L_∞-relations in WZW-like string field theory

Hiroaki Matsunaga (YITP, Kyoto)

based on upcoming paper

with Keiyu Goto (Univ. Tokyo)

China, May, 12th, 2015.

When you ask questions,

Please, speak *slowly* and use *easy* words.

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Introduction.

Witten's Cubic Theory

Witten 86'

- Interaction term includes local insertion of PCO.

  → divergent contact terms, broken gauge inv.

- To remedy these, various approaches are proposed.

In my talk, two successive formulations are picked up.

Introduction.

Witten's Cubic Theory

Witten

The small Hilbert space	The large Hilbert space (WZW-like actions)
Some attempts . . .	NS open **Berkovits 95'** NS closed **Berkovits, Okawa, Zwiebach 04'**

Gauge fixing . . . Not Yet

In 2013

Witten's Cubic Theory

Witten

The small Hilbert space (A_∞/L_∞-type actions)	The large Hilbert space (WZW-like actions)
NS open **Erler, Konopka, Sachs 13'**	NS open **Berkovits 95'** NS closed **Berkovits, Okawa, Zwiebach 04'**
Gauge fixing . . . OK!	Gauge fixing . . . Not Yet

In 2014

Witten's Cubic Theory

Witten

The small Hilbert space (A_∞/L_∞-type actions)	The large Hilbert space (WZW-like actions)
NS open	NS open **Berkovits 95'**
NS closed	NS closed **Berkovits, Okawa, Zwiebach 04'**
NS-NS closed **Erler, Konopka, Sachs 14'**	

Gauge fixing . . . OK!

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In 2014

Witten's Cubic Theory

Witten

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Gauge fixing . . . OK!

Gauge fixing . . . Not Yet

Today's Topic !!

Witten's Cubic Theory

Witten

The small Hilbert space (A_∞/L_∞-type actions)		The large Hilbert space (WZW-like actions)
NS open	⟷	NS open **Berkovits 95'**
NS closed	⟷	NS closed **Berkovits, Okawa, Zwiebach 04'**
NS-NS closed **Erler, Konopka, Sachs 14'**	⟷	NS-NS closed **H.M 14'**

Gauge fixing . . . OK

Gauge fixing . . . Possible!

Today's Topic !!

Witten's Cubic

Witten

Okaawa-san explained

The small Hilbert space (A_∞/L_∞-type actions)		large Hilbert space (WZW-like actions)
		Z_2-reversed
NS open	⟷	NS open
		Berkovits 95'
NS closed	⟷	NS closed
		Berkovits, Okawa, Zwiebach 04'
NS-NS closed	⟷	NS-NS closed
Erler, Konopka, Sachs 14'		**H.M 14'**

Gauge fixing . . . OK

Gauge fixing . . . Possible!

Today's Topic !!

Witten's Cubic

Witten

Topic in my talk

The small Hilbert space (A_∞/L_∞-type actions)		The large Hilbert space (WZW-like actions)
NS open	⟷	NS open **Berkovits 95'**
NS closed	⟷	NS closed **Berkovits, Okawa, Zwiebach 04'**
NS-NS closed **Erler, Konopka, Sachs 14'**	⟷	NS-NS closed **H.M 14'**

Gauge fixing . . . OK

Gauge fixing . . . Possible!

Plan

0. Introduction

1. Two formulations : L , Qg

2. Similarity Transformations

3. Equivalence of on-shell conditions

1. A short review of Two Formulations

- Resolving singularities -

We start with Witten's cubic superstring field theory.

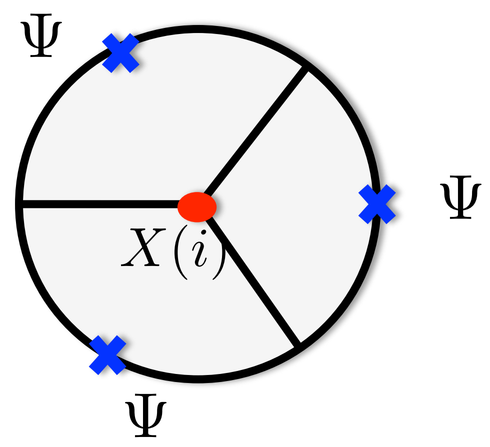
Picture # anomaly

There exists ghost & picture # anomaly !!

Witten's Action : $S = \frac{1}{2}\langle \Psi, Q\Psi \rangle + \frac{1}{3}\langle \Psi, X(i)\big(\Psi * \Psi\big)\rangle$

X(i) : Picture Changing Operator

$\langle \Psi, X(i)\big(\Psi * \Psi\big)\rangle$ =



Two formulations

- OPE of PCOs $X(i)$ is singular. . .

  ➡ Contact terms become DIVERGENT !!

- Broken gauge invariance. . .

⬇

Two Formulations for Superstring Field Theory

(A) WZW-like Formulation (the large Hilbert space)

(B) A_∞/L_∞-type Formulation (the small Hilbert space)

(A) WZW-like Formulation

(the large Hilbert space)

Changing the $\#_{\text{ghost}}$ & $\#_{\text{picture}}$ of string fields.

Large-space string fields

In the "large" Hilbert space. . .   ($\eta\xi\phi$-system)

	Vertex Op.		String Field	($\#_{gh}$ | $\#_{pic}$)
Large :	$\mathcal{V}(z) = \xi(z) c e^{-\phi} V_m$	$\rightarrow$	Φ	$(0 \vert 0)$

There exists conformal weight 1 current : η (z)

$\rightarrow$ Zero mode $\eta := \eta_0$ also becomes a derivation !!

Note that "# of Q" = (1 | 0) and "# of η" = (1 | -1)

The free Action (NS open)

Free Action	EOM	Gauge transf.
$S=\frac{1}{2}\langle\eta\Phi, Q\Phi\rangle$	$Q\eta\Phi=0$	$\delta\Phi=Q\Lambda+\eta\Omega$

→ Two generators of gauge transf. : Q & η

- Interacting terms . . . ??

Note that "# of ϕ" = (0 | 0) !

→ We can make a function of ϕ

without 'picture-changing problems'.

Berkovits' open Superstring Field Theory

Berkovits WZW-"type" Action

$$S = \frac{1}{2}\langle e^{-\Phi} Q e^{\Phi}, e^{-\Phi} \eta e^{\Phi} \rangle + \frac{1}{2}\int_0^1 dt \langle e^{-t\Phi} \partial_t e^{t\Phi}, [\![e^{-t\Phi} Q e^{t\Phi}, e^{-t\Phi} \eta e^{t\Phi}]\!] \rangle$$

EOM : $\eta\big(e^{-\Phi} Q e^{\Phi}\big) = 0$

Gauge transf. : $e^{-\Phi}\delta e^{\Phi} = Q_{\mathcal{G}}\Lambda + \eta\Omega$

$$Q_{\mathcal{G}} := Q + [\![e^{-\Phi}(Q e^{\Phi}), \quad]\!]$$

↑

A (formal) pure-gauge g is the key.

WZW-like form

- The action takes the WZW-like form

$$S = \int_0^1 dt \langle A_{\partial_t}(t), \underline{Q_{\mathcal{G}(t)}} A_\eta(t) \rangle$$

→ The g-shifted BRST operator : $Q_{\mathcal{G}(t)} = Q + [e^{-\phi}(Qe^{\Phi}), \quad]$

Associated fields : $\partial_\tau A_\eta(\tau) = \eta\Phi + [\Phi, A_\eta(\tau)]$

We can similarly construct the action for closed superstrings.

- In the Berkovits open NS theory, we can take

Z_2-reversing : $(Q, \eta, \phi) \to (\eta, Q, -\phi)$

$$S = \int_0^1 dt \langle \tilde{A}_{\partial_t}(t), \underline{Q} \, \tilde{A}_\eta(t) \rangle$$

(B) A_∞/L_∞-type Formulation

(the small Hilbert space)

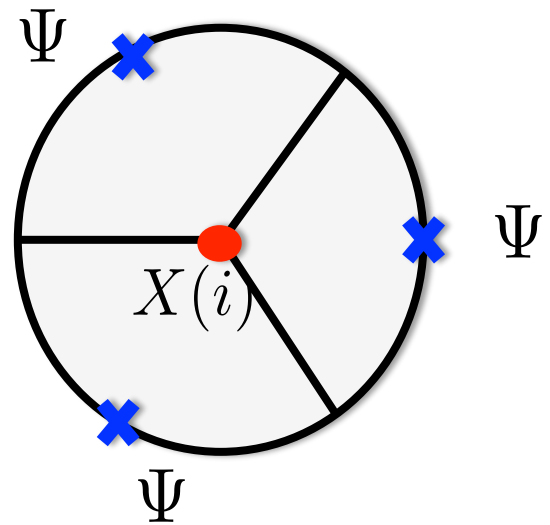
Adding the regulators satisfying A_∞/L_∞-relations.

Recall the Witten's Cubic Action

$$S = \frac{1}{2}\langle \Psi, Q\Psi \rangle + \frac{1}{3}\langle \Psi, X(i)\big(\Psi * \Psi\big)\rangle$$

$X(i)$: Picture-Changing Operator (Singular OPE !!)

$\langle \Psi, X(i)\big(\Psi * \Psi\big)\rangle$ =



This Product is Associative !!

But contact terms become DIVERGENT !!

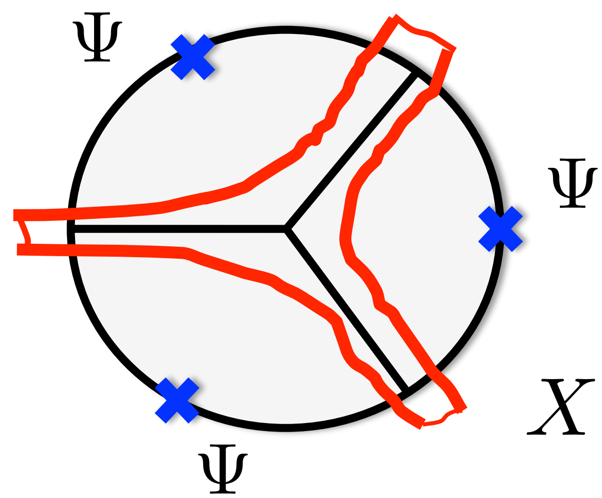
EKS's A_∞-type NS open theory

Using line integral $X = \int \frac{dz}{2\pi i} f(z) X(z)$, we can define

a new 2-string product M_2 :

$$M_2(\Psi_1, \Psi_2) = \frac{1}{3}\Big(X(\Psi_1 * \Psi_2) + (X\Psi_1) * \Psi_2 + \Psi_1 * (X\Psi_2)\Big)$$

$\langle \Psi, M_2(\Psi^2) \rangle$ =



M_2 is non-associative !!

$$M_2(M_2(A, B), C) \neq M_2(A, M_2(B, C))$$

Higher products satisfying A_∞ & η-derivation

Add appropriate '**higher products**' as '**the regulator**'!!

$$S = \frac{1}{2}\langle \Psi, Q\Psi \rangle + \frac{1}{3}\langle \Psi, M_2(\Psi^2)\rangle + \frac{1}{4}\langle \Psi, M_3(\Psi^3)\rangle + \frac{1}{5}\langle \Psi, M_4(\Psi^4)\rangle + \dots$$

→ Satisfying A_∞/L_∞-relations :

$$(Q + M_2 + M_3 + \dots)^2 = 0$$

→ Satisfying η-derivation relations :

$$\eta\, M_n(\phi \dots \phi) = \Sigma\, M_n(\phi \dots \eta\phi \dots \phi)$$

- Constructing these products, one can obtain an action.

How to construct $M = \{ M_n \}_{n=1}^{\infty}$?

- Let us introduce a generating function of the products.

$$M(\tau) = \Sigma \, \tau^n M_{n+1}$$

- The NS string products are given by the diff. eq.

$$\partial_\tau M(\tau) = [\, M(\tau) \,,\, \mu(\tau) \,]$$

Cyclicity
↓

BPZ odd

$$\mu^\dagger(\tau) = - \mu(\tau)$$

where $\mu(\tau) = \Sigma \, \tau^n \mu_{n+1}$ is the EKS's "GAUGE" PRODUCTS.

So . . . There exist Two Formulations !!

based on the large Hilbert space

WZW-like actions

based on the small Hilbert space

A_∞/L_∞-type actions

Does they relate to each other. . . ?

based on the large Hilbert space

WZW-like actions

? ? ?

based on the small Hilbert space

A_∞/L_∞-type actions

We will see that

based on the large Hilbert space

WZW-like actions

Embedding ↑ ↓ Partial Gauge Fixing

based on the small Hilbert space

A_∞/L_∞-type actions

(Ex.) NS closed string field theory

Large-space NS string field

$$V: \quad \text{ghost } \# = 1 \ , \quad \text{picture } \# = 0$$

Small-space NS string field

$$\Phi: \quad \text{ghost } \# = 2 \ , \quad \text{picture } \# = -1$$

For example . . . Free actions

Large-space Action (NS closed)

$$S_2 = \frac{1}{2}\langle \eta V, QV \rangle$$

- Q-gauge sym.
- η-gauge sym.

Small-space Action (NS closed)

$$S_2 = \frac{1}{2}\langle \xi\Phi, Q\Phi \rangle = \frac{1}{2}\langle \Phi, Q\Phi \rangle_{\rm small}$$

- Q-gauge sym.

For example . . . Free actions

Large-space Action (NS closed)

$$S_2 = \frac{1}{2}\langle \eta V, QV \rangle$$

- Q-gauge sym.
- ~~η-gauge sym.~~

Partial Gauge Fixing

$$V = \xi \Phi$$

Small-space Action (NS closed)

$$S_2 = \frac{1}{2}\langle \xi\Phi, Q\Phi \rangle = \frac{1}{2}\langle \Phi, Q\Phi \rangle_{\mathrm{small}}$$

- Q-gauge sym.

NS closed 3-point Interaction

Large-space Action (NS closed)

$$S = \frac{1}{2}\langle \eta V, QV \rangle + \frac{\kappa}{3!}\langle \eta V, [QV, V] \rangle$$

- Q-gauge sym.
- η-gauge sym.

Small-space Action (NS closed)

$$S = \frac{1}{2}\langle \xi\Phi, Q\Psi \rangle + \frac{\kappa}{3!}\langle \xi\Phi, [X\Phi, \Phi] \rangle$$

- Q-gauge sym.

NS closed 3-point Interaction

Large-space Action (NS closed)

$$S = \frac{1}{2}\langle \eta V, QV \rangle + \frac{\kappa}{3!}\langle \eta V, [QV, V] \rangle$$

- Q-gauge sym.
- ~~η-gauge sym.~~

Partial Gauge Fixing

$$V = \xi \Phi$$

Small-space Action (NS closed)

$$S = \frac{1}{2}\langle \xi \Phi, Q\Psi \rangle + \frac{\kappa}{3!}\langle \xi \Phi, [X\Phi, \Phi] \rangle$$

- Q-gauge sym.

NS closed 3-point Interaction

Large-space Action (NS closed)

$$S = \frac{1}{2}\langle \eta V, QV \rangle + \frac{\kappa}{3!}\langle \eta V, [QV, V] \rangle$$

- Q-gauge sym.
- ~~η-gauge sym.~~

Partial Gauge Fixing (up to $O(\phi^3)$)

$$V = \xi\Phi + \frac{\kappa}{3!}\xi[\xi\Phi, \Phi] + \mathcal{O}(\kappa^2)$$

Small-space Action (NS closed)

$$S = \frac{1}{2}\langle \xi\Phi, Q\Psi \rangle + \frac{\kappa}{3!}\langle \xi\Phi, [X\Phi, \Phi] \rangle$$

- Q-gauge sym.

NS closed 4-point Interaction

Large-space NS Action

$$S=\frac{1}{2}\langle \eta V, QV\rangle+\frac{\kappa}{3!}\langle \eta V,[QV,V]\rangle+\frac{\kappa^2}{4!}\langle \eta V,[QV,QV,V]+[[QV,V],V]\rangle$$

- Q-gauge sym.
- ~~η-gauge~~ sym.

Partial Gauge Fixing (up to $O(\phi^4)$)

$$V=\xi\Phi+\frac{\kappa}{3!}\xi[\xi\Phi,\Phi]$$
$$+\frac{\kappa^2}{4!}\Big(\xi\big[\xi\Phi,(Q\xi+X)\Phi,\Phi\big]+\xi\big[\xi[\Phi,\Phi],\xi\Phi\big]$$
$$+\frac{2}{3}\xi\big[\xi[\xi\Phi,\Phi],\Phi\big]+\frac{2}{3}\big[\xi[\xi\Phi,\Phi],\xi\Phi\big]\Big)$$

Small-space NS Action

$$S=\frac{1}{2}\langle \xi\Phi, Q\Phi\rangle+\frac{\kappa}{3!}\langle \xi\Phi,[X\Phi,\Phi]\rangle+\frac{\kappa^2}{4!}\langle \xi\Phi L_3(\Phi,\Phi,\Phi)\rangle$$

- Q-gauge sym.

Up to $O(\phi^4)$

Witten's Cubic Theory

Witten

Small-space theory

(A_∞/L_∞-type actions)

NS open

NS closed

NS-NS closed

Erler, Konopka, Sachs 14'

Large-space theory

(WZW-type actions)

NS open

Berkovits 95'

NS closed

Berkovits, Okawa, Zwiebach 04'

NS-NS closed

H.M 14'

Gauge fixing . . . OK

Gauge fixing . . . Possible!

Up to $O(\phi^4)$

Witten's Cubic Theory

Witten

Small-space theory (A_∞/L_∞-type actions)		Large-space theory (WZW-type actions)
NS open	⟷	NS open **Berkovits 95'**
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Gauge fixing . . . OK

Gauge fixing . . . Possible!

How to obtain a closed form expression ?

2. Similarity Transformations

- Path-ordered exponential : G , E_V

- $L = G\,Q\,G^{\dagger}$ & $Q_g = E_V\,Q\,E_V^{\dagger}$

Path-ordered exp. $A[\tau]$ of operators $O[\tau]$

- We consider a path-ordered exponential :

$$\mathcal{A}[\tau] = \overrightarrow{\mathcal{P}} \exp\left(\int_0^\tau d\tau' \mathcal{O}[\tau']\right)$$

$$= \mathbb{1} + \left(\int_0^\tau d\tau_1 \mathcal{O}[\tau_1]\right) + \sum_{n=2}^{\infty} \left(\int_0^\tau d\tau_1 \mathcal{O}[\tau_1]\right)\left(\int_0^{\tau_1} d\tau_2 \mathcal{O}[\tau_2]\right) \cdots \left(\int_0^{\tau_{n-1}} d\tau_n \mathcal{O}[\tau_n]\right)$$

- $A[\tau]$ is the solution of diff. eq. ($A[0] = 1$)

$$\partial_\tau \mathcal{A}[\tau] = \mathcal{O}[\tau] \cdot \mathcal{A}[\tau]$$

- Reversing the direction & sign, we obtain its inverse :

$$\mathcal{A}^{-1}[\tau] = \overleftarrow{\mathcal{P}} \exp\left(-\int_0^\tau d\tau' \mathcal{O}[\tau']\right)$$

$$= \mathbb{1} - \int_0^\tau d\tau_1 \mathcal{O}^\dagger[\tau_1] + \sum_{n=2}^{\infty} (-)^n \int_0^\tau d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \mathcal{O}[\tau_n] \cdots \mathcal{O}[\tau_2]\mathcal{O}[\tau_1].$$

Path-ordered exp. gives a solution.

- $\mathcal{A}[\tau] = \overrightarrow{\mathcal{P}} \exp \left(\int_0^{\tau} d\tau' \mathcal{O}[\tau'] \right)$ satisfies $\partial_\tau \mathcal{A}[\tau] = \mathcal{O}[\tau] \cdot \mathcal{A}[\tau]$ and

  $\mathcal{A}^{-1}[\tau] = \overleftarrow{\mathcal{P}} \exp \left(- \int_0^{\tau} d\tau' \mathcal{O}[\tau'] \right)$ satisfies $\partial_\tau \mathcal{A}^{-1}[\tau] = - \mathcal{A}^{-1}[\tau] \cdot \mathcal{O}[\tau]$.

- Hence, when the operator $L(\tau)$ satisfies

$$\partial_\tau L(\tau) = [\, O(\tau) , L(\tau) \,] \quad \text{with} \quad L(0) = Q ,$$

a solution is given by $L(\tau) = A(\tau)\, Q\, A^{-1}(\tau)$!!

→ Application : NS string products in the small-space.

Recall L_∞-type and WZW-like actions

- Let us introduce a generating function of the products.

Small-space L_∞-type Action (NS closed)

$$S_{\text{EKS}} = \int_0^1 dt\, \langle \boldsymbol{\pi}(\boldsymbol{\xi}_t\, e^{\wedge\Phi(t)}), \boldsymbol{\pi}\big(\mathbf{L}(e^{\wedge\Phi(t)})\big)\rangle$$

- $\mathbf{L}[\tau] = \sum_{n=1}^{\infty} \tau^{n-1}\mathbf{L}_n$: Generating fnc. of the NS products
- $e^{\wedge\Phi} = \mathrm{Id} + \Phi + \frac{1}{2}\Phi\wedge\Phi + \frac{1}{3!}\Phi\wedge\Phi\wedge\Phi + \dots$: Group-like element

→ NS string products are given by the diff. eq.

$$\partial_\tau \mathbf{L}[\tau] = [\![\mathbf{L}[\tau], \boldsymbol{\Xi}[\tau]]\!]$$

where $\boldsymbol{\Xi}[\tau] = \sum_{n=2}^{\infty} \tau^{n-2}\boldsymbol{\Xi}_n$ is EKS gauge products.

Cyclicity
↓
BPZ odd
$$\big(\boldsymbol{\Xi}(\kappa)\big)^\dagger = -\boldsymbol{\Xi}(\kappa)$$

NS products = Similarity Transformation of Q

Solution of the diff. eq. $\partial_\tau \mathbf{L}[\tau] = [\![\mathbf{L}[\tau], \mathbf{\Xi}[\tau]]\!]$

- Consider path-ordered exp. obtained from Ξ

$$\mathbf{G} = \overrightarrow{\mathcal{P}} \exp \left(- \int_0^1 d\tau \mathbf{\Xi}[\tau] \right) \qquad \mathbf{G}^\dagger[\tau] = \overleftarrow{\mathcal{P}} \exp \left(\int_0^\tau d\tau' \mathbf{\Xi}[\tau'] \right)$$

- Ξ is BPZ odd and BPZ conjugate reverse $\overrightarrow{\mathcal{P}}$ to $\overleftarrow{\mathcal{P}}$

→ [$G^\dagger$: BPZ conjugate] = [G^{-1} : Inverse] !!

- Since L(0) = Q , a solution is given by Similarity Transf.

$$\mathbf{L} = G\, Q\, G^{-1}$$

Large space WZW-like action (NS closed)

Recall the shifted bosonic string products:

$$Q_A B = QB + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} [\overbrace{A, \dots, A}^{n}, B]$$

$$[B, C]_A = [B, C] + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} [\overbrace{A, \dots, A}^{n}, B, C]$$

- (formal) pure-gauge $\mathcal{G}$ is defined by $\partial_\tau \mathcal{G}[\tau] = Q_{\mathcal{G}[\tau]} V$

- Ψ_η, Ψ_t are defined by

$$\partial_\tau \Psi_\eta[\tau] = \eta V + \kappa [V, \Psi_\eta[\tau]]_{\mathcal{G}[\tau]}$$

$$\partial_\tau \Psi_{\partial_t}[\tau] = \partial_t V + \kappa [V, \Psi_{\partial_t}[\tau]]_{\mathcal{G}[\tau]}$$

$$S = \int_0^1 dt \left\langle \Psi_t ,\, Q_{\mathcal{G}} \Psi_\eta \right\rangle$$

The g-shifted BRST op. Qg (NS closed)

- Note that

$$\frac{\partial}{\partial \tau} Q_{\mathcal{G}[\tau]} A = [Q_{\mathcal{G}[\tau]} V, A]_{\mathcal{G}[\tau]}$$
$$= [V, Q_{\mathcal{G}[\tau]} A]_{\mathcal{G}[\tau]} - Q_{\mathcal{G}[\tau]} [V, A]_{\mathcal{G}[\tau]}$$

- Using $\widehat{V}(\tau) := [\, V \, , \quad]_{g(\tau)}$, Qg satisfies

$$\partial\tau \, Q_{g(\tau)} = - [\, Q_{g(\tau)} \, , \widehat{V}(\tau) \,] \, .$$

↑

Commutator

Shifted BRST = Similarity Transformation of Q

- Q_g is also given by Similarity Transformation

$$Q_{\mathcal{G}} = \mathcal{E}_V \, Q \, \mathcal{E}_V{}^{-1}$$

Where E_V and $E_V{}^{-1}$ are given by path-ordered exponentials

$$\mathcal{E}_V := \overrightarrow{\mathcal{P}} e^{\int d\kappa \, \hat{v}(\kappa)} \qquad \mathcal{E}_V{}^{-1} := \overleftarrow{\mathcal{P}} e^{-\int d\kappa \, \hat{v}(\kappa)}$$

→ E_V : Invertible map !!

- The following choice gives our WZW-like actions

$$\hat{v}(\kappa) \equiv [V, \quad]_{\mathcal{G}}$$

Similarity Transf. F connecting L and Qg

- Since $L = G\,Q\,G^{\dagger}$ and $Q_g = E_V\,Q\,E_V^{\dagger}$,

  $F = E_V\,G^{\dagger}$ satisfies $F\,L\,F^{\dagger} = Qg$!

3. Equivalence of on-shell conditions

- Similarity transf. gives L_∞-morphism

- We derive the correspondence of fields

  preserving the on-shell condition.

Similarity Transf. connecting L and Qg

Similarity Transf. is generated by $\mathsf{F} := \mathcal{E}_V\, \mathbf{G}^\dagger$

→ Invertible map !!

$$\mathsf{F}\,\mathbf{L}\,\mathsf{F}^{-1} = \underline{(\mathcal{E}_V \mathbf{G}^\dagger)}\, \mathbf{G}\, \mathbf{Q}\, \mathbf{G}^\dagger\, \underline{(\mathbf{G}\, {\mathcal{E}_V}^\dagger)}$$

$$= \mathcal{E}_V\, Q\, {\mathcal{E}_V}^\dagger$$

$$= Q_{\mathcal{G}}.$$

- L and Q_g are connected by F : $\mathsf{F}\,\mathbf{L} = Q_{\mathcal{G}}\,\mathsf{F}$

→ F is a L_∞-morphism

Equivalence of two on-shell conditions

Small-space EOM	Large-space EOM
$\mathbf{L}(e^{\wedge\Phi}) = 0$	$Q_{\mathcal{G}}\,\Psi_\eta = 0$

The correspondence (or Field redefinition)

$$\boldsymbol{\pi}\big(\mathsf{F}(e^{\wedge\Phi})\big) = \Psi_\eta$$

provides the equivalence of two EOMs :

$$\boldsymbol{\pi}\big(\mathsf{F}\,\mathbf{L}(e^{\wedge\Phi})\big) = \boldsymbol{\pi}\big(Q_{\mathcal{G}}\,\mathsf{F}(e^{\wedge\Phi})\big)$$

$$= Q_{\mathcal{G}}\,\Psi_\eta .$$

The correspondence of fields

- Using E_V, associated string fields can be represent as

$$\Psi_{\mathbb{X}} = \mathcal{E}_V \int_0^1 d\tau \, {\mathcal{E}_V}^{\dagger}[\tau] \, (\mathbb{X} V)$$

- Therefore, $\pi\big(\mathsf{F}(e^{\wedge\Phi})\big) = \Psi_\eta$ is equivalent to

$$\boldsymbol{\pi}\big(\mathbf{G}^{\dagger}(e^{\wedge\Phi})\big) = \int_0^1 d\tau \, {\mathcal{E}_V}^{\dagger}[\tau]\big(\eta V\big)$$

function of ϕ, $\xi\phi$

function of ηV, V

How about Actions ?

L_∞-type Action

$$S_{\rm EKS} = \frac{1}{2}\langle \xi\Phi,\, Q\Phi\rangle + \sum_{n=1}^{\infty}\frac{\kappa^n}{(n+2)!}\langle \xi\Phi,\, L_{n+1}(\overbrace{\Phi,\dots,\Phi}^{n+1})\rangle$$

$$= \int_0^1 dt\,\frac{\partial}{\partial t}\Big(\sum_{n=0}^{\infty}\frac{\kappa^n}{(n+2)!}\langle \xi\Phi(t), L_{n+1}(\overbrace{\Phi(t),\dots,\Phi(t)}^{n+1})\rangle\Big)$$

$$= \int_0^1 dt\,\langle \xi\partial_t\Phi(t),\, \mathcal{F}_{\Phi(t)}\rangle$$

$\Phi(t) = t\,\phi$

E.O.M.

$$\mathcal{F}_{\Phi(t)} := Q\Phi(t) + \sum_{n=1}^{\infty}\frac{\kappa^n}{(n+1)!}L_{n+1}(\overbrace{\Phi(t),\dots,\Phi(t)}^{n+1})$$

Thus, the action becomes

$$S_{\rm EKS} = \int_0^1 dt\,\langle \boldsymbol{\pi}(\boldsymbol{\xi}_t\, e^{\wedge\Phi(t)}),\, \boldsymbol{\pi}\big(\mathbf{L}(e^{\wedge\Phi(t)})\big)\rangle$$

$$= \int_0^1 dt\,\langle \boldsymbol{\pi}(\boldsymbol{\xi}_t\, e^{\wedge\Phi(t)}),\, \boldsymbol{\pi}\big(\mathbf{G}\,\mathbf{Q}\,\mathbf{G}^{-1}(e^{\wedge\Phi(t)})\big)\rangle$$

$$= \int_0^1 dt\,\langle \boldsymbol{\pi}\big(\mathbf{G}^{-1}(\boldsymbol{\xi}_t e^{\wedge\Phi(t)})\big),\, Q\,\boldsymbol{\pi}\big(\mathbf{G}^{-1}(e^{\wedge\Phi(t)})\big)\rangle$$

How about Actions ?

WZW-like Action

$$S_{\text{WZW}} = \int_0^1 dt \, \langle \Psi_{\partial_t}(t), \, Q_{\mathcal{G}(t)} \, \Psi_\eta(t) \rangle$$

$$= \int_0^1 dt \, \langle \Psi_{\partial_t}(t), \, \big(\mathcal{E}_{V(t)} \, Q \, \mathcal{E}_{V(t)}{}^{\dagger} \big) \, \Psi_\eta(t) \rangle$$

Since $\Psi_{\mathbb{X}} = \mathcal{E}_V \int_0^1 d\tau \, \mathcal{E}_V{}^{\dagger}[\tau] \, (\mathbb{X} V)$, the action is given by

$$S_{\text{WZW}} = \int_0^1 dt \, \Big\langle \int_0^1 d\tau \, \mathcal{E}_{V(t)}{}^{\dagger}[\tau] \big(\partial_t V(t) \big), \, Q \int_0^1 d\tau \, \mathcal{E}_{V(t)}{}^{\dagger}[\tau] \big(\eta V(t) \big) \Big\rangle$$

$$\int_0^1 d\tau \, \mathcal{E}_{V(t)}{}^{\dagger}[\tau] \big(X V(t) \big) \equiv \boldsymbol{\pi} \big(\mathbf{G}^{-1} (\boldsymbol{\xi}_X \, e^{\wedge \Phi(t)}) \big)$$

(X = ∂_t , δ)

The correspondence preserving E.O.M.

$$S_{\text{EKS}} = \int_0^1 dt \, \langle \boldsymbol{\pi} \big(\mathbf{G}^{-1} (\boldsymbol{\xi}_t e^{\wedge \Phi(t)}) \big), \, Q \, \boldsymbol{\pi} \big(\mathbf{G}^{-1} (e^{\wedge \Phi(t)}) \big) \rangle$$

To preserve the action, ϕ & V must satisfy . . .

The correspondence (or Field redefinition)

$$\boldsymbol{\pi}\, \boldsymbol{G}^{-1}\big(e^{\wedge \Phi(t)}\big) \equiv \int_0^1 d\tau\, \mathcal{E}_{V(t)}{}^{\dagger}[\tau]\big(\eta V(t)\big)$$

$$\boldsymbol{\pi}\, \boldsymbol{G}^{-1}\big(\boldsymbol{\xi}_X e^{\wedge \Phi(t)}\big) \equiv \int_0^1 d\tau\, \mathcal{E}_{V(t)}{}^{\dagger}[\tau]\big(X V(t)\big)$$

provides the equivalence of two Actions :

L_∞-type Action

$$S = \int_0^1 dt\, \big\langle \xi \Phi,\, \mathbf{L}\big(e^{\wedge t\Phi}\big)\big\rangle$$

$$\mathbf{f}^{-1}\,\mathbf{f} = 1$$

⟷

WZW-like Action

$$S = \int_0^1 dt\, \big\langle \Psi_t,\, Q_{\mathcal{G}}\, \Psi_\eta \big\rangle$$

→ It is nontrivial whether a solution exists. But. . .

(We will discuss later. . .)

At least, perturbatively, we can seek a solution

We set $V = \Sigma_n \kappa^n V^{(n)}$ and impose $\xi V = 0$.

In the actions, we solve

$$\pi\, G^{-1}\big(e^{\wedge\Phi(t)}\big) \equiv \int_0^1 d\tau\, \mathcal{E}_{V(t)}{}^{\dagger}[\tau]\big(\eta V(t)\big)$$

Then, we obtain the partial gauge fixing condition

$$V = \xi\Phi + \frac{\kappa}{3!}\xi[\xi\Phi,\Phi] + \frac{\kappa^2}{4!}\Big(\xi\big[\xi\Phi,(Q\xi+X)\Phi,\Phi\big] + \xi\big[\xi[\Phi,\Phi],\xi\Phi\big]$$

$$+\frac{2}{3}\xi\big[\xi[\xi\Phi,\Phi],\Phi\big] + \frac{2}{3}\big[\xi[\xi\Phi,\Phi],\xi\Phi\big]\Big) + \ldots .$$

Summary & Discussion

NS & NS-NS sector of SFT

- String products of two formulations are given by

Similarity Transformations of Q .

$\rightarrow$ F L = Q_g F

- F preserves the space of solutions of E.O.M.

$\rightarrow$ This F induces the field redefinition :

$$\boldsymbol{\pi}\big(\mathbf{G}^{\dagger}(e^{\wedge\Phi})\big) = \int_0^1 d\tau\, \mathcal{E}_V{}^{\dagger}[\tau]\big(\eta V\big)$$

$\Rightarrow$ Equivalence of two on-shell conditions

Summary & Discussion NS & NS-NS sector of SFT

- Both actions can be represent by free-like forms:

$$S_{\mathrm{EKS}} = \int_0^1 dt \, \langle \boldsymbol{\pi}\big(\mathbf{G}^{-1}(\boldsymbol{\xi}_t e^{\wedge \Phi(t)})\big), \, Q \, \underline{\boldsymbol{\pi}\big(\mathbf{G}^{-1}(e^{\wedge \Phi(t)})\big)} \rangle$$

$$S_{\mathrm{WZW}} = \int_0^1 dt \, \Big\langle \int_0^1 d\tau \, \mathcal{E}_{V(t)}{}^{\dagger}[\tau]\big(\partial_t V(t)\big), \, Q \, \underline{\int_0^1 d\tau \, \mathcal{E}_{V(t)}{}^{\dagger}[\tau]\big(\eta V(t)\big)} \Big\rangle$$

- To obtain the equivalence of two actions, we have to find a solution of

$$\boldsymbol{\pi} \, \boldsymbol{G}^{-1}\big(e^{\wedge \Phi(t)}\big) \equiv \int_0^1 d\tau \, \mathcal{E}_{V(t)}{}^{\dagger}[\tau]\big(\eta V(t)\big)$$

$$\boldsymbol{\pi} \, \boldsymbol{G}^{-1}\big(\boldsymbol{\xi}_X e^{\wedge \Phi(t)}\big) \equiv \int_0^1 d\tau \, \mathcal{E}_{V(t)}{}^{\dagger}[\tau]\big(X V(t)\big)$$

→ Then, partial gauge fixing conditions will appear.

Discussion

- In Berkovits theory, WZW-like action is given

$$S = \int_0^1 dt \langle A_{\partial_t}(t), Q_{\mathcal{G}(t)} A_\eta(t) \rangle \qquad Q_{\mathcal{G}(t)} = Q + [e^{-\phi}(Qe^{\Phi}), \quad]$$

By Z_2-reversing : $(Q, \eta, \phi) \rightarrow (\eta, Q, -\phi)$,

$$S = \int_0^1 dt \langle \tilde{A}_{\partial_t}(t), Q\, \tilde{A}_\eta(t) \rangle$$

- Then, we can obtain a direct correspondences of

Z_2-reversed A_η , A_∂ and redefined fields

$A_\eta = \pi\, G^\dagger\, [\, 1/(1-\phi)\,]$, $A_\partial = \pi\, G^\dagger\, \xi_t\, [1/(1-\phi)]$.

Discussion

- And we can directly check that

$$A_\eta = \pi\, G^\dagger\, [1/(1-\phi)] \quad \& \quad A_\partial = \pi\, G^\dagger\, \xi_t\, [1/(1-\phi)]$$

satisfy the relation : $\eta A_{\partial_t} + \partial_t A_\eta + [A_\eta, A_{\partial_t}] = 0$.

- Then, without checking the defining eq. of asso. fields

$$\partial_\tau A_\eta(\tau) = \eta\Phi + [\Phi, A_\eta(\tau)]$$

we obtain the equivalence of actions.

Discussion

When we can't use Z_2-revesing, to obtain the equivalence,

A) Check $\partial_\tau A_\eta(\tau) = \eta\Phi + [\Phi, A_\eta(\tau)]$ in terms of redefined field.

B) Check l.h.s. & r.h.s. have the same algebraic properties.

(It may not be WZW-relations)

$$\boldsymbol{\pi}\, \boldsymbol{G}^{-1}\big(e^{\wedge\Phi(t)}\big) \equiv \int_0^1 d\tau\, \mathcal{E}_{V(t)^\dagger[\tau]}\big(\eta V(t)\big)$$

$$\boldsymbol{\pi}\, \boldsymbol{G}^{-1}\big(\boldsymbol{\xi}_X e^{\wedge\Phi(t)}\big) \equiv \int_0^1 d\tau\, \mathcal{E}_{V(t)^\dagger[\tau]}\big(XV(t)\big)$$

In the action, it is OK. But, as a state, it is not clear.

Thank you.

Appendix : Mathematics

Construction of a cyclic A_∞/L_∞-morphism

- Let $(S(H), L, \omega)$ and $(S(H)', L', \omega')$ be cyclic L_∞-algebras.

L_∞-morphism : A morphism of coalgebra $f : S(H) \rightarrow S(H)'$

satisfying $f\, L = L'\, f$.

Cyclic L_∞-morphism : L_∞-morphism f satisfying

$\omega(A, B) = \omega'(f_1(A), f_1(B))$ and

$\Sigma\, \omega'(f_j(A_1, , A_j), f_k(B_1, , B_k)) = 0$.

→ f preserve the Equation of Motion !!

Consider two EOMs

A_∞/L_∞-type EOM

$$(Q + L_2 + L_3 + \ldots)\, e^{\wedge\phi} = 0$$

$$\Updownarrow$$

$$L\, e^{\wedge\phi} = 0$$

- L_∞-algebra -

$$(Q + L_2 + L_3 + \cdots)^2 = 0$$

WZW-type EOM

$$Q_g\, \psi_\eta = 0$$

- Trivial L_∞-algebra -

$$(Q_g + 0 + \cdots)^2 = 0$$

➡ Find a isomorphism satisfying $f\, L = Q_g\, f$.