Partition function of free conformal higher spin theory

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From joint work with Matteo Beccaria & Arkady Tseytlin

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Outline & Summary

1. **Conformal higher-spin gravity**

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   - **Free theory** [Fradkin, Tseytlin, 1985]
     - Kinetic operator with $2s + d - 4$ derivatives (local for even d)
     - Higher-spin (s >2) generalization of $(d = 4)$
       Maxwell photon (s =1) and Weyl graviton (s=2)
     - *Non-unitary irreducible representation* of conformal algebra (for $s > 1$ or $d > 4$)

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     - *Non-unitary irreducible representation* of conformal algebra (for $s>1$ or $d>4$)
   - **Interacting theory** [Tseytlin, 2002; Segal, 2003; XB, Joung, Mourad, 2011; Giombi, Klebanov, Pufu, Safdi, Tarnopolsky, 2013]
     - *classical theory* as induced gravity: logarithmically divergent piece of the 1-loop effective action of a conformal scalar field minimally coupled to a background of higher-spin shadow fields
     - *quantum theory*: vanishing Weyl anomaly?

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Representation theory of algebra $\mathfrak{so}(d,2)$:

- Classification of irreducible (generalized) Verma $\mathfrak{so}(d,2)$-modules [Shaynkman, Tipunin, Vasiliev, 2004]
- Character formulae of Verma $\mathfrak{so}(d,2)$-modules [F. Dolan, 2005]
- Canonical partition function on $S^1 \times S^{d-1}$ as an $\mathfrak{so}(d,2)$-character evaluated at trivial $\mathfrak{so}(d)$ weight [Gibbons, Perry, Pope, 2006]

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- *Current* (irreducible module)
- *Shadow* (ir/reducible for odd/even d)
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classical (tree) relation [XB, Grigoriev, 2012]

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   Sum over the infinite tower of all integer spins
   - **Regularization** à la Zeta [Giombi, Klebanov, Safdi, 2014]
   - **Cancellations for unbroken Vasiliev higher-spin gravity** (Dirichlet boundary condition) at 1-loop of
     - Vacuum bubble diagrams (for boundary S^d) [Giombi, Klebanov, 2013; Giombi, Klebanov, Safdi, 2014]
     - Casimir energy [Giombi, Klebanov, Tseytlin, 2014]

   are checks (at 1-loop) that unbroken Vasiliev higher-spin gravity might be quantum exact.

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     - type-A Weyl anomaly (Euler density a-coefficient)
       [Giombi, Klebanov, Pufu, Safdi, Tarnopolsky, 2013; Tseytlin, 2013]
     - Casimir energy
       [Beccaria, XB, Tseytlin, 2014]

   suggest that (bosonic) conformal higher-spin gravity might remain consistent at quantum level (like $\mathcal{N} = 4$ conformal supergravity).

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   suggest that (bosonic) conformal higher-spin gravity might remain consistent at quantum level (like $\mathcal{N} = 4$ conformal supergravity).

   Caveat: The type-B Weyl anomaly (Weyl-squared c-coefficient) does not seem to share this remarkable cancellation property but some assumptions in the computation remain questionable [Tseytlin, 2013] e.g. the factorization of the kinetic operator in any Einstein background [Metsaev, 2014; Nutma, Taronna, 2014]

Conformal higher-spin gravity

Conformal gravity

Conformal gravity as Weyl gravity [Weyl, 1918]

- $\equiv$ gravity theory invariant under both diffeomorphisms and **Weyl transformations**:

$$g'_{\mu\nu}(x) \;=\; \Omega^2(x)\, g_{\mu\nu}(x)$$

- fourth-order action

$$S_W[g_{\mu\nu}] \;=\; \int d^4x\, \sqrt{|g|}\; C_{\mu\nu\,\rho\sigma}C^{\mu\nu\,\rho\sigma}$$

built out of the **Weyl tensor** transforming as

$$C'_{\mu\nu\,\rho\sigma}(x) \;=\; \Omega^2(x)\, C_{\mu\nu\,\rho\sigma}(x)$$

- fourth-order equation of motion [Bach, 1921]

$$(\nabla^\mu\nabla^\sigma + \frac{1}{2}R^{\mu\sigma})C_{\mu\nu\,\rho\sigma} = 0$$

Conformal gravity

Conformal gravity as induced gravity [Sakharov, 1967]

- $\equiv$ logarithmically divergent piece of the 1-loop effective action for matter, e.g.
- scalar field transforming under Weyl transformations as

$$\varphi'(x) = \Omega^{\frac{2-d}{2}}(x)\varphi(x)$$

conformally coupled to a metric background

$$S_M[\varphi\,;g_{\mu\nu}] = \frac{1}{2}\int d^dx\,\sqrt{|g|}\,\Big(\nabla_\mu\phi\nabla^\mu\phi - \frac{d-2}{4(d-1)}\,\mathcal{R}\,\phi^2\Big)$$

leads to the regularized effective action for the UV cutoff Λ.
In $d=4$:

$$\begin{aligned} W_\Lambda[g] &= -\frac{1}{2}\,\mathrm{Tr}_\Lambda \log(\nabla^2 + \mathcal{R}/6) \\ &\sim \Lambda^2\,S_{CC}[g] + \Lambda\,S_{EH}[g] + \log(\Lambda/m^2)\,S_W[g] + \text{UV finite} \end{aligned}$$

Conformal gravity

Einstein gravity from conformal gravity [Maldacena, 2011]

- *Classical solutions* [Bach, 1921]

  (conformally) Einstein $\Rightarrow$ Bach

- *Classical solutions with cosmological constant and Neumann boundary condition* [Maldacena, 2011]

  Bach $\Rightarrow$ (conformally) Einstein

- For asymptotically hyperbolic Einstein spaces, the renormalized on-shell actions of $d = 4$ Einstein gravity and conformal gravity are equal [Anderson, 2000]

Remark: In the above sense, conformal gravity with Neumann boundary condition is classically equivalent to Einstein gravity with cosmological constant (ghosts are eliminated by the boundary condition).

Conformal gravity

Conformal gravity as quantum gravity

For a conformal gravity, absence of gauge anomaly $\Rightarrow$ *UV-finiteness*

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For a conformal gravity, absence of gauge anomaly $\Rightarrow$ *UV-finiteness*

"Proof"

Quantum consistency (no gauge anomaly) of a conformal gravity
$\Rightarrow$ no conformal anomaly
$\Rightarrow$ no scale
$\Rightarrow$ UV finite

Conformal gravity

Conformal gravity as quantum gravity

- Power-counting renormalizable [Stelle, 1977]
  but Weyl anomalous [Fradkin-Tseytlin, 1984]

Conformal gravity

Conformal gravity as quantum gravity

- Power-counting renormalizable [Stelle, 1977]
  but Weyl anomalous [Fradkin-Tseytlin, 1984]
- $\exists$ examples of conformal $\mathcal{N} = 4$ supergravity coupled to $\mathcal{N} = 4$ super-Yang-Mills which are anomaly-free [Fradkin-Tseytlin, 1985]

Free theory: off-shell

The free conformal spin-s theory is described *off-shell* by the

- **Shadow field**, i.e. conformal primary field ϕ_s that is a symmetric tensor field of scale dimension $\Delta(\phi_s) = 2 - s$ and rank s, quotiented by the gauge transformation:

$$\delta\phi_{\mu_1\mu_2\ldots\mu_s} = \partial_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)} - \eta_{(\mu_1\mu_2}\alpha_{\mu_3\ldots\mu_s)}$$

- **Higher-spin Weyl tensor**, i.e. conformal primary field C_s that is the gauge-invariant irreducible tensor field of scale dimension $\Delta(C_s) = 2$ labelled by a rectangular Young diagram with rows of length s and built out of s derivatives of the shadow field:

$$C_{\mu_1\ldots\mu_s\,,\,\nu_1\ldots\nu_s} = \partial_{\nu_1}\cdots\partial_{\nu_s}\phi_{\mu_1\ldots\mu_s} + \cdots$$

- **Fradkin-Tseytlin-Segal action** [$d = 4$: Fradkin & Tseytlin, 1985; $d > 4$: Segal, 2002]

$$S[\phi_s] = \int \phi_s\,\Box^{\frac{2s+d-4}{2}}\,\phi_s + \cdots = (-1)^s \int C_s\,\Box^{\frac{d-4}{2}}\,C_s + \cdots$$

Free theory: on-shell

The free conformal spin-s theory is described *on-shell* by the

- **Higher-spin Bach tensor**, i.e. conformal primary field B_s that is the gauge-invariant symmetric tensor field of scale dimension $\Delta(B_s) = s + d - 2$ built out of $s + d - 4$ derivatives of the Weyl-like tensor:

$$B_{\mu_1 \ldots \mu_s} = \partial^{\nu_1} \cdots \partial^{\nu_s} \Box^{\frac{d-4}{2}} C_{\mu_1 \ldots \mu_s\,,\,\nu_1 \ldots \nu_s} + \cdots$$

- **Fradkin-Tseytlin field**, i.e. shadow field ϕ_s subject to the higher-spin Bach equation $B_s = 0$.

Interacting theory

Conformal higher-spin gravity as induced higher-spin gravity

[Tseytlin, 2002; Segal, 2003; XB, Joung, Mourad, 2011;
Giombi, Klebanov, Pufu, Safdi, Tarnopolsky, 2013]

Interacting theory

Conformal higher-spin gravity as induced higher-spin gravity

The interacting conformal higher-spin theory is described *off-shell* by the

- **Free scalar singleton**, i.e. free conformal scalar field φ of scale dimension $\Delta(\varphi) = d/2 - 1$ minimally coupled via

$$S_M[\varphi\,;\phi] = \int d^dx \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \sum_s \phi_s j_s\right) = \int d^dx\,\varphi(\partial^2 + \hat{\phi})\varphi$$

  to a background of higher-spin shadow fields ϕ_s through the
- **Conformal currents**, i.e. conformal primary fields j_s that are traceless conserved symmetric currents of scale dimension $\Delta(j_s) = s + d - 2$ and rank s, bilinear in the conformal scalar field

$$j_{\mu_1\dots\mu_s} = \varphi\,\partial_{\mu_1}\cdots\partial_{\mu_s}\varphi + \cdots$$

- **Induced action:** logarithmically divergent piece of the effective action (in even d)

$$W_\Lambda[\phi] = -\frac{1}{2}\,\mathrm{Tr}_\Lambda \log(\partial^2+\hat{\phi}) = \log(\Lambda/m^2)\,S_W[g] + \text{Laurent series in } \Lambda$$

Interacting theory

By dimensional analysis, one can see that the vertex involving a product of m fields with spins s_i $(i = 1, \ldots, m)$ contains a total number

$$p = d + \sum_{i=1}^{m} (s_i - 2)$$

of partial derivatives [XB, Joung, Mourad, 2011].

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$\Longrightarrow$ Minimal coupling of conformal higher-spin fields with the conformal spin-2 field.

Characters & Partition function

Classification

Exhaustive classification of irreducible (generalized) Verma $\mathfrak{so}(d,2)$-modules [Shaynkman, Tipunin, Vasiliev, 2004]

- Odd d: no subsingular module.
- Even d: $\exists$ subsingular module (due to chirality).

$\Longrightarrow$ Recursive computation of all relevant $\mathfrak{so}(d,2)$-modules

Odd dimension d

$d = 3$ realizations of irreducible $\mathfrak{so}(3,2)$-modules

CFT	Irreducible module	Verma module description
Conservation law	$\mathcal{D}(s+2, s-1)$	$\mathcal{V}(s+2, s-1)$
Conformal current	$\mathcal{D}(s+1, s)$	$\mathcal{V}(s+1, s)/\mathcal{D}(s+2, s-1)$

Conservation law $\equiv \partial_{\mu_1} j^{\mu_1 \mu_2 \cdots \mu_s} = 0$

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Pure gauge shadow	$\mathcal{D}(2-s,s)$	$\mathcal{V}(2-s,s)/\mathcal{D}(s+1,s)$

Higher-spin Cotton tensor $\partial^{2s-1}\phi_s=0 \Leftrightarrow \phi_s=\partial\varepsilon_{s-1}+\eta_2\alpha_{s-2}$

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Pure gauge shadow	$\mathcal{D}(2-s, s)$	$\mathcal{V}(2-s, s)/\mathcal{D}(s+1, s)$
Conformal Killing	$\mathcal{D}(1-s, s-1)$	$\mathcal{V}(1-s, s-1)/\mathcal{D}(2-s, s)$

Conformal Killing tensor $\equiv \partial_{(\mu_1}\varepsilon_{\mu_2\dots\mu_s)} = \eta_{(\mu_1\mu_2}\alpha_{\mu_3\dots\mu_s)}$

Odd dimension d

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Pure gauge shadow	$\mathcal{D}(2-s, s)$	$\mathcal{V}(2-s, s)/\mathcal{D}(s+1, s)$
Conformal Killing	$\mathcal{D}(1-s, s-1)$	$\mathcal{V}(1-s, s-1)/\mathcal{D}(2-s, s)$
CFT	Reducible module	Contragredient module
Shadow field	$\mathcal{S}(2-s, s)$	$\sim \mathcal{V}(2-s, s)/\mathcal{D}(2-s, s)$

Even dimension d

$d = 4$ realizations of irreducible $\mathfrak{so}(4,2)$-modules

CFT	Irreducible module	Verma module description
Conservation law	$\mathcal{D}(s+3, s-1)$	$\mathcal{V}(s+3, s-1)$
Conformal current	$\mathcal{D}(s+2, s)$	$\mathcal{V}(s+2, s)/\mathcal{D}(s+3, s-1)$
Chiral FT field	$\mathcal{D}(2; s, \pm s)$	$\mathcal{V}(2; s, \pm s)/\mathcal{D}(s+2, s)$
FT field	$\mathcal{D}(2, s, s)$ $\oplus\mathcal{D}(2, s, -s)$	$\mathcal{U}(2, s, s)/\mathcal{V}^*(s+2, s)$
Pure gauge fixed shadow	$\mathcal{D}(2-s, s)$	$\mathcal{V}(2-s, s)/\mathcal{U}(2, s, s) \cong$ $\mathcal{U}(2-s, s)/\mathcal{V}^*(s+3, s-1)$
Conformal Killing	$\mathcal{D}(1-s, s-1)$	$\mathcal{V}(1-s, s-1)/\mathcal{U}(2-s, s)$
CFT	**Reducible module**	**Contragredient module**
Shadow field	$\mathcal{S}(2-s, s)$	$\sim \mathcal{V}(2-s, s)/\mathcal{U}(2-s, s)$

Partition function

Canonical partition function

- **Generating function**

$$\mathcal{Z}(q) \equiv \sum_n d_n q^{\Delta+n} = Tr(\exp(-\beta \hat{H}))$$

of the auxilliary variable $q = \exp(-\beta)$ defined in terms of the inverse temperature β, i.e. the inverse of the perimeter of the circle S^1 in radial quantization. It encodes the number d_n of descendants of level n associated with a primary field of scale dimension Δ.

- **Character** $\chi(q, x_1, \cdots, x_{[d/2]})$ of $\mathfrak{so}(d,2)$ evaluated at trivial $\mathfrak{so}(d)$-weight:

$$\mathcal{Z}(q) = \chi(q, 0, \cdots, 0)$$

Partition function

$d = 4$ canonical partition functions

- ***Verma module***

$$\mathcal{V}_{[\Delta;(\ell_1,\ell_2)]}(q) = \frac{d_{(\ell_1,\ell_2)}q^{\Delta}}{(1-q)^4}$$

Partition function

$d = 4$ canonical partition functions

- ***Conformal current* $\Leftrightarrow$ *Massless fields (Dirichlet)***

$$\mathcal{Z}_{(s+2,s)}(q) = \frac{(s+1)^2\, q^{s+2} - s^2\, q^{s+3}}{(1-q)^4}$$

- ***Shadow field* $\Leftrightarrow$ *Massless fields (Neumann)***

$$\mathcal{Z}_{(2-s,s)}(q) = \frac{2(2\,s+1)\,q^2 - (s+1)^2\, q^{s+2} + s^2\, q^{s+3}}{(1-q)^4}$$

- ***Fradkin-Tseytlin field***

$$\mathcal{Z}_{(2,s,s)}(q) = \frac{2(2\,s+1)\,q^2 - 2(s+1)^2\, q^{s+2} + 2s^2\, q^{s+3}}{(1-q)^4}$$

Regularization & Cancellation

Sum over all integer spins

Sum of partition functions over the infinite tower of
Conformal currents ⇔ Massless fields (Dirichlet)

- = Partition function of $d = 4$ free higher-spin gravity
- is finite in agreement with the Flato-Fronsdal theorem

$$\sum_{s=0} \mathcal{Z}_{(s+2,s)}(q) = \frac{q^2(1+q)^2}{(1-q)^6} = \left(\mathcal{Z}_{(1,0)}(q)\right)^2$$

where $\mathcal{Z}_{(1,0)}(q)$ is the partition function of the $d = 4$ scalar singleton.

Sum over all integer spins

Sum of partition functions over the infinite tower of **Fradkin-Tseytlin fields**

- = Partition function of $d = 4$ free conformal higher-spin gravity
- is infinite since

$$\mathcal{Z}_{(2,s,s)}(q) = \frac{4(s+1/2)\,q^2}{(1-q)^4} - 2\mathcal{Z}_{(s+2,s)}(q)$$

and the series $\sum_{s=0}(s+1/2)$ is obviously divergent

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$$\zeta_H(z,\alpha) \equiv \sum_{n=0}(n+\alpha)^{-z}$$

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$$\sum_{s=0}(s+1/2) = \zeta_H(-1,1/2) = 1/24$$

$$\Rightarrow \quad \sum_{s=0}\mathcal{Z}_{(s+2,s)}(q) = -\frac{q^2(11+26q+11q^2)}{6(1-q)^6}$$

Casimir Energy

Casimir Energy

- **Vacuum energy**

$$E_C \equiv \frac{1}{2} \sum_n d_n \left(\Delta + n\right) = \frac{1}{2} Tr(\hat{H})$$

in radial quantization

- can be computed from the canonical partition function via the Hamiltonian zeta function

$$E_C = \frac{1}{2}\zeta_E(-1) \,, \qquad \zeta_E(z) = \frac{1}{\Gamma(z)} \int_0^\infty d\beta \, \beta^{z-1} \, \mathcal{Z}(e^{-\beta})$$

- vanishes when $\mathcal{Z}(q) = \mathcal{Z}(1/q)$.

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- vanishes when $\mathcal{Z}(q)=\mathcal{Z}(1/q)$.

$\Rightarrow$ The total Casimir energy of free bosonic higher-spin gravity vanishes both for Vasiliev theory [Giombi, Klebanov, Tseytlin, 2014] and Segal theory [Beccaria, XB, Tseytlin, 2014].

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     - massless (boundary) higher-spin fields, with Neumann (shadow) or Dirichlet (current) boundary conditions.
     - conformal higher-spin fields, either off-shell (shadow) or on-shell (Fradkin-Tseytlin).

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   - Total (regularized) partition functions and Casimir energy for
     - conformal higher-spin gravity
3. Indication that no scale is generated at 1-loop for conformal higher-spin gravity.