Rényi Entropy in AdS_3/CFT_2 (with W symmetry)

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AdS_3/CFT_2 correspondence

- 3D AdS_3 Einstein gravity is special: No dynamical d.o.f
- In 1986, Brown and Heanneaux: there exists boundary d.o.f.
- More precisely they found that under appropriate boundary conditions the asymptotic symmetry group (ASG) of AdS_3 Einstein gravity is a 2D CFT with central charge

$$c_L = c_R = \frac{3l}{2G}$$

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- In modern understanding: quantum gravity in AdS_3 is dual to a 2D CFT at AdS boundary
- A new window to study AdS/CFT without resorting to string theory

AdS_3/CFT_2: a perfect platform

- AdS_3 gravity is solvable: all classical solutions are quotients of AdS_3 such that a path-integral is possible in principle E. Witten (1988) ...
- 2D conformal symmetry is infinitely dimensional so that 2D CFT has been very well studied Belavin et.al. (1984) ..., even though the explicit construction of dual 2D CFT is unknown

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- However, it is not clear
  1. how to define the quantum AdS_3 gravity?
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- Let us consider the semiclassical gravity, which correspond to the large central charge limit

$$c = \frac{3l}{2G}$$

HS/CFT correspondence

- The higher spin theory in AdS_3 is relatively easy
- It could be defined in terms of Chern-Simons theory with gauge group SL(n,R), describing the interacting fields with spin from 2 to n
- With generalized Brown-Henneaux b.c., spin n gravity in AdS_3 has W_n asym. symmetry algebra, with the same central charge $c_L = c_R = 3l/2G$ M. Henneaux and S.J. Rey 1008.4579, A. Campoleoni et.al. 1008.4744
- Dictionary
  1. massless graviton $\leftrightarrow$ stress tensor
  2. massless spin 3 field $\leftrightarrow$ W_3 field with conformal weight $(3,0)$ (holomorphic sector)
  3. massless spin 4 field $\leftrightarrow$ W_4 field with conformal weight $(4,0)$ (holomorphic sector)

In this talk, I will introduce the recent developments of AdS_3/CFT_2 in the light of entanglement entropy

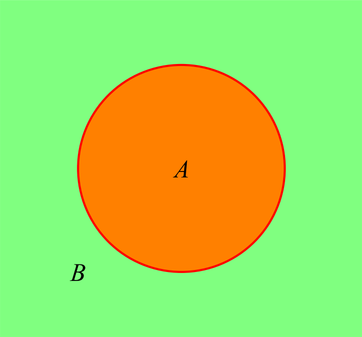
Based on the following works

- with J.-j. Zhang, "On short interval expansion of Rényi entropy," arXiv:1309.5453 [hep-th].
- With J. Long and J.-j. Zhang, " Holographic Rényi entropy for CFT with W symmetry" , arXiv: 1312.5510 [hep-th].
- with J.-q. Wu, "Single Interval Rényi Entropy At Low Temperature," arXiv:1405.6254 [hep-th].
- with J.-q. Wu, "Universal relation between thermal entropy and entanglement entropy in CFT," arXiv:1412.0761 [hep-th].
- with J.-q. Wu, "Large Interval Limit of Rényi Entropy At High Temperature," arXiv:1412.0763 [hep-th].
- with J.-q. Wu, "Holographic Calculation for Large Interval Rényi Entropy at High Temperature", to appear

- Related works by many others...

Entanglement entropy

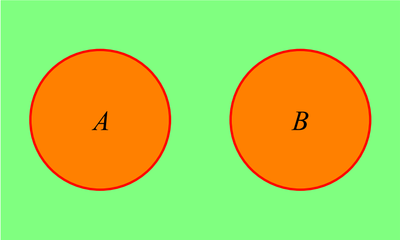
Entanglement entropy is an important notion in quantum world.



For A and its complement B

- $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{tot} = |\Psi\rangle\langle\Psi|$
- Reduced density matrix: $\rho_A = \mathrm{tr}_B \rho_{tot}$
- Entanglement entropy $S_A = -\mathrm{tr}_A \rho_A \ln \rho_A$
- Rényi entropy $S_A^{(n)} = -\frac{\ln \mathrm{tr}_A \rho_A^n}{n-1}$
- $S_A = \lim_{n \to 1} S_A^{(n)}$

Rényi mutual information



- Choose two subsystems A and B which are not necessarily each other's complement
- Define the Rényi mutual information of A and B

$$I^{(n)}_{A,B} = S^{(n)}_A + S^{(n)}_B - S^{(n)}_{A+B}.$$

- For $n = 1$, it is called mutual information, which measures the entanglement between A and B
- From subadditivity, we know $I(A, B) \geq 0$

EE in QFT

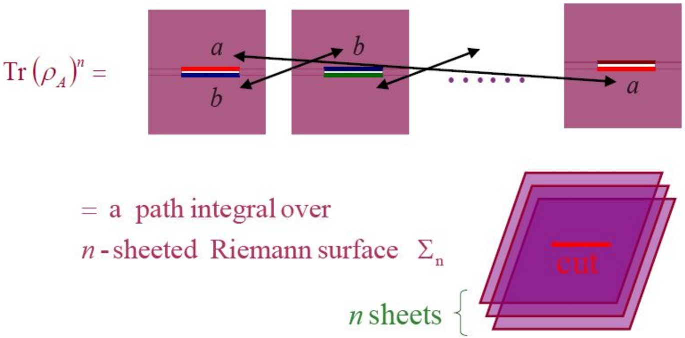
- Consider a QFT on a $(d+1)$-dim. manifold $R \times M$, where R is time direction
- Subsystem: a d-dim. submanifold $A \in M$ at a fixed time
- In this case, the EE S_A is called the geometric entropy as it depends on the geometry of A L.Bombelli et.al. 1986, M. Srednicki 9304048

$$S_A = \gamma \frac{\text{Area}(\partial A)}{\epsilon^{d-1}} + \text{subleading terms}$$

  where ∂A is the boundary of A, ϵ is the UV cutoff and γ is a constant depending on the system
- For 2D QFT, logarithmic law rather than area law
- It is notoriously difficult to compute, even for free field theory

Replica trick

- The standard way is to use the replica trick J. Callan et.al. 9401072
- Here, we only focus on the 2D CFT, which provides more analytic results Figures from T. Takayanagi's lecture



Replica trick II

- Picture 1: partition function on a n-sheeted Riemann surface
- Picture 2: multi-point function in a product orbifold $(\mathrm{CFT})^n/Z_n$
- Branch points: twist operators with dimension

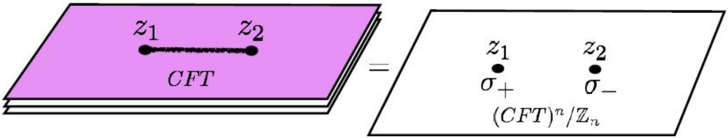
$$h=\bar{h}=\frac{c}{24}\left(n-\frac{1}{n}\right).$$

- One interval case

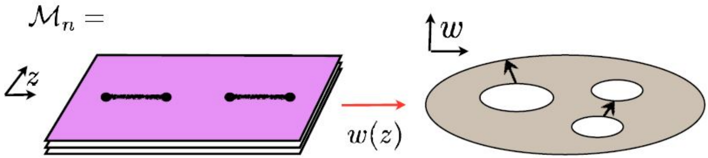
$$\mathrm{Tr}\rho_A^n=\langle\sigma(\ell,\ell)\tilde{\sigma}(0,0)\rangle_C=c_n\ell^{-\frac{c}{6}\left(n-\frac{1}{n}\right)},$$

from which the Rényi entropy for one interval could be read P. Calabrese and J.L. Cardy 0405152

$$S_n=\frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{\ell}{\epsilon},$$



Multi-intervals



- In the case of N intervals, there are more branch cuts so that the Riemann surface is of genus $(n-1)(N-1)$, where n is the number of replica
- It is very difficult to compute (partition function on higher genus RS)
- If we have multiple intervals $A = [z_1, z_2] \cup \cdots \cup [z_{2N-1}, z_{2N}]$,

$$\mathrm{Tr}\rho_A^n = \langle \sigma(z_{2N}, \bar{z}_{2N})\tilde{\sigma}(z_{2N-1}, \bar{z}_{2N-1}) \cdots \sigma(z_2, \bar{z}_2)\tilde{\sigma}(z_1, \bar{z}_1)\rangle_C .$$

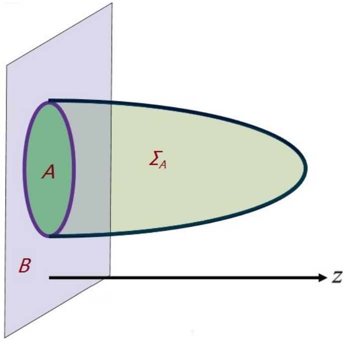
- Very few exact results: free boson on a torus, free boson in two-interval case, Ising model, ...
- In higher dim., very limited knowledge

Holographic entanglement entropy Ryu and Takayanagi 2006

- AdS/CFT: A field theory could be holographically described by a higher-dim. gravity
- Ryu and Takayanagi(2006): Find a codimension two minimal surface Σ_A in the bulk that is homogeneous to A
- The holographic entanglement entropy (for Einstein gravity)

$$S_A = \frac{\mathrm{Area}(\Sigma_A)}{4G_N}$$

- It is called the RT formula
- The area law is reminiscent of black hole entropy
- For 2D CFT, the HEE is just the length of the bulk geodesic ending at the branch points



Remarks on HEE

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  1. Covariant RT for dynamical spacetime Hubeny et.al. 2007
  2. Higher curvature case Huang et.al. 2011, de Boer et.al. 2011
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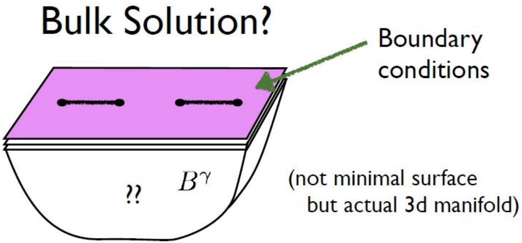
- In higher dimensions ($\geq 4D$ gravity), it is not clear if HEE $=$ EE at strong coupling
- In $2+1$ dimension, the RT formula has been proven to give the EE in CFT (for multi-intervals) by T. Hartman (1303.6955) and T. Faulkner (1303.7221) independently

HRE in AdS_3: A sketch T. Faulkner 1303.7221



- Find the bulk gravity solutions B^γ such that $\partial B^\gamma = \Sigma_n$
- Key point: all solutions of AdS_3 gravity could be obtained by $B^\gamma = H_3/\Gamma_\gamma$, where Γ_γ is the subgroup of isometry $SL(2, C)$
- For the handlebody solution, Γ_γ is the Schottky group
- Γ_γ acts on C such that $C/\Gamma_\gamma = \Sigma_n$

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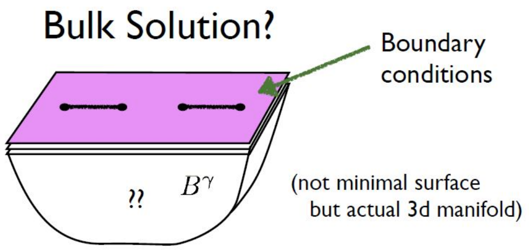

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- For a fixed Riemann surface, find its Schottky uniformization
- Extend the uniformization to the bulk to find the gravitational solution
- The classical regulated bulk action reproduces RT formula

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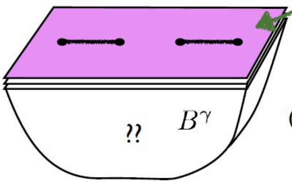

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- The classical regulated bulk action reproduces RT formula
- For the same Σ_n, there could be more than one B^γ
- In the classical gravity limit, keep only the solution of least action

Schottky uniformization

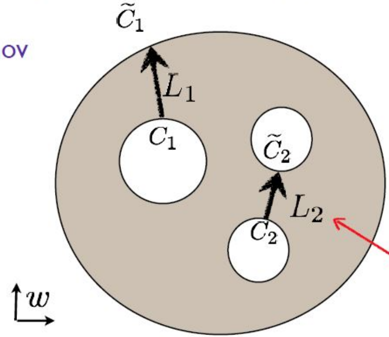
draw 2g circles and identify pair wise:

Krasnov

$SL(2,\mathbb{C})$

$\tilde{C}_m = L_m(C_m)$

Γ_γ : freely generated by $L_1, L_2, \ldots L_g$



Fundamental domain of quotient

Figure: c.f. Faulkner

Schottky uniformization

- Every compact Riemann surface could be obtained by the Schottky uniformization "Retrosection theorem" by Koebe (1914)
- The Schottky uniformization is determined by a differential equation

$$\psi''(z) + \frac{1}{2} T_{zz}\psi(z) = 0 \tag{2.1}$$

- Two independent solutions: ψ_1 and ψ_2
- Their ratio $w = \frac{\psi_1}{\psi_2}$ gives the quotient map
- More importantly, T_{zz} is the stress tensor of Liouville CFT. Its explicit form depends on $(3g-3)$ complex accessory parameters with respect to the holomorphic quadratic differentials on the Riemann surface.
- Imposing the monodromy conditions on the cycles allows us to solve this ode

Bulk action

- The essential point is that the on-shell regulated bulk action of gravitational configurations in pure AdS_3 gravity is a Liouville type action on the boundary K. Krasnov (2000)
- More importantly, the dependence of this so-called Zograf-Takhtadzhyan action on the accessory parameters is determined by the differential equation Zograf and Takhtadzhyan (1988)

$$\frac{\partial S_n}{\partial z_i} = -\frac{cn}{6(n-1)}\gamma_i. \tag{2.2}$$

- γ_i are accessory parameters, being fixed by the monodromy problem of an ordinary differential equation
- For a general Riemann surface of high genus, it is a difficult problem to determine this regulated action, even perturbatively
- Nevertheless, for the Riemann surface in computing the Rényi entropy, the problem is simplified due to the replica symmetry
  1. Two-interval case: one cross ratio
  2. Single interval in a torus (finite temperature, finite size)

Two-interval case

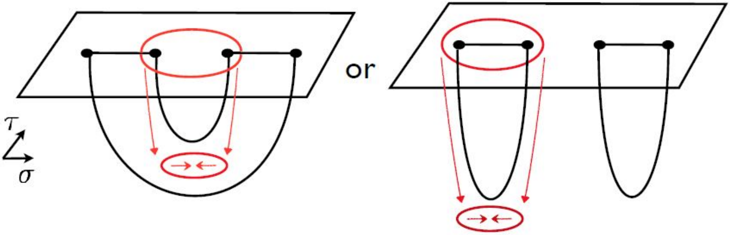
- In this case

$$T_{zz} = \sum_i \frac{\Delta}{(z-z_i)^2} + \frac{\gamma_i}{z-z_i}, \quad (2.3)$$

where

$$\Delta = \frac{1}{2}(1-\frac{1}{n^2}), \quad (2.4)$$

- There is only one accessory parameter
- The accessory parameters are determined by requiring trivial monodromy at infinity and on either the A-cycle or B-cycle



Single interval on a torus

$$T_{zz} = \sum_i (\Delta \wp(z - z_i) + \gamma_i \zeta(z - z_i)) + \delta, \qquad (2.5)$$

where $\wp$ is the doubly periodic Weierstrass function, and

$$\zeta(z) = \sum_m \pi T \coth[\pi T(z + mL)] + \sum_{m \neq 0} \frac{\pi^2 T^2 z}{\sinh^2 \pi m T L} - \frac{\pi^2 T^2 z}{3}. \qquad (2.6)$$

- Torus: $z \sim z + mL + in\beta \Rightarrow$ thermal circle and spatial circle
- We can set trivial monodromy along one circle and the cycle enclosing two branch points, so that the identification of the other circle gives the generator of Schottky group

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- Torus: $z\sim z+mL+in\beta \Rightarrow$ thermal circle and spatial circle
- We can set trivial monodromy along one circle and the cycle enclosing two branch points, so that the identification of the other circle gives the generator of Schottky group
- At high temperature above the Hawking-Page transition, the bulk spacetime is actually a black hole, so the time direction is of trivial monodromy.
- At low temperature, the dual bulk is the thermal AdS spacetime, so the spatial direction is of trivial monodromy.

Finite size effect with J.q. Wu 1405.6254

- For the torus at high temperature, we have to consider the effect of its finite size
- The regulated action depends not only on the accessory parameter, but also on the size of the torus

$$\frac{\partial S_n}{\partial L} = \frac{c}{12\pi}\frac{n}{n-1}\beta(\tilde{\delta} - \tilde{\delta}_{n=1}). \tag{2.7}$$

where $\tilde{\delta}$ includes all the constant contribution in $T(z)$.

- This could be derived from the variation of the Liouville action

Remarks

- The above treatment is universal, even for other 3D gravity theory with a AdS_3 vacuum CB et.al. 1401.0261

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- Simply speaking, the holographic Renyi entropy(HRE) is given by the classical action of the corresponding gravitational configurations
- Caution: this picture is only true in the large central charge limit

Large central charge limit

- In the AdS_3/CFT_2 correspondence for pure gravity J. Brown and M. Henneaux (1986)

$$c = \frac{3l}{2G}$$

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- The subleading corrections in 2D CFT being independent of c should correspond to the 1-loop partition function around the configurations
- There are good reasons to consider the quantum correction: mutual information, thermal correction, ...

1-loop correction to HRE

- The gravitational configurations for HRE are generated by the Schottky group
- Consider the fluctuations around the configuration and compute their functional determinants

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- Consider the fluctuations around the configuration and compute their functional determinants
- 1-loop partition function Giombi et.al. 0804.1773, Yin 0710.2129

$$Z^{1-loop} = \prod_{\gamma\in\mathcal{P}} \prod_{s} \prod_{m=s}^{\infty} \frac{1}{|1-q_\gamma^m|}. \tag{2.8}$$

  Here the product over s is with respect to the spins of massless fields and $\mathcal{P}$ is a set of representatives of primitive conjugacy classes of the Schottky group Γ. q_γ is defined by writing the two eigenvalues of $\gamma\in\Gamma$ as $q_\gamma^{\pm 1/2}$ with $|q_\gamma|<1$.
- The contributions of the fields with different spins could be separated

Strategy

- Find the Schottky group Γ corresponding to $\mathcal{M}_n$
- Generate $\mathcal{P} = \{$non-repeated words up to conjugation$\}$, e.g.

$$\mathcal{P} = \{L_1, L_2, L_1^{-1}, L_2^{-1}, L_1 L_2 \sim L_2 L_1, ...\}$$

- Compute eigenvalues of these words and sum over their contributions
- For two intervals with small cross ratio x, only finitely many words contribute to each order in x
  1. Metric fluctuations, up to x^8 Barrella et.al. 1306.4682
  2. Spin 3 and/or 4 fluctuations, up to x^8 BC et.al. 1312.5510
- For single interval on a circle at finite temperature, similar strategy works in the low temperature and high temperature limits Barrella et.al. 1306.4682, BC et.al. 1405.6254

CFT computation

General prescription

M. Headrick 1006.0047, P. Calabrese et.al. 1011.5482, BC and J-j Zhang 1309.5453

The replica trick requires us to study a orbifold CFT: $(\mathrm{CFT})_n/Z_n$. When the intervals are short, we have the OPE of the twist operators

$$\sigma(z,\bar{z})\tilde{\sigma}(0,0)=c_n\sum_K d_K\sum_{m,r\geq 0}\frac{a_K^m}{m!}\frac{\bar{a}_K^r}{r!}\frac{1}{z^{2h-h_K-m}\bar{z}^{2\bar{h}-\bar{h}_K-r}}\partial^m\bar{\partial}^r\Phi_K(0,0),$$

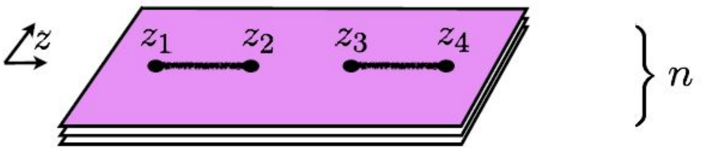
with the summation K being over all the independent quasiprimary operators of CFT_n.

Short interval expansion

We are interested in the two-interval case, then

$$\begin{aligned}\mathrm{Tr}\rho_A^n &= \langle \sigma(1+y,1+y)\tilde{\sigma}(1,1)\sigma(y,y)\tilde{\sigma}(0,0)\rangle_C \\ &= c_n^2 x^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\left(\sum_K \alpha_K d_K^2 x^{h_K} F(h_K,h_K;2h_K;x)\right)^2\end{aligned}$$

where x is the cross ratio and $F(h_K,h_K;2h_K;x)$ is the hypergeometric function. α_K is the normalization factor of Φ_K, and d_K is the OPE coefficients.



Remarks

- For a concrete CFT model, the summation should be over all the conformal blocks
- For pure AdS_3 gravity, it is enough to consider the vacuum Verma module T. Hartman 1303.6955
- For HS AdS_3 gravity, it is necessary to include the quasi-primary operators from W fields

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- Step 1: find the quasi-primary operators (in holomorphic sector) level by level
- Step 2: work out their coefficients in OPE
- In the small x limit, to each order only finite number of the quasi-primary operators contribute
- The holo. and anti-holo. sectors are decoupled but share the similar structure, so we can focus on the holo. one
- The key ingredients in the OPE of twist operators is the coefficients α_K and d_K.

Rényi mutual information

- We are interested in the mutual Renyi entropy

$$I^{(n)}_{A_1,A_2} = S^{(n)}_{A_1} + S^{(n)}_{A_2} - S^{(n)}_{A_1+A_2}$$

In the following, we write I_n for $I^{(n)}_{A_1,A_2}$.

- The Rényi mutual information is

$$\begin{aligned} I_n &= \frac{c}{3}(1+\frac{1}{n})\log\frac{y}{\epsilon} + \frac{1}{n-1}\log \mathrm{Tr}\rho_A^n, \\ &= I_n^{tree} + I_n^{1-loop} + I_n^{2-loop} + \cdots . \end{aligned}$$

- Here we have classified the contributions according to the order of the inverse of central charge $\frac{1}{c}$, which in the large c limit corresponds to tree, 1-loop, and 2-loop contributions in the gravity side
  1. $I_n^{tree} \sim \mathcal{O}(c)$ terms
  2. $I_n^{1-loop} \sim \mathcal{O}(c^0)$ terms
  3. $I_n^{2-loop} \sim \mathcal{O}(1/c)$ terms
- After some highly nontrivial summations...

Useful formulae I

Define

$$f_m(n) \equiv \sum_{j=1}^{n-1} \frac{1}{\left(\sin \frac{\pi j}{n}\right)^{2m}},$$

we need

$$f_1(n) = \frac{n^2-1}{3}, \quad f_2(n) = \frac{(n^2-1)\left(n^2+11\right)}{45},$$

$$f_3(n) = \frac{(n^2-1)\left(2n^4+23n^2+191\right)}{945},$$

$$f_4(n) = \frac{(n^2-1)\left(n^2+11\right)\left(3n^4+10n^2+227\right)}{14175},$$

$$f_5(n) = \frac{(n^2-1)\left(2n^8+35n^6+321n^4+2125n^2+14797\right)}{93555},$$

$$f_6 = \frac{(n^2-1)\left(1382n^{10}+28682n^8+307961n^6+2295661n^4+13803157n^2+92427157\right)}{638512875}.$$

$$\vdots$$

Useful formulae II

$$h^{j_1j_2j_3}_{mpq} = \frac{1}{s^{2m}_{j_1j_2} s^{2p}_{j_2j_3} s^{2q}_{j_3j_1}} + \text{possible cyc.}$$

with $s_{j_1j_2} \equiv \sin\frac{\pi(j_1-j_2)}{n}$ and $c_{j_1j_2} \equiv \cos\frac{\pi(j_1-j_2)}{n}$.

$$\sum_{0\le j_1<j_2<j_3\le n-1} h^{j_1j_2j_3}_{111} = \frac{n(n^2-1)(n^2-4)\left(n^2+47\right)}{2835},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1} h^{j_1j_2j_3}_{211} = \frac{n(n^2-1)(n^2-4)\left(n^4+40n^2+679\right)}{14175},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1} h^{j_1j_2j_3}_{220} = \frac{2n(n^2-1)(n^2-4)\left(n^2+11\right)\left(n^2+19\right)}{14175},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1} h^{j_1j_2j_3}_{320} = \frac{2n(n^2-1)(n^2-4)\left(6n^6+173n^4+2084n^2+12137\right)}{467775},$$

$$\sum_{0\le j_1<j_2<j_3\le n-1} h^{j_1j_2j_3}_{330} = \frac{n(n^2-1)(n^2-4)\left(739n^8+20075n^6+355677n^4+2953625n^2+14813884\right)}{638512875},$$

$\vdots$

Mutual information: classical part

The tree-level part, or the classical part, being proportional to the central charge c, originates only from the vacuum module

$$
\begin{aligned}
I_n^{tree} = & \frac{c(n-1)(n+1)^2x^2}{144n^3} + \frac{c(n-1)(n+1)^2x^3}{144n^3} \\
& + \frac{c(n-1)(n+1)^2\left(1309n^4-2n^2-11\right)x^4}{207360n^7} \\
& + \frac{c(n-1)(n+1)^2\left(589n^4-2n^2-11\right)x^5}{103680n^7} \\
& + \frac{c(n-1)(n+1)^2(805139n^8-4244n^6-23397n^4-86n^2+188)x^6}{156764160n^{11}} \\
& + (\text{the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}\left(x^9\right)
\end{aligned}
$$

It matches exactly the result in M. Headrick 1006.0047, T. Hartman 1303.6955, T. Faulkner 1303.7221 up to order x^8.

Mutual information: 1-loop correction from graviton

The quantum 1-1oop part from the stress tensor, being proportional to c^0, is

$$I_n^{(2)1-loop} = \frac{(n+1)\left(n^2+11\right)\left(3n^4+10n^2+227\right)x^4}{3628800n^7}$$
$$+\frac{(n+1)\left(109n^8+1495n^6+11307n^4+81905n^2-8416\right)x^5}{59875200n^9}$$
$$+\frac{(n+1)\left(1444050n^{10}+19112974n^8+140565305n^6+1000527837n^4-167731255n^2-14142911\right)x^6}{523069747200n^{11}}$$
$$+\text{(the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}\left(x^9\right).$$

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It matches exactly the result in M. Headrick 1006.0047, T. Barrella 1306.4682 up to order x^8.

- The classical mutual information $(n = 1)$ is vanishing
- Its quantum correction is nonvanishing

Mutual information: 1-loop correction in W_3

The quantum 1-1oop part in CFT with W_3 symmetry, being proportional to c^0, is

$$I_n^{(2,3)1-loop} = \cdots + \frac{(n+1)x^6(3610816n^{10}+47796776n^8+351567243n^6+2502467423n^4-412426559n^2+10856301)}{1307674368000n^{11}} + (\text{the terms proportional to } x^7 \text{ and } x^8) + \mathcal{O}(x^9),$$

- the "$\cdots$" being the x^4, x^5 parts of $I_n^{(2)1-loop}$
- The extra contributions start to appear from order x^6, as the conformal weight of W_3 field is three
- It exactly matches the 1-loop correction to HRE to order x^8

Mutual information: 2-loop correction

Remarkably there is also the quantum 2-loop contribution, being proportional to $1/c$,

$$I_n^{2-loop} = \frac{(n+1)\left(n^2-4\right)(19n^8+875n^6+22317n^4+505625n^2+5691964)x^6}{70053984000n^{11}c} \frac{(n+1)\left(n^2-4\right)\left(276n^{10}+12571n^8+317643n^6+7151253n^4+79361381n^2-9428724\right)x^7}{326918592000n^{13}c} +(\text{the terms proportional to } x^8) + \mathcal{O}\left(x^9\right),$$

This is novel, expected to be confirmed by 2-loop computation in gravity

- When $n=2$, the two-loop correction is vanishing, as $S^{(2)}$ being genus 1 partition function is 1-loop exact A. Maloney and E. Witten 0712.0155
- When $n>2$, there are nonvanishing 2-loop corrections Xi Yin, 0710.2129
- Actually there is nonvanishing quantum 3-loop contribution, being proportional to $1/c^2$, for $S^{(n)}, n>3$.

Short summary: two-interval case

- Q1: Is the holographic computation of quantum correction of Renyi entropy correct?
- We considered the two disjoint intervals with small cross ratio x, which allows us to use well-established CFT techniques with J.J. Zhang 1309.5453
- We showed that for two disjoint intervals with small cross ratio x, the CFT result matches exactly with 1-loop HRE to order x^8.

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- Q2: How about the situation with matter coupling?
- We discussed the case with higher spin fields, the results are remarkably good ...with J. Long and J.J. Zhang 1312.5510
- Q3: How about other 3D gravity theories, CTMG or CNMG?
- They may dual to logarithmic CFT, which requires some cares
- Quite interesting, all in good agreement with F.y. Song and J.J. Zhang 1401.0261

Single interval on a torus

- When the interval is not very large, HRE could be computed perturbatively for both high and low temperatures
- We studied the Rényi entropy of single interval on a circle in a 2D CFT at a low temperature, whose thermal density matrix could be expanded level by level

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr} e^{-\beta H}} = \frac{1}{\mathrm{Tr} e^{-\beta H}} \sum \mid \phi\rangle\langle\phi \mid e^{-\beta E_\phi}$$

- We find exact agreement between CFT and bulk results at low temperature with J.q. Wu 1405.6254
  1. Classical contribution up to order $e^{-\frac{8\pi\beta}{L}}$
  2. Quantum correction up to order $e^{-\frac{6\pi\beta}{L}}$
- Similar agreement at high temperature if the interval is not very large

Large interval: holographic result

- The entanglement entropy of single interval at high temperature is

$$S_{EE} = \frac{c}{3} \log \sinh(\pi T l) \tag{4.1}$$

- From holographic point of view, it is given by the geodesic in the BTZ background ending on the interval

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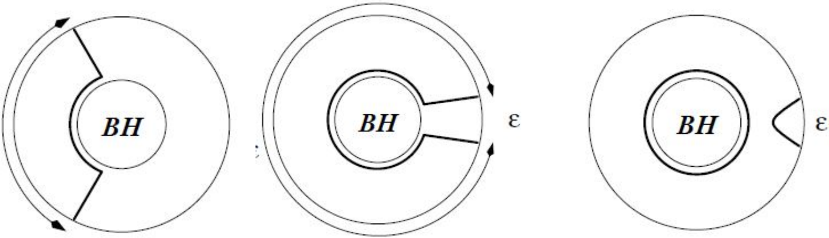
- From holographic point of view, it is given by the geodesic in the BTZ background ending on the interval
- However, it is only true when the interval is not very large



- When the interval is very large, the disconnected curve gives smaller length T. Azeyanagi et.al. 0710.2956

Entanglement and thermal entropy

- Moreover, it suggests the relation

$$S_{EE}(1-\epsilon) = S_{BH} + S_{EE}(\epsilon)$$

- Or change it into a field theory relation between thermal entropy and entanglement entropy

$$S_{thermal} = S_{EE}(1-\epsilon) - S_{EE}(\epsilon)$$

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$$S_{thermal} = S_{EE}(1-\epsilon) - S_{EE}(\epsilon)$$

- Conjecture: this should be true for any CFT, not only for the CFT with a holographic dual
- We have proved this universal relation for any CFT with discrete spectrum with J-q Wu, 1412.0761

HRE: large interval limit

- The study of HEE suggests that in the large interval limit, the gravitational configuration for the Rényi entropy should be different
- This means we should impose a different set of monodromy conditions on the cycles

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- This means we should impose a different set of monodromy conditions on the cycles

- On the CFT side, we proposed a novel expansion based on the twist sector of symmetric orbifold with J-q Wu, 1412.0763
- We tested our proposal in the case of free boson, after correcting some errors in the literature with J-q Wu, 1412.0763,1501.00373
- We find quite good agreement between bulk and CFT, both on classical and 1-loop quantum levels with J-q Wu, to appear

Discussion

- Rényi entropy opens a new window to study the AdS_3/CFT_2 correspondence
- Compelling evidence that the holographic computation of Rényi entropy is correct, beyond the RT formula
- How to prove the HRE beyond classical level?

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- For pure AdS_3 quantum gravity, it is the vacuum conformal module in the dual CFT which dominate the contribution
- How about the other modules in the dual CFT? light spectrum, heavy spectrum ....
- What's the CFT dual of quantum AdS_3 gravity? E. Witten 1988, S. Carlip 050302, A. Maloney and E. Witten 0712.0155,H. Verlinde et.al. 1412.5205,...

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Rényi entropy in HS gravity

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- The high spin field has been taken as the fluctuation, giving the quantum correction to RE
- A more interesting question: EE in the case that the high spin fields presents as hair, say high spin black hole
- In this case, a prescription based on WL has been proposed to compute HS EE holographically de Boer et.al. 2013,2015, Ammon et.al. 2013
- On the field side, one has to consider the deformation of W-field to the lagrangian
- J. Long did a quite remarkable job to compute the HS RE in CFT under the large central charge limit J. Long, 1408.1298
  1. Reproduce the holographic HSEE
  2. 1-loop quantum correction
- How to compute HSRE and its quantum correction in HS gravity?

Other questions

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- From entanglement to gravity?

Thanks for your attention!

Quasiprimary fields in 2D CFT

- In a 2D CFT, the field $\phi(z)$ is called quasi-primary if

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n}, \quad \text{for } m = \pm 1, 0 \tag{6.1}$$

- We write the quasiprimary operators as ϕ_i with conformal weights h_i and $\bar{h}_i$
- The correlation functions of two and three quasiprimary operators on complex plane C are determined by the global $SL(2,C)/Z_2$ conformal symmetry

$$\langle \phi_i(z_i,\bar{z}_i)\phi_j(z_j,\bar{z}_j)\rangle_C = \frac{\alpha_i\delta_{ij}}{z_{ij}^{2h_i}\bar{z}_{ij}^{2\bar{h}_i}},$$

$$\langle \phi_i(z_i,\bar{z}_i)\phi_j(z_j,\bar{z}_j)\phi_k(z_k,\bar{z}_k\rangle_C$$

$$= \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j} \bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_k} \bar{z}_{jk}^{\bar{h}_j+\bar{h}_k-\bar{h}_i} \bar{z}_{ik}^{\bar{h}_i+\bar{h}_k-\bar{h}_j}},$$

with $z_{ij} \equiv z_i - z_j$ and $\bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j$.

OPE in 2D CFT

The OPE of two quasiprimary operators could be generally written as

$$\phi_i(z,\bar{z})\phi_j(0,0)=\sum_k C_{ij}^k \sum_{m,r\geq 0} \frac{a_{ijk}^m}{m!}\frac{\bar{a}_{ijk}^r}{r!}\frac{1}{z^{h_i+h_j-h_k-m}\bar{z}^{\bar{h}_i+\bar{h}_j-\bar{h}_k-r}}\partial^m\bar{\partial}^r\phi_k(0,0),$$

where the summation k is over all quasiprimary operators and

$$a_{ijk}^m \equiv \frac{C^m_{h_k+h_i-h_j+m-1}}{C^m_{2h_k+m-1}}, \quad \bar{a}_{ijk}^r \equiv \frac{C^r_{\bar{h}_k+\bar{h}_i-\bar{h}_j+r-1}}{C^r_{2\bar{h}_k+r-1}}, \quad C_{ij}^k \equiv \frac{C_{ijk}}{\alpha_k}$$

with the binomial coefficient being $C_x^y=\frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$.

Holomorphic quasiprimary operators in CFT_1

Explicitly the holomorphic quasiprimary operators of first few levels from vacuum module are listed as follows.

- At level 0, it is the identity operator 1
- At level 2, there is one quasiprimary operator the stress tensor T.
- At level 4, it is $\mathcal{A} = (TT) - \frac{3}{10}\partial^2 T$.
- At level 6, they are $\mathcal{B} = (\partial T \partial T) - \frac{4}{5}(T\partial^2 T) + \frac{23}{210}\partial^4 T$ and $\mathcal{D} = \mathcal{C} + \frac{93}{70c+29}\mathcal{B}$, with $\mathcal{C} = (T(TT)) - \frac{9}{10}(T\partial^2 T) + \frac{4}{35}\partial^4 T$.
- At level 8, more complicated construction

Quasiprimaries from vacuum module in CFT$_n$

The quasiprimary operators from vacuum module are listed as below

L_0	quasiprimary operators	degeneracies	#
0	1	1	1
2	$T(z_j)$	n	n
4	$T(z_{j_1})T(z_{j_2})$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$
	$\mathcal{A}(z_j)$	n	
5	$\mathcal{J}_{j_1 j_2}(z)$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
6	$T(z_{j_1})T(z_{j_2})T(z_{j_3})$ with $j_1 < j_2 < j_3$	$\frac{n(n-1)(n-2)}{6}$	$\frac{n(n+1)(n+5)}{6}$
	$T(z_{j_1})\mathcal{A}(z_{j_2})$ with $j_1 \neq j_2$	$n(n-1)$	
	$\mathcal{K}_{j_1 j_2}(z)$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$	
	$\mathcal{B}(z_j)$	n	
	$\mathcal{D}(z_j)$	n	
...	...	...	...

Note that the j's listed above vary as $0 \leq j \leq n-1$, and also the operators

$$\mathcal{J}_{j_1 j_2}(z) = T(z_{j_1}) i\partial T(z_{j_2}) - i\partial T(z_{j_1}) T(z_{j_2}),$$
$$\mathcal{K}_{j_1 j_2} = \partial T_{j_1} \partial T_{j_2} - \frac{2}{5}\left(T_{j_1}\partial^2 T_{j_2} + \partial^2 T_{j_1} T_{j_2}\right)$$

can not be factorized into the operators at different copies.
The coefficients α_K for these operators could be calculated easily

$$\alpha_{TT} = \frac{c^2}{4}, \quad \alpha_{\mathcal{J}} = 2c^2, \quad \alpha_{TTT} = \frac{c^3}{8},$$
$$\alpha_{T\mathcal{A}} = \frac{c^2(5c+22)}{20}, \quad \alpha_{\mathcal{K}} = \frac{36c^2}{5}.$$

The coefficient d_K

To compute d_K we consider the multivalued transformation

$$z \to f(z) = \left(\frac{z-\ell}{z}\right)^{1/n},$$

which maps the Riemann surface $\mathcal{R}_{n,1}$ to the complex plane C. With some efforts, we find d_K's for various operators listed above,

$$d_1 = 1, \quad d_T = \frac{n^2-1}{12n^2}, \quad d_{\mathcal{B}} = -\frac{(n^2-1)^2\left(2(35c+61)n^2-93\right)}{10368n^6(70c+29)},$$

$$d_{\mathcal{A}} = \frac{(n^2-1)^2}{288n^4}, \quad d_{\mathcal{D}} = \frac{(n^2-1)^3}{10368n^6}, \quad d_{\mathcal{J}}^{j_1j_2} = \frac{1}{16n^5c}\frac{c_{j_1j_2}}{s_{j_1j_2}^5},$$

$$d_{TTT}^{j_1j_2j_3} = -\frac{1}{8n^6c^2}\frac{1}{s_{j_1j_2}^2 s_{j_2j_3}^2 s_{j_1j_3}^2} + \frac{n^2-1}{96n^6c}\left(\frac{1}{s_{j_1j_2}^4}+\frac{1}{s_{j_2j_3}^4}+\frac{1}{s_{j_1j_3}^4}\right) + \frac{(n^2-1)^3}{1728n^6},$$

$$d_{T\mathcal{A}}^{j_1j_2} = \frac{n^2-1}{96n^6c}\frac{1}{s_{j_1j_2}^4} + \frac{(n^2-1)^3}{3456n^6}, \quad d_{TT}^{j_1j_2} = \frac{1}{8n^4c}\frac{1}{s_{j_1j_2}^4} + \frac{(n^2-1)^2}{144n^4},$$

Here $s_{j_1j_2} \equiv \sin\frac{\pi(j_1-j_2)}{n}$ and $c_{j_1j_2} \equiv \cos\frac{\pi(j_1-j_2)}{n}$.

Quasiprimaries from W_3 field in CFT$_n$

The quasiprimary operators from W_3 field with nonvanishing coefficients are listed as below

L_0	quasiprimary operators	degeneracies
6	$W_{j_1} W_{j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
7	$\mathcal{U}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
8	$W_{j_1} \mathcal{S}_{j_2}$ with $j_1 \neq j_2$	$n(n-1)$
	$\mathcal{V}_{j_1 j_2}$ with $j_1 < j_2$	$\frac{n(n-1)}{2}$
	$T_{j_1} W_{j_2} W_{j_3}$ with $j_1 \neq j_2$, $j_1 \neq j_3$ and $j_2 < j_3$	$\frac{n(n-1)(n-2)}{2}$
$\cdots$	$\cdots$	$\cdots$

Here we have

$$\mathcal{S} = (TW) - \frac{3}{14}\partial^2 W,$$
$$\mathcal{U}_{j_1 j_2} = W_{j_1} i\partial W_{j_2} - i\partial W_{j_1} W_{j_2},$$
$$\mathcal{V}_{j_1 j_2} = \partial W_{j_1} \partial W_{j_2} - \frac{2}{7}\left(W_{j_1}\partial^2 W_{j_2} + \partial^2 W_{j_1} W_{j_2}\right).$$

Normalizations and coefficients

$$\alpha_{\mathcal{S}} = \frac{c(7c+114)}{42}, \quad \alpha_{WW} = \frac{c^2}{9}, \quad \alpha_{\mathcal{U}} = \frac{4c^2}{3},$$
$$\alpha_{W\mathcal{S}} = \frac{c^2(7c+114)}{126}, \quad \alpha_{\mathcal{V}} = \frac{52c^2}{7}, \quad \alpha_{TWW} = \frac{c^3}{18}.$$

$$d_{WW}^{j_1j_2} = -\frac{3}{(2n)^6 c}\frac{1}{s_{j_1j_2}^6}, \quad d_{\mathcal{U}}^{j_1j_2} = -\frac{3}{(2n)^7 c}\frac{c_{j_1j_2}}{s_{j_1j_2}^7}, \quad d_{W\mathcal{S}}^{j_1j_2} = -\frac{n^2-1}{(2n)^8 c}\frac{1}{s_{j_1j_2}^6},$$
$$d_{\mathcal{V}}^{j_1j_2} = \frac{1}{26(2n)^8 c}\left(\frac{6(n^2+13)}{s_{j_1j_2}^6} - \frac{91}{s_{j_1j_2}^8}\right), \quad d_{TWW}^{j_1j_2j_3} = \frac{18}{(2n)^8 c^2}\frac{1}{s_{j_1j_2}^2 s_{j_2j_3}^4 s_{j_3j_1}^2} - \frac{n^2-1}{(2n)^8 c}\frac{1}{s_{j_2j_3}^6}.$$