Ramond Equations of Motion in Superstring Field Theory

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Problems with the Ramond Sector

We'd like to write a free action

$$S = \frac{1}{2}\langle \Psi, Q\Psi \rangle$$

but $\langle , \rangle$ requires picture -2. Thus Ψ must have picture -1.

OK for NS sector, but not OK for Ramond sector.

Let's find EOM instead

Review of NS Sector

(Open Superstring with Witten Vertex)

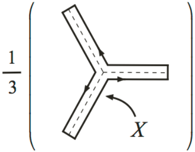
NS string field: Φ_{N}, picture -1.

EOM:

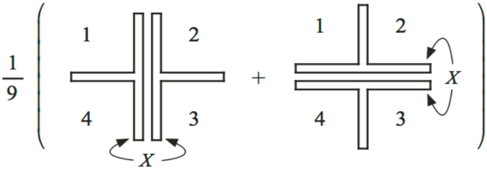
$$0 = Q\Phi_{\mathrm{N}} + M_2(\Phi_{\mathrm{N}}, \Phi_{\mathrm{N}}) + ...$$

M_2 must carry picture $+1$.

Witten inserts picture changing operator at midpoint of open string star product. This is problematic. Instead we use contour integral of picture changing operator:



Compute associator of M_2:



This is not zero!

We need an A_∞ algebra

EOM:

$$0 = Q\Phi_\mathrm{N} + M_2(\Phi_\mathrm{N}, \Phi_\mathrm{N}) + M_3(\Phi_\mathrm{N}, \Phi_\mathrm{N}, \Phi_\mathrm{N}) + ...$$

M_ns satisfy A_∞ relations.

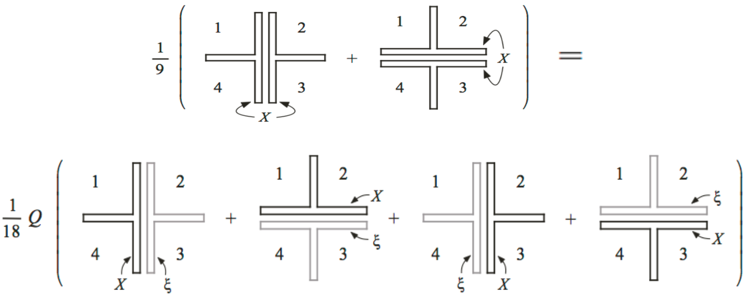
A_∞ relations require:

$$\text{Associator of } M_2 = Q(\text{3-string product } M_3)$$

Pretend picture changing operator is BRST exact.

$$X = [Q, \xi]$$

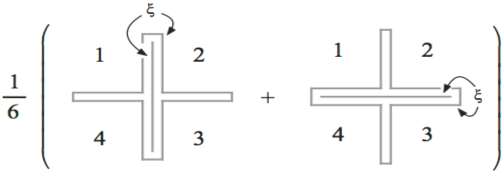
Then we can simply factor Q out of M_2 associator to find M_3



Oops. X is not BRST exact.

At least not in the small Hilbert space.

Have to make sure that M_3 is in the small Hilbert space. Upshot is that we have to add to the stuff under the parentheses of Q the BRST variation of



This works! We have a solution to the A_∞ relations, and therefore the EOM, out to third order.

Need equations to go along with these pictures

Signs: degree(Ψ) = Grassmann parity(Ψ) +1 (Don't ask...)

Act string products on any number of copies of state space $\mathcal{H}$:

$$\begin{aligned}
Q\mathcal{H} &= Q\mathcal{H}\\
Q(\mathcal{H}\otimes\mathcal{H}) &= (Q\mathcal{H})\otimes\mathcal{H}+\mathcal{H}\otimes(Q\mathcal{H})\\
Q(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) &= (Q\mathcal{H})\otimes\mathcal{H}\otimes\mathcal{H}+\mathcal{H}\otimes(Q\mathcal{H})\otimes\mathcal{H}\\
&\quad +\mathcal{H}\otimes\mathcal{H}\otimes(Q\mathcal{H})\\
&\ \vdots\\
M_2(\mathcal{H}) &= 0\\
M_2(\mathcal{H}\otimes\mathcal{H}) &= M_2(\mathcal{H},\mathcal{H})\\
M_2(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}) &= M_2(\mathcal{H},\mathcal{H})\otimes\mathcal{H}+\mathcal{H}\otimes M_2(\mathcal{H},\mathcal{H})\\
&\ \vdots
\end{aligned}$$

For example:

$$m_2 = \text{Witten's open string star product}$$

Bosonic SFT axioms can be expressed:

* BRST operator is nilpotent: $[Q, Q] = 0$

* BRST operator is derivation: $[Q, m_2] = 0$

* Star product is associative: $[m_2, m_2] = 0$

More generally, and A_∞ algebra is defined by a sequence of products $Q, M_2, M_3, ...$ which satisfy A_∞ relations:

$$[Q, M_n] + [M_2, M_{n-1}] + ... + [M_{n-1}, M_2] + [M_n, Q] = 0$$

Or, even more simply, we can take the sum

$$M = Q + M_2 + M_3 + M_4 + ...$$

The A_∞ relations imply that M is nilpotent:

$$[M, M] = 0$$

Now let's put equations to the pictures.

Since we're pretending that X is BRST exact, the product M_2 in the EOM is BRST exact:

$$M_2 = [Q, \mu_2]$$
$$m_2 = [\eta, \mu_2]$$

μ_2 is the same as M_2 with the replacement $X \to \xi$. Again, m_2 is the ordinary star product.

Note

$$[\eta, M_2] = -[Q, m_2] = 0$$

Now let's derive the 3-product M_3. (Bear with me.) Pretending X is BRST exact, we can pull Q out of third A_∞ relation:

$$\begin{aligned} 0 &= 2[Q, M_3] + [M_2, M_2] \\ &= [Q, 2M_3 - [M_2, \mu_2]] \end{aligned}$$

Therefore

$$M_3 = \frac{1}{2}\Big([Q, \mu_3] + [M_2, \mu_2]\Big)$$

$[Q, \mu_3]$ is the extra term needed to make sure M_3 is in small Hilbert space. Defining $m_3 = [\eta, \mu_3]$ we must have

$$0 = [\eta, M_3] = [Q, m_3] + [M_2, m_2] = [Q, m_3 - [m_2, \mu_2]]$$

so

$$m_3 = [m_2, \mu_2]$$

Note $[\eta, m_3] = 0$. Surrounding m_3 with ξ defines μ_3, and therefore the product that we want!

This is how it works at all orders: Defining

$$\begin{aligned} M(t) &= Q + tM_2 + t^2 M_3 + t^3 M_4 + \ldots \\ \mu(t) &= \mu_2 + t\mu_3 + t^2\mu_4 + \ldots \\ m(t) &= m_2 + tm_3 + t^2 m_4 + \ldots \end{aligned}$$

The products are defined by solution of the equations

$$\begin{aligned} \frac{d}{dt}M(t) &= [M(t), \mu(t)] \\ \frac{d}{dt}m(t) &= [m(t), \mu(t)] \\ [\eta, \mu(t)] &= m(t) \end{aligned}$$

Finally, we have the EOM for the NS open superstring!

Ramond Equations of Motion

(Open Superstring with Witten Vertex)

Include Ramond string field Ψ_{R} with picture $-1/2$.

EOM:

$$
\begin{aligned}
0 &= Q\Phi_{\mathrm{N}} + M_2(\Phi_{\mathrm{N}},\Phi_{\mathrm{N}}) + m_2(\Psi_{\mathrm{R}},\Psi_{\mathrm{R}}) + \ldots \\
0 &= Q\Psi_{\mathrm{R}} + M_2(\Psi_{\mathrm{R}},\Phi_{\mathrm{N}}) + M_2(\Phi_{\mathrm{N}},\Psi_{\mathrm{R}}) + \ldots
\end{aligned}
$$

Define composite string field $\tilde{\Phi} = \Phi_{\rm N} + \Psi_{\rm R}$. EOM can be written

$$0 = Q\tilde{\Phi} + \tilde{M}_2(\tilde{\Phi}, \tilde{\Phi}) + ...$$

with

$$\begin{aligned}
\tilde{M}_2(N, N) &= M_2(N, N), \\
\tilde{M}_2(N, R) &= M_2(N, R), \\
\tilde{M}_2(R, N) &= M_2(R, N), \\
\tilde{M}_2(R, R) &= m_2(R, R)
\end{aligned}$$

Want $\tilde{M}_3$

$$0 = Q\tilde{\Phi} + \tilde{M}_2(\tilde{\Phi}, \tilde{\Phi}) + \tilde{M}_3(\tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) + ...$$

so that third A_∞ relation is obeyed:

$$2[Q, \tilde{M}_3] + [\tilde{M}_2, \tilde{M}_2] = 0$$

Again we just want to pull a Q out of the associator to find the 3-product.

Since $\tilde{M}_2$ is different depending on the number of R states being multiplied, we have to do this separately for the 8 possible ways three NS and R states can multiply. Upshot:

$$\begin{aligned}
\tilde{M}_3(N,N,N) &= M_3(N,N,N)\\
\tilde{M}_3(N,N,R) &= M_3(N,N,R)\\
\tilde{M}_3(N,R,N) &= M_3(N,R,N)\\
\tilde{M}_3(R,N,N) &= M_3(R,N,N)\\
\tilde{M}_3(N,R,R) &= m_2(\mu_2(N,R),R) - \mu_2(N,m_2(R,R))\\
\tilde{M}_3(R,N,R) &= m_2(\mu_2(R,N),R) + m_2(R,\mu_2(N,R))\\
\tilde{M}_3(R,R,N) &= -\mu_2(m_2(R,R),N) + m_2(R,\mu_2(R,N))\\
\tilde{M}_3(R,R,R) &= -\mu_2(m_2(R,R),R) - \mu_2(R,m_2(R,R))
\end{aligned}$$

Whew!

Note $\tilde{M}_3(R, R, R) \neq 0$.

What about $\tilde{M}_4(R, R, R, R)$? Must vanish by ghost and picture counting.

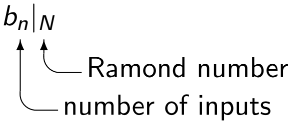
EOM is precisely cubic in the Ramond string field.

Finding all ways NS and R states multiply seems like a pain

Key technical idea: Ramond number:

$$\begin{aligned}\text{Ramond number} = &\ \text{Number of Ramond inputs}\\ &-\text{Number of Ramond outputs}\end{aligned}$$

Denote



$$b_n|_N$$

Ramond number

number of inputs

Can decompose products into components of definite Ramond number:

$$b_n = b_n|_{-1} + b_n|_0 + b_n|_1 + ... + b_n|_n$$

We can write the composite 2-product:

$$\tilde{M}_2 = M_2|_0 + m_2|_2$$

We can write the composite 3-product:

$$\tilde{M}_3 = M_3|_0 + m_3'|_2$$

with

$$m_3'|_2 = [m_2|_2, \mu_2|_0]$$

Ramond number restriction of products in commutator automatically takes care of different NS and R multiplications

All composite products have components at Ramond number 0 and 2:

$$\tilde{M}_n = M_n|_0 + m_n'|_2$$

Products of 4 or more Ramond states vanish.

This is how it works at all orders: Defining

$$\begin{aligned}
M(t) &= Q + tM_2|_0 + t^2 M_3|_0 + t^3 M_4|_0 + \dots \\
\mu(t) &= \mu_2|_0 + t\mu_3|_0 + t^2\mu_4|_0 + \dots \\
m(t) &= m_2|_0 + tm_3|_0 + t^2 m_4|_0 + \dots \\
m'(t) &= m_2|_2 + tm'_3|_2 + t^2 m'_4|_2 + \dots
\end{aligned}$$

The products are defined by solution of the equations

$$\begin{aligned}
\frac{d}{dt}M(t) &= [M(t), \mu(t)] \\
\frac{d}{dt}m(t) &= [m(t), \mu(t)] \\
\frac{d}{dt}m'(t) &= [m'(t), \mu(t)] \\
[\eta, \mu(t)] &= m(t)
\end{aligned}$$

Finally, we have the NS+R equations of motion for the NS open superstring!

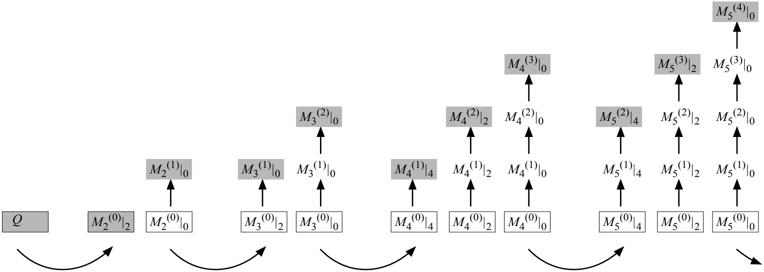
Solving the differential equations gives a set of recursive equations for the products. The recursion is solved by following the diagram:

$$\begin{tikzcd}
& M_2|_0 & & {m'}_3|_2 & & M_3|_0 & & {m'}_4|_2 & & M_4|_0 & & {m'}_5|_2 & & M_5|_0 & & {m'}_6|_2 & \\
& & \mu_2|_0 \arrow[ul] \arrow[ur] \arrow[drr] & & & & \mu_3|_0 \arrow[ul] \arrow[ur] \arrow[drr] & & & & \mu_4|_0 \arrow[ul] \arrow[ur] \arrow[drr] & & & & \mu_5|_0 \arrow[ul] \arrow[ur] \arrow[drr] & & \cdots \\
m_2|_0 \arrow[urr] & & & & m_3|_0 \arrow[urr] & & & & m_4|_0 \arrow[urr] & & & & m_5|_0 \arrow[urr] & & & & m_6|_0
\end{tikzcd}$$

Ramond Equations of Motion

(heterotic string)

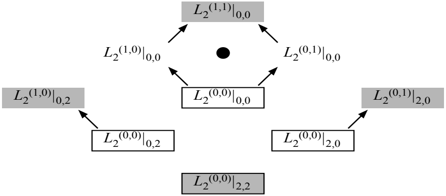
The story for the NS+R equations of motion of the heterotic string is similar but more intricate. In the end you solve for a bunch of products by following a recursion illustrated by the diagram:



Ramond Equations of Motion

(type II closed superstring)

The story for the NS-NS+R-NS+NS-R +R-R equations of motion for the type II closed superstring is similar but even more complicated. You find the 2-string products by following the diagram:

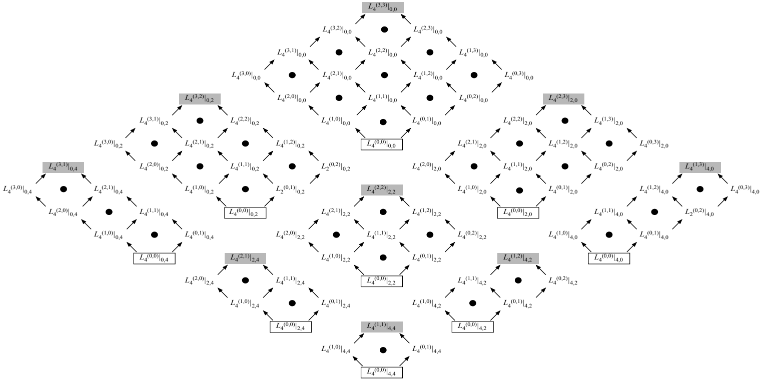


$$\begin{tikzcd}
 & & \boxed{L_2^{(1,1)}|_{0,0}} & & \\
 & L_2^{(1,0)}|_{0,0} \arrow[ur] & \bullet & L_2^{(0,1)}|_{0,0} \arrow[ul] & \\
\boxed{L_2^{(1,0)}|_{0,2}} & & \boxed{L_2^{(0,0)}|_{0,0}} \arrow[ul] \arrow[ur] & & \boxed{L_2^{(0,1)}|_{2,0}} \\
 & \boxed{L_2^{(0,0)}|_{0,2}} \arrow[ul] & & \boxed{L_2^{(0,0)}|_{2,0}} \arrow[ur] & \\
 & & \boxed{L_2^{(0,0)}|_{2,2}} & &
\end{tikzcd}$$

Then the 3-string products by following the diagram:

$$\begin{tikzcd}
 & & & & \colorbox{lightgray}{$L_3^{(2,2)}|_{0,0}$} & & & & \\
 & & & L_3^{(2,1)}|_{0,0} \arrow[ur] & \bullet & L_3^{(1,2)}|_{0,0} \arrow[ul] & & & \\
 & & L_3^{(2,0)}|_{0,0} \arrow[ur] & \bullet & L_3^{(1,1)}|_{0,0} \arrow[ul] \arrow[ur] & \bullet & L_3^{(0,2)}|_{0,0} \arrow[ul] & & \\
 & \colorbox{lightgray}{$L_3^{(2,1)}|_{0,2}$} & & L_3^{(1,0)}|_{0,0} \arrow[ul] \arrow[ur] & \bullet & L_3^{(0,1)}|_{0,0} \arrow[ul] \arrow[ur] & & \colorbox{lightgray}{$L_3^{(1,2)}|_{2,0}$} & \\
L_3^{(2,0)}|_{0,2} \arrow[ur] & \bullet & L_3^{(1,1)}|_{0,2} \arrow[ul] & & \boxed{L_3^{(0,0)}|_{0,0}} \arrow[ul] \arrow[ur] & & L_3^{(1,1)}|_{2,0} \arrow[ur] & \bullet & L_3^{(0,2)}|_{2,0} \arrow[ul] \\
 & L_3^{(1,0)}|_{0,2} \arrow[ul] \arrow[ur] & \bullet & L_3^{(0,1)}|_{0,2} \arrow[ul] & & L_3^{(1,0)}|_{2,0} \arrow[ur] & \bullet & L_3^{(0,1)}|_{2,0} \arrow[ul] \arrow[ur] & \\
 & & \boxed{L_3^{(0,0)}|_{0,2}} \arrow[ul] \arrow[ur] & & \colorbox{lightgray}{$L_3^{(1,1)}|_{2,2}$} & & \boxed{L_3^{(0,0)}|_{2,0}} \arrow[ul] \arrow[ur] & & \\
 & & & L_3^{(1,0)}|_{2,2} \arrow[ur] & \bullet & L_3^{(0,1)}|_{2,2} \arrow[ul] & & & \\
 & & & & \boxed{L_3^{(0,0)}|_{2,2}} \arrow[ul] \arrow[ur] & & & &
\end{tikzcd}$$

Then the 4-string products by following the diagram:



And so on.

Supersymmetry

Easiest to describe SUSY in another set of field variables $\tilde{\Phi} \to \tilde{\phi}$ where the equations of motion take the form

$$0 = (Q - \eta)\tilde{\phi} + \tilde{\phi}^2$$

(See Okawa's talk)

SUSY transformation takes the form:

$$\begin{aligned} \delta\phi_{\mathrm{N}} &= q\psi_{\mathrm{R}} + [\psi_{\mathrm{R}}, q_{\xi}\phi_{\mathrm{N}}] \\ \delta\psi_{\mathrm{R}} &= q_X\phi_{\mathrm{N}} + q_{\xi}(\psi_{\mathrm{R}})^2 \end{aligned}$$

Now we can ask whether SFT solutions are supersymmetric. For example, can translate reference BPS D-brane with analytic solution

$$\phi_{\mathrm{N}} = \Psi_{\mathrm{tv}} - \Sigma\Psi_{\mathrm{tv}}\overline{\Sigma}$$

SUSY invariance is easy to check.

Thank you!