ϕ^3 theory on the lattice

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Work in progress w. *Francis Bursa*

Outline

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Original motivation

Putting string field theory on the lattice can provide *in principle*
a complete non-perturbative *definition* of string theory,
as well as a *practical framework* for addressing some hard questions.

It seems sensible to start with the simplest string field theory at hand,
namely *Witten's bosonic open SFT*.

Problems

- Too many dimensions for a lattice approach to be practical.
- The theory is cubic and is thus unbounded from below.
- The theory has infinitely many modes.

Resolving the problems

Suggested resolution

- Reduce dimensionality by either of the following:
  - Working with a linear dilaton background.
  - Compactifying space-time on a very small torus.
- Use analytical continuation from imaginary coupling values (Witten 10) in order to define the theory: $\Psi \star \Psi \star \Psi \to i\Psi \star \Psi \star \Psi$
- Use level truncation obtaining at the lowest level a non-local version of ϕ^3 *theory*.

Analytical continuation probably ruins the reality of observables.
Expectation: Reality of the observables is restored in the continuum limit.

Lowest level with $d = 0$ – analytical results

The action is $S = -i\lambda\big(-\frac{1}{2}\phi^2 + \frac{1}{3}\phi^3\big)$ with $\lambda = \frac{i}{g^2}$.

Evaluate analytically for real λ:

$$Z = \int_{-\infty}^{\infty} dT e^{-S} = \frac{2\pi e^{-\frac{i\lambda}{12}} \mathrm{Ai}\big(-\frac{\lambda^{2/3}}{4}\big)}{\sqrt[3]{\lambda}}$$

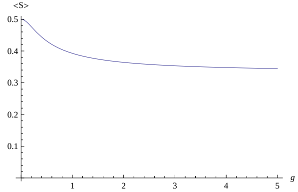
Use the action as an example for an observable $\langle S\rangle$ (substituting back $\lambda = \frac{i}{g^2}$):

$$\langle S\rangle = -\lambda\partial_\lambda \log Z = \frac{1}{3} - \frac{1}{12g^2} - \frac{e^{-\frac{2\pi i}{3}} \mathrm{Ai}'\big(\frac{e^{-\frac{2\pi i}{3}}}{4g^{\frac{4}{3}}}\big)}{6g^{\frac{4}{3}} \mathrm{Ai}\big(\frac{e^{-\frac{2\pi i}{3}}}{4g^{\frac{4}{3}}}\big)}$$

There are three possible cubic roots of g^4.
Hence, there are three ways to perform the analytical continuation.

A real $\langle S \rangle$

There are three possible definitions of $\langle S \rangle$ for real g^2.
Two of the possibilities diverge as $g \to 0$.
They are also complex, and are complex conjugate to each other.
The third $\langle S \rangle$ is real and has good limits as $g \to 0$ and as $g \to \infty$.



Some questions

- How could $\langle S \rangle$ be real? We are far away from the continuum limit.
- Which choice of $\langle S \rangle$ is the correct one?
- Is there indeed a unique "correct" choice?

All that must be understood in order to advance the lattice SFT approach.

Proper motivation

Should one infer from these results that a definition of ϕ^3 theory exists in which it is a regular field theory?
(renormalizable, non-perturbatively well defined, real expectation values, sensible limits, unitary...)

Questions regarding field theories

- *What is the space of renormalizable theories in d dimensions*?
- Does ϕ^3 theory constitute a renormalizable non-perturbative theory?
- If so, for which values of d? (only $d = 0$ maybe?)
- Can one obtain using analytical continuation of a regular theory another regular theory?
- If so, how are different such theories related?
- *What should be considered a definition of a field theory?*

Outline

Schwinger-Dyson equations

The expressions defining field theories are formal.
Several ways to attempt a proper definition.
From the action we can formally obtain the Schwinger-Dyson equations.

A possible approach for defining a field theory:
The theory is defined by solving the Schwinger-Dyson equations.

A zero-dimensional example (Guralnik, Guralnik 07)

The path integral reduces to an ordinary integral.
All observables can be defined using the generating functional:
$Z(J) = \int d\phi e^{-S(\phi)+J\phi}$.
From $\int d\phi \partial_\phi e^{-S(\phi)+J\phi} = 0$, we can deduce $\big(S'(\partial_J) - J\big)Z(J) = 0$.
If S depends on couplings g_i, then similarly $\big(\partial_{g_i} + \frac{\partial S}{\partial g_i}(\partial_J)\big)Z(J, g_j) = 0$.
The first equation gives the J dependence for fixed couplings.
The second equation evolves Z as the couplings vary.
Similar equations with functional derivatives in the $d > 0$ case.

Solutions of the Schwinger-Dyson equations in $d = 0$

Characterizing the space of solutions

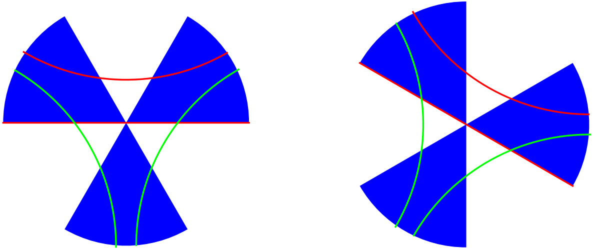
For polynomial $S(\phi)$, *deg(S) = n*, the equation $\big(S'(\partial_J) - J\big)Z(J) = 0$ is a linear ODE of degree $n-1$.
The solutions form an *$n-1$ dimensional linear space*.

How can we find a basis of solutions?

- An integral in the complex plane for which the integrand vanishes at the boundaries of the integration region is a solution.
- The integrand vanishes in asymptotic regions in the complex plane in which the real part of the action becomes indefinitely large.
- There are exactly n such regions.
- Small contour deformations do not change the solution.
- There are *$n-1$ independent families of integration contours* that start and end at the various n regions.
- These families form a basis of solutions.

Zero dimensional ϕ^3 theory

The $n = 3$ *asymptotic regions* (blue) in the complex ϕ plane before and after the analytical continuation (Left – for $\lambda > 0$, Right – for $g^2 > 0$).



Rotation by θ in the λ plane leads to a rotation by $-\frac{\theta}{3}$ of the convergence-regions in the ϕ-plane.

Starting with what might seems as the natural choice in the positive λ case seem to lead to the straight red curve on the right. Both straight lines can be deformed to any other representative in the same family.

This can be described by a *relative cohomology in the space of curves*.

Two other cohomology elements. Only $n - 1 = 2$ *independent ones*.

The monodromy

A 2π rotation in the complex ϕ plane gives three rotations of the g plane.
The three basic contours are interchanged in each such rotation.
In other words, *$\mathbb{Z}_3$-monodromy shuffles the cohomology elements*.

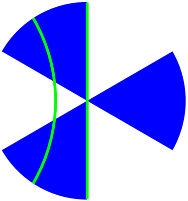
It is possible to choose a continuous family of solutions with respect to the couplings, but only locally.
The equation $\left(\partial_g + \frac{\partial S}{\partial g}(\partial_J)\right) Z(J, g) = 0$ is defined over a triple cover of the complex g plane punctured at the origin.
The points $g = 0, \infty$ are essential singularities.

What does a cohomology element define (up to normalization)?

- A field theory?
- A symmetry breaking of a vacuum of a single theory?
- A "D-brane" in the same theory?
  (changing contours is like D-brane condensation)

An integration contour with real observables

Choose the cohomology class represented by the green curves.



It turns out that *all the vevs* $\langle \phi^k \rangle$ *are now real*. *Why?*

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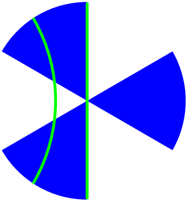

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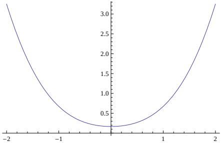
Represent the cohomology using the *Lefschetz Thimble* (Witten 10)

parametrized as: $\phi = \frac{-(1+\sqrt{s^2+1})+i\sqrt{3}s}{2}$.

The action becomes: $S = \frac{1}{g^2}\left(\frac{\phi^2}{2} + \frac{\phi^3}{3}\right) = \frac{1}{12g^2}\left(1 + (4s^2+1)\sqrt{s^2+1}\right)$.



The measure factor

The action includes the familiar $\frac{1}{6g^2}$ constant, which can be ignored. Even then, the action is *strictly positive and even*:
$S = \frac{1}{12g^2}\Big(-1 + (4s^2+1)\sqrt{s^2+1}\,\Big)$.

We must not forget the measure factor: $\frac{d\phi}{ds} = \frac{i\sqrt{3}}{2}\Big(1 + i\frac{s}{\sqrt{3(s^2+1)}}\Big)$.
The prefactor is a constant and can be ignored. Now evaluate vevs:

$$\left\langle \phi^k \right\rangle = \frac{1}{Z}\int_{-\infty}^{\infty} ds\Big(1 + i\frac{s}{\sqrt{3(s^2+1)}}\Big)\Big(\frac{-(1+\sqrt{s^2+1}) + i\sqrt{3}s}{2}\Big)^k e^{-S}.$$

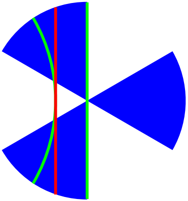
Here Z is the same integral with $k=0$.
All factors of i multiply odd powers of s. They either appear in pairs with real contributions, or drop out upon integration.

There might still be a somewhat strange result of $\left\langle \phi^{2k} \right\rangle$ being negative. The origin of it is that the total measure is not positive definite:
The residual sign problem.

$\mathcal{PT}$-symmetric field theories (Bender, Boettcher 98; Bender 07)

A renormalizable theory with real vevs. But maybe only for $d = 0$?

Consider now another integration contour from the same family.



This contour passes near the perturbative vacuum at $\phi = -1$, around which the action (disregarding the constant) is $S = \frac{1}{g^2}(-\frac{\phi^2}{2} + \frac{\phi^3}{3})$.
Sending $\phi \to -i\phi$ gives only constant for the measure factor.
The action becomes $S = \frac{1}{g^2}(\frac{\phi^2}{2} + i\frac{\phi^3}{3})$.
This theory and its higher dimensional generalizations were studied by Bender and others, under the category of *$\mathcal{PT}$-symmetric field theories*.
Such theories have real spectra and should be *acceptable quantum theories* (proved for $d = 1$, conjectured and studied for $d > 1$).

Outline

1. Motivation
2. Defining field theories
3. Lattice
4. Outlook

Numerical methods for $d > 0$

Lefschetz Thimbles (Cristoforetti et. al. 12)

- An analytical function that goes to zero fast enough in the complex plane in n directions has $n-1$ cycles for defining an integral.
- An alternative basis for the cycles can be obtained by considering downward gradient flow of $Re(S)$ from stationary points.
- The gradient flow has $Im(S)$ as a conserved quantity. But this does not distinguish downward and upward flows.
- Efficient integration cycles. Useful for complex actions.

Langevin method (Parisi 83; Aarts 13)

- A method especially suitable for complex actions.
- Add an auxiliary "time" θ and consider stochastic process along it: For $Z = \int D\phi e^{-S}$ the process is $\partial_\theta \phi = -\frac{\delta S}{\delta \phi} + \eta$. η – Gaussian noise.
- Evaluate everything at equilibrium (if it is obtained) as $\theta \to \infty$.

Both methods probe similar distributions in the complex plane. (Aarts 13)

Lefschetz Thimbles

The good

- A natural basis for the relative cohomology.
- Follows steepest descent – Convergence is best along these lines.
- A constant phase along the curve – Only a residual sign problem.
- Coefficients can be found by intersection number with dual basis.

The bad

- Finding the exact curve adds some complexity to the numerics.
- Integrating along the exact curve adds some more complexity.

In many cases "the bad" is not too bad and it is well worth to obtain and evaluate along the exact Lefschetz Thimbles.

Integration cycles for $d > 0$

For a lattice of n points (regardless of d), integration should be performed over an *n-dimensional real sub-space of $\mathbb{C}^n$*.
Over this space $Re(S)$ must tend asymptotically to infinity.

Since mass and kinetic terms are quadratic, asymptotic regions are defined by tensoring the asymptotic regions of $d = 0$.
One can define 3^n such regions, however, only 2^n are independent.

Choosing a cycle

- Tensoring the cycle we used for $d = 0$ should lead to real observables.
- Is it the only cycle among the 2^n with this property?
- Is it possible to define a *continuum limit* using a set of different choices?

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Outlook

Possible outcome of simulations

- Regular continuum limit for $d \leq 6$. No such limit for $d > 6$.
- Something else.

Other questions

- New fundamental theories?
- Other real continuum limits?
- Non-trivial fixed-point for $d < 6$? (Bender, Branchinab, Messina 12)
- Generalization to ϕ^k for $k > 3$, to other theories?
- Back to lattice string field theory:
  - Analytical continuation for higher levels?
  - Ambiguities?
  - Other problems.

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THANK YOU