4.3 Renormalization Group Flow, Effective Field Theory

4.3.1 Integrating over high (UV) momentum mode.

We observed that, in conventional renormalization program, the divergence "problem" is caused by high momentum mode in the loops.

We systematically study such effect following the pioneering work by Kadanoff and Wilson.

- Starting with a theory with a cut off Λ_{UV}, the partition function is

$$Z = \int [D\Phi]_{\Lambda_{UV}} \exp\left(-\int d^dx \left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4\right]\right)$$

(It's understood that we have performed Wick rotation and in Euclidean space)

$$[D\Phi]_{\Lambda_{UV}} = \prod_{|k|<\Lambda_{UV}} d\phi(k).$$

Now, we decompose

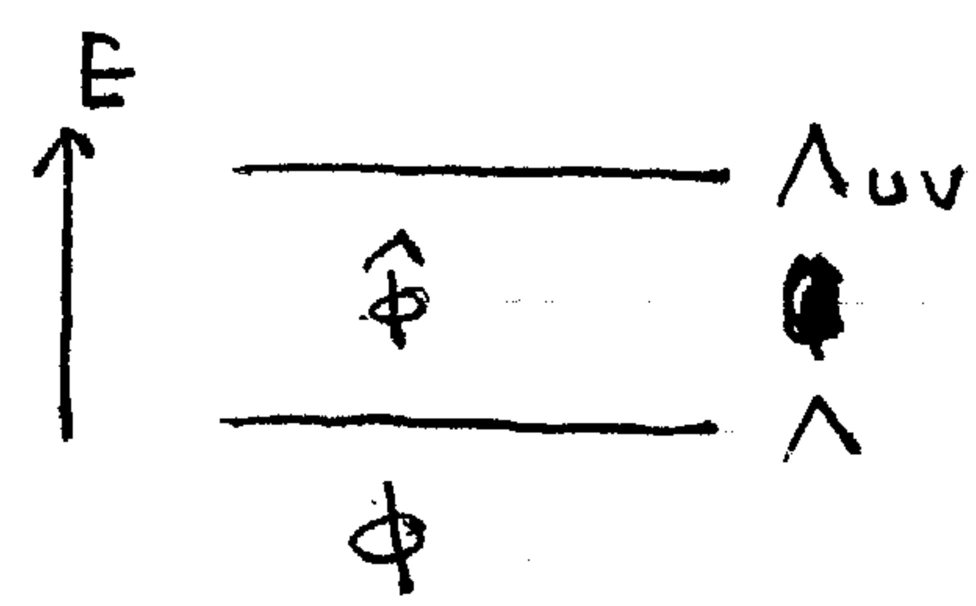
$$\Phi = \phi + \hat{\phi}$$

where

$$\hat{\phi} = \begin{cases} \phi(k) & \text{for } \Lambda < |k| < \Lambda_{UV} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi = \begin{cases} 0 & \text{otherwise} \\ \phi(k) & \text{for } |k| \le \Lambda \end{cases}$$



$$\frac{\Lambda}{\Lambda_{UV}} = b$$

- Performing the path integral over $\hat{\phi}$

$$Z = \int [D\Phi]_{\Lambda_{UV}} \, e^{-\int d^dx \, \mathcal{L}(\Phi)}$$
$$= \int [D\phi]_\Lambda \, [D\hat{\phi}] \, e^{-\int d^dx \, \mathcal{L}(\phi + \hat{\phi})}.$$
$$= \int [D\phi]_\Lambda \, e^{-\int d^dx \, \mathcal{L}_{eff}(\phi)}.$$

where $\mathcal{L}_{eff}(\phi)$ is the Lagragian of the Effective theory of ϕ with a lower cut off Λ.

→ Important note:
We have only carried out the integral of $\hat{\phi}$, and we have <u>not</u> changed the theory at all. In particular, theory $\mathcal{L}(\Phi)$ with cut off Λ_{UV} and theory $\mathcal{L}(\phi)$ with cut off Λ give <u>IDENTICAL</u> predictions for any physical observable ~~below~~ with energy $< \Lambda$.

In general, it's not possible to carry out the integral over $\hat{\phi}$ analytically. However, in weakly coupled theories, we can do this with standard perturbation theory.

In our case

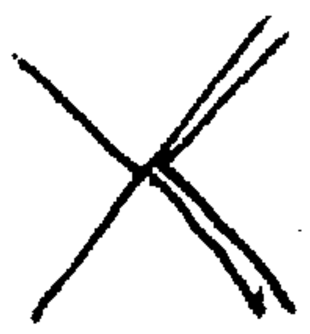
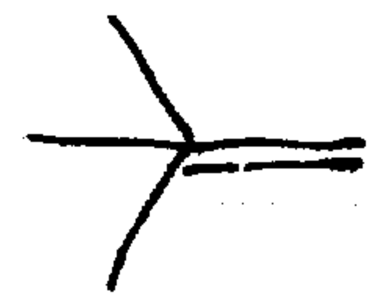
$$Z = \int D\phi \int D\hat{\phi} \, \exp\left(-\int d^dx \left[\frac{1}{2}(\partial_\mu \phi + \partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 (\phi + \hat{\phi})^2 + \frac{\lambda}{4!}(\phi + \hat{\phi})^4\right]\right)$$
$$= \int D\phi \, e^{-S_0(\phi)} \int D\hat{\phi} \, \exp\left(-\int d^dx \left[\frac{1}{2}(\partial\hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2 + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right]\right)$$

where we just separated out the $\hat{\phi}$ dependent piece.

Assuming $m^2 \ll \Lambda_{UV}^2$. The relevant Feynman rules are:

denote ϕ : ——— , Φ : ═══

propagator: ═══ $= \frac{1}{k^2}$ for $\Lambda < |k| < \Lambda_{UV}$

 : λ  : λ

and so on.

Perturbative expansion in λ

order λ: 

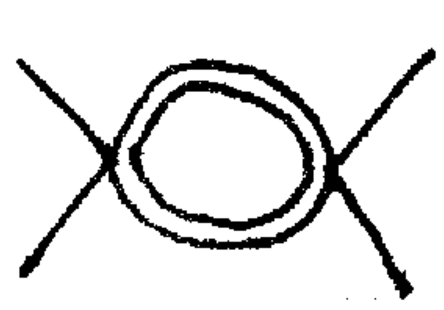
→ new contribution to $\mathcal{L}_{eff}$

$$\delta \mathcal{L}_{eff} = \frac{1}{2} \mu^2 \phi^2$$

$$\mu^2 = \frac{\lambda}{2} \int_{\Lambda}^{\Lambda_{UV}} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{\lambda}{(4\pi)^{d/2}\,\Gamma(\frac{d}{2})} \frac{1}{d-2} \left(\Lambda_{UV}^2 - \Lambda^2 \right)$$

order λ^2:

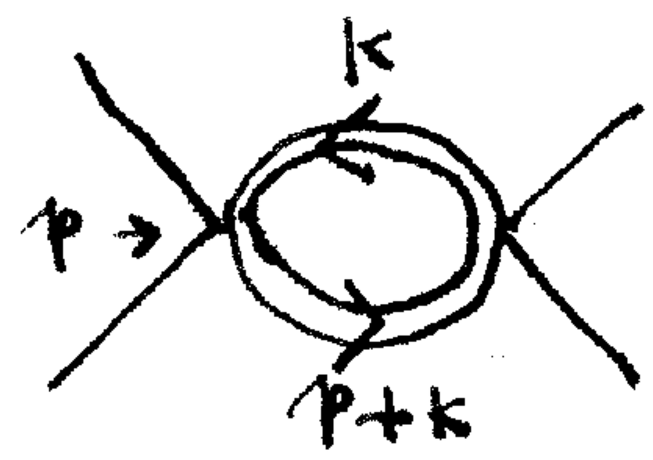
There are many contributions. For example.

 $\longrightarrow \quad \delta \mathcal{L}_{eff} = \frac{1}{4!} \xi \phi^4.$

$$\xi = \frac{3}{2} \lambda^2 \int_{\Lambda < |k| < \Lambda_{UV}} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 = \frac{-3\lambda^2}{(4\pi)^{d/2}\,\Gamma(\frac{d}{2})} \left(\frac{\Lambda_{UV}^{d-4} - \Lambda^{d-4}}{d-4} \right)$$

$$\xrightarrow{d \to 4} -\frac{3\lambda^2}{16\pi^2} \log\left(\frac{\Lambda_{UV}}{\Lambda} \right)$$

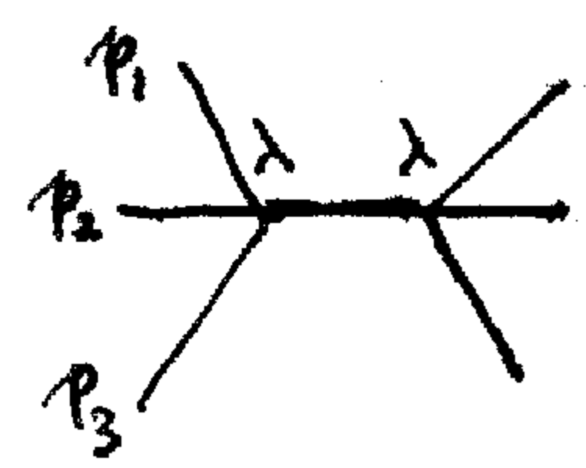
Note that in the calculation of ξ above, we have assumed external momentum p is much smaller than the loop momentum k. A more careful treatment would start from



and Taylor expand in p^2/k^2. This would lead to terms in the effective action with derivative couplings, such as

$$-\frac{1}{4}\frac{1}{\Lambda^2}\int d^dx\, \eta\, \phi^2 (\partial_\mu \phi)^2$$

At order λ^2, we also ~~have~~ would have a ϕ^6 operator. We can see why we should have this as follows. Consider $3\to 3$ scattering. In the orignal theory with cut off Λ_{UV}, we have from $\lambda\Phi^4$ coupling



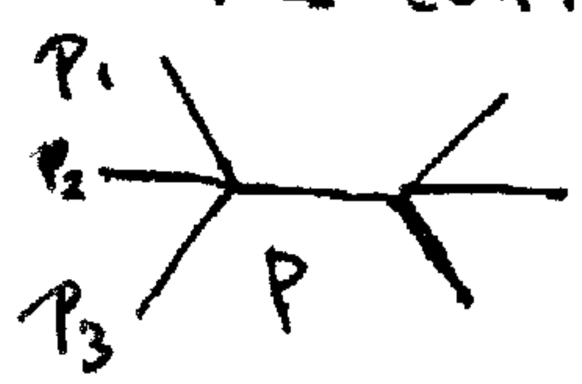
$$\propto \lambda^2 \frac{1}{P^2} \qquad \text{with } P = p_1 + p_2 + p_3 .$$

~~Conditi~~

Consider the case of $\Lambda < P < \Lambda_{UV}$. The amplitude is

$$\frac{\lambda^2}{\Lambda^2}\left(1 + O\left(\frac{\delta^2}{\Lambda^2}\right)\right) \qquad \text{where } \delta = (P - \Lambda).$$

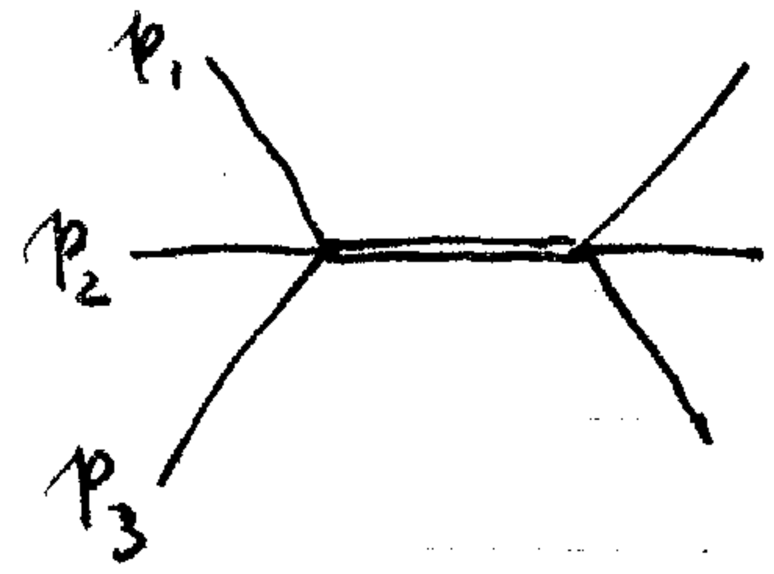
Now consider the same process in the theory with cut off Λ. The contribution from diagram



is 0, since all propagator in this theory have momentum $P < \Lambda$.

Therefore, to keep physics unchanged, theory below cut off Λ must contain a ϕ^6 operator with coefficient $\frac{\lambda^2}{\Lambda^2}$, so that $3\to 3$ scattering with $p_1+p_2+p_3 = P > \Lambda$ ~~with~~ will have the same amplitude as the theory with cut off Λ_{UV}.

Indeed, using our procedure of integrating out modes between Λ and Λ_{UV}, we will generate such an operator with the correct coefficient through diagram



$$\propto \frac{\lambda^2}{P^2}\ ; \ \Lambda < P < \Lambda_{UV}$$

$$\to \quad \delta \mathcal{L}_{eff} \propto \frac{\lambda^2}{\Lambda^2}\phi^6$$

So, we see our procedure shifts the existing operators ~~More generally, are~~ in the Lagrangian.

$$m^2 \to m^2 + \mu^2, \qquad \lambda \to \lambda + \xi \ .$$

It also generates new operators

$$\phi^2 (\partial\phi)^2, \quad \phi^6, \quad \ldots$$

→ More generally, all possible operators consistent with the symmetry (in our case, Lorentz invariance ⊕ ($\phi \Leftrightarrow -\phi$)) will be generated, even if they are not present in the original Lagrangian.

Taking this argument one step further, since the original theory at Λ_{UV} can also be viewed as an effective theory after integrating out physics degrees of freedom above Λ_{UV}, we should expect all possible operators consistent with symmetry to be there to begin with! In this sense, integrating down to lower cut off only shifts the coefficients of the complete collection of possible operators.

→ Therefore, as the <u>principle of effective field theory</u>:

We should always start with a Lagrangian will ALL possible operators, involving the degrees of freedom we want to describle and consistent with the symmetries of our choice.

Back to out example of $\lambda\phi^4$. Starting with the theory with cut off Λ_{uv}, we can schematically write the theory with cut off Λ as

$$\int d^dx\, \mathcal{L}_{eff} = \int d^dx \Big[\frac{1}{2}(1+\Delta Z)(\partial\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + (D+\Delta D)\phi^6 + \cdots + (C+\Delta C)(\partial\phi)^4 + \cdots \Big]$$

But this is not quite appropriate yet. Imposing a cut off on the highest possible momentum implies that we are ignoring details smaller that a minimal size $l_{min} \sim \frac{1}{\Lambda_{uv}}$

We can view this as discreticizing the space-time into a lattice with lattice spacing of l_{min}, then approximate the sum ~~with integral~~ over lattice points with integral, the difference between Lattice points with derivative, etc.

Going to a lower cut off means we have a larger l_{min}. Therefore, we also need to rescale

$$dx \to dx\left(\frac{\Lambda_{uv}}{\Lambda}\right), \quad \partial \to \left(\frac{\Lambda}{\Lambda_{uv}}\right)\partial$$

Therefore,

$$\int d^dx\, \mathcal{L}_{\text{eff}} = \int d^dx \left(\frac{\Lambda_{UV}}{\Lambda}\right)^d$$

$$\times \left[\frac{1}{2}(1+\Delta Z)\left(\frac{\Lambda}{\Lambda_{UV}}\right)^2 (\partial\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 \right.$$

$$\left. + (D+\Delta D)\phi^6 + (c+\Delta c)\left(\frac{\Lambda}{\Lambda_{UV}}\right)^4 (\partial\phi)^4 + \cdots \right]$$

It's more convenient to work ~~wit~~ in a basis in which the propagator does not have additional factors in it, or the kinetic term has coefficient 1 (so called canonically normalized fields). To go to this basis, we rescale the field as

$$\phi' = \left[\left(\frac{\Lambda}{\Lambda_{UV}}\right)^{2-d}(1+\Delta Z)\right]^{1/2}\phi .$$

We have

$$\int d^dx\, \mathcal{L}_{\text{eff}} = \int d^dx \left[\frac{1}{2}(\partial\phi')^2 + \frac{1}{2}m'^2(\phi')^2 + \frac{1}{4!}\lambda'\phi'^4 \right.$$

$$\left. + D'\phi'^6 + c'(\partial\phi')^4 + \cdots \right]$$

with

$$m'^2 = (m^2+\Delta m^2)(1+\Delta Z)^{-1}\left(\frac{\Lambda_{UV}}{\Lambda}\right)^2$$

$$\lambda' = (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}\left(\frac{\Lambda}{\Lambda_{UV}}\right)^{d-4}.$$

$$c' = (c+\Delta c)(1+\Delta Z)^{-2}\left(\frac{\Lambda}{\Lambda_{UV}}\right)^d$$

$$D' = (D+\Delta D)(1+\Delta Z)^{-3}\left(\frac{\Lambda}{\Lambda_{UV}}\right)^{2d-6}$$

This is a key result. Many important consequences follow.

- We begin with a connection (and a better understanding) of the conventional renormalization program.

In BPHZ renormalization, we write

physical parameter = bare parameter − counter term (from loops).

where physical parameter is defined by some low energy renormalization condition.

We see that this is indeed the form of m'^2 and λ' ("low scale physical parameters"), which are the sum of a term from the original theory ("bare para.") and a term ($\propto \Delta m^2$, $\Delta\lambda$) from the loop.

To make this more precise, we first clarify the meaning of bare parameter in this Wilsonian approach.

For simplicity, we focus on λ.

We define coupling constant λ by performing a $2 \to 2$ scattering and measure its amplitude at some specific kinematical point $s = s^*$, $t = t^*$, $u = u^*$. This is of course just the renormalization condition

$$i\mathcal{M}\Big|_{s=s^*,\, t=t^*,\, u=u^*} = -i\lambda.$$

We can call λ the coupling constant measured at energy scale specified by s^*, t^* and u^*.

Said this way, we see coupling constants are not constants, but rather functions of energy scale at which we measure them.

It's not a surprise. After all, as we have already seen, changing the cut-off (or energy range allowed) of the theory already induces a shift in the coupling constant.

We can repeat our calculation of counter term within the theory with cut off Λ_{UV}.

We get

$$\delta_\lambda = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{\Lambda_{UV}^2}{m^2 - x(1-x)s^*}\right) + (s^* \to t^*) + (s^* \to u^*)\right]$$

(Heuristically, we can obtain this from the dim-reg result by replacing $\frac{2}{\epsilon} - \log(p^2) = \log\left(\frac{\Lambda}{p^2}\right)$)

Now, we have ~~sat~~ scattering amplitude at general s, t, u

$$i\mathcal{M} = -i\Big[\lambda + \frac{\lambda^2}{32\pi^2}\int dx \times \left(\log\left(\frac{m^2 - x(1-x)s}{m^2 - x(1-x)s^*}\right) + (s^* \to t^*) + (s^* \to u^*)\right)\Big]$$

Now, the coupling constant measured at Λ_{UV} is

$$-i\lambda(\Lambda_{UV}) = i\mathcal{M}(s = \Lambda_{UV}^2) = -i(\lambda + \delta_\lambda + \underset{\text{independent of } \Lambda_{UV}}{\text{finite term}})$$

Using $\lambda_0 = \lambda + \delta_\lambda$,
We see that $\lambda(\Lambda_{UV}) = \lambda_0 + \text{finite}$.

In this sense, bare parameter λ_0 is the coupling constant measured at cut off Λ_{UV}.

So we have the following picture for BPHZ

Λ_{UV} ———————— $\mathcal{L}_{\Lambda_{UV}}(\lambda_0 \;\cdots\;)$.

bare parameters.

Λ - - - - - - - - - Scale of experiment. we want to describe

$\lambda = \lambda(\Lambda) = \lambda_0 - \delta\lambda$.

The shift between $\lambda(\Lambda)$ and λ_0, $\delta\lambda$, is produced by integrating out momentum modes between Λ_{UV} and Λ.

This is what BPHZ is actually doing.

A renormalizable theory (with only $m^2\phi^2$ and $\lambda\phi^4$ terms here) is predictive because

a) Every field theory we know in nature has a cut off. So they are all effective field theories.

b) More-over, the dependence of what's happening near the cut off Λ_{UV} can all be wrapped into low energy physical parameters through a finite number of counter terms. Therefore, we don't need to know precisely about the physics near or above the cut-off. We get predictive power once we measured low energy couplings.

Comment: What about non-renormalizable terms $(\phi^6, \phi^2(\partial\phi)^2 \ldots)$? We have just argued that they must be there too. But, we will see that they don't spoil the predictive power.

4.3.2. Power counting, scaling dimension of local operators.

Following Wilson's approach of integrating out momentum shells, we have seen that Lagrangian parameters change with the energy scale at which experiments are performed to measure them. Different terms, i.e., local operators, become more or less important as we lower the energy scale (towards IR).

Now, we develop a systematical power counting method and classify various scaling behavior of the operators.

- Consider an action of the form, defined with cut-off Λ_{UV}

$$\int d^dx \sum_i g_i O_i = \int d^dx \mathcal{L}_{\Lambda_{UV}}$$

where $\{O_i\}$ are all possible local operators consistent with the symmetries of the theory, ~~and $\{g_i\}$ are~~ made of field variables and derivatives. $\{g_i\}$ are coefficients of these operators in the Lagrangian.

Considering physical process with energy E. The contribution of operator O_i to the path integral which determines the amplitude of this process can be parameterized as E^{δ_i}. δ_i is called the energy (or scaling) dimension of operator O_i. We can denote this as

$$[O_i] = \delta_i .$$

We also have $[d^dx] = -d$. Therefore, to keep $\int d^dx\, g_i O_i$ dimensionless, we must have $[g_i] = d - \delta_i$

Operator energy dimension is in general hard to compute. However, in weakly coupled theories, it's possible to calculate ~~this~~ these dimensions perturbatively.

We begin ~~by~~ with the operator dimensions in the zeroth order of the perturbation theory.

In this case, we will use kinetic term to fixed the scaling dimension of field variable ϕ. This is the reasonable thing to do, since in weakly coupled theory, kinetic term will be the largest contribution to the action and sets the typical size of the quantum fluctuation which contributes to a certain process.

(Actually, the point of the study of scaling dimension is to compare the relative importance ~~of~~ between an operator and the kinetic term).

From kinetic term $\int d^dx \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi)$, we obtain

$$[\phi] = -1 + d/2 .$$

(Note we are using $[\phi^M] = M[\phi]$, which is only strickly true for free theory. However in weakly coupled theories, this is a good estimate).

Therefore, the zeroth order estimate for the dimension of an operator of the form $O_i = \partial^N\phi^M$ is

$$[\partial^N\phi^M] \doteq M\left(\frac{d}{2}-1\right) + N.$$

Now, g_i is dimensionful. It's useful to redefine

$$g_i = \frac{\lambda_i}{\Lambda_{UV}^{\delta_i - d}}, \text{ where } \lambda_i \text{ is dimensionless.}$$

We have argued that in an effective field theory, all possible operators (consistent with symmetry) should be expected to be present, this is called naturalness.

In our current Laguage, this means we expect $\lambda_i \sim O(1)$ as required by naturalness.

We remark that naturalness is not a consistency condition. Unnatural theory is not inconsistent.
Naturalness is an expectation of the behavior of a typical quantum field theory without choosing parameters too carefully.

— Now we are ready to classify operators according to their importance at an energy scale $E \ll \Lambda_{UV}$.

Consider a process of typical energy E. The contribution of an action term is

$$\underbrace{\int d^d x}_{E^{-d}} \; g_i \; \underbrace{\mathcal{O}_i}_{E^{\delta_i}} \longrightarrow \lambda_i \left(\frac{E}{\Lambda_{UV}} \right)^{\delta_i - d} = \lambda(E)$$

We can define this as measured coupling at E.

Therefore, the importance at low energy depends on $\delta_i - d$. We have

δ_i	effect as $E \to 0$	classification	examples.
$< d$	grows important at low E	relevant	ϕ^2, ϕ^3 super-renormalizable
$=$	$\sim$ constant. important at all E	marginal	ϕ^4 (strickly)-renormalizable.
$> d$	decreases. important at high E	irrelevant	non-renormalizable

From this perspectives, renormalizable theory are those which the low energy physics can be well captured by a finite number of relevant and marginal operators.

We now look at the "flow" (going to lower energy) of different operators.

1. Flow of relevant operator.

For example, we consider mass term $\lambda_{\phi^2} \Lambda_{UV}^2 \phi^2$.

Naturally, we expect $\lambda_{\phi^2} \sim 1$. This is already a problem, since this means the mass of ϕ is around the cut-off. Therefore, this field should belong to the physics above the cut-off (or, should not be present below the cut-off).

Let's imagine we choose $\lambda_{\phi^2} \ll 1$ so that
$\lambda_{\phi^2} \Lambda_{UV}^2 = m^2(\Lambda_{UV}) \ll \Lambda_{UV}$.

However, there still can be a problem. From our discussion of the ~~change in~~ measured coupling at E, we have

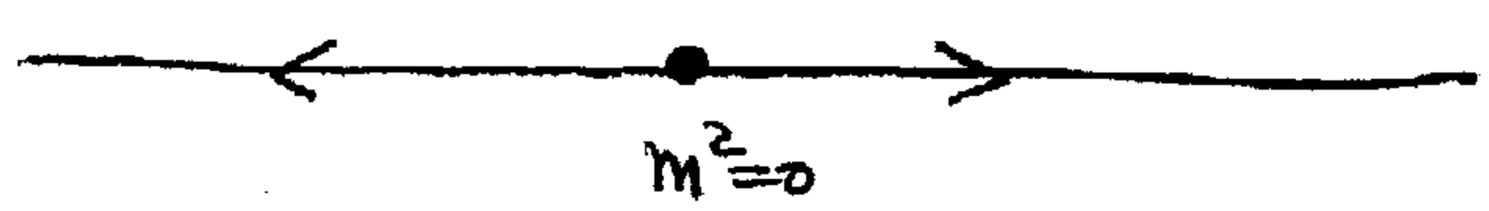
$$\lambda_{\phi^2}(E) \doteq \lambda_{\phi^2} \left(\frac{\Lambda_{UV}}{E} \right)^2.$$

Therefore,

$$m^2(E) \doteq m^2(\Lambda_{UV}) \left(\frac{\Lambda_{UV}}{E} \right)^2.$$

Notice that this is the same scaling behavior we obtained in the Wilsonian approach [with $m'^2 = m^2(E)$].
$E = \Lambda$.

This "flow" can be pictured as



with the arrows denotes "going to lower E".

This still can be a problem. Since even if the mass term starts small, it can still quickly grow to ~~to~~ be very large. To have a renormalizable theory with $m \ll \Lambda_{UV}$, we need to take into account the quantum corrections. In this case, at 1-loop for $\lambda\phi^4$ theory, this is

$$m^2(E) \sim \underbrace{m^2(\Lambda_{UV})\left(\frac{\Lambda_{UV}}{E}\right)^2}_{m_0^2} + \frac{\lambda}{32\pi^2}\left(\Lambda_{UV}^2 - E^2\right)$$

In our conventional renormalization, we call the first term bare ~~parame~~ mass, the second counter term. In Wilsonian approach, we see the counter term comes from integrating out momentum modes between Λ_{UV} and E.

This is possible. However, for $\Lambda_{UV} \gg E$, ~~this requires~~ and requiring $\Lambda_{UV}^2 \gg m^2(E)$, this requires a ~~p~~ very precise fine tuning or cancellation. This is not natural and it's called the technical naturalness problem.

For example, for a Higgs boson with mass ~ 100 GeV, and a cut-off at Planck-scale, the tuning is 1 part in 10^{32}.

Now, symmetries can be used to forbid both the bare mass term and the quantum corrections. Supersymmetry is such an example. Motivated by naturalness, it's one of the candidates for new physics beyond the SM.

Actually, the most relevant operator is the vacuum energy. $\int d^4x\sqrt{-g}\,\Lambda$. It's measured to be tiny $\Lambda \sim (10^{-3}\text{eV})^4$. There is a huge finetuning problem without a very good solution (cosmological constant problem)

2. Flow of marginal operator., ~~for~~ example ϕ^4.

- We expect, in a weakly coupled theory,

$$\delta_{\phi^4} = 4 + \text{small corrections.}$$

Therefore, the coupling constant of this operator won't be exactly as a constant, but varying slowly.

In fact, in our calculation by integrating out a shell in momentum space, we already see in $\lambda\phi^4$ theory

$$\lambda'(\Lambda) = \lambda(\Lambda_{UV}) - \frac{3\lambda^2(\Lambda_{UV})}{16\pi^2}\log\left(\frac{\Lambda_{UV}}{\Lambda}\right)$$

Therefore, taking $E = \Lambda$ and going to lower energies, we have the flow of $\lambda(E)$ governed by differetial equation

$$E\frac{\partial\lambda(E)}{\partial E} = \frac{3}{16\pi^2}\lambda^2(E)$$

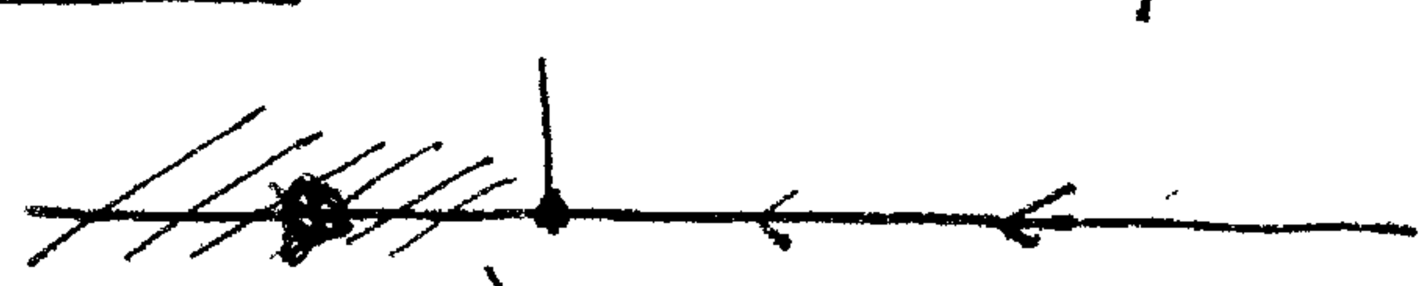
This is called the renormalization group equation (RGE) of $\lambda(E)$, at 1-loop level.

The solution is $$\lambda(E) = \frac{\lambda(\Lambda_{UV})}{1 + \beta_1\lambda(\Lambda_{UV})\log\left(\frac{\Lambda_{UV}}{E}\right)}$$

where $\beta_1 = \frac{3}{16\pi^2}$ in $\lambda\phi^4$ theory.

We see that $\lambda(E)$ indeed changes slowly (logarithically) as we lower E. Moreover, it becomes less and less important at lower E. Therefore, we call ϕ^4 operator <u>marginally irrelevant</u> in $\lambda\phi^4$ theory.

RG flow:



Now, in other field theories, the RGE of ~~the~~ coupling of a marginal operators ~~could~~ will have the generic form

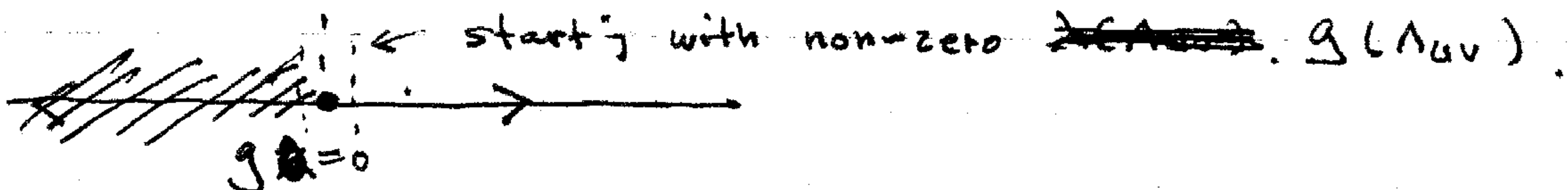
$$E \frac{\partial}{\partial E} g(E) = b\, g^2(E)$$

with solution

$$g(E) = \frac{g(\Lambda_{UV})}{1 + b\, g(\Lambda_{UV}) \log \frac{\Lambda_{UV}}{E}}$$

b could in general be either positive or negative.
If b is negative, the coupling $g(E)$ will become larger, the operator more and more important going toward lower E.
In this case, we say this operator is <u>marginally relevant</u>.

The flow of this coupling looks like



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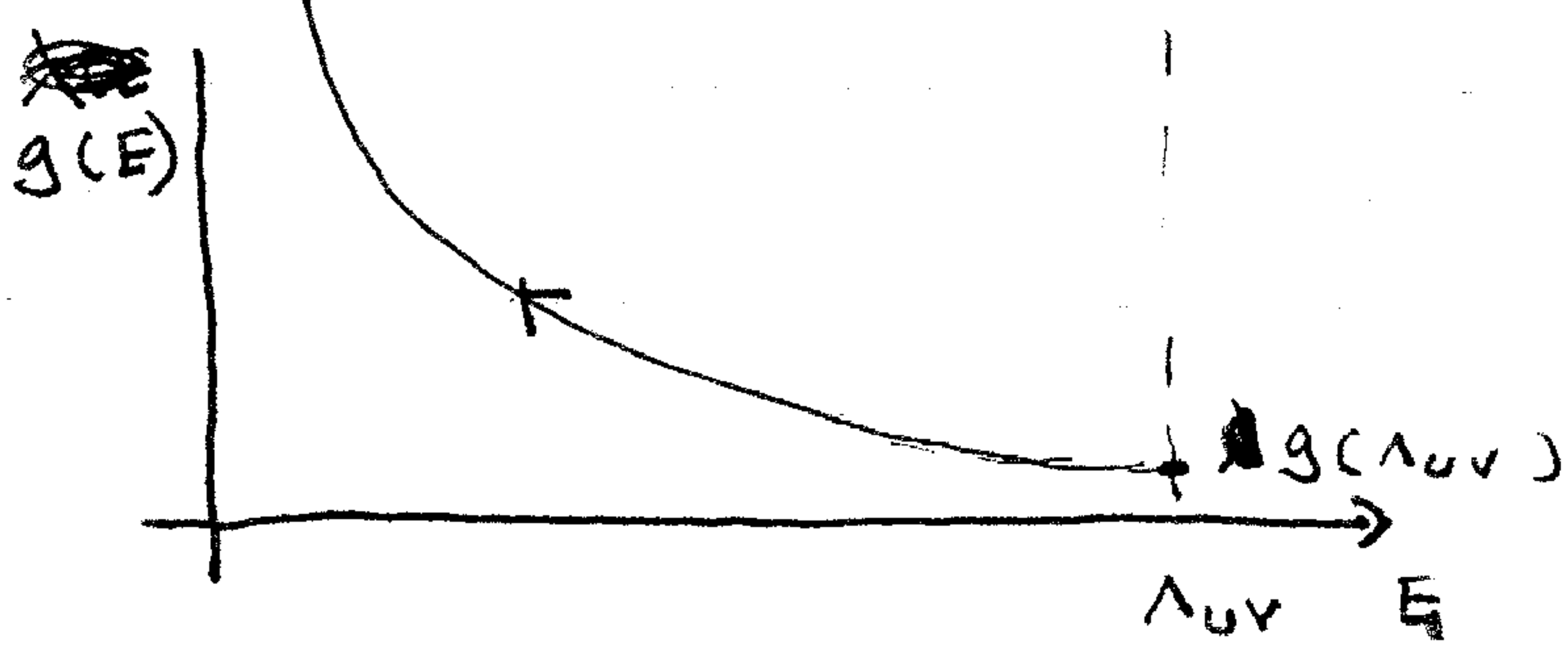


"dimensional transmutation"

At some point, $E=E_0$ $g(E_0)$ will become very large. Usually, in physics, something new happens when coupling becomes large, such as a phase transition. Therefore, starting with Λ_{UV} and small $g(\Lambda_{UV})$, marginally relevant operator can generate a new, and very different, energy scale E_0 at ~~it~~ which new physics will happen.

$$\frac{E_0}{\Lambda_{UV}} \sim e^{\frac{1}{b\, g(\Lambda_{UV})}}$$

3. Flow of irrelevant operators.

The effective dimensionless coupl; $\lambda_i(E)$ of irrelevant operators scales like

$$\lambda_i(E) \sim \left(\frac{E}{\Lambda_{UV}}\right)^{\# \in \text{positive power}}.$$

Therefore, going to lower energy, it becomes less important.

~~However~~

a) Irrelevant operators, in the spirit of effective field theory, should be present in the low energy theory. Such non-renormalizable operators do not spoil the predictive power of effective field theory. Since their contributi to any low energy variables are always suppressed by $(E/\Lambda_{UV})^{\#}$, they give a well defined perturbative expansion in terms of E/Λ_{UV}. As long as we include sufficient number of terms, we can achieve any desirable accuracy in principle. Of course, the premise is $E/\Lambda_{UV} \ll 1$, i.e., there is a scale separation between the cut-off and E (the scale of the experiment/measurement).

b) Irrelevant operator can give important effects.

Measurement of their small but finite effect at low energies can provide invaluable information, since it tells us about the scale of the cut-off, i.e., the scale at which the current theory breaks down and new physics theory must set in.

Measurements of irrelevant operators have played important and crucial roles.

- Example 1. β-decay neutron $\to$ proton + e + neutrino
  β-decay is described by a dimension 6 operator

$$\frac{1}{M^2} \mathcal{O}^{(6)}$$

  Measurements of this decay rate leads to the ~~[illegible]~~ conclusion that $M \sim 100$ GeV, i.e., new physics (new particle) must come in at that scale.
  Indeed, such a new particle, the W-boson, was discovered in the 80'. $M_W \sim 80$ GeV, and it's the central piece of the Standard Model.

- Example 2. ~~All interaction~~ Gravity.

  All interactions in GR are irrelevant, suppressed by Planck Scale, $M_{Pl} \sim 10^{19}$ GeV.

There fore, theory of gravity as we know it has a cut-off at M_{Pl}. New gravity theory, the yet unknown quantum gravity, is supposed to set in around M_{Pl}.

4. fixed point.

It's also possible that the parameters of effective Lagrangian does not evolve. This of course can only happen at specially point(s) of parameter space (if it happens at all). Such special point(s) are called fixed point(s).

A theory at a fixed point ~~does~~ has parameters which does not change with energy. Performing measurements at different energy (or length) scales gives the same result. Such a theory at the fixed point is called <u>scale invariant</u>.

A ~~slightly~~ larger ~~set~~ class of invariance is called conformal invariance. All conformal field theory (CFT) are also scale invariant. Although no obivious exceptions has been established clearly, it's not known whether the converse is true.

The most obvious, and perhaps the most important, fixed point is the trivial fixed point of massless free theory.

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2, \quad m=0, \quad \lambda=0, \quad C, D, \cdots = 0$$

where C, D etc. are coefficients of non-renormalizable operators $\phi^2(\partial\phi)^2$, ϕ^6 ....

5.

- Combining our discussions so far, we discuss the general RG flow pattern of $\lambda\phi^4$ theory.

We starts at Λ_{UV} with $m^2(\Lambda_{UV})$ and $\lambda(\Lambda_{UV})$. We ~~know~~ have

a) $\lambda(E)$ flow ~~to~~ towards the trivial fixed point $\lambda = 0$.

b) However, we also know that it requires large fine tuning (i.e., very careful choice of $m^2(\Lambda_{UV})$ and $\lambda(\Lambda_{UV})$) to get a small mass $m^2(E) \ll \Lambda_{UV}^2$. More often than not, we ~~go~~ flow to a large m^2.

c) ~~The~~ We generically also have irrelevant operators $C(\Lambda_{UV})$, $D(\Lambda_{UV})$ and so on. ~~The~~ Their coefficients scales like $\left(\frac{E}{\Lambda_{UV}}\right)^{\#}$, where # is some positive number.

d) "Renormalizable" theory.
A "renormalizable" theory requires

1) A finite number of relevant and marginal couplings, such as m, λ in $\lambda\phi^4$.
2) A large scale separation, $m \ll \Lambda_{UV}$, so that
$E \sim m \ll \Lambda_{UV}$
and the contribution from irrelevant operators are small (not necessarily vanishing)

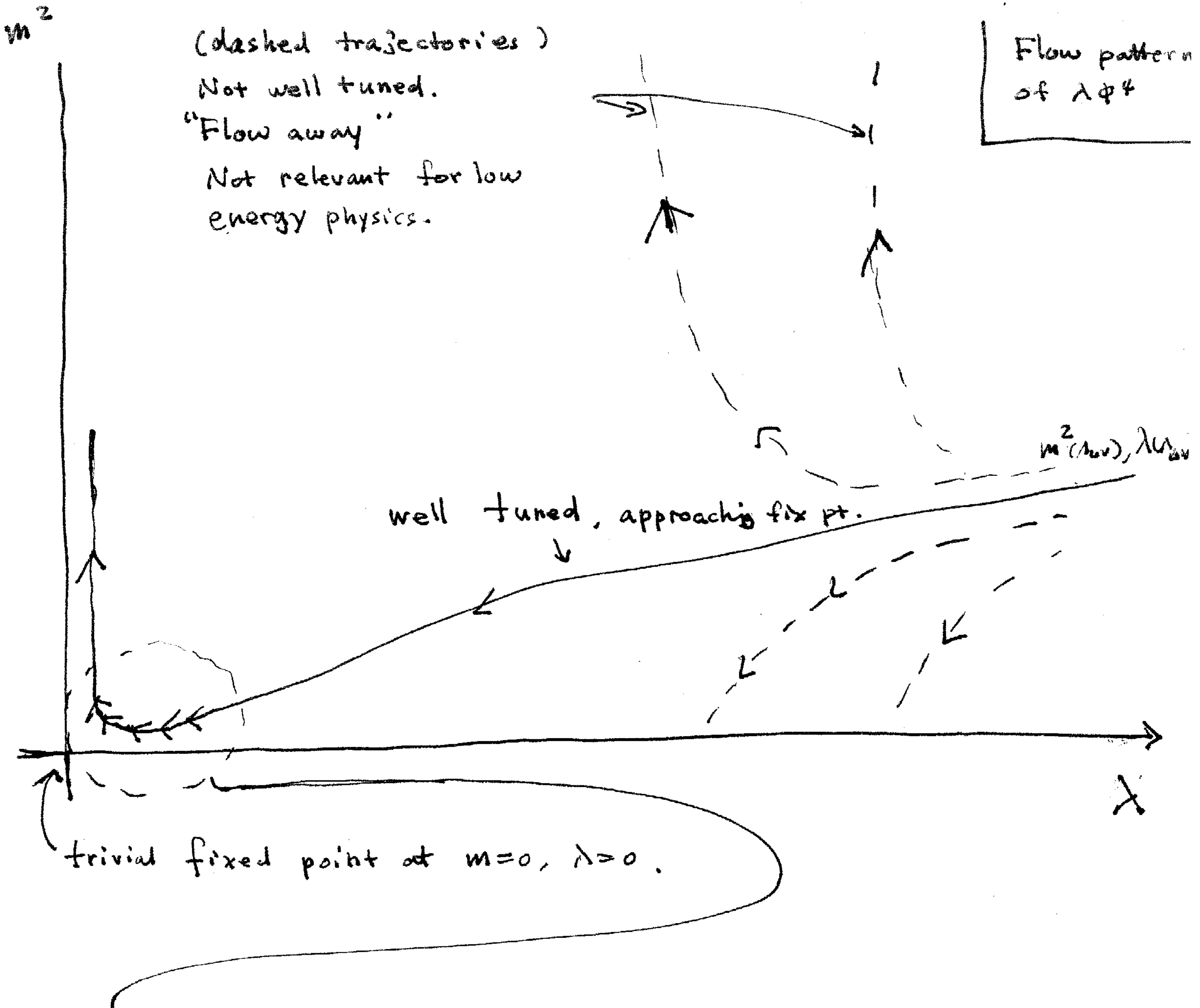
e) We observe that

1) ~~o~~ Once we choose the parameters at Λ_{UV} so that ~~m and~~ relevant and marginal operators approach ~~exact~~ close to (not reaching) a fixed point ~~at~~ some energy scale Λ', it will stay close to the fixed point for "a long time", i.e., for a large range of energy $E < \Lambda'$. This is because, by definition, a fixed point is wher everything stops moving. Therefore, everything is mov'g slowly if ~~are~~ they are sufficiently close to a fixed point.
Again, the one obvious example is the neighborhoo of trivial fixed point of $\lambda\phi^4$ theory where $m^2 \ll \Lambda_{UV}^2$, $\lambda \sim 0$.

2) Getting close to a fixed point (at some scale) is exactly what we need to have a "renormalizable" theory.

The theory their will have a finite # of relevant and marginal operators. So if we tuned m^2 to be small, it will stay small.
Moreover, since we are there "for a long time", we are still close to the fix point at a much smaller energy $E \ll \Lambda'$ or Λ_{UV}. This is exactly the large scale separation we need to make sure the sizes of the contribution from irrelevant operators, $\left(\frac{E}{\Lambda_{UV}}\right)^{\#}$ are not large.

3) The fact that our world seems to be described well by renormalizable theories means we are not far away from a fixed point (actually the trivial one).



theory spends "a lot of time" close to the fixed point. This is our renormalizable $\lambda\phi^4$ theory.

4.4 Systematics of Renormalization Group Equation (RGE).

So far, we have followed Wilsonian evolution of parameter as we moving down the cut off scale

$$
\begin{array}{lcl}
\Lambda_{UV} & \text{———————} & \lambda(\Lambda_{UV}) \\
 & \downarrow & \\
\Lambda_1 & \text{———————} & \lambda(\Lambda_1) \\
\vdots & & \\
\Lambda_i & \text{———————} & \lambda(\Lambda_i). \\
\vdots & & \\
 & \text{———————} & \lambda(\Lambda).
\end{array}
$$

The RG flow is governed by Wilsonian RGE

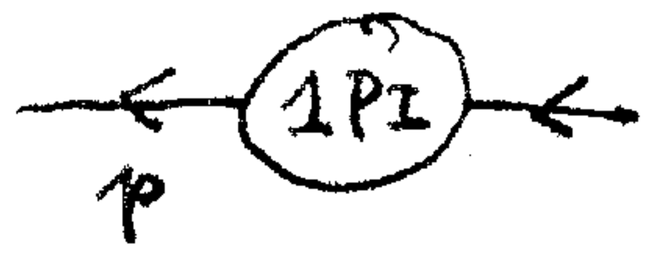
$$\Lambda \frac{\partial}{\partial \Lambda} \lambda = \beta$$

In practice, integrating out momentum shells is often inconvenient. Introducing a cut-off usually breaks too many symmetries and leaves the calculation unnecessarily complicated.

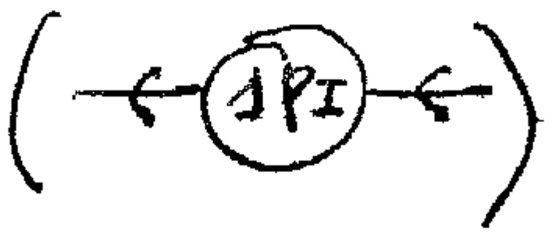
It's possible to extract the same information using the "standard" renormalization procedure, based on the following:

$\lambda(\Lambda)$ can be interpretted as the coupling constant measured at scale Λ. Similarly, renormalization condition at scale M is defining the coupling constant at this scale, $\lambda(M)$. Therefore, the evolution of $\lambda(M)$ as we varying M should carry the same information as the RGE flow of $\lambda(\Lambda)$, differring only by finite terms independent of Λ.

Renormalization condition (Focusing on $m = 0$ for simplicity)



$= 0$ for $p^2 = -M^2$

$\frac{d}{dp^2}\Big($ 

$\Big) = 0$ for $p^2 = -M^2$

$= -i\lambda$ for $s = t = u = -M^2$

Consider correlation function of bare field variables.

$$\langle \Omega | T \phi_0(x_1) \cdots \phi_0(x_n) | \Omega \rangle.$$

This is independent of M by definition. Now we change variable

$$\phi = Z^{-\frac{1}{2}} \phi_0$$

$$Z^{n/2} \underbrace{\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle}_{G^{(n)}} = \langle \Omega | T \phi_0(x_1) \cdots \phi(x_n) | \Omega \rangle$$

Both Z and $G^{(n)}$ depend on M, both explicitly in the counter terms and implicitly through $\lambda(M)$.

Define:

$$M \frac{\partial}{\partial M}(Z) = 2\gamma, \qquad M \frac{\partial}{\partial M} \lambda = \beta(\lambda).$$

We have

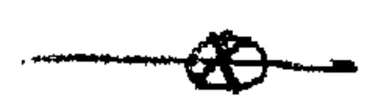
$$M \frac{\partial}{\partial M} \left(\langle \Omega | T \phi_0(x_1) \cdots \phi_0(x_n) | \Omega \rangle \right) = 0$$

$\Rightarrow$

$$\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + n \gamma \right] G^{(n)}(x_1 \cdots x_n) = 0 .$$

This is the Callan–Symanzik equation.
It shows that the fact that physics does not change with the choice of renormalization scale implies non-trivial relations between the scale dependence of the Green's function and the scale dependence of the parameters of the theory.

Example 1. $G^{(2)}(p)$ in $\lambda \phi^4$

order λ: $G^{(2)}(p) =$ $+$  $+$ 

Explicit dependence on M can only come from counter-term

$\delta_m = -\frac{\lambda}{32\pi^2} \Lambda^2$ since $m^2 = 0$.

$\delta_Z = 0$ at this order.

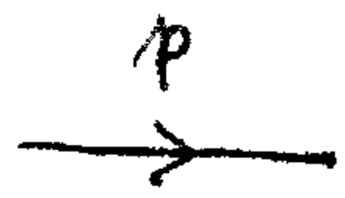
Therefore, $M \frac{\partial}{\partial M} G^{(2)}(p) = 0 + \mathcal{O}(\lambda^2)$.

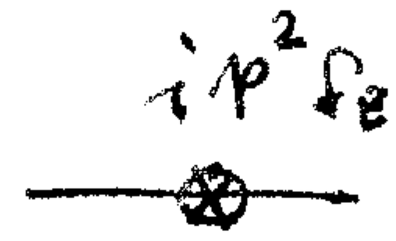
Also, $\gamma = \frac{1}{2} M \frac{\partial}{\partial M} (Z) = \frac{1}{2} M \frac{\partial}{\partial M} (\delta_Z) = 0 + \mathcal{O}(\lambda^2)$.

As we have seen in Wilsonian evolution, and we will justify again, $\beta \sim \mathcal{O}(\lambda^2)$. Therefore, $\beta \frac{\partial}{\partial \lambda} G^{(2)} \sim \mathcal{O}(\lambda^2)$.

Hence, at orde λ, Callan–Symanzik eq. is statisfied trivially.

In general:

$$G^{(2)}(p) = $$ 

$$+ \;(\text{loops})\; +$$ 

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A \log \frac{\Lambda^2}{-p^2} + \text{finite}\right) + \frac{i}{p^2}(ip^2\delta_Z)\frac{i}{p^2}.$$

For $\lambda\phi^4$, A starts at 2-loop order since 1-loop graph is cancelled by ~~const~~ 1-loop counter term to keep $m=0$.

Therefore, $A \leftarrow$  $+$  $+ \cdots$

To cancel the divergence in A, the counter term must be of the form

$$\delta_Z = A \log \frac{\Lambda^2}{M^2} + \text{finite}.$$

Note that

$$\beta \frac{\partial}{\partial \lambda} G^{(2)} \sim O(\lambda^3),$$

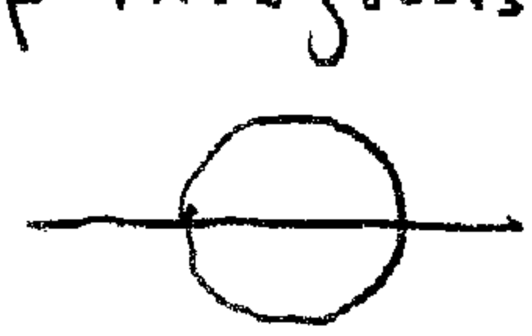
so we can neglect it at ~~this~~ order. λ^2.

→ ~~Therefore~~ At order λ^2: Callan-Symanzik eq. implies

$$-\frac{i}{p^2} M \frac{\partial}{\partial M}\delta_Z + 2\gamma \frac{i}{p^2} = 0$$

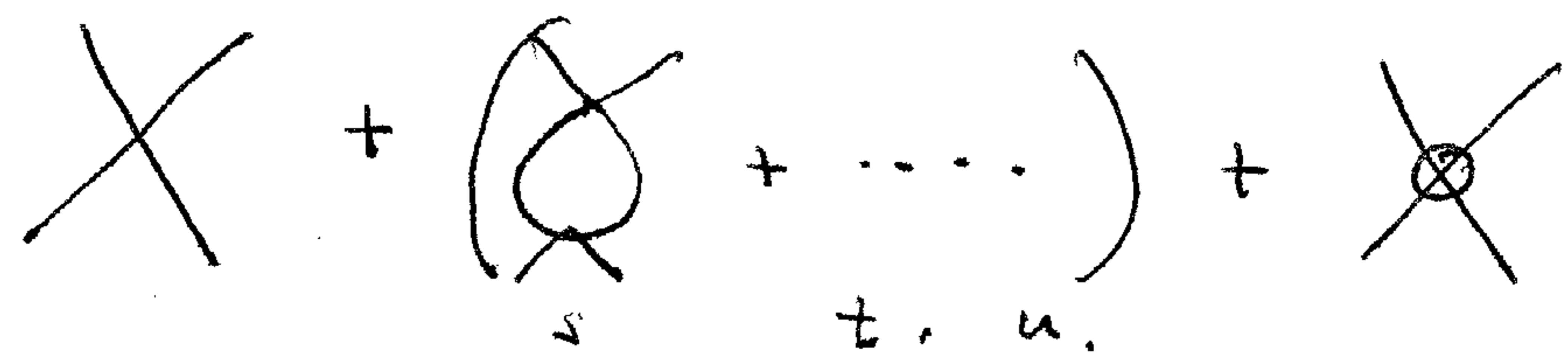
$$\Rightarrow \quad \gamma = -A.$$

So, we see that we can compute γ ~~for~~ from 2-loop integrals ~~of the form~~ from diagrams

at order λ^2.

Example 2. 4-point function in $\lambda \phi^4$.

$G^{(4)} =$ 

$+ \quad O(\lambda^3).$

$$= \left(\prod_{i=1}^{4} \frac{i}{p_i^2} \right) \times \Big[-i\lambda + (-i\lambda)^2 \left(i V(s) + i V(t) + i V(u) \right) - i \delta_\lambda \Big]$$

$$\delta_\lambda = (-i\lambda)^2 \, 3 V(-M^2) \overset{d \to 4-\epsilon}{=} \frac{3\lambda^2}{16\pi^2} \left[\frac{1}{\epsilon} - \log M + \text{finite} \right]$$

$$M \frac{\partial}{\partial M} G^{(4)} = \frac{3i}{(4\pi)^2} \left(\prod_i \frac{1}{p_i^2} \right)$$

$$\frac{\partial}{\partial \lambda} G^{(4)} = -i + O(\lambda).$$

$$\gamma G^{(4)} = O(\lambda^3)$$

Therefore, Callan-Symanzik.

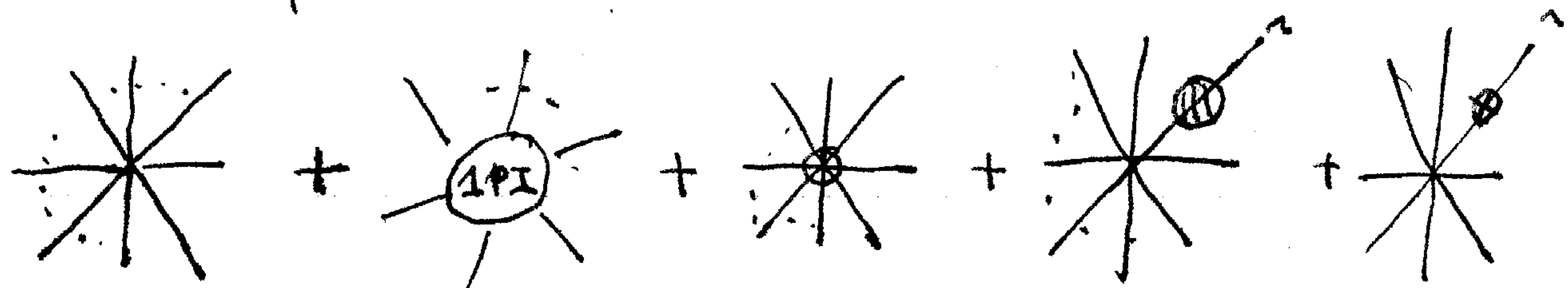
$$\left[M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 4\gamma \right] G^{(4)} = 0$$

$$\rightarrow \quad \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3).$$

which agrees with Wilsonian RGE at this order.

Example 3. General n-point vertex.

Consider n-point function



n-point vertex	n-point loop.	n-point counterterm	loop on ith external leg	counter term
$-ig$	B	$-i\delta g$	A_i	δ_{z_i}

$$G^{(n)} = \left[\prod_{i=1}^{n}\left(\frac{i}{p_i^2}\right)\right] \times \left[-ig - iB\log\frac{\Lambda^2}{-p^2} - i\delta_g + (-ig)\sum_i\left(A_i\log\frac{\Lambda^2}{-p_i^2} - \delta_{z_i}\right)\right]$$

+ finite terms.

The counter terms are of the form

$$\delta_g = -B\log\frac{\Lambda^2}{M^2} + \text{finite}, \quad \delta_{z_i} = A_i\log\frac{\Lambda^2}{M^2} + \text{finite}$$

Therefore, Callan-Symanzik equation

$$M\frac{\partial}{\partial M}\left(\delta_g - g\sum_i \delta_{z_i}\right) + \beta(g) + g\sum_i \frac{1}{2}M\frac{\partial}{\partial M}\delta_{z_i} = 0$$

To lowest order, this means

$$M\frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2}g\sum_i \delta_{z_i}\right) = \beta(g)$$

$$\rightarrow \beta(g) = -2B - g\sum_i A_i$$

- Callan - Symanzik equation also puts strong constraints on possible dependence of n-point function on external momenta.

Consider 4-point function with ~~$p_1^2 = p_2^2 = p_3^2 = p_4^2$~~.

$$s = t = u = -p^2.$$

p is therefore the characteristic energy scale.

It can be shown that the solution of Callan-Symanzik equation

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + 4\gamma(\lambda) \right] G^{(4)}(p;\lambda) = 0$$

has the following general form:

a) define running coupling constant at the scale p, $\lambda(p)$, as the solution of

$$\frac{d}{d\log(\frac{p}{M})} \lambda(p) = \beta(\lambda(p))$$

with boundary condition $\lambda(p=M) = \lambda$.

Conceptually, $\lambda(p)$ is the coupling constant measured at p, given it's measured at M to be λ.

b) Now, the solution of Callan-Symanzik eq. has the form

$$G^{(4)}(p;\lambda) = \left(\prod_{i=1}^{4} \frac{i}{p_i^2} \right) g^{(4)}[\lambda(p)] \exp\left[4 \int_M^p d\log\left(\frac{p'}{M}\right) \times \gamma(\lambda(p')) \right]$$

Comments

1) the form of $g^{(4)}(\lambda(p))$ has to be determined by actual computation using perturbation theory. However, the essential constraint is that this function <u>only</u> depends on the <u>running coupling</u> $\lambda(p)$.

2) To appreciate the importance of the statement in 1), we compute $g^{(4)}(\lambda(p))$ to the lowest order in λ.

With LSZ reduction formula, we see that $g^{(4)}$ is the S-Matrix of $2 \to 2$ scattering.

To the lowest order in λ, $\quad g^{(4)}(p) = -i\lambda$

To reproduce this using runing coupling constant $\lambda(p)$,

~~we note $\lambda(p) = \lambda + a\lambda^2 + \cdots$~~

~~Therefore,~~

we note at 1-loop order

$$\frac{d}{d\log\left(\frac{p}{M}\right)} \lambda(p) = \frac{3}{16\pi^2} \lambda^2(p)$$

$$\rightarrow \quad \lambda(p) = \frac{\lambda}{1 + \frac{3\lambda}{16\pi^2}\log\left(\frac{M}{p}\right)} = \lambda + O(\lambda^2)$$

Therefore, to match at lowest order in λ, we must have

$$g^{(4)}(p) = -i\lambda(p) + O(\lambda^2(p)).$$

In general, doing perturbation theory in λ, we can expect the scattering amplitude to be of the form

$$g^{(4)}(p) = -i\lambda \Big(1 + A_1 \left(\lambda \log\left(\tfrac{M}{p}\right)\right) + A_2 \left(\lambda \log \tfrac{M}{p}\right)^2 + \cdots$$

$$+ B_1 \lambda \left[\lambda \log\left(\tfrac{M}{p}\right)\right] + B_2 \lambda \left[\lambda \log\left(\tfrac{M}{p}\right)\right]^2 + \cdots$$

$$+ \cdots \Big).$$

We would have expected that A_i and B_i depend on multiple loop integrals in general.

First of all, notice that the first line is an expansion in parameter $\lambda \log\left(\frac{M}{p}\right)$, the second line is a similar expansion only with one more power of λ in front. Therefore, in doing perturbation theory, it's always good to choose a renormalization scale not too different from the typical energy scale of the problem, so the $\lambda \log\left(\frac{M}{p}\right)$ is not too large and perturbation theory is valid.

Now, here is the important point.
Callan - Symanzik equation tells us at leading order

$$g^{(4)}(p) = -i\lambda(p). \qquad \text{with} \quad \lambda(p) = \frac{\lambda}{1 + \frac{3\lambda}{16\pi^2} \log\left(\frac{M}{p}\right)}$$

Taylor expanding in λ, we find.

$$g^{(4)}(p) = -i\lambda \left(1 - \frac{3}{16\pi^2} \lambda \log\left(\frac{M}{p}\right) + \frac{1}{2} \left(\frac{3}{16\pi^2}\right)^2 \left(\log \frac{M}{p}\right)^2 + \cdots \right).$$

Therefore, $A_1, A_2 \cdots$ are complete determined by <u>1-loop β-function</u>! ~~Nothing to do with~~

$$A_1 = \frac{-3}{16\pi^2} \qquad A_2 = \frac{1}{2}\left(\frac{3}{16\pi^2}\right)^2 \quad \cdots$$

In other words, by using running coupling determined by 1-loop β-function, we have resummed a ~~complete~~ full subset of loop corrections (called leading log) into a single parameter $\lambda(p)$, to all orders of $\lambda \log(\frac{M}{p})$.

It turns out this is the case to all orders of perturbation theory, once we include higher order terms in β-function and ~~compare~~ use it in the solution of Callan-Symanzik equation.

For example, next leading Log corrections $B_1 \cdots B_i \cdots$ and completely determined by β_2, if we writn β-function as $\beta(\lambda) = \beta_1 \lambda^2 + \beta_2 \lambda^3$.

Impractice, with renormalized coupling defined at scale M ($\lambda(M) = \lambda$), we should use RGE, i.e., the solution of running coupling constant $\lambda(p)$ to calculate at energy scale p. This is especially important if p is very different from M, since using $\lambda(p)$ resums large logs and great improves the convergence of perturbation theory.