Viscous hydrodynamics

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Conservation laws

Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

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$\Longleftrightarrow$ **Hydrodynamics!**

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Local conservation of charge and energy-momentum.

$\Longleftrightarrow$ **Hydrodynamics!**

This can be generalized to systems with several conserved charges:

$$\partial_\mu N_i^\mu = 0,$$

$i =$ **baryon number**, **strangeness**, **charge**. . .

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Consider only baryon number conservation, $i = B$.

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Consider only baryon number conservation, $i = B$.

- $\Rightarrow$ **5 equations contain 14 unknowns!**
- $\Rightarrow$ **The system of equations does not close.**
- $\Rightarrow$ **Provide 9 additional equations or**
  Eliminate 9 unknowns.

So what are the components of $T^{\mu\nu}$ and N^{μ}?

- **N^{μ} and $T^{\mu\nu}$ can be decomposed with respect to arbitrary, normalized, time-like 4-vector u^{μ},**

$$u_{\mu}u^{\mu} = 1$$

- **Define a projection operator**

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}, \quad \Delta^{\mu\nu}u_{\nu} = 0,$$

which projects on the 3-space orthogonal to u^{μ}.

- **Then**

$$\boxed{N^\mu = nu^\mu + \nu^\mu}$$

where

$n = N^\mu u_\mu$ **is (baryon) charge density in the frame where** $u = (1, \mathbf{0})$, **local rest frame, LRF**

$\nu^\mu = \Delta^{\mu\nu} N_\nu$ **is charge flow in LRF,**

and

$$\boxed{T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}}$$

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$ **energy density in LRF**

$P \equiv -\frac{1}{3}\Delta^{\mu\nu} T_{\mu\nu}$ **isotropic pressure in LRF**

$W^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$ **energy flow in LRF**

$\pi^{\mu\nu} \equiv [\frac{1}{2}(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\nu{}_\beta \Delta^\mu{}_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}] T^{\alpha\beta}$

(traceless) shear-stress tensor in LRF

- **The 14 unknowns in 5 equations:**

$$\left.\begin{matrix} N^\mu & 4 \\ T^{\mu\nu} & 10 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} n,\ \epsilon,\ P & 3 \\ W^\mu & 3 \\ \nu^\mu & 3 \\ \pi^{\mu\nu} & 5 \end{matrix}\right.$$

- **So far u^μ is arbitrary. It attains a physical meaning by relating it to N^μ or $T^{\mu\nu}$:**

  1. **Eckart frame:**
  $$u_E^\mu \equiv \frac{N^\mu}{\sqrt{N_\nu N^\nu}}$$
  u^μ is 4-velocity of charge flow, $\nu^\mu = 0$.
  The 14 unknowns are n, ϵ, P, W^μ, $\pi^{\mu\nu}, u^\mu$.

  2. **Landau frame:**
  $$u_L^\mu \equiv \frac{T^{\mu\nu} u_\nu}{\sqrt{u_\alpha T^{\alpha\beta} T_{\beta\gamma} u^\gamma}}$$
  u^μ is 4-velocity of energy flow, $W^\mu = 0$.
  The 14 unknowns are n, ϵ, P, ν^μ, $\pi^{\mu\nu}, u^\mu$.

- **In general, the hydrodynamical equations are not closed and cannot be solved uniquely.**

- **14 unknowns $\Leftrightarrow$ 5 equations**

Viscous hydrodynamics

In Landau frame,

$$W^\mu \equiv 0, \qquad \nu^\mu = -\frac{q^\mu}{h} = -\frac{n}{\epsilon + P} q^\mu,$$

where q^μ is heat flow:

$$\begin{aligned} N^\mu &= n u^\mu + \nu^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P_{\rm eq} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} \end{aligned}$$

Need 9 additional equations to determine

$$\Pi,\ \pi^{\mu\nu},\ q^\mu,\ P_{\rm eq}$$

Equation of state

$$P_{\rm eq} = P(T,\mu)$$

Matching conditions

ideal fluid $\Longleftrightarrow$ **exact local kinetic equilibrium**

dissipation $\Longleftrightarrow$ **deviations from thermal distribution**

Non-equilibrium thermodynamics?

- **What are entropy and pressure?**
- **EoS? Temperature?**

Matching conditions

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Non-equilibrium thermodynamics?

- **What are entropy and pressure?**
- **EoS? Temperature?**

Energy and particle number defined for arbitrary system:

$$\epsilon = u_\mu T^{\mu\nu} u_\nu \qquad \text{and} \qquad n = N^\mu u_\mu$$

apply equilibrium EoS:

$$s = s_0(\epsilon, n) \qquad \text{and} \qquad P = P_0(\epsilon, n)$$

i.e. **we match the system to an equilibrium system of** ***the same*** ϵ **and** n

relativistic Navier-Stokes

Entropy four-current:

$$S^\mu = su^\mu + \frac{\mu}{T}\frac{q^\mu}{h}$$

where

$$h = \frac{\epsilon + P}{n}$$

Require non-decrease of entropy:

$$0 \leq \partial_\mu S^\mu = -\Pi \nabla^\mu u_\mu - q_\mu \frac{T}{e+p} \nabla^\mu \frac{\mu}{T} + \pi_{\mu\nu} \nabla^{\langle\mu} u^{\nu\rangle}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^\mu_\sigma \Delta^\nu_\tau + \Delta^\nu_\tau \Delta^\mu_\sigma\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right] A^{\sigma\tau}$$

and

$$\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

relativistic Navier-Stokes

$$0 \leq \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

is always valid if we identify

$$\Pi \propto X, \qquad q^\mu \propto X^\mu, \qquad \pi^{\mu\nu} \propto X^{\mu\nu}$$

dissipative currents small corrections linear in gradients

$$\begin{aligned} \Pi &= -\zeta \nabla^\mu u_\mu \\ q^\mu &= -\kappa \frac{T}{e+p} \nabla^\mu \frac{\mu}{T} \\ \pi^{\mu\nu} &= 2\eta \nabla^{\langle\mu} u^{\nu\rangle} \end{aligned}$$

η, ζ **shear and bulk viscosities,** κ **heat conductivity**

Navier-Stokes equations of motion

$$
\begin{aligned}
Dn &= -n\partial_\mu u^\mu - \partial_\mu \left(\kappa \frac{Tn}{h^2} \nabla^\mu \frac{\mu}{T}\right) \\
D\epsilon &= -(\epsilon + P - \zeta \nabla^\alpha u_\alpha)\partial_\mu u^\nu + 2\eta \nabla^{\langle\alpha} u^{\beta\rangle} \nabla_{\langle\alpha} u_{\beta\rangle} \\
(\epsilon + P - \zeta \nabla^\alpha u_\alpha) D u^\mu &= \nabla^\mu (P - \zeta \nabla^\alpha u_\alpha) - 2\Delta^\mu_\alpha \partial_\beta (\eta \nabla^{\langle\alpha} u^{\beta\rangle})
\end{aligned}
$$

where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu} \partial_\nu$$

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\end{aligned}
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where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

but these are parabolic. . .

Parabolic partial differential equations

PDE of the form

$$A\frac{\partial^2}{\partial x^2}u + B\frac{\partial^2}{\partial x \partial y}u + C\frac{\partial^2}{\partial y^2}u + D\frac{\partial}{\partial x}u + E\frac{\partial}{\partial y}u + F = 0$$

is parabolic if

$$B^2 - AC = 0$$

Such equations provide infinite speed for signal propagation

Müller ('76), Israel & Stewart ('79) ...

Solutions are unstable

Hiscock & Lindblom, PRD31, 725 (1985) ...

Hyperbolic partial differential equations

PDE of the form

$$A\frac{\partial^2}{\partial x^2}u + B\frac{\partial^2}{\partial x \partial y}u + C\frac{\partial^2}{\partial y^2}u + D\frac{\partial}{\partial x}u + E\frac{\partial}{\partial y}u + F = 0$$

is hyperbolic if

$$B^2 - AC > 0$$

For example one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Solutions stable and with finite propagation speed.

Causal viscous hydro

To obtain causal equations we have to replace

$$\Pi = -\zeta \nabla^\mu u_\mu$$

by

$$\tau_\Pi D\Pi + \Pi = -\zeta \nabla^\mu u_\mu + \cdots$$

or something similar.

Causal viscous hydro

Israel & Stewart:

Entropy four-flow including terms second order in dissipative fluxes:

$$S^\mu = \quad s u^\mu + \frac{\mu}{T}\frac{q^\mu}{h} \quad - \left(\beta_0 \Pi^2 - \beta_1 q_\nu q^\nu + \beta_2 \pi_{\lambda\nu}\pi^{\lambda\nu}\right)\frac{u^\mu}{2T}$$
$$- \frac{\alpha_0 q^\mu \Pi}{T} + \frac{\alpha_1 q_\nu \pi^{\nu\mu}}{T}$$

$\Rightarrow$ **"Second order theory"**

or, rather, Transient fluid dynamics

Evolution equation for shear

Require non-decrease of entropy:

$$0 \leq \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

Identify $\pi^{\mu\nu} = 2\eta X^{\langle\mu\nu\rangle}$:

$$\begin{aligned}\pi^{\mu\nu} &= 2\eta \left[\nabla^{\langle\mu} u^{\nu\rangle} - \beta_2 \langle u^\lambda \partial_\lambda \pi^{\mu\nu}\rangle - \frac{1}{2}\pi^{\mu\nu} T \partial_\lambda \left(\frac{\tau_\pi u^\lambda}{2\eta T}\right)\right] \\ &\quad + 2\eta \left[\alpha_1 \nabla^{\langle\mu} q^{\nu\rangle} + a_1' q^{\langle\mu} u^\lambda \partial_\lambda u^{\nu\rangle}\right]\end{aligned}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^\mu_\sigma \Delta^\nu_\tau + \Delta^\nu_\tau \Delta^\mu_\sigma\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right] A^{\sigma\tau}$$

and

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu.$$

Israel-Stewart evolution equations

$$
\begin{aligned}
D\Pi &= -\frac{1}{\tau_\Pi}(\Pi + \zeta\nabla_\mu u^\mu) - \frac{1}{2}\Pi\left(\nabla_\mu u^\mu + D\ln\frac{\beta_0}{T}\right) \\
&\quad + \frac{\alpha_0}{\beta_0}\partial_\mu q^\mu - \frac{a_0'}{\beta_0} q^\mu D u_\mu \\
Dq^\mu &= -\frac{1}{\tau_q}\left[q^\mu + \kappa_q \frac{T^2 n}{\varepsilon + p}\nabla^\mu\left(\frac{\mu}{T}\right)\right] - u^\mu q_\nu D u^\nu \\
&\quad - \frac{1}{2} q^\mu \left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) - \omega^{\mu\lambda} q_\lambda \\
&\quad - \frac{\alpha_0}{\beta_1}\nabla^\mu \Pi + \frac{\alpha_1}{\beta_1}(\partial_\lambda \pi^{\lambda\mu} + u^\mu \pi^{\lambda\nu}\partial_\lambda u_\nu) + \frac{a_0}{\beta_1}\Pi D u^\mu - \frac{a_1}{\beta_1}\pi^{\lambda\mu} D u_\lambda \\
D\pi^{\mu\nu} &= -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu}u^{\nu\rangle}\right) - (\pi^{\lambda\mu}u^\nu + \pi^{\lambda\nu}u^\mu) D u_\lambda \\
&\quad - \frac{1}{2}\pi^{\mu\nu}\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_2}{T}\right) - 2\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} \\
&\quad - \frac{\alpha_1}{\beta_2}\nabla^{\langle\mu} q^{\nu\rangle} + \frac{a_1'}{\beta_2} q^{\langle\mu} D u^{\nu\rangle}
\end{aligned}
$$

Israel-Stewart evolution. . .

bulk pressure Π, shear stress $\pi^{\mu\nu}$ heat flow q^{μ} treated as independent dynamical quantities that relax to their Navier-Stokes value on time scales $\tau_{\Pi}(e,n)$, $\tau_{\pi}(e,n)$, $\tau_{q}(e,n)$

Equations of motion	5 equations
evolution of bulk	1 equation
evolution of heat flow	3 equations
evolution of shear stress	5 equations
14 equations, 14 unknowns	

These equations are causal and stable

But what are the parameters α_0, α_1, $\beta_0, \beta_1, \beta_2$?

Or how to obtain ζ, κ, η?

$\Longrightarrow$ use kinetic theory

Or some other microscopic theory

more terms. . .

- **Kinetic theory derivation (see Denicol et al., PRD85, 114047 (2012)) or gradient expansion (see Romatschke et al., JHEP 0804, 100 (2008)) lead to even more terms (all possible in second order in products of gradients)**

- **Do not contribute to entropy, may affect the evolution**

- **What is usually solved is**

$$\pi^{\mu\nu} + \tau_\pi \left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\nabla_\alpha u^\alpha \right] = \eta \nabla^{\langle\mu} u^{\nu\rangle}$$

η/s

Ideal:

$$(\epsilon + P)Du^\mu = \nabla^\mu P$$

c.f.

$$ma = F$$

η/s

Ideal:

$$(\epsilon + P) D u^\mu = \nabla^\mu P$$

Viscous:

$$(\epsilon + P) D u^\mu = \nabla^\mu P - \Delta^\mu{}_\alpha \partial_\beta \pi^{\alpha\beta}$$

$$D u^\mu = \frac{1}{\epsilon + P} \nabla^\mu P - \frac{2\eta}{\epsilon + P} \Delta^\mu{}_\alpha \partial_\beta \left[\nabla^{\langle\alpha} u^{\beta\rangle} + \cdots \right] + \cdots$$

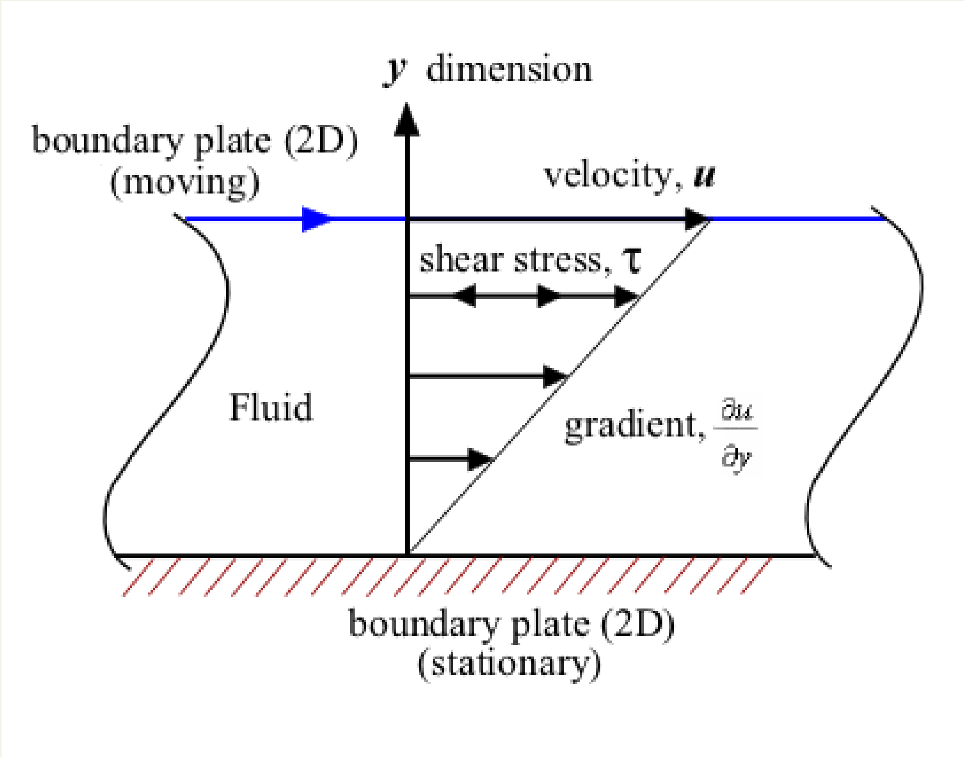
$\mu = 0 \Longrightarrow Ts = \epsilon + P :$

$$D u^\mu = \frac{1}{\epsilon + P} \nabla^\mu P - \frac{2}{T} \frac{\eta}{s} \Delta^\mu{}_\alpha \partial_\beta \left[\nabla^{\langle\alpha} u^{\beta\rangle} + \cdots \right] + \cdots$$

Shear viscosity

Newton:

$$T_{xy} = -\eta \frac{\partial u_x}{\partial y}$$



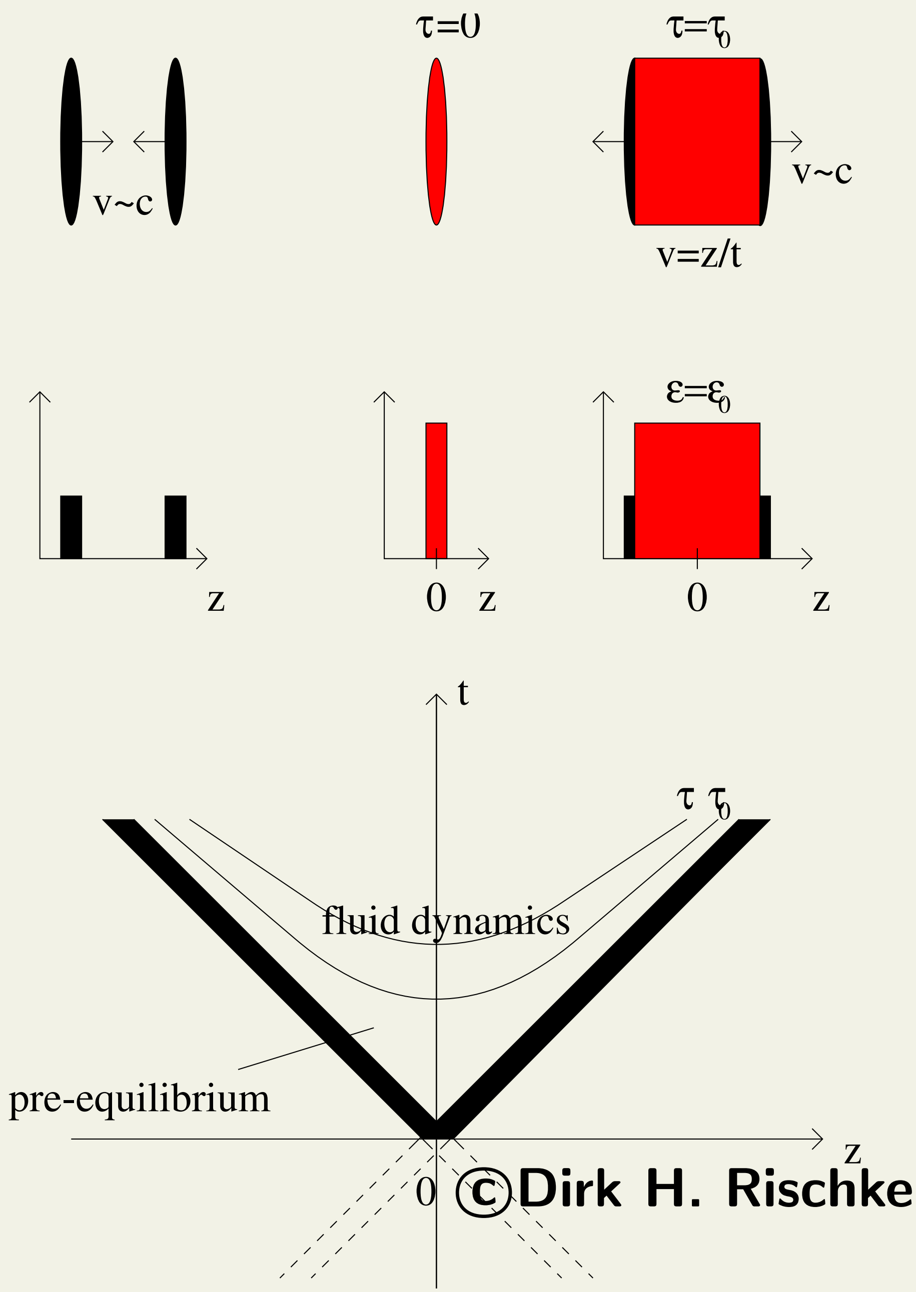
acts to reduce velocity gradients

in closed system: **energy conserved**

kinetic energy gets converted to internal energy

$\Rightarrow$ **dissipation**

Bjorken hydrodynamics



- **At very large energies, $\gamma \to \infty$ and "Landau thickness" $\to 0$**
- **Lack of longitudinal scale**
  $\Rightarrow$ scaling flow

$$v = \frac{z}{t}$$

Shear in 1D-bjorken

Navier-Stokes stress

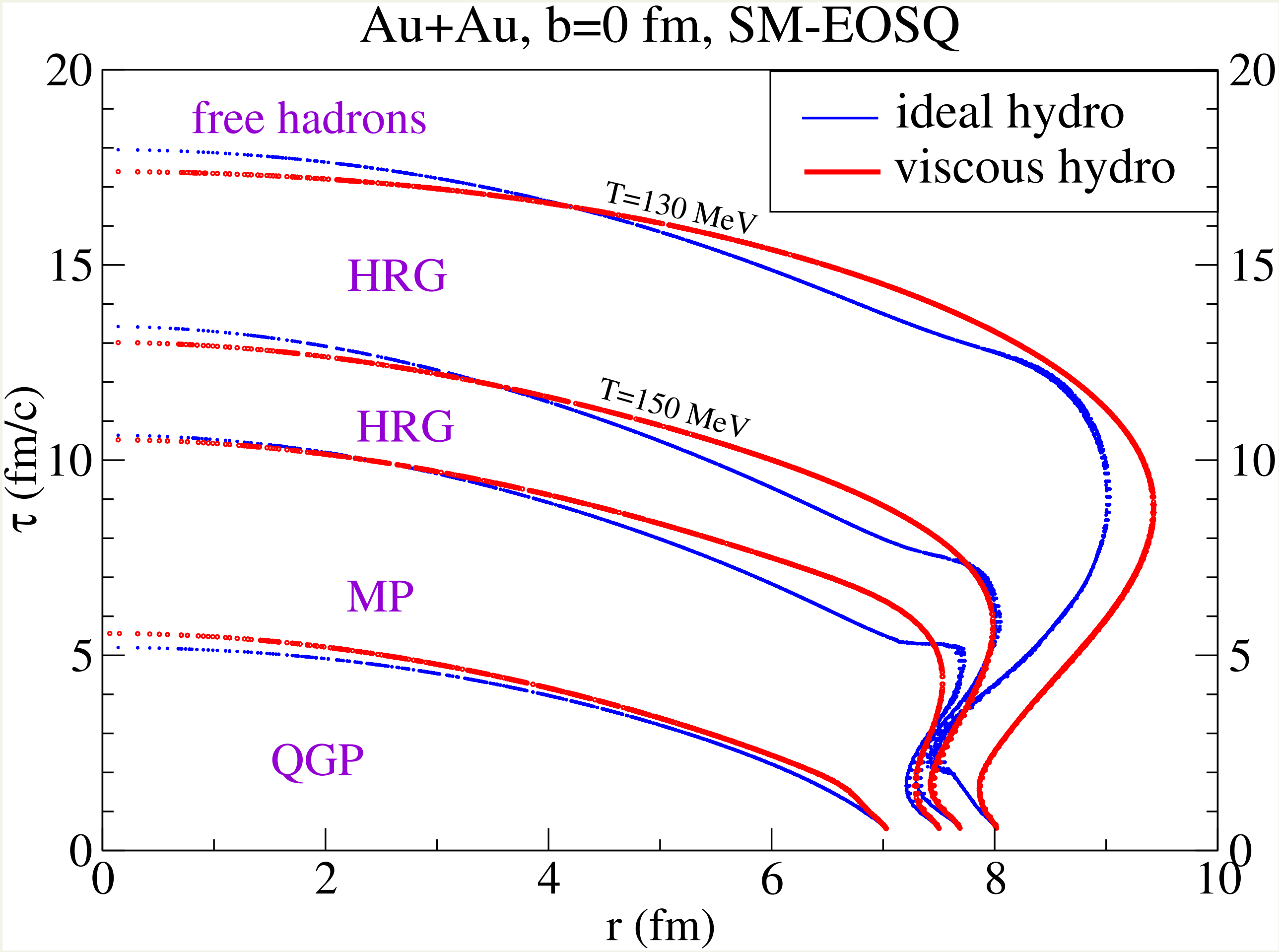
$$\begin{aligned}\pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} &= \mathrm{diag}(0, \frac{2\eta}{3\tau}, \frac{2\eta}{3\tau}, -\frac{4\eta}{3\tau}) \\ T^{\mu\nu} &= \mathrm{diag}(\epsilon, P - \frac{\pi_L}{2}, P - \frac{\pi_L}{2}, P + \pi_L)\end{aligned}$$

where $\pi_L = \pi^{\eta\eta} = -\frac{4\eta}{3\tau}$

Effective longitudinal pressure $P + \pi_L < P$
Effective transverse pressure $P - \pi_L/2 > P$

Shear slows down longitudinal expansion and accelerates transverse expansion

Effect on temperature



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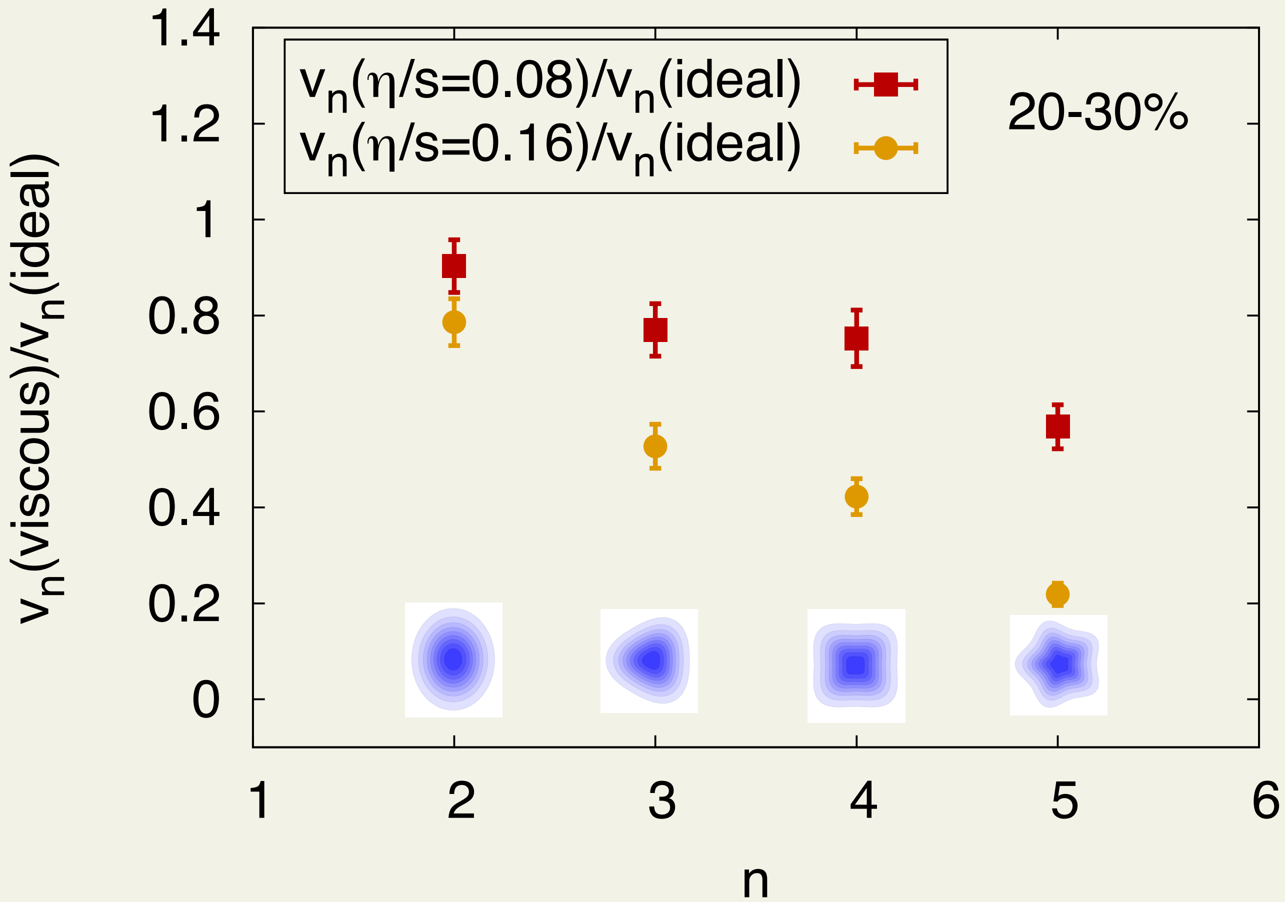
- Edges expand further and stay hotter
- At first core cools slower, later faster

Sensitivity to η/s

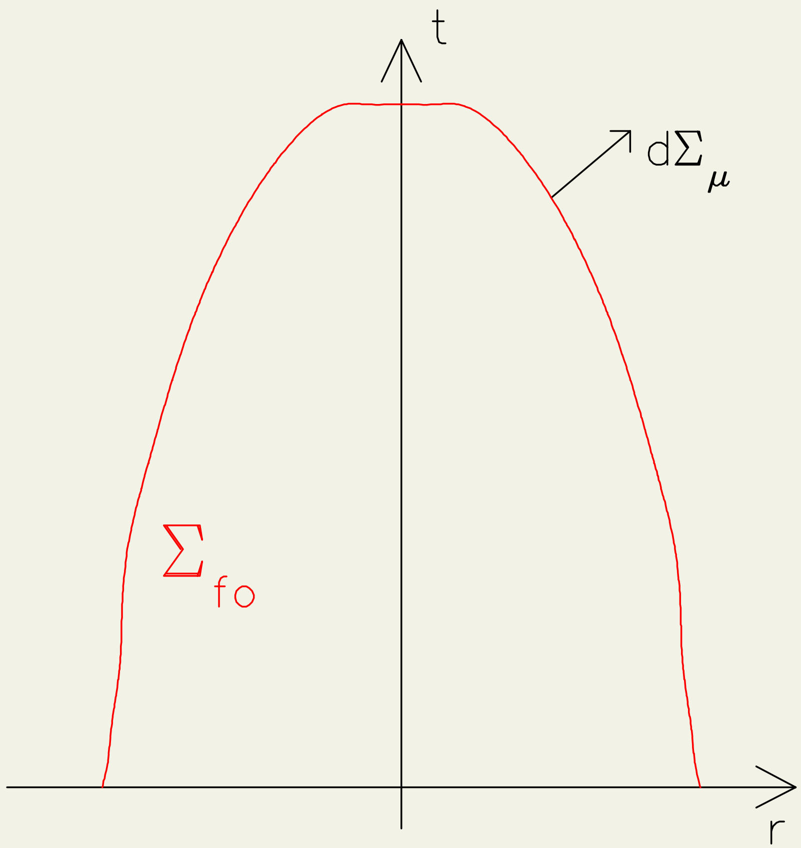
Schenke *et al.* Phys.Rev.C85:024901,2012



- **higher coefficients are suppressed more by dissipation**

When to end?

- How far is hydro valid?
- How and when to convert fluid to particles?



- Kinetic equilibrium requires scattering rate $\gg$ expansion rate
- Scattering rate $\tau_{\rm sc}^{-1} \sim \sigma n \propto \sigma T^3$
- Expansion rate $\theta = \partial_\mu u^\mu$
- Fluid description breaks down when $\tau_{\rm sc}^{-1} \approx \theta$

$\rightarrow$ momentum distributions freeze-out

- $\tau_{\rm sc}^{-1} \propto T^3 \rightarrow$ rapid transition to free streaming
- Approximation: decoupling takes place on constant temperature hypersurface $\Sigma_{\rm fo}$, at $T = T_{\rm fo}$

Cooper-Frye

- **Number of particles emitted = Number of particles crossing** $\Sigma_{\rm fo}$

$$\Rightarrow \quad N = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, N^\mu$$

- **Frozen-out particles do not interact anymore: kinetic theory**

$$\Rightarrow \quad N^\mu = \int \frac{\mathrm{d}^3\mathbf{p}}{E} p^\mu f(x, p\cdot u)$$

$$\Rightarrow \quad N = \int \frac{\mathrm{d}^3\mathbf{p}}{E} \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)$$

- **Invariant single inclusive momentum spectrum: (Cooper-Frye formula)**

$$\boxed{E\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}^3} = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)}$$

Cooper and Frye, PRD 10, 186 (1974)

Freeze-out from viscous fluid

Cooper-Frye still works

$$E\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}^3} = \int_{\Sigma_{\mathrm{fo}}} \mathrm{d}\Sigma_\mu\, p^\mu f(x, p\cdot u) = \int_{\Sigma_{\mathrm{fo}}} \mathrm{d}\Sigma_\mu\, p^\mu f_0[1+\delta f]$$

Grad 14-moment approximation (Boltzmann distribution)

$$\delta f = \varepsilon(x) + \varepsilon_\mu(x) k^\mu + \varepsilon_{\mu\nu} k^\mu k^\nu$$

Shear only, require Landau matching:

$$\epsilon = \int \frac{\mathrm{d}^3\mathbf{p}}{E} u_\mu p^\mu u_\nu p^\nu f(x, p\cdot u) \quad = \quad \int \frac{\mathrm{d}^3\mathbf{p}}{E} u_\mu p^\mu u_\nu p^\nu f_0[1+\delta f] = \epsilon$$

$$n = \int \frac{\mathrm{d}^3\mathbf{p}}{E} u_\mu p^\mu f(x, p\cdot u) \quad = \quad \int \frac{\mathrm{d}^3\mathbf{p}}{E} u_\mu p^\mu f_0[1+\delta f] = n$$

i.e. δf **does** ***not*** **contribute to** ϵ **or** n

Freeze-out from viscous fluid

Cooper-Frye still works

$$E\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}^3} = \int_{\Sigma_{\mathrm{fo}}} \mathrm{d}\Sigma_\mu\, p^\mu f(x, p\cdot u) = \int_{\Sigma_{\mathrm{fo}}} \mathrm{d}\Sigma_\mu\, p^\mu f_0[1+\delta f]$$

Grad 14-moment approximation (Boltzmann distribution)

$$\delta f = \varepsilon(x) + \varepsilon_\mu(x)k^\mu + \varepsilon_{\mu\nu}k^\mu k^\nu$$

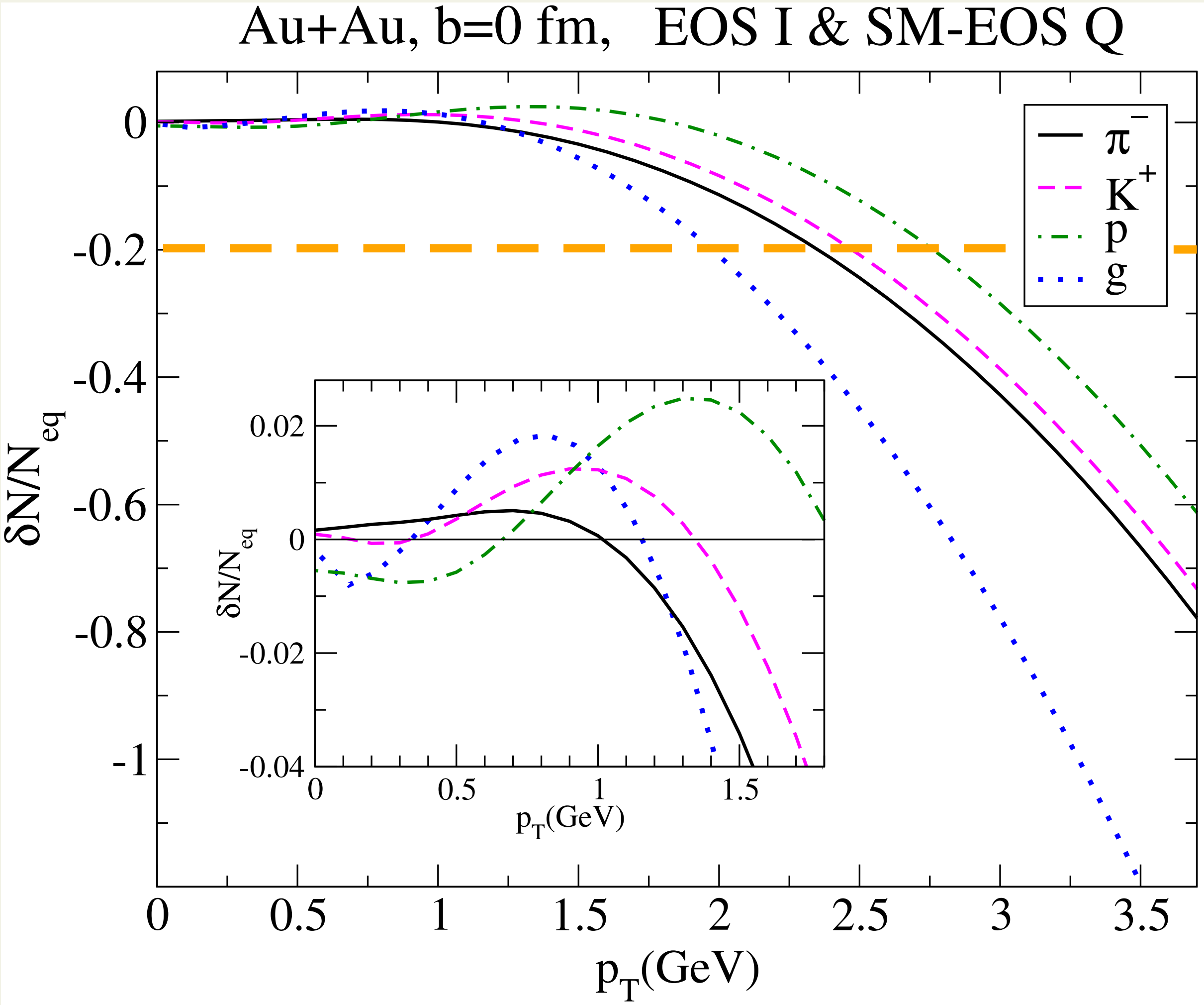
Shear only, Landau matching gives

$$\delta f = \varepsilon_{\mu\nu}k^\mu k^\nu = \frac{1}{2T^2(\epsilon+P)}\pi^{\mu\nu}k_\mu k_\nu$$

Thus, even if velocity and temperature are the same, finite shear causes different particle distributions

How to share $\pi^{\mu\nu}$ for each particle species?

Region of validity



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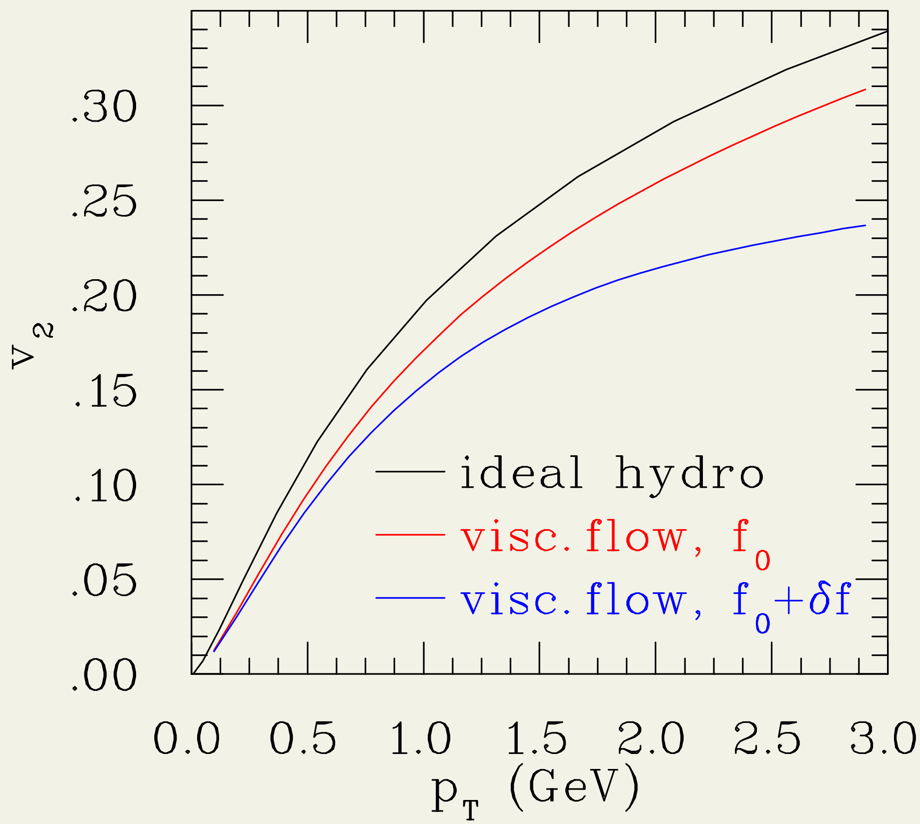
Corrections to thermal distributions "uncomfortably large" when $p_T \gtrsim 2$ **GeV**

$\delta f \propto p^2$

Effect on v_2



- **massless particles**
- **Note: both change in flow and distributions affect v_2**

A set of partial differential equations. . .

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We need

- **Boundary conditions**

A set of partial differential equations. . .

We need

- **Boundary conditions**
  - **Initial state $\longrightarrow$ Bjoern**
  - **Final state $\longrightarrow$ Piotr**

A set of partial differential equations...

We need

- **Boundary conditions**
  - **Initial state $\longrightarrow$ Bjoern**
  - **Final state $\longrightarrow$ Piotr**
- **Equation of state $\longrightarrow$ my next talk**
- **Transport coefficients $\longrightarrow$ Piotr**