partial \epsilon(\boldsymbol{p})}{\partial \boldsymbol{p}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} - \frac{\partial L}{\partial \boldsymbol{x}} = 0$$

$$\to \dot{\boldsymbol{p}} + \boxed{\frac{d}{dt}\boldsymbol{A}(\boldsymbol{x})} = -\frac{\partial \phi(\boldsymbol{x})}{\partial \boldsymbol{x}} + \dot{\boldsymbol{x}}_i \frac{\partial \boldsymbol{A}_i(\boldsymbol{x})}{\partial \boldsymbol{x}}$$

$$\frac{d}{dt}\boldsymbol{A}(\boldsymbol{x}) = \frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{x}) + \dot{\boldsymbol{x}}_j \frac{\partial \boldsymbol{A}_i(\boldsymbol{x})}{\partial \boldsymbol{x}_j}$$

$$\to \dot{\boldsymbol{p}} = \boldsymbol{E} + \dot{\boldsymbol{x}} \times \boldsymbol{B}$$

Phase space description of charged fermion

- **Re-defining variables $\xi^i = x_i, \xi^{i+3} = p_i$ with ($i = 1,2,3$)**

$$S(\xi) = \int dt[\gamma_a(\xi)\dot{\xi}^a - H(\xi)]$$

No conjugate variables for $\dot{p}$

- **where** $H(\xi) = \phi(\boldsymbol{x}) + \epsilon(\boldsymbol{p})$ **and** $\gamma_a(\xi) = (\boldsymbol{p} + \boldsymbol{A}(\boldsymbol{x}), \boldsymbol{0})$
- **EOM is from Euler-Lagrange Equation**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}^a} = \frac{\partial L}{\partial \xi^a} \to \frac{\partial \gamma_a(\xi)}{\partial \xi^b}\dot{\xi}^b = \frac{\partial \gamma_b(\xi)}{\partial \xi^a}\dot{\xi}^b - \frac{\partial H(\xi)}{\partial \xi^a}$$

$$\gamma_{ab}\dot{\xi}^b = -\frac{\partial H(\xi)}{\partial \xi^a}$$

EOM

$$\gamma_{ab} = \frac{\partial \gamma_a(\xi)}{\partial \xi^b} - \frac{\partial \gamma_b(\xi)}{\partial \xi^a}$$

$$[\gamma_{ab}] = \begin{pmatrix} 0 & -B_3 & B_2 & 1 & 0 & 0 \\ B_3 & 0 & -B_1 & 0 & 1 & 0 \\ -B_2 & B_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Phase space description of charged fermion

- **EOM can be cast to a normal Poisson Bracket in Hamiltonian dynamics**

$$\gamma_{ab}\dot{\xi}^b = -\frac{\partial H(\xi)}{\partial \xi^a} \Longrightarrow \dot{\xi}^b = -[\gamma^{-1}]_{ba}\frac{\partial H(\xi)}{\partial \xi^a} \longleftarrow \text{from Lagrangian}$$

$$\dot{\xi}^a = -\{\xi^a, H\} \longleftarrow \text{from Hamiltonian}$$

- **Now we add the Berry connection term $-\vec{a}(\vec{p})\cdot\dot{\vec{p}}$ to Lagrangian**

$$S(\boldsymbol{x},\boldsymbol{p}) = \int dt[(\boldsymbol{p}+\boldsymbol{A}(\boldsymbol{x}))\cdot\dot{\boldsymbol{x}} - \boldsymbol{a}(\boldsymbol{p})\cdot\dot{\boldsymbol{p}} - \phi(\boldsymbol{x}) - \epsilon(\boldsymbol{p})]$$

$$S(\xi) = \int dt[\gamma_a(\xi)\dot{\xi}^a - H(\xi)]$$

$$\longrightarrow \gamma_a(\xi) = (\boldsymbol{p}+\boldsymbol{A}(\boldsymbol{x}), -\boldsymbol{a}(\boldsymbol{p}))$$

Phase space description of charged fermion

- **EOM can be rewritten into a form which can be compared to Hamiltonian representation**

$$\gamma_{ab}\dot{\xi}^b = -\frac{\partial H(\xi)}{\partial \xi^a} \Longrightarrow \dot{\xi}^b = -[\gamma^{-1}]_{ba}\frac{\partial H(\xi)}{\partial \xi^a}$$

$$[\gamma_{ab}] = \begin{pmatrix} 0 & -B_3 & B_2 & 1 & 0 & 0 \\ B_3 & 0 & -B_1 & 0 & 1 & 0 \\ -B_2 & B_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \Omega_3 & -\Omega_2 \\ 0 & -1 & 0 & -\Omega_3 & 0 & \Omega_1 \\ 0 & 0 & -1 & \Omega_2 & -\Omega_1 & 0 \end{pmatrix}$$

$$\Rightarrow$$

$$\det(\gamma) = (1 + \boldsymbol{B}\cdot\boldsymbol{\Omega})^2$$

$$\boldsymbol{\Omega} = \nabla_{\boldsymbol{p}} \times \boldsymbol{a}(\boldsymbol{p})$$

Berry curvature

Phase space description of charged fermion

- **The inverse matrix $[\gamma^{-1}]$ and EOM**

$$\gamma^{-1} = \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 & -1-B_1\Omega_1 & -B_1\Omega_2 & -B_1\Omega_3 \\ -\Omega_3 & 0 & \Omega_1 & -B_2\Omega_1 & -1-B_2\Omega_2 & -B_2\Omega_3 \\ \Omega_2 & -\Omega_1 & 0 & -B_3\Omega_1 & -B_3\Omega_2 & -1-B_3\Omega_3 \\ 1+B_1\Omega_1 & B_2\Omega_1 & B_3\Omega_1 & 0 & -B_3 & B_2 \\ B_1\Omega_2 & 1+B_2\Omega_2 & B_3\Omega_2 & B_3 & 0 & -B_1 \\ B_1\Omega_3 & B_2\Omega_3 & 1+B_3\Omega_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\dot{\xi}^a = -[\gamma^{-1}]^{ab}\frac{\partial H(\xi)}{\partial \xi^b} \longrightarrow \text{from Euler-Lagrange equation}$$

Hamiltonian dynamics

- **Comparing Lagranian and Hamiltonian, we arrive at**

$$\{\xi^a, \xi^b\} = [\gamma^{-1}]^{ab} \quad \Rightarrow \quad \begin{aligned} \{x_i, x_j\} &= \frac{\epsilon_{ijk}\Omega_k}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \\ \{p_i, p_j\} &= \frac{-\epsilon_{ijk}B_k}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \\ \{x_i, p_j\} &= -\frac{\delta_{ij}+B_i\Omega_j}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}} \end{aligned}$$

- **where**

$$\begin{aligned} \dot{x}_i &= -\{x_i, x_j\}\frac{\partial H}{\partial x_j} - \{x_i, p_j\}\frac{\partial H}{\partial p_j} \\ &= \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}}[\boldsymbol{E}\times\boldsymbol{\Omega}+\boldsymbol{v}+\boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{\Omega})]_i \end{aligned}$$

$$\begin{aligned} \dot{p}_i &= -\{p_i, x_j\}\frac{\partial H}{\partial x_j} - \{p_i, p_j\}\frac{\partial H}{\partial p_j} \\ &= \frac{1}{1+\boldsymbol{B}\cdot\boldsymbol{\Omega}}[\boldsymbol{E}+(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}+\boldsymbol{v}\times\boldsymbol{B}]_i \end{aligned}$$

Chiral Kinetic equation in 3D

- **The above can be re-written as ($\sqrt{\det(\gamma)} = 1 + \boldsymbol{B}\cdot\boldsymbol{\Omega}$)**

$$\begin{aligned}\sqrt{\det(\gamma)}\dot{\boldsymbol{x}} &= \boldsymbol{v} + \boldsymbol{E}\times\boldsymbol{\Omega} + \underline{\boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{\Omega})} \\ \sqrt{\det(\gamma)}\dot{\boldsymbol{p}} &= \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} + \underline{(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}}\end{aligned}$$

symmetry

$$\boldsymbol{x} \leftrightarrow \boldsymbol{p}$$
$$\boldsymbol{v} \leftrightarrow \boldsymbol{E}$$
$$\boldsymbol{B} \leftrightarrow \boldsymbol{\Omega}$$

- **Then we can evaluate**

$$\begin{aligned}\frac{d\sqrt{\det(\gamma)}}{dt} &= \frac{\partial\sqrt{\det(\gamma)}}{\partial t} + \frac{\partial \dot{x}_i\sqrt{\det(\gamma)}}{\partial x_i} + \frac{\partial \dot{p}_i\sqrt{\det(\gamma)}}{\partial p_i} \\ &= \boldsymbol{\Omega}\cdot\dot{\boldsymbol{B}} + (\nabla_x\times\boldsymbol{E})\cdot\boldsymbol{\Omega} + (\boldsymbol{E}\cdot\boldsymbol{B})(\nabla_p\cdot\boldsymbol{\Omega}) \\ &= \boxed{(\boldsymbol{E}\cdot\boldsymbol{B})}(\nabla_p\cdot\boldsymbol{\Omega})\end{aligned}$$

anomaly

- **where we have used Maxwell equations**

$$\begin{aligned}\nabla_x\cdot\boldsymbol{B} &= 0 \\ \nabla_x\times\boldsymbol{E} + \dot{\boldsymbol{B}} &= 0\end{aligned}$$

Chiral Kinetic equation in 3D

- **Then we can prove the conservation of invariant phase space volume is violated by anomaly (with $\nabla_p \cdot \Omega = 2\pi\delta^3(\vec{p})$)**

$$\frac{d^3x d^3p}{(2\pi)^3}\sqrt{\det(\gamma)} \quad \Rightarrow \quad \frac{d}{dt}\int \frac{d^3x d^3p}{(2\pi)^3}\sqrt{\det(\gamma)} = 2\pi \int \frac{d^3x}{(2\pi)^3}(\boldsymbol{E}\cdot\boldsymbol{B})$$

- **If $f(x,p)$ is conserved in normal phase space**

$$\frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = 0$$

- **Then we have chiral kinetic equation in 3D**

anomaly

$$\frac{\partial f\sqrt{\det(\gamma)}}{\partial t} + \frac{\partial f \dot{x}_i\sqrt{\det(\gamma)}}{\partial x_i} + \frac{\partial f \dot{p}_i\sqrt{\det(\gamma)}}{\partial p_i} = 2\pi\delta^3(\boldsymbol{p})\underline{(\boldsymbol{E}\cdot\boldsymbol{B})}f(t,\boldsymbol{x},\boldsymbol{p})$$

Covariant Chiral Kinetic Equation (CCKE) with Berry curvature and monopole

Question:

Can we find a covariant form of chiral kinetic equation from quantum kinetic theory in terms of Wigner function?

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Wigner Function

Gauge invariant Wigner operator/function

$$W(x,p) = <:\hat{W}(x,p):>$$

$$\hat{W}_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \bar{\psi}_\beta\left(x+\frac{1}{2}y\right) \mathcal{P}U\left(A, x+\frac{1}{2}y, x-\frac{1}{2}y\right) \psi_\alpha\left(x-\frac{1}{2}y\right)$$

Gauge link $\mathcal{P}U\left(A, x+\frac{1}{2}y, x-\frac{1}{2}y\right) \equiv \mathcal{P}\mathrm{Exp}\left(-iey^\mu \int_0^1 ds A_\mu\left(x-\frac{1}{2}y+sy\right)\right)$

Dirac equation in electromagnetic field

$$[i\gamma^\mu D_\mu(x) - m]\,\psi(x) = 0, \qquad \bar{\psi}(x)\left[i\gamma^\mu D_\mu^\dagger(x) + m\right] = 0$$

Quantum Kinetic Equation for Wigner function for massless fermion in homogeneous electromagnetic field

$$\gamma_\mu\left(p^\mu + \frac{1}{2} i\nabla^\mu\right) W(x,p) = 0$$

phase space derivative

$$\nabla^\mu \equiv \partial_x^\mu - QF^{\mu\nu}\partial_\nu^p$$

Wigner Function

Decomposition of Wigner function (16 generators of Clifford algebra):

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu\mathscr{V}_\mu + \gamma^5\gamma^\mu\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

scalar p-scalar vector axial-vector tensor

Coupled equations for vector and axial-vector

$$\begin{aligned}
p^\mu\mathscr{V}_\mu &= 0, \quad p^\mu\mathscr{A}_\mu = 0, \\
\nabla^\mu\mathscr{V}_\mu &= 0, \quad \nabla^\mu\mathscr{A}_\mu = 0, \\
\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\mathscr{A}^\sigma &= -2\left(p_\mu\mathscr{V}_\nu - p_\nu\mathscr{V}_\mu\right), \\
\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\mathscr{V}^\sigma &= -2\left(p_\mu\mathscr{A}_\nu - p_\nu\mathscr{A}_\mu\right).
\end{aligned}$$

Vasak, Gyulassy and Elze,
Annals Phys. 173, 462 (1987);
Elze, Gyulassy and Vasak,
Nucl. Phys. B 276, 706(1986).

16 equations for 8 components of $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ $\Rightarrow$ these equations must be highly consistent with each other. At this point, $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ can be any functions of x and p that satisfy the above equations.

Vector/Axial-vector current and energy-momemtum tensor

Charge and axial-charge currents and stress tensor can be obtained from $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ by integrating over momenta:

$$j^\mu = \int d^4p \mathscr{V}^\mu = nu^\mu + \xi\omega^\mu + \xi_B B^\mu,$$

⇨ CME/CVE

$$j_5^\mu = \int d^4p \mathscr{A}^\mu = n_5 u^\mu + \xi_5\omega^\mu + \xi_{B5} B^\mu,$$

$$\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}u^\nu\partial^\rho u^\sigma$$

$$B_\sigma = \frac{1}{2}\epsilon_{\sigma\mu\nu\rho}u^\mu F^{\nu\rho}$$

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2}\int d^4p(p^\mu\mathscr{V}^\nu + p^\nu\mathscr{V}^\mu) \\ &= (\epsilon+P)u^\mu u^\nu - Pg^{\mu\nu} + n_5(u^\mu\omega^\nu + u^\nu\omega^\mu) \\ &\quad + \frac{1}{2}Q\xi(u^\mu B^\nu + u^\nu B^\mu) \end{aligned}$$

All coefficients $\xi, \xi_B, \xi_5, \xi_{5B}$ are functions of T, μ, μ_5.

With more flavors

Consider 3-flavor quark matter (u,d,s), the vector current can be electromagnetic or baryonic

$$\xi^{\text{baryon}} = \frac{N_c}{\pi^2}\mu\mu_5, \quad \boxed{\xi_B^{\text{baryon}} = \frac{N_c}{6\pi^2}\mu_5 \sum_f Q_f,}$$

$$\boxed{\xi^{\text{EM}} = \frac{N_c}{\pi^2}\mu\mu_5 \sum_f Q_f,} \quad \xi_B^{\text{EM}} = \frac{N_c}{2\pi^2}\mu_5 \sum_f Q_f^2.$$

Since $\sum_f Q_f = 0$ for the three-flavor quark matter, we have

$$\xi_B^{\text{baryon}} = \xi^{\text{EM}} = 0$$

Baryonic current is blind to B
EM-current is blind to ω

D.Kharzeev and D.T.Son, PRL 106, 062301(2011);
J.H.Gao, Z.T.Liang, S.Pu, Q.Wang, X.N. Wang, PRL109, 232301(2012)

Local Polarization Effect

Consider 3-flavor quark matter (u,d,s), the axial baryonic current

$$j_5^\sigma = n_5 u^\sigma + \xi_5 \omega^\sigma + \xi_{5B} B^\sigma$$

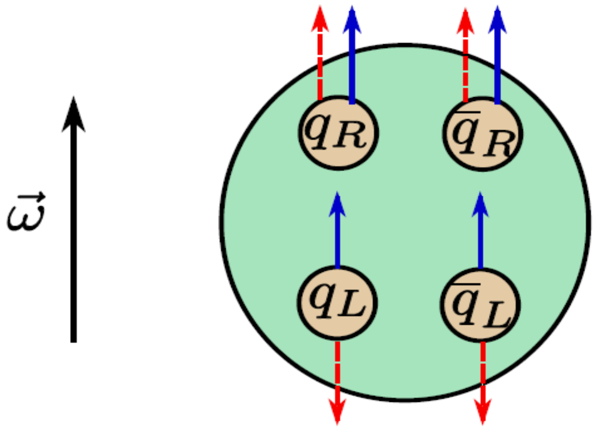
$$\xi_5 = N_c \left[\frac{1}{6} T^2 + \frac{1}{2\pi^2} (\mu^2 + \mu_5^2) \right],$$

$$\xi_{B5} = \frac{N_c}{6\pi^2} \mu \sum_f Q_f = 0.$$

Quadratic in temperature, chemical potential, chiral chemical potential
→
No cancellation!

Leading to Local Polarization Effects!
(either for high or low energy HIC)

The LPE can be measured in heavy ion collisions by the hadron (e.g. hyperon) polarization along the vorticity direction once it is fixed in the event.



J.H.Gao, Z.T.Liang, S.Pu, Q.Wang, X.N. Wang, PRL109, 232301(2012)

Covariant Chiral Kinetic Equation in 4D (CCKE)

- Re-arrange the equations for vector and axial vector components of Wigner functions, we obtain the Chiral Kinetic Equation with manifest Lorentz covariance in 4D

J.W. Chen, S.Pu, Q.Wang, X.N. Wang, PRL 110, 262301(2013)

$$\frac{1}{2}\nabla_{\mu}(\mathscr{V}^{\mu} \pm \mathscr{A}^{\mu}) = 0$$

$$\rightarrow \delta(p^2)\left[\frac{dx^{\sigma}}{d\tau}\partial_{\sigma}^{x} + \frac{dp^{\sigma}}{d\tau}\partial_{\sigma}^{p}\right] f_{R/L} = 0$$

- where

$$\begin{aligned} m_0\frac{dx^{\sigma}}{d\tau} &= p^{\sigma} \pm Q\left[(u\cdot b)B^{\sigma} - (b\cdot B)u^{\sigma} + \epsilon^{\sigma\alpha\beta\gamma}u_{\alpha}b_{\beta}E_{\gamma}\right] \\ &\quad \pm \left[\frac{1}{2}\omega^{\sigma} + \omega^{\sigma}(p\cdot u)(b\cdot u) - 2u^{\sigma}(p\cdot\omega)(b\cdot u)\right] \\ m_0\frac{dp^{\sigma}}{d\tau} &= -Qp_{\rho}F^{\rho\sigma} \mp Q^2(E\cdot B)b^{\sigma} \\ &\quad \pm Q\frac{1}{2}(\omega\cdot E)u^{\sigma} \mp Q(p\cdot\omega)b_{\eta}F^{\sigma\eta} \end{aligned}$$

$$\boxed{b^{\sigma} \equiv -p^{\sigma}/p^2}$$

Continuity and Liouville Equation

- Taking space-time and momentum divergence, we obtain

$$\partial_\sigma\left[\frac{dx^\sigma}{d\tau}\delta(p^2)\right]f_{R/L}+\partial^p_\sigma\left[\frac{dp^\sigma}{d\tau}\delta(p^2)\right]f_{R/L}$$
$$=\ \mp Q^2(E\cdot B)\partial^p_\sigma[b^\sigma\delta(p^2)]f_{R/L}.$$

- With Covariant Chiral Kinetic Equation, we obtain the continuity or Liouville equation with anomalous source

space-time current

momentum current

phase space flow

$$\partial_\sigma\left[\frac{dx^\sigma}{d\tau}\delta(p^2)f_{R/L}\right]+\partial^p_\sigma\left[\frac{dp^\sigma}{d\tau}\delta(p^2)f_{R/L}\right]$$
$$=\mp Q^2(E\cdot B)\partial^p_\sigma[b^\sigma\delta(p^2)]f_{R/L}.$$

singular

Derivation of 3D Chiral Kinetic Equation

- The chiral kinetic equation in 3-dimensions by integration over p0 for the Lorenz covariant chiral kinetic equation as

$$\int dp_0 \delta(p^2) \left[\frac{dx^\sigma}{d\tau} \partial_\sigma^x f_{R/L} + \frac{dp^\sigma}{d\tau} \partial_\sigma^p f_{R/L} \right] = 0$$

- which amounts to calculating the following integrals (n=0,1,2)

$$I_n = \int dp_0 \delta(p^2) \frac{p_0^n}{p^2} F(x,p)$$

- Using the following formula to evaluate I_n by enclosing the pole $p_0 = |\vec{p}| - i\epsilon$ in the lower half plane

$$\delta(x) \mathscr{P} \frac{1}{x} = -\frac{1}{2\pi} \mathrm{Im} \frac{1}{(x+i\epsilon)^2}$$

Derivation of 3D Chiral Kinetic Equation

- Then we can derive the chiral kinetic equation in 3-dimension

$$\frac{dt}{d\tau}\partial_t f_{R/L} + \frac{d\mathbf{x}}{d\tau}\cdot\nabla_{\mathbf{x}} f_{R/L} + \frac{d\mathbf{p}}{d\tau}\cdot\nabla_{\mathbf{p}} f_{R/L} = 0$$

- where $\boldsymbol{\Omega} = \mathbf{p}/(2|\mathbf{p}|^3)$ Berry curvature in 3D

$$\begin{aligned}
\frac{dt}{d\tau} &= 1 \pm Q\boldsymbol{\Omega}\cdot\mathbf{B} \pm 4|\mathbf{p}|(\boldsymbol{\Omega}\cdot\boldsymbol{\omega}),\\
\frac{d\mathbf{x}}{d\tau} &= \hat{\mathbf{p}} \pm Q(\hat{\mathbf{p}}\cdot\boldsymbol{\Omega})\mathbf{B} \pm Q(\mathbf{E}\times\boldsymbol{\Omega}) \pm \frac{1}{|\mathbf{p}|}\boldsymbol{\omega},\\
\frac{d\mathbf{p}}{d\tau} &= Q(\mathbf{E}+\hat{\mathbf{p}}\times\mathbf{B}) \pm Q^2(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}\\
&\quad \mp Q|\mathbf{p}|(\mathbf{E}\cdot\boldsymbol{\omega})\boldsymbol{\Omega} \pm 3Q(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})(\mathbf{p}\cdot\mathbf{E})\hat{\mathbf{p}},
\end{aligned}$$

Setting
$\omega = 0$

D.T. Son,
N. Yamamoto,
PRL 109 (2012)
181602

M.A. Stephanov,
Y. Yin,
PRL 109 (2012)
162001

4D monopole in momentum space

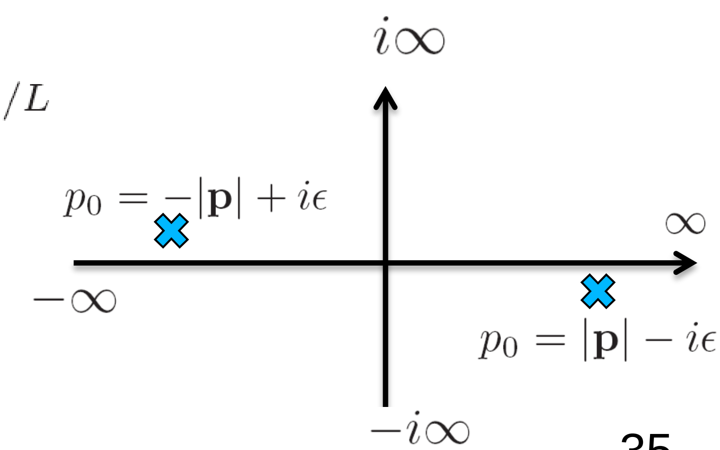
Understand the chiral anomaly from the perspective of a 4-dimensional Berry monopole $\delta(p^2)b^\sigma = -\delta(p^2)p^\sigma/p^2$

$$\boxed{\pi\delta(x) = -\mathrm{Im}[1/(x+i\epsilon)]}$$

$$\boxed{p_4 = ip_0}$$

$$\boxed{\begin{aligned}&\partial_\sigma^{p_E}(p_E^\sigma/p_E^4)\\&=2\pi^2\delta^{(4)}(p_E^\sigma)\end{aligned}}$$

$$\begin{aligned}
\partial_\sigma j^\sigma_{R/L} &= \mp Q^2(E\cdot B)\int d^4p\,\partial^p_\sigma[b^\sigma\delta(p^2)]f_{R/L}\\
&= \mp Q^2(E\cdot B)\frac{1}{\pi}\mathrm{Im}\int_{-\infty}^{\infty}dp_0 d^3p\,\partial^p_\sigma\left[\frac{p^\sigma}{p^2}\frac{1}{p^2+i\epsilon}\right]f_{R/L}\\
&= \mp Q^2(E\cdot B)\frac{1}{\pi}\mathrm{Im}\int_{-i\infty}^{i\infty}dp_0 d^3p\,\partial^p_\sigma\left[\frac{p^\sigma}{p^2}\frac{1}{p^2+i\epsilon}\right]f_{R/L}\\
&= \pm Q^2(E\cdot B)\frac{1}{\pi}\int_{-\infty}^{\infty}dp_4 d^3p\,\partial^{p_E}_\sigma\left[\frac{p_E^\sigma}{p_E^4}\right]f_{R/L}\\
&= \pm Q^2(E\cdot B)2\pi f_{R/L}(p_E=0)\\
&= \pm\frac{Q^2}{4\pi^2}(E\cdot B)
\end{aligned}$$



Quantum Kinetic Theory: a unified description

CME/CVE and LPE

Quantum Kinetic Theory

Wigner function

Berry phase and monopole in 4D

Covariant Chiral Kinetic Equation

Simulation of such quantum phenomena in a semi-classical transport approach possible and reliable

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