Generalized Higher Spin Algebras

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Motivation

- Many technical difficulties in Vasiliev theory are connected to the HS algebra. Therefore analyzing the structure of HS algebra can give us tools to deal with physical problems related to Vasiliev theory, HS AdS/CFT…
- The structure of HS algebra in higher dimensions is not explicit and the structure constants were not available.
- There was no explicit trace formula available for HS algebra for dimensions greater than four.

HS algebras: A first look

Spin s massless symmetric field is a gauge field, described by a rank s double traceless symmetric tensor, with a gauge parameter, that is a rank s-1 traceless symmetric tensor.

$$\delta_\varepsilon \, \varphi_{\mu_1 \cdots \mu_s} = \bar{\nabla}_{(\mu_1} \, \varepsilon_{\mu_2 \cdots \mu_s)} + t_{\mu_1 \cdots \mu_s}(\varphi, \varepsilon)$$

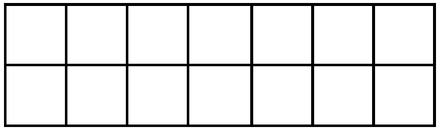
We restrict ourselves to global symmetries, imposing Killing tensor equations

$$0 = \left[\, \delta_\varepsilon \, \varphi_{\mu_1 \cdots \mu_s} \, \right]_{\varphi = 0} = \bar{\nabla}_{(\mu_1} \, \varepsilon_{\mu_2 \cdots \mu_s)}$$

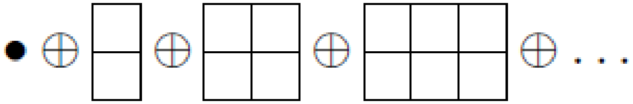
Vector space of the HS algebra

We start with SO(D-1,2) or SO(D,1) algebra of isometries of (A)dS$_D$ space.

HS algebras are span by Killing tensors, that are parameterized by two row rectangular Young diagrams of SO(D+1)



Vector space of HS algebra is the space of two row rectangular Young diagrams of SO(D+1)



Requirements on HS algebra

- Spin two part is gravity

$$[\![M_{ab} \, , M_{cd}]\!] = 2 \, (\eta_{a[c} \, M_{d]b} - \eta_{b[c} \, M_{d]a})$$

- All the fields couple to gravity minimally (equivalence principle)

$$[\![M_{a_1 b_1 , \ldots , a_r b_r} \, , M_{cd}]\!] = 2 \sum_{k=1}^{r} \eta_{a_k [c} \, M_{\ldots , d] b_k , \ldots} - \eta_{b_k [c} \, M_{\ldots , d] a_k , \ldots}$$

Higher Spin algebra can be defined as a quotient algebra. Take the Universal Enveloping Algebra of (A)dS isometry algebra, which is defined as a quotient of the tensor algebra of $\mathfrak{so}_{D+1}$ with the ideal generated by

$$I_{abcd} = M_{ab} \otimes M_{cd} - M_{cd} \otimes M_{ab} - [\![M_{ab} \, , \, M_{cd}]\!]$$

Class representatives are *GL(D+1)* tensors

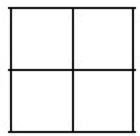
$$M_{a_1 b_1 | \cdots | a_n b_n} := \frac{1}{n!} \sum_{\sigma \in S_n} M_{a_{\sigma(1)} b_{\sigma(1)}} \otimes \cdots \otimes M_{a_{\sigma(n)} b_{\sigma(n)}}$$

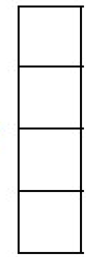
They contain Killing tensors, but not only.

Gunaydin ’89, C. Iazeolla and P. Sundell ‘08,

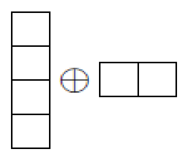
N. Boulanger and E. Skvortsov ’11

At the quadratic level we have decomposition to GL(D+1) tensors of the following symmetry type:

$$M^{(a_1}{}_{(b_1|}{}^{a_2)}{}_{b_2)} \sim$$ 

$$M_{[a_1 b_1 | a_2 b_2]} \sim$$ 

The first one has traceless $\mathfrak{so}_{D+1}$ tensor part (which is a Killing tensor), and trace part. The algebra, that is a quotient by the two sided ideal, generated by the two elements



contains only (all the) Killing tensors and therefore is a proper Higher Spin algebra (Vasiliev algebra).

Ideal generating elements are

$$J_{ab} = M_{(a}{}^{c}{}_{|b)c} - \frac{1}{D+1}\,\eta_{ab}\,M^{cd}{}_{|cd}\,, \qquad J_{abcd} := M_{[ab|cd]}$$

Quotienting these elements automatically fixes all the Casimirs

$$C_{2n} = M_{a_1}{}^{a_2}{}_{|\,a_2}{}^{a_3}{}_{|\cdots|\,a_{2n}}{}^{a_1}$$

In D=5, we have one-parameter family of ideal generating elements, and corresponding HS algebras

$$V^{\lambda}_{abcd} = M_{[ab|cd]} - \lambda\,\epsilon_{abcdef}\,M^{ef}$$

Formal definition of HS algebras

The kernel of the minimal representation in the UEA is the Joseph ideal. HS algebra is the algebra of the symmetries of the minimal representation:

$$hs(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/\mathcal{J}(\mathfrak{g})$$

As a vector space, it is the dual of the space of polynomials in the minimal coadjoint orbit.

This definition of HS algebra can be generalized to any Lie algebra.

Joseph, ‘74

We will discuss HS algebras of classical Lie algebras

$$\mathfrak{sp}_{2N} \qquad [\![N_{AB}\,,\ N_{CD}]\!] = \Omega_{A(C}\,N_{D)B} + \Omega_{B(C}\,N_{D)A}$$

$$\mathfrak{sl}_N \qquad [\![L^a_b\,,\ L^c_d]\!] = \delta^a_d\,L^c_b - \delta^c_b\,L^a_d$$

$$\mathfrak{so}_N \qquad [\![M_{ab}\,, M_{cd}]\!] = 2\,(\eta_{a[c}\,M_{d]b} - \eta_{b[c}\,M_{d]a})$$

We introduce dual vector space elements that satisfy

$$U^{[AB]} = 0 \qquad V_a{}^a = 0 \qquad W^{(ab)} = 0$$

Minimal coadjoint orbits for classical Lie algebras are defined by following quadratic equations

$$\mathcal{O}_{\min}(\mathfrak{sp}_{2N}) \quad : \qquad U^{A[B}\,U^{D]C} = 0\,,$$

$$\mathcal{O}_{\min}(\mathfrak{sl}_{N}) \quad : \qquad V_{[a}^{\ b}\,V_{c]}^{\ d} = 0 = V_{a}^{\ b}\,V_{b}^{\ c}\,,$$

$$\mathcal{O}_{\min}(\mathfrak{so}_{N}) \quad : \qquad W^{a[b}\,W^{cd]} = 0 = W^{ab}\,W_{b}^{\ c}$$

which are solved by

$$U^{AB} = u^A\,u^B\,, \quad V_b^a = v_+^a\,v_{-b} \quad [v_+\cdot v_- = 0]\,, \quad W^{ab} = w_+^{[a}\,w_-^{b]} \quad [w_\alpha\cdot w_\beta = 0]$$

The generating functions of the elements of HS algebra, corresponding to classical Lie algebra, are given by

$$hs(\mathfrak{sp}_{2N}) \quad : \qquad N(U) = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n\, n!}\, N_{A_1B_1,\dots,A_nB_n}\, U^{A_1B_1} \dots U^{A_nB_n} \,,$$

$$hs_\lambda(\mathfrak{sl}_N) \quad : \qquad L(V) = 1 + \sum_{n=0}^{\infty} \frac{1}{n!}\, L^{a_1 \cdots a_n}_{b_1 \cdots b_n}\, V^{b_1}_{a_1} \cdots V^{b_n}_{a_n} \,,$$

$$hs(\mathfrak{so}_N) \quad : \qquad M(W) = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n\, n!}\, M_{a_1b_1,\dots,a_nb_n}\, W^{a_1b_1} \dots W^{a_nb_n} \,,$$

Corresponding HS generators are

$$N_{A_1B_1,\dots,A_nB_n} \,, \qquad L^{a_1 \cdots a_n}_{b_1 \cdots b_n} \,, \qquad M_{a_1b_1,\dots,a_nb_n}$$

We are interested in structure constants $C_{ab}{}^{c}$ of HS algebras, defined by

$$T_a \star T_b = C_{ab}{}^{c}\, T_c$$

Where T_a - s are generators of HS algebra $hs(\mathfrak{g})$.

We define the trace of the element of HS algebra as the coefficient of the identity

$$\mathrm{Tr}\,[\,c_0 + c^a\, T_a] = c_0$$

We want to compute symmetric invariant bilinear form

$$B_{\boldsymbol{ab}} = \mathrm{Tr}\,[\,T_{\boldsymbol{a}} \star T_{\boldsymbol{b}}\,]$$

and the trilinear form, that has cyclic symmetry and is simply connected to the structure constant

$$C_{\boldsymbol{abc}} = \mathrm{Tr}\,[\,T_{\boldsymbol{a}} \star T_{\boldsymbol{b}} \star T_{\boldsymbol{c}}\,] = C_{\boldsymbol{ab}}{}^{\boldsymbol{d}}\; B_{\boldsymbol{dc}}$$

generating functions for bilinear and trilinear forms are given by

$$\mathcal{B}(A_1, A_2) \,=\, \mathrm{Tr}\,[\,T(A_1) \star T(A_2)\,]$$

$$\mathcal{C}(A_1, A_2, A_3) \,=\, \mathrm{Tr}\,[\,T(A_1) \star T(A_2) \star T(A_3)\,]$$

$\mathfrak{sp}_{2N}$ series

$$\mathcal{B}(U) = \frac{1}{\sqrt{1 + \frac{\langle U_1\, U_2\rangle}{4}}}\,, \qquad \mathcal{C}(U) = \frac{1}{\sqrt{1 + \frac{\langle U_1\, U_2\rangle + \langle U_2\, U_3\rangle + \langle U_3\, U_1\rangle + \langle U_1\, U_2\, U_3\rangle}{4}}}\,.$$

$\mathfrak{sl}_N$ series

$$\mathcal{B}(V) = {}_3F_2\Big(\tfrac{N}{2}\,(1+\lambda)\,,\ \tfrac{N}{2}\,(1-\lambda)\,,\ 1\,;\ \tfrac{N}{2}\,,\ \tfrac{N+1}{2}\,;\ -\tfrac{1}{4}\,\langle V_1\, V_2\rangle\Big)\,,$$

$$\mathcal{C}(V) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k\binom{k}{\ell}\frac{\big(\frac{N(1+\lambda)}{2}\big)_{2\,k-\ell}\big(\frac{N(1-\lambda)}{2}\big)_{k+\ell}}{(N)_{3k}}\times$$

$$\times\left[\langle V_1\, V_2\rangle + \langle V_2\, V_3\rangle + \langle V_3\, V_1\rangle + \langle V_1\, V_2\, V_3\rangle\right]^{k-\ell}$$

$$\times\left[\langle V_1\, V_2\rangle + \langle V_2\, V_3\rangle + \langle V_3\, V_1\rangle - \langle V_3\, V_2\, V_1\rangle\right]^{\ell}\,.$$

Minimal representation of $\mathfrak{sp}_{2N}$ is described by oscillators y_A

$$N_{AB} = y_A \, y_B$$

Endowed with Moyal $\star$ - product

$$(f \star g)(y) = \exp\left(\Omega_{AB} \, \partial_{y_A} \, \partial_{z_B} \right) f(y) \; g(z) \Big|_{z=y}$$

$hs(\mathfrak{sp}_{2N})$ is generated by powers of $y_A \, y_B$

$$N_{A_1 B_1, \ldots, A_n B_n} = y_{A_1} \, y_{B_1} \, \cdots \, y_{A_n} \, y_{B_n}$$

Gaussian generating function

$$N(U) = \exp\left(\frac{1}{2}\, y_A\, U^{AB}\, y_B\right)$$

For $hs(\mathfrak{sp}_{2N})$ the trace is given simply by

$$\mathrm{Tr}\left[f(y)\right] = f(0)$$

In order to derive generating functions for bilinear and trilinear forms we need to evaluate star-product of Gaussian functions and take trace. We define:

$$\frac{1}{\sqrt{G^{(n)}(U_1,\ldots,U_n)}} := \mathrm{Tr}\left[\, N(U_1) \star \cdots \star N(U_n) \,\right]$$

We adopt index notation

$$A = \alpha\, a\,, \qquad \alpha = \pm\,, \quad a = 1, 2, \ldots, N$$

such that

$$\Omega_{\alpha a\ \beta b} = \epsilon_{\alpha\beta}\,\eta_{ab}\,, \qquad \epsilon_{\pm\mp} = \pm 1$$

$\mathfrak{sl}_N$ $\qquad L^a_b := \eta^{ac} N_{+c\,-b} - \frac{1}{N}\,\delta^a_b\,\eta^{cd}\,N_{+c\,-d}$

$\mathfrak{so}_N$ $\qquad M_{ab} := N_{+a\,-b} - N_{+b\,-a}$

In order to address the algebraic structure of $hs_\lambda(\mathfrak{sl}_N)$ we consider the dual pair

$$(\,GL_1\,,\,GL_N\,) \subset Sp_{2N}$$

The $\mathfrak{u}_1$ - centralizer in $hs(\mathfrak{sp}_{2N})$ consists of elements, satisfying the following condition

$$\left[\,y_+ \cdot y_-\,,\; f(y)\,\right]_\star = \left(y_+ \cdot \partial_{y_+} - y_- \cdot \partial_{y_-}\right) f(y) = 0$$

The solution space consists of generators of the type

$$\tilde{L}^{a_1 \cdots a_n}_{b_1 \cdots b_n} = y_-^{\;a_1}\, y_{+b_1} \cdots\, y_-^{\;a_n}\, y_{+b_n}$$

The traceless part of this element can be identified with the HS generator $L^{a_1 \cdots a_n}_{b_1 \cdots b_n}$ of $hs_\lambda(\mathfrak{sl}_N)$.

generating function of the solution space is given by:

$$\tilde{L}(\tilde{V}) = \exp\left(y_+ \cdot \tilde{V} \cdot y_-\right)$$

with

$$\tilde{V}^b_{[a}\,\tilde{V}^d_{c]} = 0 \quad \Leftrightarrow \quad \tilde{V}^b_a = \tilde{v}_{+a}\,\tilde{v}^b_-$$

We call this algebra $hs(\mathfrak{gl}_N)$. It makes use of the same Moyal product as original $hs(\mathfrak{sp}_{2N})$.

HS algebra of $\mathfrak{sl}_N$ is a quotient of the $hs(\mathfrak{gl}_N)$ by a relation

$$K_\lambda := y_+ \cdot y_- - \frac{N}{2}\,\lambda \sim 0$$

corresponding HS algebra is what we call

$$hs_\lambda(\mathfrak{sl}_N)$$

Consider following isomorphism

$$\begin{aligned} \rho_\lambda \ : \ hs(\mathfrak{gl}_N) \quad &\to \quad hs_\lambda(\mathfrak{gl}_N)\,, \\ f(y) \quad &\mapsto \quad \rho_\lambda(f)(y) = e^{\lambda\,\partial_{y_+}\cdot\,\partial_{y_-}}\, f(y) \end{aligned}$$

$hs_\lambda(\mathfrak{gl}_N)$ admits a deformed star product

$$\begin{aligned} &(f \star_\lambda g)(y) = \rho_\lambda\big(\,\rho_\lambda^{-1}(f) \star \rho_\lambda^{-1}(g)\,\big)(y) \\ &= \exp\Big[\,\big(\partial_{y_+}\cdot\partial_{z_-} - \partial_{z_+}\cdot\partial_{y_-}\big) + \lambda\,\big(\partial_{y_+}\cdot\partial_{z_-} + \partial_{z_+}\cdot\partial_{y_-}\big)\,\Big]\, f(y)\ g(z)\Big|_{z=y} \end{aligned}$$

$hs_\lambda(\mathfrak{sl}_N)$ is a quotient of $hs_\lambda(\mathfrak{gl}_N)$ by a relation

$$y_+ \cdot y_- \sim 0$$

In order to deal with the structure of the quotient algebras, we choose *class representatives*

$[\![a]\!]$ is a class representative for $a \in hs(\mathfrak{gl}_N)$

The product in HS algebra $hs_\lambda(\mathfrak{sl}_N)$ is defined by

$$[\![a]\!] \bigstar [\![b]\!] := [\![a \star b]\!]$$

As a simple example, we look at the class representative for the following element

$$\tilde{L}^{a_1\, a_2}_{b_1\, b_2} = y_-{}^{a_1}\, y_{+b_1} y_-{}^{a_2}\, y_{+b_2}$$

$$\tilde{L}^{a_1\,a_2}_{b_1\,b_2} = L^{a_1\,a_2}_{b_1\,b_2} + \frac{4}{N+2}\, y_+\cdot y_-\ \delta^{(a_1}_{(b_1}\, L^{a_2)}_{b_2)} + \frac{2}{N(N+1)}(\,y_+\cdot y_-)^2\delta^{(a_1}_{(b_1}\,\delta^{a_2)}_{b_2)}$$

We use star product and the relation

$$K_\lambda := y_+\cdot y_- - \frac{N}{2}\,\lambda \sim 0$$

Trace

to show

$$\tilde{L}^{a_1\,a_2}_{b_1\,b_2} = L^{a_1\,a_2}_{b_1\,b_2} + \frac{2N\lambda}{N+2}\ \delta^{(a_1}_{(b_1}\, L^{a_2)}_{b_2)} + \frac{N\lambda^2+1}{2(N+1)}\delta^{(a_1}_{(b_1}\,\delta^{a_2)}_{b_2)}$$

More generally:

$$[\![\, \tilde{L}^{(n)}(\tilde{V})\,]\!] = \sum_{m=0}^{n} s^{(n)}_m\, \frac{\langle \tilde{V}\rangle^m}{m!}\, L^{(n-m)}(\tilde{V})$$

For the algebra $hs_\lambda(\mathfrak{sl}_N)$ we compute the trace of a Gaussian function

$$\mathrm{Tr}\big[\exp(y_- \cdot B \cdot y_+)\big] = \frac{\Gamma(N)}{\Gamma\big(\frac{N(1+\lambda)}{2}\big)\,\Gamma\big(\frac{N(1-\lambda)}{2}\big)} \int_0^1 dx\, \frac{x^{P-1}\,(1-x)^{Q-1}}{\det_N\left[1+\frac{1}{2}(1-2x)\,B\right]}$$

From which we deduce

$$\mathrm{Tr}\big[\,f(y)\,\big] = (\Delta_\lambda \star f\,)(0)$$

with

$$\Delta_\lambda(y) = \frac{\Gamma(N)}{\Gamma\big(\frac{N(1+\lambda)}{2}\big)\,\Gamma\big(\frac{N(1-\lambda)}{2}\big)} \int_0^1 dx\; x^{\frac{N(1+\lambda)}{2}-1}\,(1-x)^{\frac{N(1-\lambda)}{2}-1}\, e^{2(1-2x)\,y_+\cdot y_-}$$

In order to compute the generating function for structure constants of $hs_\lambda(\mathfrak{sl}_N)$, we take composition of Gaussian generating functions of HS generators with additional factor, coming from projector. We define:

$$\frac{1}{G^{(n)}(\rho, V_1, \ldots, V_n)} := e^{2\rho\, y_+ \cdot y_-} \star L(V_1) \star \cdots \star L(V_n) \Big|_{y=0}$$

With some computation one can show

$$G^{(n)}(\rho, V) = \det{}_N \left[\frac{1+\rho}{2} \prod_{k=1}^{n} (1 + V_k) + \frac{1-\rho}{2} \right]$$

we compute this determinant analogously to $\mathfrak{sp}_{2N}$ case

For the algebra $\mathfrak{so}_N$, we consider the dual pair

$$(Sp_2, O_N) \subset Sp_{2N}$$

The $\mathfrak{sp}_2$ - centralizer of the $hs(\mathfrak{sp}_{2N})$ consists of elements, satisfying

$$\left[\, y_\alpha \cdot y_\beta \,,\, f(y) \,\right]_\star = \left(y_\alpha \cdot \partial_{y^\beta} + y_\beta \cdot \partial_{y^\alpha} \right) f(y) = 0$$

All the generators of HS algebra commute with generators of Howe dual $\mathfrak{sp}_2$.

The solution space is generated by

$$\tilde{M}(\tilde{W}) = \exp\left(\frac{1}{2}\, y_{+a}\, \tilde{W}^{ab}\, y_{-b}\right)$$

with

$$\tilde{W}^{a[b}\,\tilde{W}^{cd]} = 0 \quad \Leftrightarrow \quad \tilde{W}^{ab} = \tilde{w}_+^{[a}\,\tilde{w}_-^{b]}$$

We call this algebra $\widetilde{hs}(\mathfrak{so}_N)$.
It makes use of the same Moyal product as $hs(\mathfrak{sp}_{2N})$.
HS algebra $hs(\mathfrak{so}_N)$ is a quotient of the algebra $\widetilde{hs}(\mathfrak{so}_N)$ by

$$K_{\alpha\beta} := y_\alpha \cdot y_\beta \sim 0$$

The trace in the algebra $hs(\mathfrak{so}_N)$ is given by

$$\mathrm{Tr}\big[\, f(y)\,\big] = (\Delta \star f)(0)$$

with

$$\Delta(y) = \frac{\Gamma\big(\frac{N-1}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\,\Gamma\big(\frac{N-4}{2}\big)} \int_0^1 dx\, x^{\frac{1}{2}}\,(1-x)^{\frac{N-6}{2}}\, e^{-2\sqrt{x}\, y_+ \cdot y_-}$$

Generating functions of structure constants are

$$\mathcal{B}(W) = {}_2F_1\Big(2\,,\ \frac{N-4}{2}\,;\ \frac{N-1}{2}\,;\ -\tfrac{1}{8}\,\langle W_1\, W_2\rangle\Big)$$

$$\mathcal{C}(W) = \frac{\Gamma\big(\frac{N-1}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\,\Gamma\big(\frac{N-4}{2}\big)} \int_0^1 dx\, \frac{(1-x)^{\frac{1}{2}}\, x^{\frac{N-6}{2}}}{\big(1+\frac{x}{8}\,\Lambda(W)\big)^2 - \frac{(1-x)\,x^2}{32}\,\Sigma(W)}$$

with

$$\Sigma(W) := \big\langle (W_1\, W_{(2}\, W_{3)})^2 \big\rangle$$

$$\Lambda(W) := \langle W_1\, W_2\rangle + \langle W_2\, W_3\rangle + \langle W_3\, W_1\rangle + \langle W_1\, W_2\, W_3\rangle\,,$$

3d HS algebra $hs[\lambda]$

Structure constants are given by following generating functions for bilinear and trilinear forms

$$\mathcal{B}(V) = {}_2F_1\Big(1+\lambda\,,\,1-\lambda\,;\,\tfrac{3}{2}\,;\,-\tfrac{1}{4}\,\langle V_1\,V_2\rangle\Big)$$

$$\mathcal{C}(V) = {}_2F_1\Big(1+\lambda\,,\,1-\lambda\,;\,\tfrac{3}{2}\,;\,-\tfrac{1}{4}\,\Lambda(V)\Big)$$

Associative star product is given by a compact formula

$$L(V_1)\star L(V_2) = \sum_{n=0}^{\infty} {}_2F_1\big(n+1+\lambda\,,\,n+1-\lambda\,;\,n+\tfrac{3}{2}\,;\,-\tfrac{1}{4}\,\langle V_1\,V_2\rangle\,\big)\,L^{(n)}(V_1+V_2+V_1\,V_2)$$

Structure constants of $hs[\lambda]$ (Pope, Romans, Shen 1990)

$$V^i_m \star V^j_n \equiv \frac{1}{2} \sum_{a=-1}^{\infty} q^a g^{ij}_a(m,n;\mu) V^{i+j-a}_{m+n}$$

$$g^{ij}_{2r}(m,n;\mu) = \frac{\phi^{ij}_{2r}(\mu)}{2(2r+1)!} N^{ij}_{2r}(m,n) , \qquad (2)$$

where the $N^{ij}_{2r}(m,n)$ are given by

$$N^{ij}_{2r}(m,n) = \sum_{k=0}^{2r+1} (-1)^k \binom{2r+1}{k} [i+1+m]_{2r+1-k}$$

$$\times [i+1-m]_k [j+1+n]_k [j+1-n]_{2r+1-k} , \qquad (3)$$

the $\phi^{ij}_{2r}(\mu)$ can be expressed as $\mu = s(s+1)$

$$\phi^{ij}_{2r}(\mu) = {}_4F_3\left[\begin{matrix} -\frac{1}{2}-2s,\ \frac{3}{2}+2s,\ -r-\frac{1}{2},\ -r \\ -i-\frac{1}{2},\ -j-\frac{1}{2},\ i+j-2r+\frac{5}{2} \end{matrix} ; 1\right] , \qquad (5)$$

Another description of $hs[\lambda]$ is given through deformed oscillators

$$\left[\, \hat{y}_a \,,\, \hat{y}_b \,\right] = 2\, \epsilon_{ab} \left(\hat{1} + \hat{\nu}\, \hat{k} \right)$$

$$\hat{k}^2 = 1\,, \qquad \{\, \hat{k}\,,\, \hat{y}_a \,\} = 0\,, \qquad [\, \hat{\nu}\,,\, \hat{y}_a \,] = 0\,, \qquad [\, \hat{\nu}\,,\, \hat{k} \,] = 0$$

we make contact between this description and ours, introducing matrix valued oscillators:

$$\hat{y}_a := 2 \begin{pmatrix} 0 & y_{+a} \\ y_-{}^c\, \epsilon_{ca} & 0 \end{pmatrix} \qquad \hat{k} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\nu} := 2\, \lambda\, \hat{1} + \hat{k}$$

with product rule

$$\hat{y}_a\ \hat{y}_b := 2 \begin{pmatrix} 0 & y_{+a} \\ y_-{}^c\, \epsilon_{ca} & 0 \end{pmatrix} \star 2 \begin{pmatrix} 0 & y_{+b} \\ y_-{}^d\, \epsilon_{db} & 0 \end{pmatrix}$$

Further generalizations

The HS algebras discussed so far correspond to the symmetries of *massless symmetric* fields. There are two obvious generalizations - to incorporate symmetries of:

1. Non-massless (partially massless) fields
2. Non-symmetric (mixed symmetry) fields
   (In dimensions d>5)

UEA admits corresponding quotients:

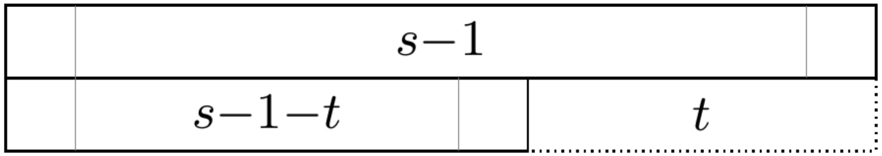
$$\mathcal{I}(\nu) = \left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \oplus (C_2 - \nu) \right), \qquad \mathcal{J}(\nu) = \left(\begin{array}{|c|c|}\hline & \\ \hline \end{array} \oplus (C_2 - \nu) \right).$$

X. Bekaert, M. Grigoriev ‘13, J.P. Michel ‘11,
N. Boulanger, E. Skvortsov ‘11

Partially Massless HS algebra

Partially massless fields appear in the spectrum of conformal fields in (A)dS background. Therefore conformal HS algebra is an example of partially massless HS algebra.
Partially massless generators are two row non-rectangular Young tableaux representations of (A)dS isometry algebra SO(N).

$$M_{M_1 \cdots M_{s-1}, N_1 \cdots N_{s-t}} \sim$$



Partially Massless HS algebra

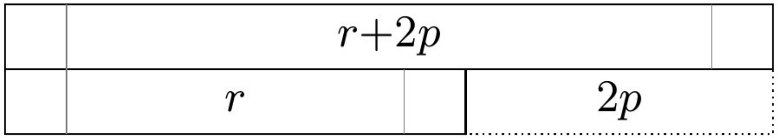
One parameter family of partially massless algebras is given by

$$\mathcal{A}(\nu) = \mathcal{U} / \mathcal{I}(\nu)$$

Which has the following vector space

$$\mathcal{A}(\nu) \simeq \bigoplus_{p=0}^{\infty} K_p \,,$$

$$K_p = \bigoplus_{r=0}^{\infty}$$



For special values of the parameter

$$\nu_\ell = -\frac{(D-1-2\,\ell)(D-1+2\,\ell)}{4}\,, \qquad \ell = 1, 2, \cdots$$

This algebra develops an ideal, which consists of partially massless generators of even depth not smaller than 2ℓ and the coset algebra describes partially massless generators of even depth less than 2ℓ

$$\mathfrak{p}_\ell = \mathcal{A}(\nu)/\mathcal{A}_\ell \simeq \bigoplus_{p=0}^{\ell-1} K_p$$

These algebras describes the symmetries of higher order singletons described by the action

$$S[\phi] = \int d^{D-1}x\,\phi\,\Box^\ell\,\phi$$

Further speculations on partially massless theories

Partially massless (PM) Higher Spin fields are part of the conformal HS spectrum, where PM fields of all depths are present. The even depth and odd depth PM fields in that spectrum are relatively ghost. Turning off odd depth fields by boundary conditions gives a unitary theory for PM and massless HS fields. Corresponding free actions for the PM fields up to certain depth are also available.

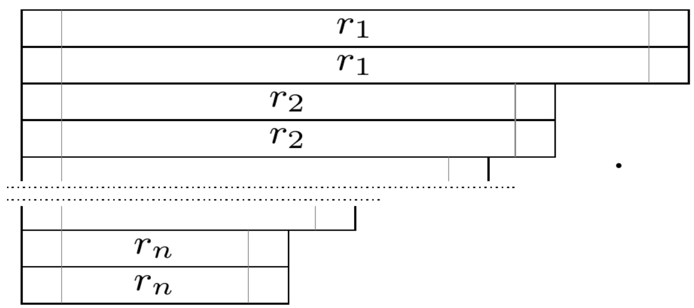
Xavier Bekaert's talk,
R. Metsaev ‘11, J. Maldacena ‘11,
E. Joung, K.M. ‘12

Mixed Symmetry HS Algebra

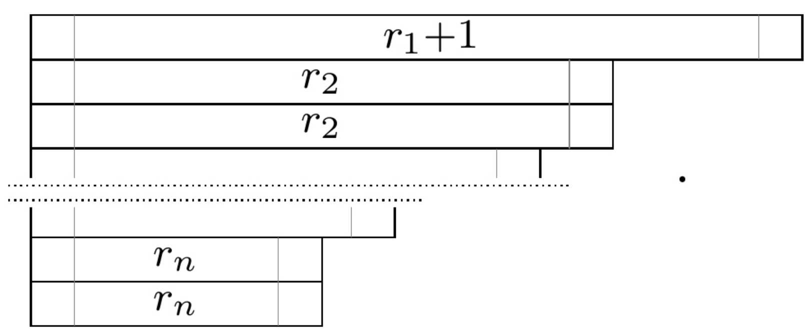
The vector space of mixed symmetry HS algebra

$$\mathcal{B}(\nu) = \mathcal{U} / \mathcal{J}(\nu)$$

is given by generators of the type



They correspond to the symmetries of the fields



Mixed Symmetry HS Algebra

The landscape of mixed symmetry algebras is very rich. In particular, for any odd dimensions, there are finite dimensional mixed symmetry HS algebras, which correspond to symmetric fields only in three and five dimensions.

Generalization of Flato – Fronsdal theorem to any dimensions for any spin states naturally suggests that corresponding HS theories contain mixed symmetry fields.

M. Vasiliev ‘04, F. Dolan ‘06, N. Boulanger, E. Skvortsov ‘11,
X. Bekaert's talk
E.Joung, K.M. work in progress

Thank you for your attention!

Using Cayley transform

$$\mathscr{C}(U) = \frac{2+U}{2-U}\,, \qquad \mathscr{C}^{-1}(S) = 2\,\frac{S-1}{S+1}$$

one can simplify the Gaussian composition rule.
For the Gaussian functions

$$\mathcal{G}(S) = \frac{1}{\sqrt{\det\left(\frac{1+S}{2}\right)}}\,\exp\left[y_A\left(\frac{S-1}{S+1}\right)^{AB} y_B\right]$$

the star product is manifestly associative

$$\mathcal{G}(S_1) \star \mathcal{G}(S_2) = \mathcal{G}(S_1\,S_2)$$

For the minimal coadjoint orbit the following identities hold

$$U_k^2 = 0 \quad \text{and} \quad U_1\, U_2\, U_1 = \langle U_1\, U_2 \rangle\, U_1$$

Using the first of them, we compute

$$G^{(n)}(U) = \det{}_{2N}\left[\, \frac{1}{2} \prod_{k=1}^{n} (1 + U_k) + \frac{1}{2}\, \right]$$

using the second one we get

$$G^{(2)}(U_1, U_2) = 1 + \tfrac{1}{4}\, \langle\, U_1\, U_2\, \rangle \quad \text{and} \quad G^{(3)}(U) = 1 + \tfrac{1}{4}\, \Lambda(U)$$

with $\Lambda(U) := \langle\, U_1\, U_2\, \rangle + \langle\, U_2\, U_3\, \rangle + \langle\, U_3\, U_1\, \rangle + \langle\, U_1\, U_2\, U_3\, \rangle$

With some effort, the $2N \times 2N$ determinant

$$G^{(n)}(U) = \det{}_{2N}\left[\frac{1}{2}\prod_{k=1}^{n}(1+U_k) + \frac{1}{2}\right]$$

can be shown to be equal to $n \times n$ determinant

$$G^{(n)}(U) = \det{}_n\left[1 + \frac{1}{2}U_{ij}\right]$$

with

$$U_{ij} = \Omega_{AB}\,(u_i)^A\,(u_j)^B$$

One can show that the $N \times N$ determinant

$$G^{(n)}(\rho, V) = \det{}_N \left[\frac{1+\rho}{2} \prod_{k=1}^{n} (1 + V_k) + \frac{1-\rho}{2} \right]$$

is equal to $n \times n$ one

$$G^{(n)}(\rho, V) = \det{}_n \left[1 + \frac{1-\rho}{2} \, \mathsf{Up}(V_{ij}) + \frac{1+\rho}{2} \, \mathsf{Lo}(V_{ij}) \right]$$

with

$$\big[\mathsf{Up}(A)\big]_{ij} = \delta_{i<j} \, A_{ij} \, , \qquad \big[\mathsf{Lo}(A)\big]_{ij} = \delta_{i>j} \, A_{ij}$$

$$V_{ij} = (v_i)_+ \cdot (v_j)_-$$

HS algebras of isomorphic classical algebras.

The simplest example is the isomorphism

$$\mathfrak{sl}_2 = \mathfrak{sp}_2$$

From the identity

$${}_3F_2\left(\tfrac{N}{2}(1+\lambda)\,,\ \tfrac{N}{2}(1-\lambda)\,,\ 1\,;\ \tfrac{N}{2}\,,\ \tfrac{N+1}{2}\,;\ -z\right) = \frac{1}{\sqrt{1+z}} \qquad \left[N=2, \lambda=\tfrac{1}{2}\right]$$

we see the isomorphism

$$hs_{\frac{1}{2}}(\mathfrak{sl}_2) \simeq hs(\mathfrak{sp}_2)$$

From

$${}_2F_1\left(2\,,\ \tfrac{N-4}{2}\,;\ \tfrac{N-1}{2}\,;\ -z\right) = \frac{1}{\sqrt{1+z}} \qquad [N=5]$$

we conclude

$$hs(\mathfrak{so}_5) \simeq hs(\mathfrak{sp}_4)$$

From yet another identity we find the connection between $hs(\mathfrak{so}_6)$ and $hs_\lambda(\mathfrak{sl}_4)$

$$
{}_2F_1\left(2\,,\ \tfrac{N-4}{2}\,;\ \tfrac{N-1}{2}\,;\ -z\right) = {}_3F_2\left(\tfrac{N'}{2}\,(1+\lambda)\,,\ \tfrac{N'}{2}\,(1-\lambda)\,,\ 1\,;\ \tfrac{N'}{2}\,,\ \tfrac{N'+1}{2}\,;\ -z\right)
$$
$$
\left[N=6,\ N'=4,\ \lambda=0\right].
$$

this implies

$$
hs(\mathfrak{so}_6) \simeq hs_0(\mathfrak{sl}_4)
$$

For the special values of λ , satisfying

$$N(1 \pm \lambda) = -2\,M \qquad M \in \mathbb{N}$$

$hs_\lambda(\mathfrak{sl}_N)$ develops an infinite dimensional ideal, while the factor algebra becomes finite dimensional HS algebra with generators of spin $2, \ldots, M$.

These algebras are special linear algebras

$$\mathfrak{sl}_{(M)_{N-1}}$$

Where

$$(M)_{N-1} := M(M+1)\cdots(M+N-2)$$

Spin 1 generator can be added: it forms the $\mathfrak{u}_1$ center of the algebra.

R. Manvelyan, R. Mkrtchyan, K.M., S. Theisen ‘13

E. Joung, K.M. ‘14

One can use reduced set of unconstrained oscillators to describe $hs_\lambda(\mathfrak{sl}_N)$. The cost is the loss of $\mathfrak{sl}_N$ covariance:

$$L^N{}_j = y_{+j}\,, \qquad L^i{}_j = y_-{}^i\, y_{+j} - \frac{\lambda}{N}\,\delta^i_j\,, \qquad L^i{}_N = -\,y_-{}^i\,(y_-{}^j\, y_{+j} - \lambda)$$

$$i, j = 1, \ldots, N-1$$

Higher spin generators are all possible star-polynomials of $\mathfrak{sl}_N$ generators in this representation. For $\mathfrak{so}_N$ the corresponding unconstrained oscillator realization is much more complicated. The corresponding generators are expressed by non-polynomial expressions through oscillators.