Infinitesimal moduli of G2 structure manifolds with instanton bundles

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X. de la Ossa, ML, E. Svanes (1607.03473 & work in progress),
See also X. de la Ossa, ML, E. Svanes (1409.7539) and J. Gray, ML, D. Lüst (1205.6208)
Motivation and summary

Long history of heterotic string compactifications
- Minkowski 4D $\mathcal{N} = 1$ vacua with good particle physics from Calabi–Yau manifolds with vector bundles.
- Minkowski 4D $\mathcal{N} = 1$ vacua also from Strominger–Hull compactifications with flux and bundles.

Stabilising moduli is an open problem.

This talk: heterotic compactifications on $G_2$ structure manifolds
- Heterotic 4D $\mathcal{N} = 1/2$ domain wall solutions. May “uplift” non-perturbatively to non-SUSY AdS solutions.
- Moduli of $G_2$ holonomy manifolds with bundles.
- Some comments on deformations of integrable $G_2$ structures (relevant also for non-SUSY M-theory compact.)
Outline

1 Motivation and summary

2 Heterotic supersymmetric vacua
   - 4D Heterotic $\mathcal{N} = 1$ Minkowski vacua
   - 4D Heterotic $\mathcal{N} = 1/2$ DW vacua

3 Manifolds with $G_2$ structure

4 Infinitesimal Moduli
   - $\mathcal{N} = 1$
   - $\mathcal{N} = 1/2$

5 Conclusions and outlook
Heterotic supersymmetric vacua

**Heterotic supergravity**
- Bosonic fields: Metric $G$, B-field $B$, dilaton $\phi$, gauge field $A$
- Fermionic fields: Gravitino, dilatino, gaugino

Fermionic SUSY variations vanish $\iff$

\[
\begin{align*}
\left(\nabla_M + \frac{1}{8} H_M \right) \epsilon &= 0 \\
\left(\hat{\nabla} \phi + \frac{1}{12} H \right) \epsilon &= 0 \\
\tilde{F} \epsilon &= 0
\end{align*}
\]

where $\hat{\nabla} = \Gamma^M \nabla_M$, etc.

**Compactifications**

$M_{10} = M_E \times X$: SUSY $\iff$ nowhere vanishing spinor $\eta$ on $X$: $\epsilon = \rho_E \otimes \eta$

$\iff$ $X$ has reduced structure group

Hitchin:02, Gualtieri:04, Grana et al:05, ...
4D Heterotic $\mathcal{N} = 1$ Minkowski vacua

Geometry: 6D manifold $X$ with $SU(3)$ structure

Candelas, et.al.:85, Hull:86; Strominger:86, Ivanov, Papadopoulos:00; Gauntlett, et.al.:03,...

SUSY $\Rightarrow$ globally defined spinor $\eta$ on $X$: $\nabla_H \eta = 0$
$\iff$ complex decomposable $(3,0)$-form $\Psi$ and real $(1,1)$ form $\omega$ such that

$$\omega \wedge \Psi = 0, \quad \omega \wedge \omega \wedge \omega \sim \Psi \wedge \bar{\Psi}$$

- No $H$-flux $\iff$ $X$ is Calabi–Yau $\quad d\Psi = 0 = d\omega.$
- $H \neq 0$ $\iff$ $X$ is complex and conformally balanced

$$d(e^{-2\phi} \Psi) = 0 = d(e^{-2\phi} \omega \wedge \omega).$$

* Need $\alpha'$ corrections to avoid no-go theorem for flux if $X$ is compact without boundary
4D Heterotic $\mathcal{N} = 1$ Minkowski vacua

Gauge fields $\rightarrow$ vector bundle $V$

Candelas, et.al.:85, Donaldson:85, Uhlenbeck, Yau:86, Li, Yau:87, ...

SUSY $\implies$

- $F$ holomorphic $F^{(0,2)} = F^{(2,0)} = 0$
- $F$ satisfies Hermitian Yang-Mills equation $F\omega = 0$

polystable holomorphic $V$: $\exists! A$ satisfying HYM.
4D Heterotic $\mathcal{N} = 1/2$ DW vacua

Lukas et al:10, 11; Gray, ML, Lüst:12, de la Ossa, ML, Svanes:14

**Geometry**

4D domain wall vacuum: $\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{W} X(r) \equiv \mathcal{M}_3 \times Y$

$\mathcal{M}_4 = \mathcal{M}_3 \times \mathbb{R}$, $\mathcal{M}_3$ AdS or Minkowski

**SUSY and BI at $\mathcal{O}(\alpha'^0)$**

Restricts torsion, DW flow of the SU(3) structure, and the flux.

SU(3) torsion: $X(r)$ conformally balanced, but otherwise generic

SU(3) flow: $\partial_r \omega$ fixed in terms of $\phi$ and SU(3) torsion

$\partial_r \Psi$ fixed up to primitive $(2,1)+(1,2)$-form $\gamma$.

Flux $\hat{H}$: fixed by SUSY in terms of $\phi$ and SU(3) torsion up to $\gamma$

$\Rightarrow$ can check Bianchi identity.
Embed $SU(3)$ in $G_2$: \[ \varphi = dr \wedge \omega(r) + \text{Re}(\Psi(r)) \].

$H$-flux components allowed by symmetry: $H_{\alpha\beta\gamma}$, $H_{tmn}$ and $H_{mnp}$.

- $H_{tmn} = 0 = H_{\alpha\beta\gamma}$ allows non-perturbative “uplift” to 4D AdS.
**4D Heterotic $\mathcal{N} = 1/2$ DW vacua**

Gray, ML, Lüst:12, de la Ossa, ML, Svanes:14, Fernandez–Gray:82, Chiossi–Salamon:02

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**SUSY $\iff$ $Y$ has $G_2$ structure determined by 3-form $\varphi$ ($\psi = *\varphi$)**

\[
\text{d}\varphi = \tau_0 \psi + 3 \tau_1 \wedge \varphi + *\tau_3 , \\
\text{d}\psi = 4 \tau_1 \wedge \psi + *\tau_2 .
\]

with torsion classes (in $G_2$ reps)

\[
\tau_0 = \text{dvol}_{\mathcal{M}_3} \, H , \quad \tau_1 = \frac{1}{2} \, d\phi , \\
\tau_2 = 0 , \quad \tau_3 = -H + \frac{1}{6} \tau_0 \varphi - \tau_1 \phi \, \psi
\]

This is an integrable $G_2$ structure.
4D Heterotic $\mathcal{N} = 1/2$ DW vacua

Gray, ML, Lüst:12, de la Ossa, ML, Svanes:14, Fernandez–Gray:82, Chiossi–Salamon:02

Gauge vector bundle

**SUSY**

- $F \wedge \psi = 0 \implies F$ is a $G_2$ instanton
- Embed $SU(3) \implies$ recover $F^{(2,0)} = 0 = F \perp \omega$
Manifolds with $G_2$ structure

Fernandez–Gray:82, Chiossi–Salamon:02

Decomposition of forms

$\Lambda^k(Y)$ decomposes into $\Lambda^k_p(Y)$, $p$ denotes $G_2$ irrep. Find these using $\varphi$:

Example: $\Lambda^1 = \Lambda^1_7 = T^*Y \cong TY$

$\implies$ any $\beta \in \Lambda^2$ decomposes as $\beta = \alpha \downarrow \varphi + \gamma$, where $\alpha \in \Lambda^1$ and $\gamma \downarrow \varphi = 0$

\begin{align*}
\Lambda^0 &= \Lambda^0_1 , \\
\Lambda^1 &= \Lambda^1_7 = T^*Y \cong TY , \\
\Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{14} , \\
\Lambda^3 &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} .
\end{align*}
Variational studies

- Moduli of heterotic $\mathcal{N} = 1$ vacua: deformations of conformally balanced complex 3-fold $X$ with holomorphic gauge bundle $V$

- Moduli of heterotic $\mathcal{N} = 1/2$ vacua: deformations of 7D $Y$ with integrable $G_2$ structure and an instanton gauge bundle $V$

- Flow of heterotic $\mathcal{N} = 1/2$ solutions at $O(\alpha'^0)$: the moduli spaces of different $SU(3)$ structures may connect via flow along domain wall direction in $Y$ with integrable $G_2$ structure.
Infinitesimal Moduli

Variational studies

- **Moduli of heterotic $N = 1$ vacua**: deformations of conformally balanced complex 3-fold $X$ with holomorphic gauge bundle $V$
  

- **Moduli of heterotic $N = 1/2$ vacua**: deformations of 7D $Y$ with integrable $G_2$ structure and an instanton gauge bundle $V$
  
  de la Ossa, ML, Svanes:16 + in progress, Clarke, et al:16

- Flow of heterotic $N = 1/2$ solutions at $O(\alpha'^0)$: the moduli spaces of different $SU(3)$ structures may connect via flow along domain wall direction in $Y$ with integrable $G_2$ structure.
  
  de la Ossa, ML, Svanes:14
Infinitesimal Moduli: $\mathcal{N} = 1$

Moduli space building blocks

$X$: conformally balanced complex 3-fold ($H = 0$: Calabi–Yau)  
$V$: holomorphic gauge bundle

- $\partial_t \Psi$: Complex structure moduli $H^{(2,1)}_{\bar{\partial}}(X) \cong H^{(0,1)}_{\bar{\partial}}(X, TX)$
- $\partial_t \omega$: $H = 0$ Kähler moduli $H^{(1,1)}_{d}(X) \cong H^{(0,1)}_{\bar{\partial}}(X, T^*X)$  
  $H \neq 0$ Hermitian moduli
- $\partial_t A$: Vector bundle moduli $H^1(X, \text{End}(V))$

Kodaira, Spencer: 58,60, Candelas, de la Ossa:91, Becker, et.al:05,06,...
Infinitesimal Moduli: $\mathcal{N} = 1$

Varying full set of $\mathcal{N} = 1$ constraints

“Atiyah class stabilization” restricts the moduli space:

- Gaugino variation $F \wedge \Psi = 0 \Rightarrow$ cs moduli $\in \ker(\mathcal{F})$
- Instanton connection: $R(\nabla) \wedge \Psi = 0 \Rightarrow$ cs moduli $\in \ker(\mathcal{R})$
  - Extra moduli for connection variations.
  - Related to field redefinitions.

SUSY + BI $\implies$ EOM $\iff$ $\theta$ is an instanton
dela Ossa, Svanes:14

- Gaugino $F \wedge \omega \wedge \omega$: Use Donaldson–Uhlenbeck–Yau and Li–Yau theorems:
  stable under first order deformations; no constraint.
  CY $\leadsto$ D-terms in 4D

  Anderson et.al:11

- Anomaly cancellation constraint together with $H = i(\partial - \bar{\partial})\omega$:
  (cs+bundle moduli) $\in \ker(\mathcal{H})$; Herm. moduli space: quotiented by $\text{Im}\mathcal{H}$

Atiyah:57, Fu, Yau:11, Anderson,Gray, Lukas, Ovrut:10,11,13, Anderson, Gray, Sharpe:14,
dela Ossa, Svanes:14, Garcia-Fernandez, et.al:13,15...
Infinitesimal Moduli: $\mathcal{N} = 1$

**End result**

Finite dimensional moduli space for all heterotic $\mathcal{N} = 1$ vacua.

Infinitesimal moduli equivalent to deformations of the holomorphic structure on an extension bundle of Atiyah type.

Atiyah:57, Fu, Yau:11, Anderson, Gray, Lukas, Ovrut:10,11,13, Anderson, Gray, Sharpe:14, de la Ossa, Svanes:14, Garcia-Fernandez, et.al:13,15...
Infinitesimal Moduli: $\mathcal{N} = 1/2$

Moduli space building blocks

$Y$: integrable $G_2$ structure manifold ($H = 0$: $G_2$ holonomy)
$V$: instanton gauge bundle

- $\partial_t \psi, \partial_t \varphi$: geometric moduli

Joyce:96, Dai–Wang–Wei:03, de Boer–Naqvi–Shomer:05,...

- $\partial_t A$: Vector bundle moduli $H^1(Y, \text{End}(V))$
Infinitesimal Moduli: $\mathcal{N} = 1/2$

### Comment on connections on $G_2$ holonomy manifolds

- **Unique $G_2$ invariant connection**: the Levi-Civita connection
- **Variations of $G_2$ structure** $\sim d\theta$
  
  - $d\theta V^a = dV^a + \theta^a_b V^b$, $\theta^a_b = \Gamma^{a}_{bc} d\chi^c$
  
  - $\Gamma^{a}_{bc}$: connection symbols of Levi-Civita connection
  
  - $d\theta$ is $G_2$ metric compatible.
  
  - $R(\theta)$ is an instanton: $R(\theta) \wedge \psi = 0$

### Comment on connections on $G_2$ structure manifolds

Can generalise to $G_2$ structure manifolds, but then

- **Two-parameter family of metric compatible connections.** Bryant:03

- **Connection with totally antisymmetric torsion** $\iff \tau_2 = 0$ Friedrich:06

- **Unique connection** $\nabla^S = \nabla_{LC} + H$ with $Hol(\nabla^S) = G_2$ and totally antisymmetric torsion Bryant:03
Infinitesimal Moduli: $\mathcal{N} = 1/2$

Decomposition of de Rham cohomology

Reyes-Carrion:93, Fernandez–Ugarte:98

Analogue of Dolbeault operator on a complex manifold: project $d$ onto $G_2$ irreps.

- The differential operator $\mathcal{\tilde{d}}$ is defined by
  
  \[
  \mathcal{\tilde{d}}_0 = d, \quad \mathcal{\tilde{d}}_1 = \pi_7 \circ d, \quad \mathcal{\tilde{d}}_2 = \pi_1 \circ d.
  \]

- $\tau_2 = 0 \iff \mathcal{\tilde{d}}^2 = 0$, so can construct differential, elliptic complex
  
  \[
  0 \to \Lambda^0(Y) \xrightarrow{\mathcal{\tilde{d}}} \Lambda^1(Y) \xrightarrow{\mathcal{\tilde{d}}} \Lambda^2_7(Y) \xrightarrow{\mathcal{\tilde{d}}} \Lambda^3_1(Y) \to 0
  \]

- $H^*_\mathcal{\tilde{d}}(Y)$ is “canonical $G_2$-cohomology of $Y$”.

This generalizes to $TY$-valued forms: Elliptic complex

\[
0 \to \Lambda^0(TY) \xrightarrow{\mathcal{\tilde{d}}_\theta} \Lambda^1(TY) \xrightarrow{\mathcal{\tilde{d}}_\theta} \Lambda^2_7(TY) \xrightarrow{\mathcal{\tilde{d}}_\theta} \Lambda^3_1(TY) \to 0
\]

with finite-dim cohomology groups $H^p_{\mathcal{\tilde{d}}_\theta}(Y, TY)$, if $R(\theta) \wedge \psi = 0$
Infinitesimal Moduli: $\mathcal{N} = 1/2$

Geometric moduli for $G_2$ holonomy

\[ \partial_t \psi = \frac{1}{3!} M_t^a \wedge \psi_{bcda} \, dx^{bcd} , \quad M_t^a = M_t b^a \, dx^b \]

\[ \partial_t \varphi = -\frac{1}{2} M_t^a \wedge \varphi_{bca} \, dx^{bc} \]

- Diffeomorphisms:
  \[ \mathcal{L}_V \psi = -\frac{1}{3!} (d_\theta V^a) \wedge \psi_{bcda} \, dx^{bcd} \]

  where $d_\theta$ is a connection for $TY$-valued forms.

- Preserve $d\psi = 0 = d\varphi$:

  \[ d_\theta \Delta_t^a \wedge \psi_{bcda} \, dx^{bcd} = 0 , \]

  \[ d_\theta \Delta_t^a \wedge \varphi_{bca} \, dx^{bc} = 0 . \]

  where $\Delta_t b^a = M_t b^a - \frac{1}{7} (\text{tr} M_t)$

- Compact $G_2$ manifold:

  \[ TM_{G_2\text{hol}} \cong H^3_d (Y) \subset H^1_{d_\theta} (Y , TY) \]
Geometric moduli for integrable $G_2$ structure

- Diffeomorphisms:
  \[ \mathcal{L}_V \psi = -\frac{1}{3!} (d_\theta V^a) \wedge \psi_{bcda} \, dx^{bcd} \]
  where $d_\theta$ is a connection for $TY$-valued forms.

- Preserve $\tau_2 = 0$:
  \[ (\tilde{d}_\theta \Delta_t^a) \wedge \psi_{bcda} \, dx^{bcd} = 0 \]

- $\partial_t \varphi \implies$ Variational constraints on torsion
Infinitesimal Moduli: $\mathcal{N} = 1/2$

$G_2$ “Atiyah” class stabilization

Instanton condition $F \wedge \psi = 0$: couples bundle and geometric moduli

\[
0 = \partial_t (F \wedge \psi) \iff \bar{d}_A (\partial_t A) = -\bar{\mathcal{F}}(\Delta_t).
\]

where $\Delta_t \in \Lambda^1(Y, TY)$, $d_A$ covariant derivative, and $\bar{d}_A$, $\bar{\mathcal{F}}$: project to $G_2$ irreps

- “Atiyah” map

\[
\mathcal{F} : \Lambda^p(Y, TY) \to \Lambda^{p+1}(Y, \text{End}(V))
\]

\[
\Delta \to \mathcal{F}(\Delta) = -F_{ab} dx^b \wedge \Delta^a.
\]

$\bar{\mathcal{F}}$ is a map in cohomology:

Bianchi identity $d_A F = 0 \implies \bar{\mathcal{F}}(\bar{d}_\theta(\Delta)) + \bar{d}_A(\bar{\mathcal{F}}(\Delta)) = 0$.

- In fact, $\bar{\mathcal{F}}$ also maps geometric modulus of integrable $G_2$ structure to a $\bar{d}_A$-closed form

Corrected moduli space for bundle and geometric moduli:

\[
H^1_{d_A}(\text{End}(V)) \oplus \ker(\bar{\mathcal{F}})
\]
Infinitesimal Moduli: $\mathcal{N} = 1/2$

Corrected moduli space for bundle and geometric moduli:

$$H^1_{dA}(\text{End}(V)) \oplus \ker(\tilde{F})$$

Remark 1: $B$-field deformations

- Infinitesimal moduli of $G_2$-holonomy metrics only spans part of $H^1_{d\theta}(Y, TY)$:
  $$H^1_{d\theta}(Y, TY) \cong \tilde{H}^1(Y, TY) = \tilde{S}^1(Y, TY) \oplus \tilde{A}^1(Y, TY)$$
- On compact $G_2$ manifolds, the rest is spanned by $\partial_t B$
- All $\partial_t B$ are in the kernel of $\tilde{F}$.
- Thus easily incorporate $B$-field deformations in the infinitesimal moduli space.
Corrected moduli space for bundle and geometric moduli:
\[
H_{dA}^1(\text{End}(V)) \oplus \ker(\check{F})
\]

Remark 2: Extension bundle
- Use the $G_2$ Atiyah map to define a new bundle

\[
0 \longrightarrow \text{End}(V) \longrightarrow E \longrightarrow TY \longrightarrow 0,
\]

- $E$ has connection $\mathcal{D}_E$:

\[
\mathcal{D}_E = \begin{pmatrix}
\check{d}_A & \check{F} \\
0 & \check{d}_\theta
\end{pmatrix}.
\]

- $\mathcal{D}^2_E = 0 \iff \check{F}(\check{d}_\theta(\Delta)) + \check{d}_A(\check{F}(\Delta)) = 0$

- $H^1_{\mathcal{D}_E}(Y, E)$ is the moduli space

Proof: construct long exact sequence in cohomology.
Infinitesimal Moduli: $\mathcal{N} = 1/2$ in progress

**Integrable $G_2$ structure**

Corrected moduli space for bundle and geometric moduli:

$$H^1_{d_A}(\text{End}(V)) \oplus H^1_{d_\theta}(\text{End}(TY)) \oplus \ker(\mathcal{F} + \mathcal{K})$$

**Integrable $G_2$ structure**

SUSY + BI $\implies$ EOM $\leadsto$ expect instanton condition: $R(\theta) \wedge \psi = 0$:

- $\theta$ must vary with $\psi$
- Extra moduli for connection variations.
- Related to field redefinitions as in $\mathcal{N} = 1$?
- $\mathcal{R}$ map

$$\mathcal{R} : \Lambda^p(Y, TY) \quad \mapsto \quad \Lambda^{p+1}(Y, \text{End}(TY))$$

$$\Delta \quad \mapsto \quad \mathcal{R}(\Delta) = -R_{ab} \, dx^b \wedge \Delta^a.$$  

$\mathcal{K}$: map in cohomology that maps any $\Delta \in \mathcal{T}M_0(Y, \varphi)$ to a $d_\theta$-closed form

- $\ker(\mathcal{F} + \mathcal{K})$ may still be infinite-dimensional.
Anomaly cancellation condition: $\mathcal{N} = 1/2$

**Integrable $G_2$ structure**

Corrected moduli space for bundle, instanton and geometric moduli:

$$H^1_{dA}(\text{End}(V)) \oplus H^1_{d\theta}(\text{End}(TY)) \oplus \ker(\mathcal{F} + \mathcal{R})$$

Last equation to bring in: anomaly cancellation condition

$$H = dB + \frac{\alpha'}{4} (\mathcal{C}S[A] - \mathcal{C}S[\theta]) \quad (\star)$$

- Recall from $\mathcal{N} = 1$: Vary $(\star)$ together with $H = d^c \omega \leadsto \text{map } \mathcal{H}: \Lambda^p(X, E) \longrightarrow \Lambda^{p+1}(X, T^*X)$
  - well-defined in cohomology
  - finite-dim Hermitian moduli space
- $\mathcal{N} = 1/2$
  - Vary $(\star)$ together with $H = \frac{1}{6} \tau_0 \varphi - \tau_1 \psi - \tau_3 \leadsto \text{map } \mathcal{H}$
  - $\mathcal{H}$ well-defined in cohomology? Finiteness of moduli space? In progress...
Conclusions and outlook

Conclusions

- 4D heterotic $\mathcal{N} = 1/2$ DW solutions
  - $\mathcal{Y}$ Integrable $G_2$ structure $\supset G_2$ holonomy
  - $\mathcal{X}$ Conformally balanced (non-complex) $SU(3)$ structure

- Infinitesimal def. space of $G_2$ holonomy manifold w. instanton bundle $V$:
  \[
  H^1_{d_A}(\text{End}(V)) \oplus \ker(\tilde{\mathcal{F}}), \\
  \ker(\tilde{\mathcal{F}}) \subset H^1_{d_0}(Y, TY)
  \]

Atiyah’s deformation space of holomorphic structures on extension bundles applies to (real) $G_2$ holonomy instanton bundles.

- Infinitesimal def. space of integrable $G_2$ structures w. instanton bundle $V$:
  \[
  H^1_{d_A}(\text{End}(V)) \oplus H^1_{d_0}(\text{End}(TY)) \oplus \ker(\tilde{\mathcal{F}} + \tilde{\mathcal{R}})
  \]
Conclusions and outlook

Conclusions
Atiyah’s deformation space of holomorphic structures on extension bundles applies to (real) $G_2$ holonomy instanton bundles.

Outlook
- Include constraint from Bianchi identity $dH = ...$ (in progress).
- Determine conditions for finite-dimensional infinitesimal moduli space.
- Relation of $SU(3)$ and $G_2$ structure moduli spaces for domain wall solutions.
- Relevance for compactifications of M-theory and type II string theory.