# Equation of state

**Pasi Huovinen**
**Uniwersytet Wrocławski**

**Collective Flows and Hydrodynamics in High Energy Nuclear Collisions**

**Dec 14, 2016, University of Science and Technology of China, Hefei, China**

The speaker has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 665778 via the National Science Center, Poland, under grant Polonez DEC-2015/19/P/ST2/03333

**Equation of state in form**

$$P = P(\epsilon, n)$$

**needed to close the system of hydrodynamic equations**

**Remark:** $P = P(\epsilon, n)$ **is not a complete equation of state**
**in a thermodynamical sense.**

**A complete equation of state allows to compute all thermodynamic variables.**

**For example,** $s = s(\epsilon, n)$**:** $\mathrm{d}s = 1/T\mathrm{d}\epsilon - \mu/T\mathrm{d}n$ **(1st law of thermod.)**

$$\frac{1}{T} = \frac{\partial s}{\partial \epsilon}|_n, \qquad \frac{\mu}{T} = -\frac{\partial s}{\partial n}|_\epsilon, \qquad P = Ts + \mu n - \epsilon$$

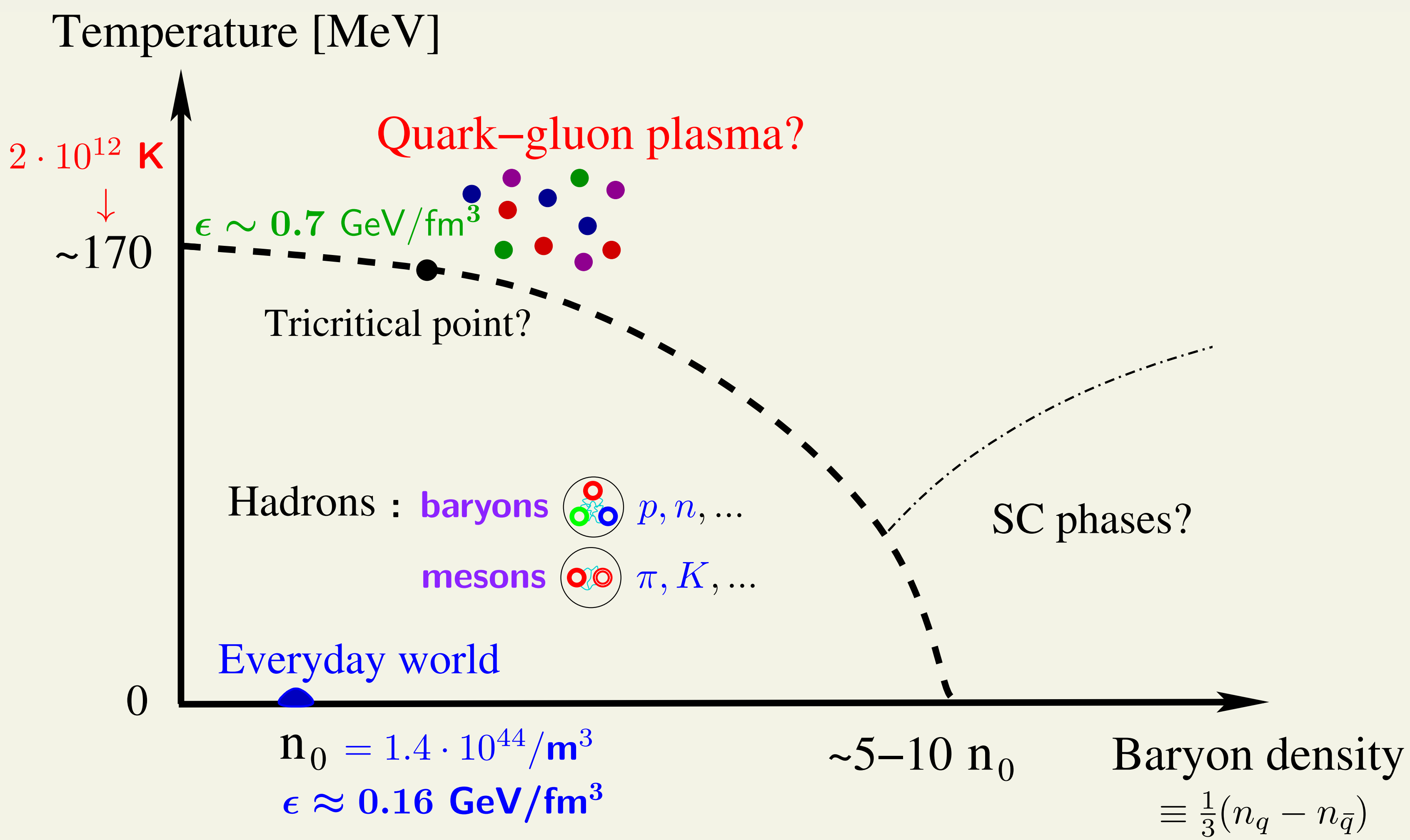
$P = P(\epsilon, n)$ **does not work!**

$$\frac{\partial P}{\partial \epsilon}|_n =? \qquad \frac{\partial P}{\partial n}|_\epsilon =?$$

**However,** $P = P(T, \mu)$ **does work!**

$$\mathrm{d}P = s\mathrm{d}T + n\mathrm{d}\mu \quad \Rightarrow \quad s = \frac{\partial P}{\partial T}|_\mu, \qquad n = \frac{\partial P}{\partial \mu}|_T$$

# Nuclear phase diagram



- **0th approximation for equation of state at $\mu_B = n_B = 0$:**
  - **Hadronic phase: ideal gas of massless (boltzmann) pions, $g_\pi = 3$**

$$\begin{aligned} \epsilon_\pi &= \frac{3g_\pi}{\pi^2} T^4 \\ P_\pi &= \frac{g_\pi}{\pi^2} T^4 = \frac{1}{3}\epsilon_\pi \end{aligned}$$

  - **Partonic phase: ideal gas of partons**
    **+ bag constant of the bag model, $B$**

$$\begin{aligned} \epsilon_{QGP} &= \frac{3g_{QGP}}{\pi^2} T^4 + B \\ P_{QGP} &= \frac{g_{QGP}}{\pi^2} T^4 - B = \frac{1}{3}\epsilon_{QGP} - \frac{4}{3}B \end{aligned}$$

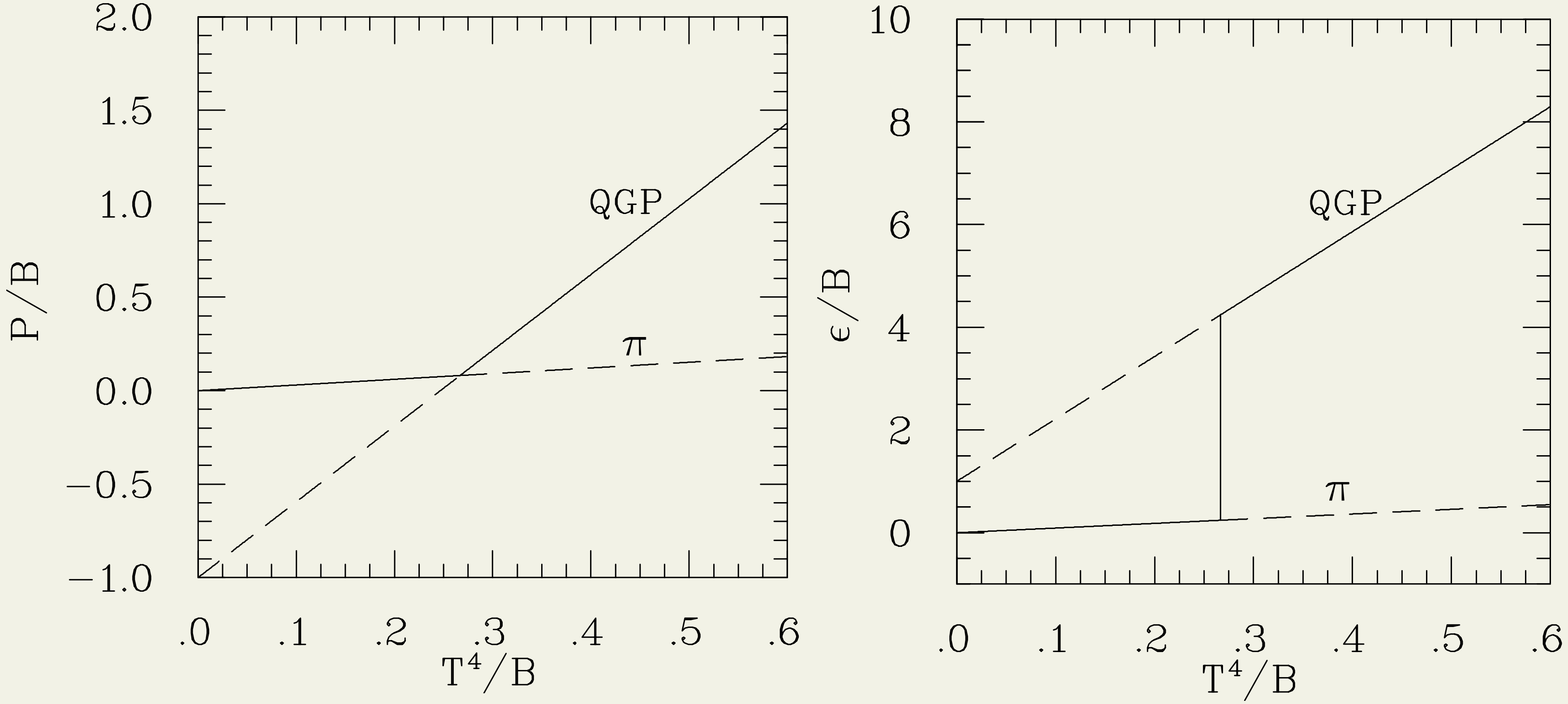
  - **Number of DOF in partonic phase:**
    **2 quark flavours and gluons $\Rightarrow g_{QGP} = 40$**

– **Gibbs criterion:** $P_{QGP}(T_c) = P_\pi(T_c)$

$$\Rightarrow T_c = \left(\frac{\pi^2}{g_{QGP} - g_\pi}\right)^{\frac{1}{4}} B^{\frac{1}{4}}$$

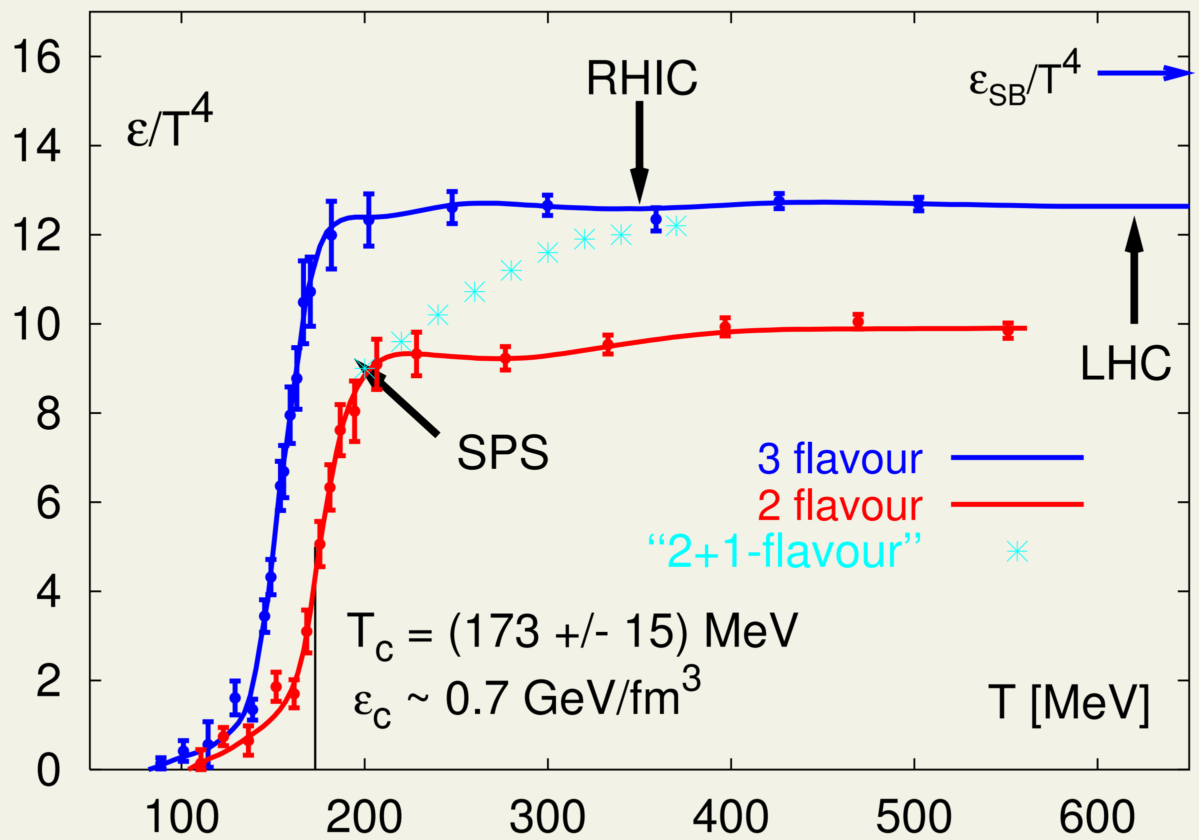
– **First order phase transition:**

**Pressure:** **Energy density:**



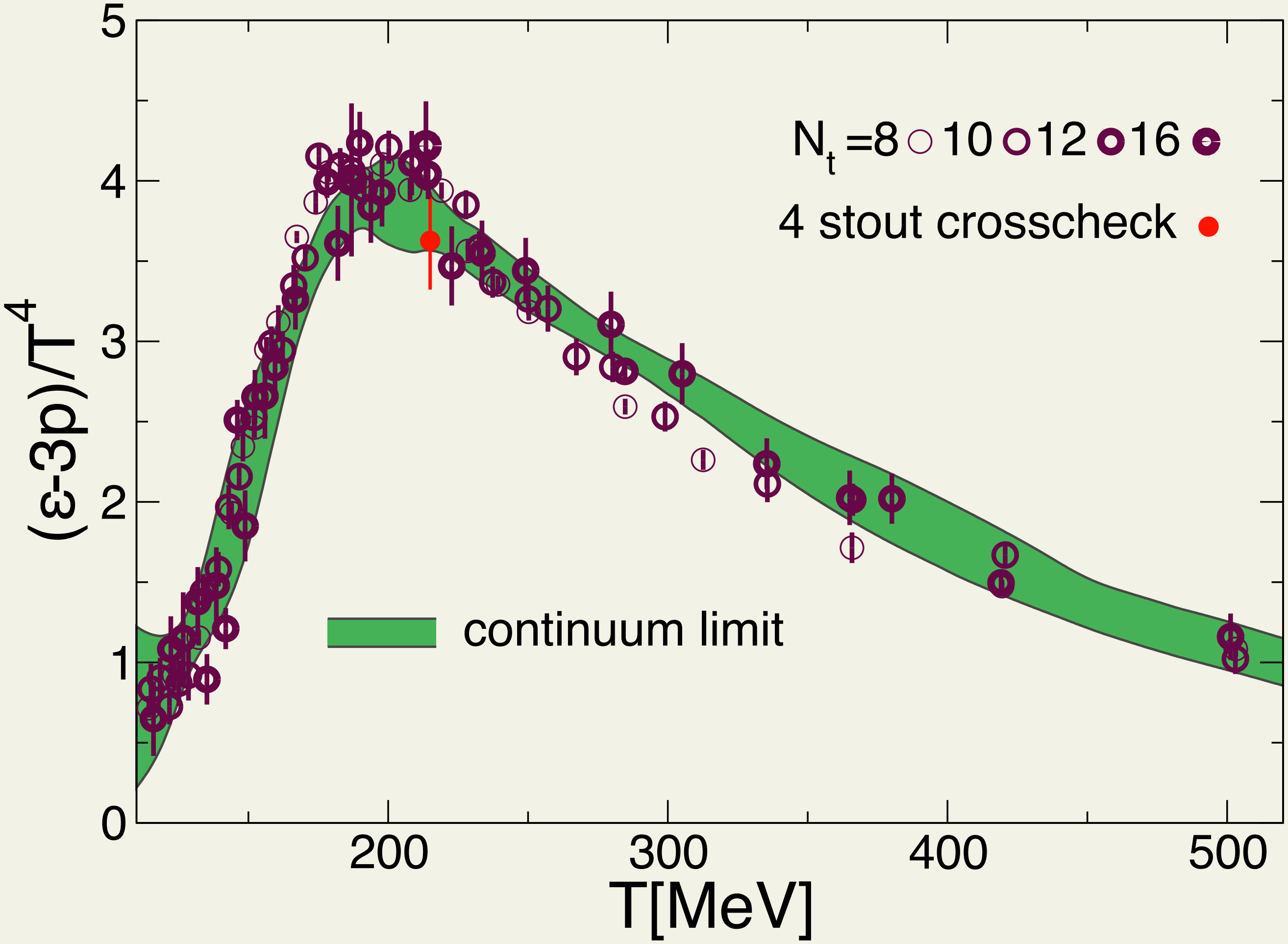
# QCD equation of state

**lattice QCD** **(Karsch & Laermann, hep-lat/0305025):**




- **EoS from first principles**

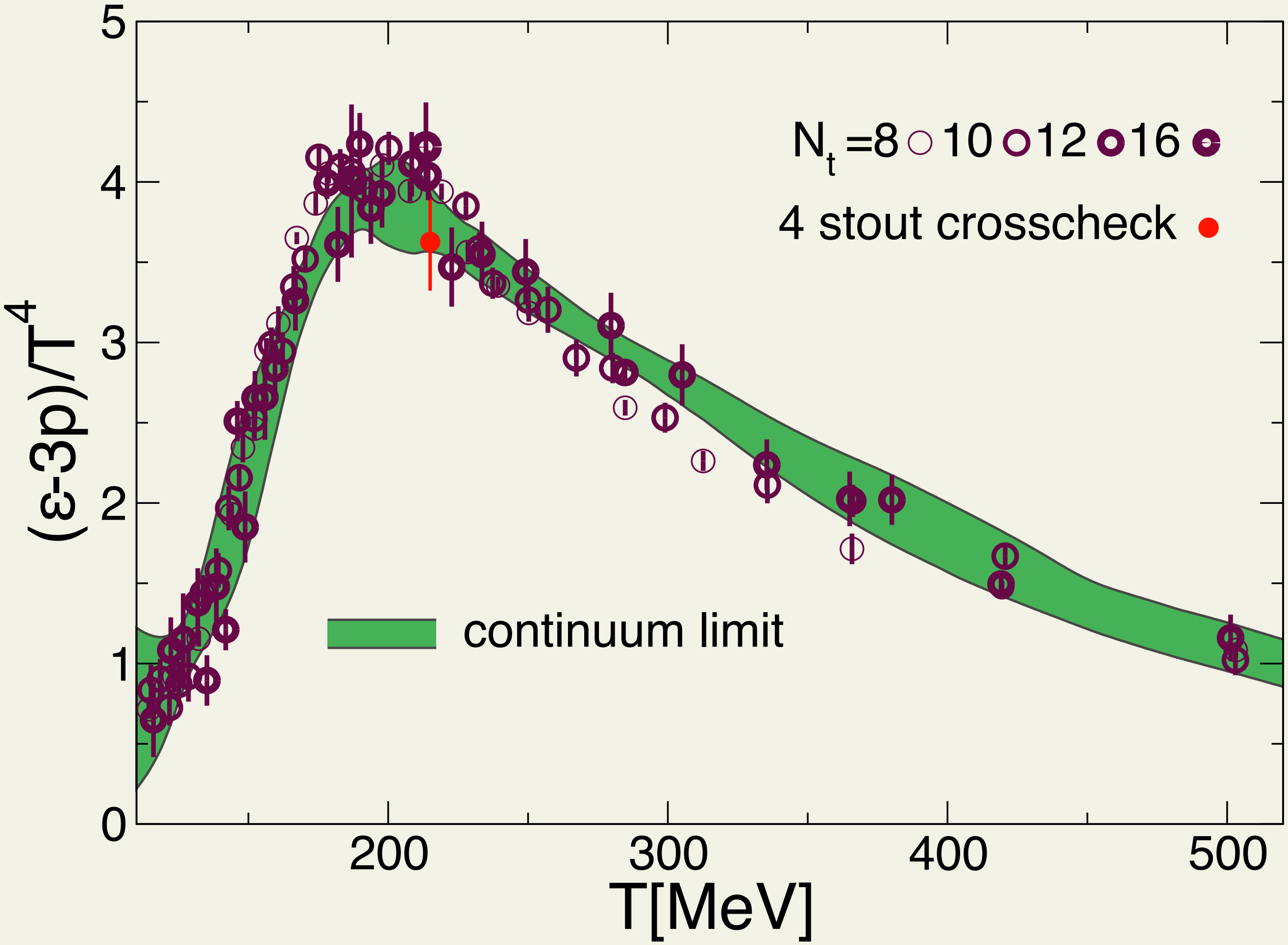
## lattice QCD: Budapest-Wuppertal collaboration, arXiv:1309.5258:




- **Trace anomaly**

$$\frac{\epsilon - 3P}{T^4}$$

# lattice QCD: Budapest-Wuppertal collaboration, arXiv:1309.5258:

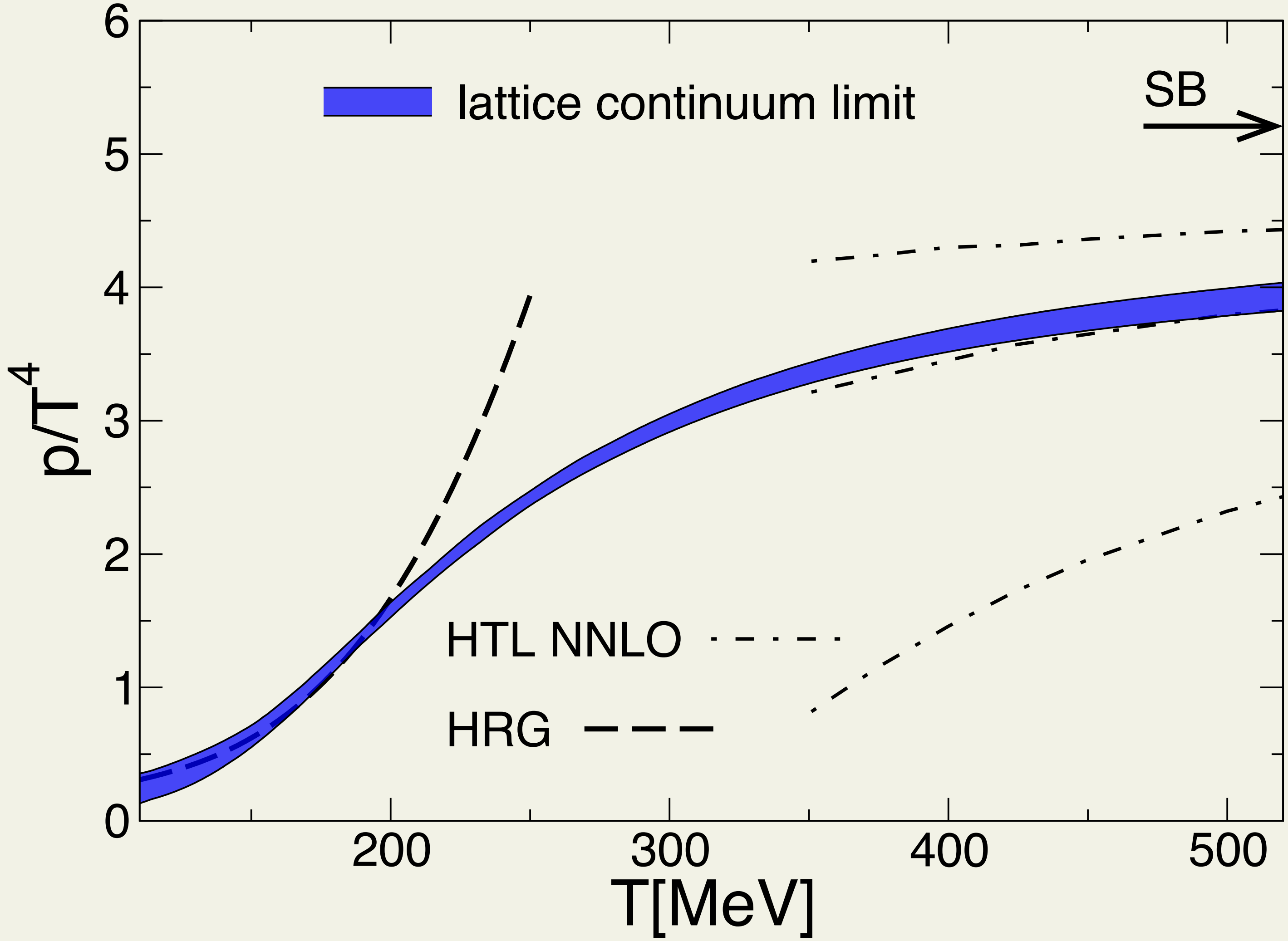



- **obtain pressure via**

$$\frac{P}{T^4} - \frac{P_0}{T_0^4} = \int_{T_0}^{T} \mathrm{d}T' \, \frac{\epsilon - 3P}{T'^5}$$

- **What is $P(T_0)$?**

# Lattice vs. HRG




- **Lattice agrees with Hadron Resonance Gas at low T**

# Hadron Resonance Gas model

- **Dashen-Ma-Bernstein:** Phys. Rev. 187, 345 (1969)

  EoS of interacting hadron gas well approximated
  by non-interacting gas of hadrons and resonances

$$P(T,\mu) = \sum_i \frac{\pm g_i}{(2\pi)^3} T \int \mathrm{d}^3p \,\ln\left(1 \pm e^{-\frac{\sqrt{p^2+m^2}-\mu_i}{T}}\right)$$

- valid when

- interactions mediated by resonances
- resonances have zero width

- Prakash & Venugopalan, NPA546, 718 (1992): experimental phase shifts
- Gerber & Leutwyler, NPB321, 387 (1989): chiral perturbation theory

$\Rightarrow$ HRG good approximation at low temperatures

$\rightarrow$ lattice should reproduce HRG at $T \leq 120 - 140$ MeV

$\rightarrow$ and it does

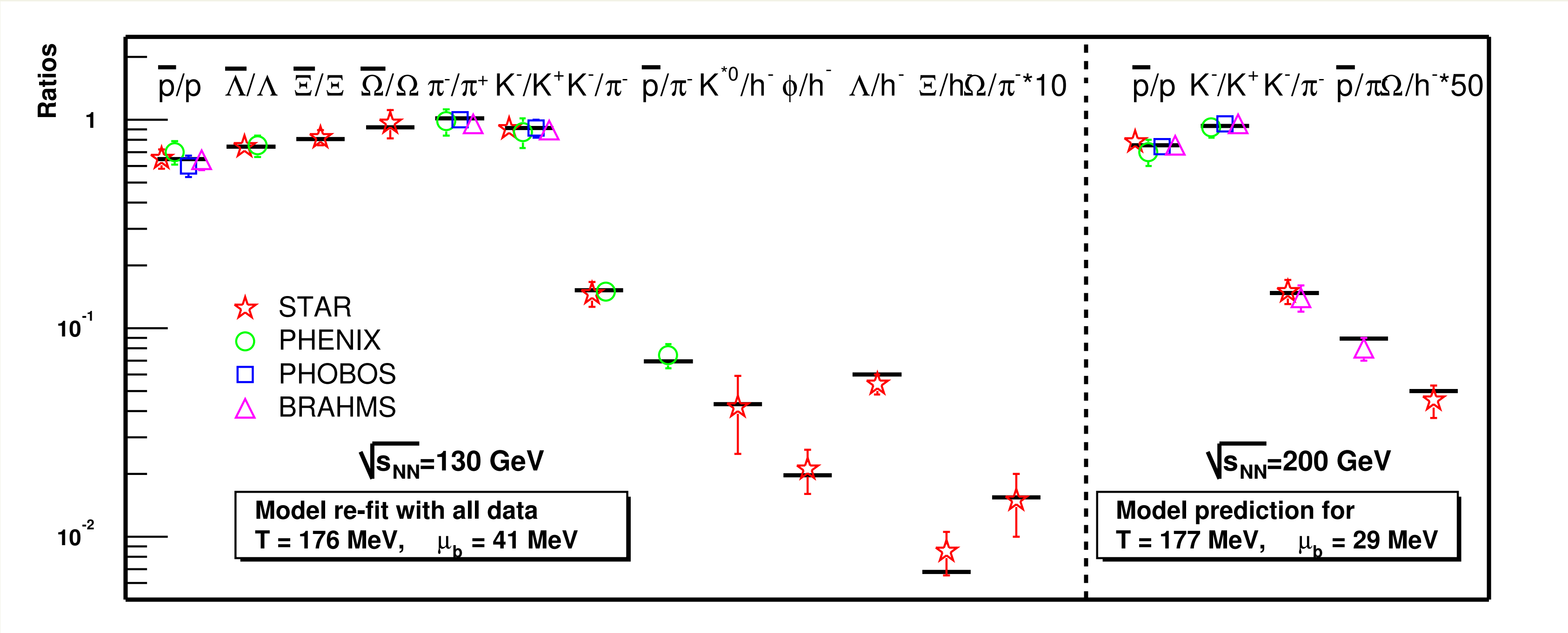
# End of evolution I

- **when fluid dynamical description breaks down, so-called freeze-out**

$\rightarrow$ **convert fluid to particles**

- **energy conservation iff EoS is the same before and after freeze-out**
- **in HRG this is by definition true**

# End of evolution II




- **Particle ratios** $\Longleftrightarrow$ $T \approx 160\text{–}170$ **MeV temperature**
- $p_T$**-distibutions** $\Longleftrightarrow$ $T \approx 100\text{–}140$ **MeV temperature**

$\Rightarrow$ **Evolution out of chemical equilibrium**

# Chemical non-equilibrium

- **Treat number of pions, kaons etc. as conserved quantum numbers below $T_{ch}$ (Bebie et al, Nucl.Phys.B378:95-130,1992)**

- **number of pions: thermal pions + everything from decays:**

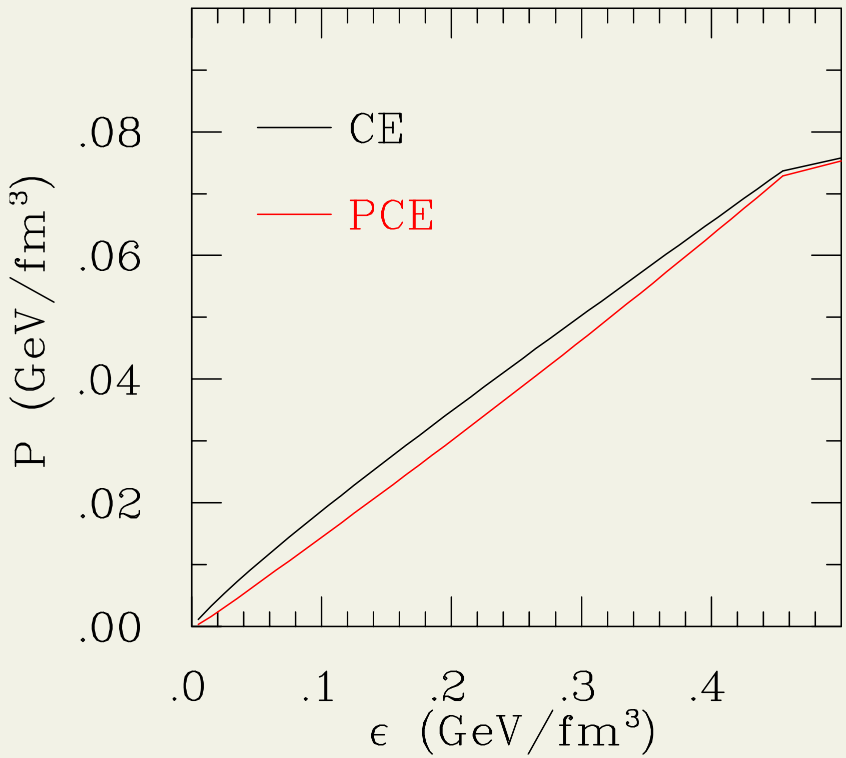
$$\hat{n}_\pi = n_\pi + 2n_\rho + n_\Delta + \cdots$$

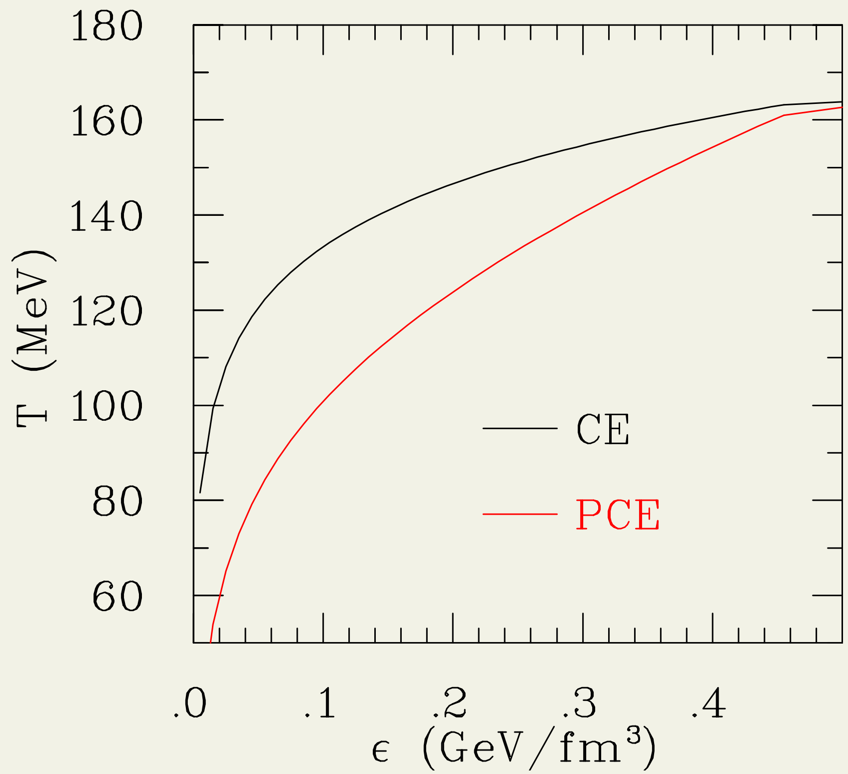
- **entropy per "pion", "kaon" etc. conserved**

$$\begin{aligned} \frac{\hat{n}_\pi}{s}(T,\{\mu_i\}) &= \frac{\hat{n}_\pi}{s}(T_{ch},0) \\ \frac{\hat{n}_K}{s}(T,\{\mu_i\}) &= \frac{\hat{n}_K}{s}(T_{ch},0) \\ &\vdots \end{aligned}$$

# Chemical non-equilibrium

- **Treat number of pions, kaons etc. as conserved quantum numbers below $T_{ch}$ (Bebie et al, Nucl.Phys.B378:95-130,1992)**
- $P = P(\epsilon, n_b)$ **changes very little,** **but** $T = T(\epsilon, n_b)$ **changes. . .**

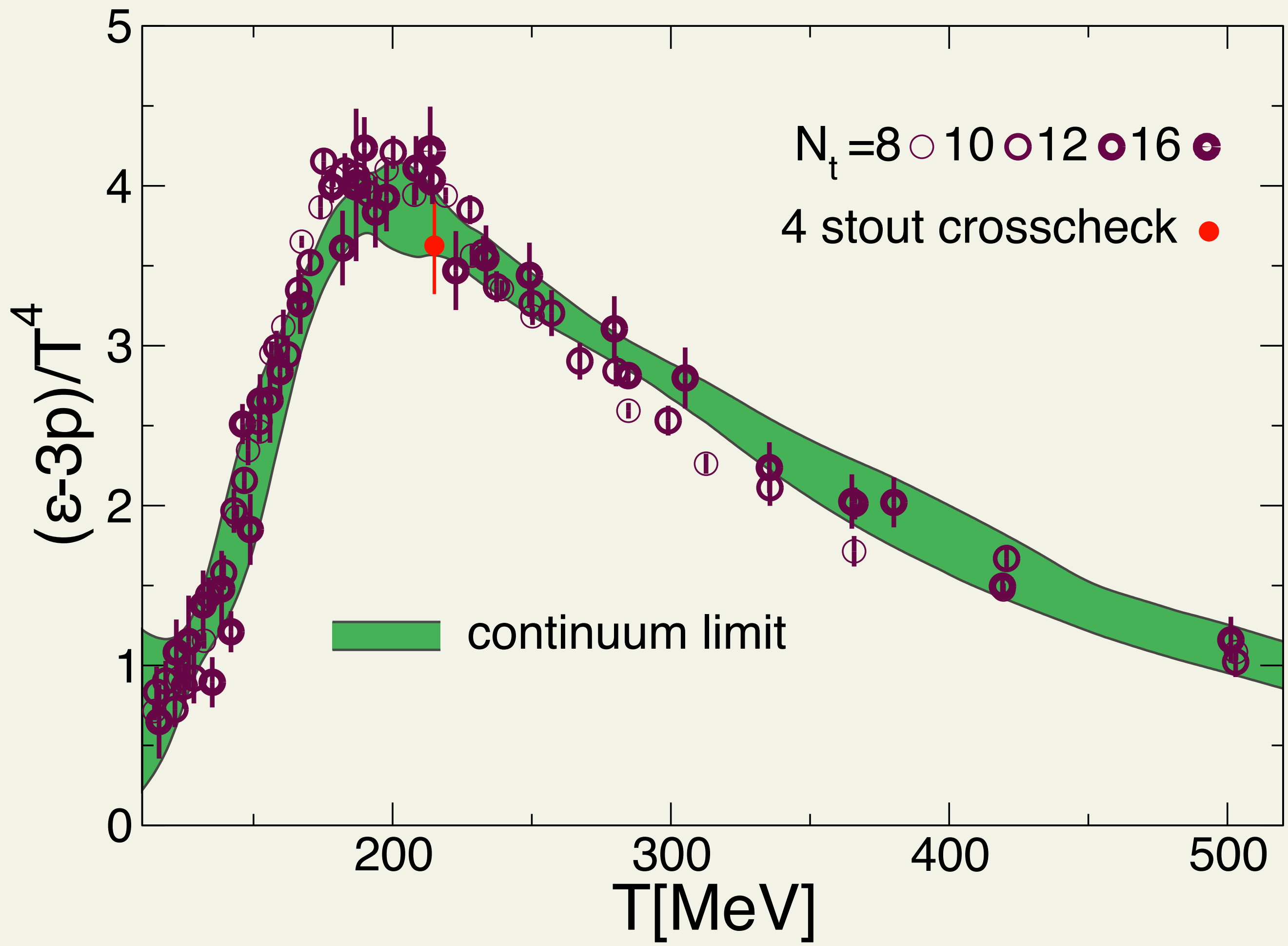


# Procedure for EoS

- **HRG below $T \approx 160 - 170$ MeV**
- **Parametrize lattice using:**

$$\frac{\epsilon - 3P}{T^4} = \frac{d_2}{T^2} + \frac{d_4}{T^4} + \frac{c_1}{T^{n_1}} + \frac{c_2}{T^{n_2}}$$



# Procedure for EoS

- **HRG below $T \approx 160 - 170$ MeV**
- **Parametrize lattice using:**

$$\frac{\epsilon - 3P}{T^4} = \frac{d_2}{T^2} + \frac{d_4}{T^4} + \frac{c_1}{T^{n_1}} + \frac{c_2}{T^{n_2}}$$

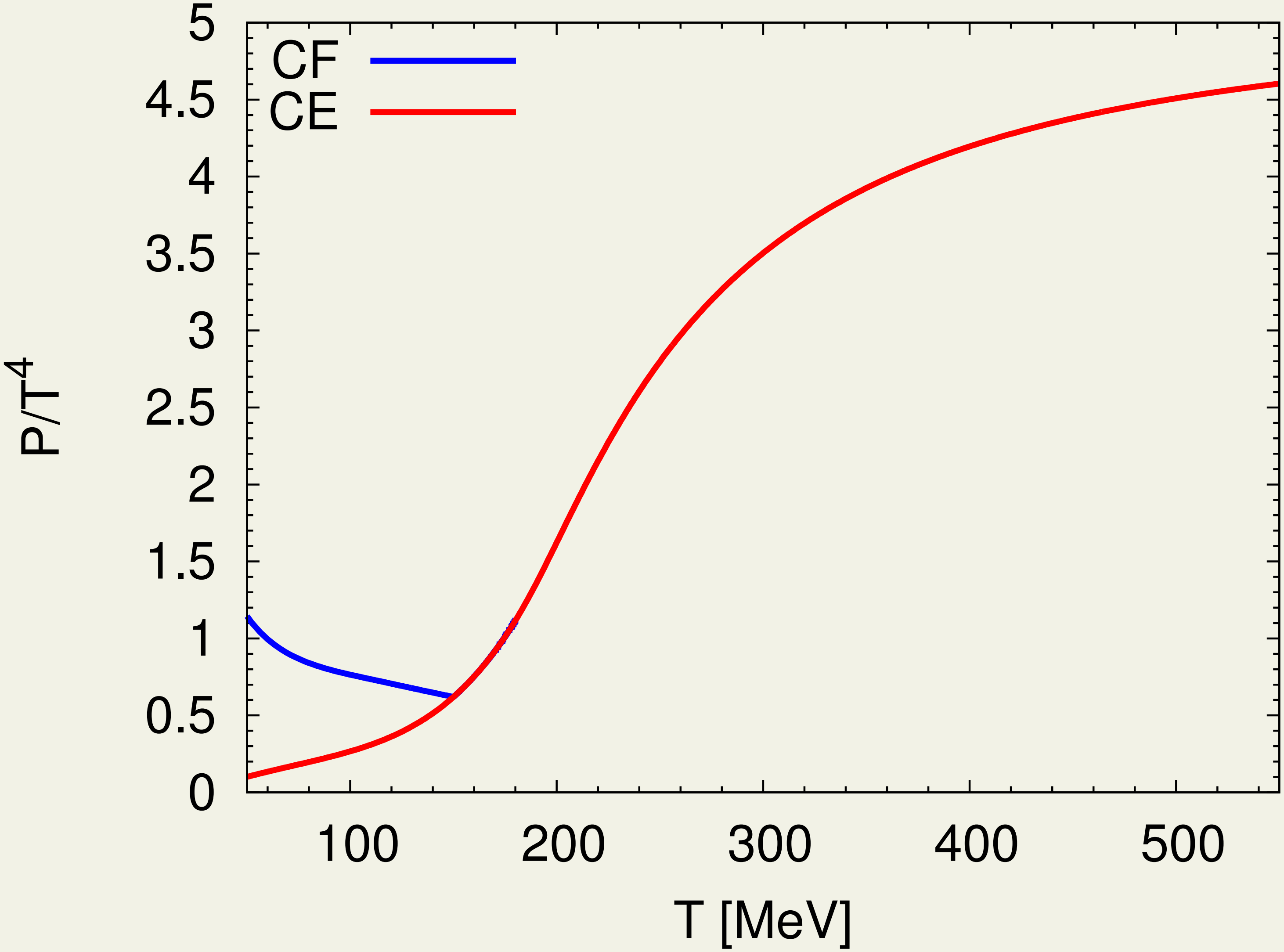
- **Require that:**

$$\left.\frac{\epsilon - 3P}{T^4}\right|_{T_0}, \quad \left.\frac{\mathrm{d}}{\mathrm{d}T}\frac{\epsilon - 3P}{T^4}\right|_{T_0}, \quad \left.\frac{\mathrm{d}^2}{\mathrm{d}T^2}\frac{\epsilon - 3P}{T^4}\right|_{T_0} \qquad \text{are continuous}$$

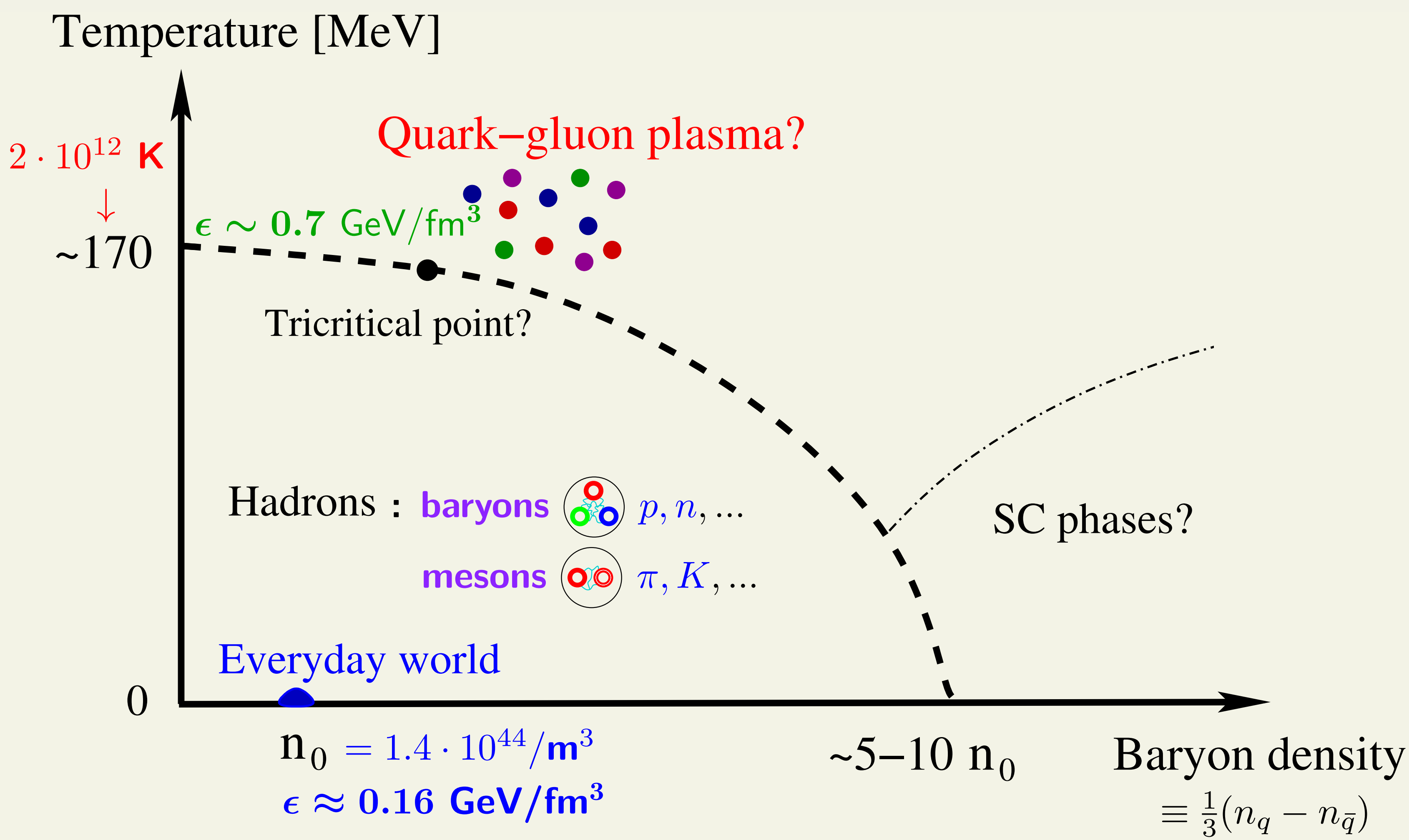
$\Longrightarrow T_0$, $c_1$, $c_2$ **fixed**

- $\chi^2$ **fit to lattice above $T_0$ MeV**

# Final result, $P/T^4$



# Nuclear phase diagram



# Taylor expansion for pressure

$$\frac{P}{T^4} = \Sigma_{i,j}\, c_{ij}(T) \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_S}{T}\right)^j ,$$

**where**

$$c_{ij}(T) = \frac{1}{i!j!} \frac{\partial^i}{\partial(\mu_B/T)^i} \frac{\partial^j}{\partial(\mu_S/T)^j} \frac{P}{T^4},$$

**i.e. moments of baryon number and strangeness fluctuations and correlations**

- **an EoS based on lattice calculations of these?**

**But: Only limited set extrapolated to continuum**

# Parametrization

$$c_{ij}(T) = \frac{a_{1ij}}{\hat{T}^{n_{1ij}}} + \frac{a_{2ij}}{\hat{T}^{n_{2ij}}} + \frac{a_{3ij}}{\hat{T}^{n_{3ij}}} + \frac{a_{4ij}}{\hat{T}^{n_{4ij}}} + \frac{a_{5ij}}{\hat{T}^{n_{5ij}}} + \frac{a_{6ij}}{\hat{T}^{n_{6ij}}} + c_{ij}^{SB},$$

**where** $n_{kij}$ **are integers with** $1 < n_{kij} < 42$**,**
**and**

$$\hat{T} = \frac{T - T_s}{R},$$

**with** $T_s = 0.1$ **or** $0$ **GeV, and** $R = 0.05$ **or** $0.15$ **GeV.**

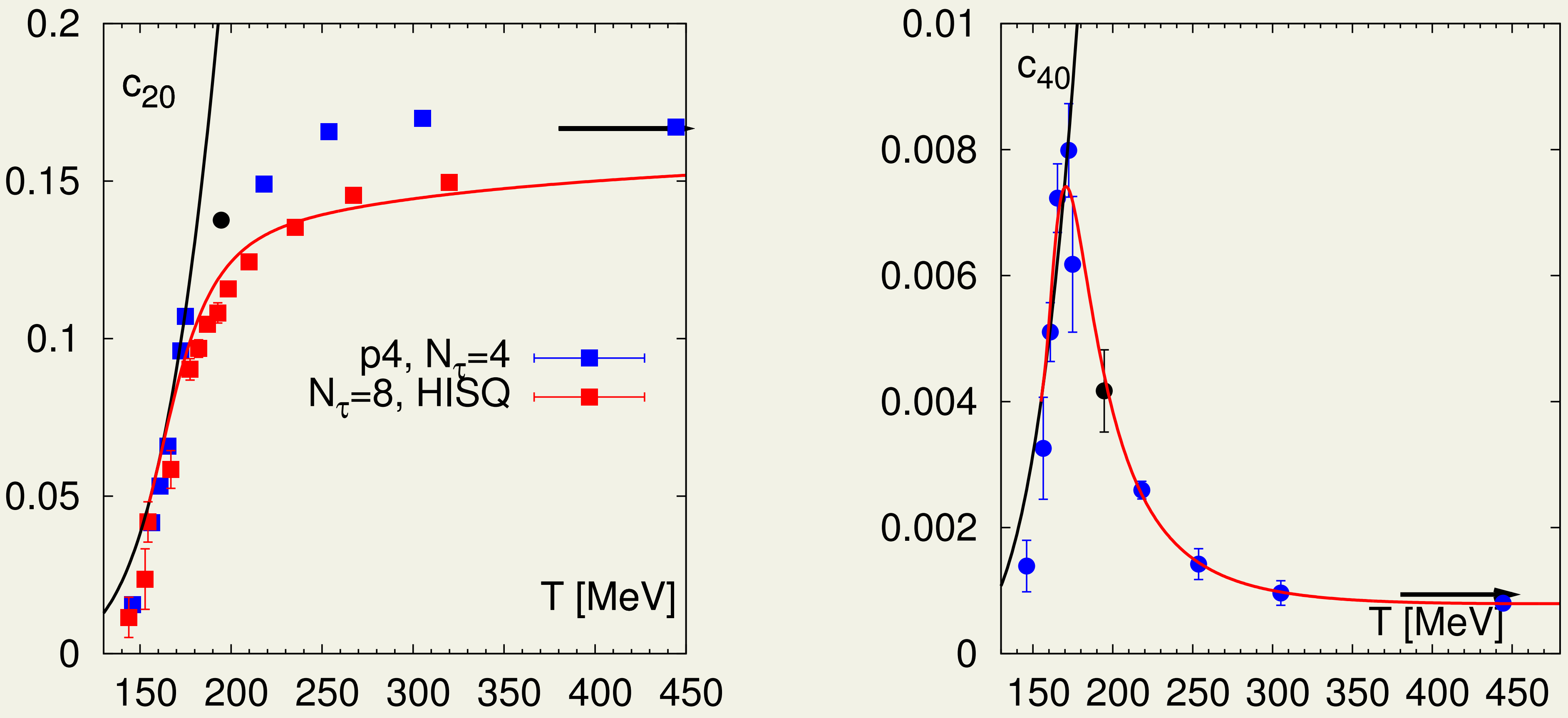
## Constraints:

$$
\begin{aligned}
c_{ij}(T_{\mathrm{sw}}) &= c_{ij}^{\mathrm{HRG}}(T_{\mathrm{sw}}) \\
\frac{\mathrm{d}}{\mathrm{d}T}\, c_{ij}(T_{\mathrm{sw}}) &= \frac{\mathrm{d}}{\mathrm{d}T}\, c_{ij}^{\mathrm{HRG}}(T_{\mathrm{sw}}) \\
\frac{\mathrm{d}^2}{\mathrm{d}T^2}\, c_{ij}(T_{\mathrm{sw}}) &= \frac{\mathrm{d}^2}{\mathrm{d}T^2}\, c_{ij}^{\mathrm{HRG}}(T_{\mathrm{sw}}) \\
\frac{\mathrm{d}^3}{\mathrm{d}T^3}\, c_{ij}(T_{\mathrm{sw}}) &= \frac{\mathrm{d}^3}{\mathrm{d}T^3}\, c_{ij}^{\mathrm{HRG}}(T_{\mathrm{sw}})
\end{aligned}
$$

**at $T_{\mathrm{sw}} = 155$ MeV**

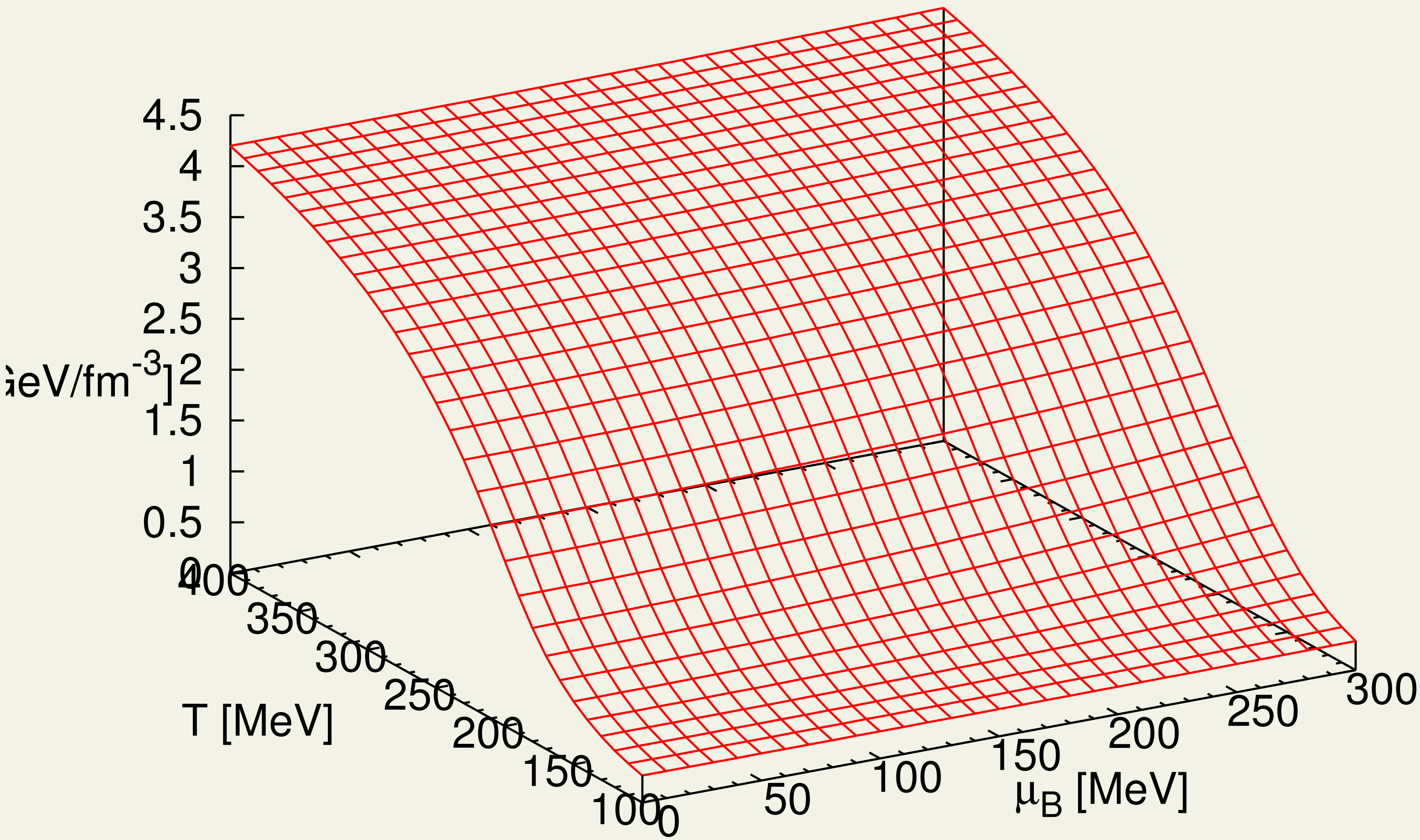
**3rd derivative to quarantee smooth behaviour of speed of sound:**

$$
c_s^2 \propto \frac{\mathrm{d}^2}{\mathrm{d}T^2}\, c_{ij}
$$



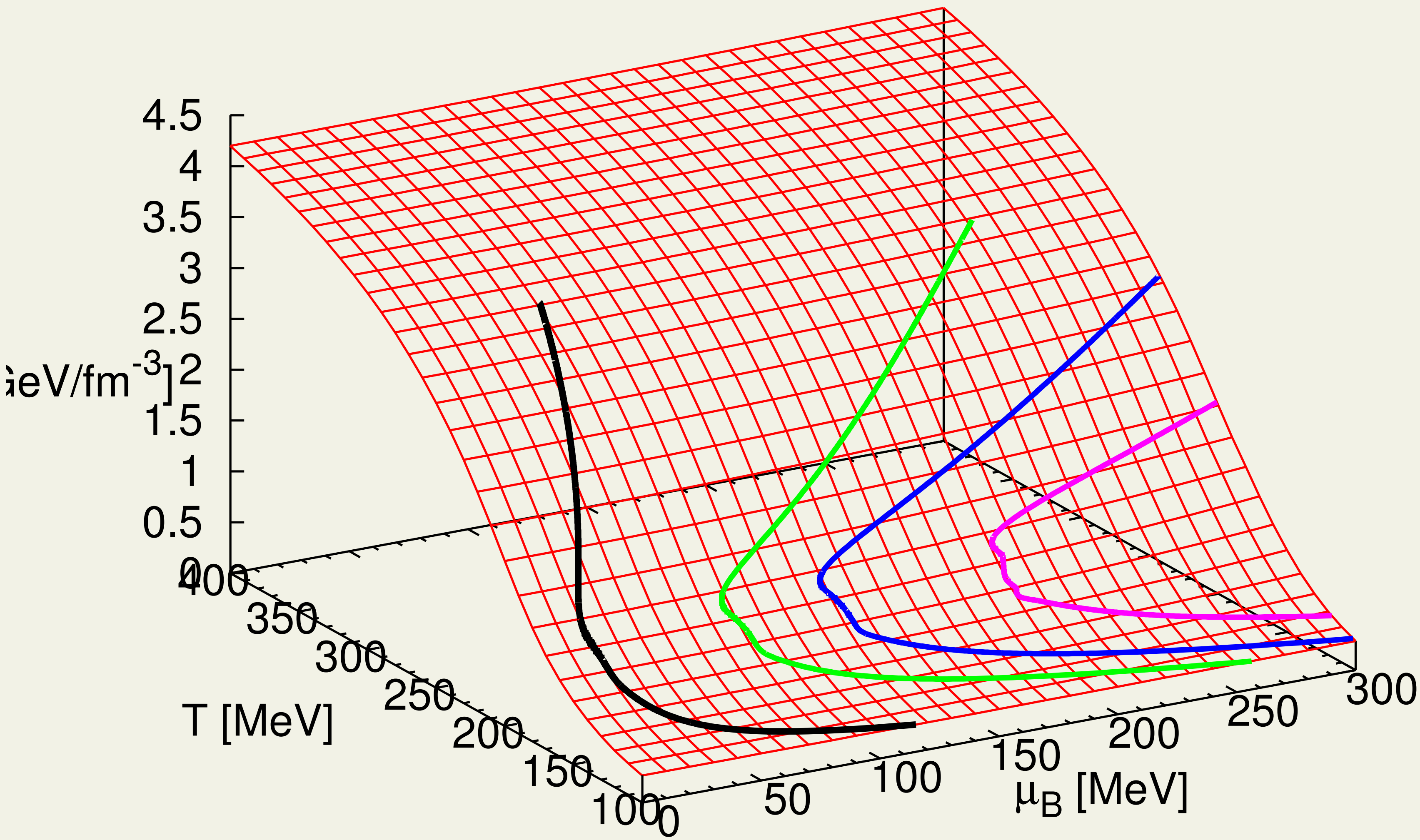
c20
p4, Nτ=4
Nτ=8, HISQ
T [MeV]
c40
T [MeV]

# $P/T^4$

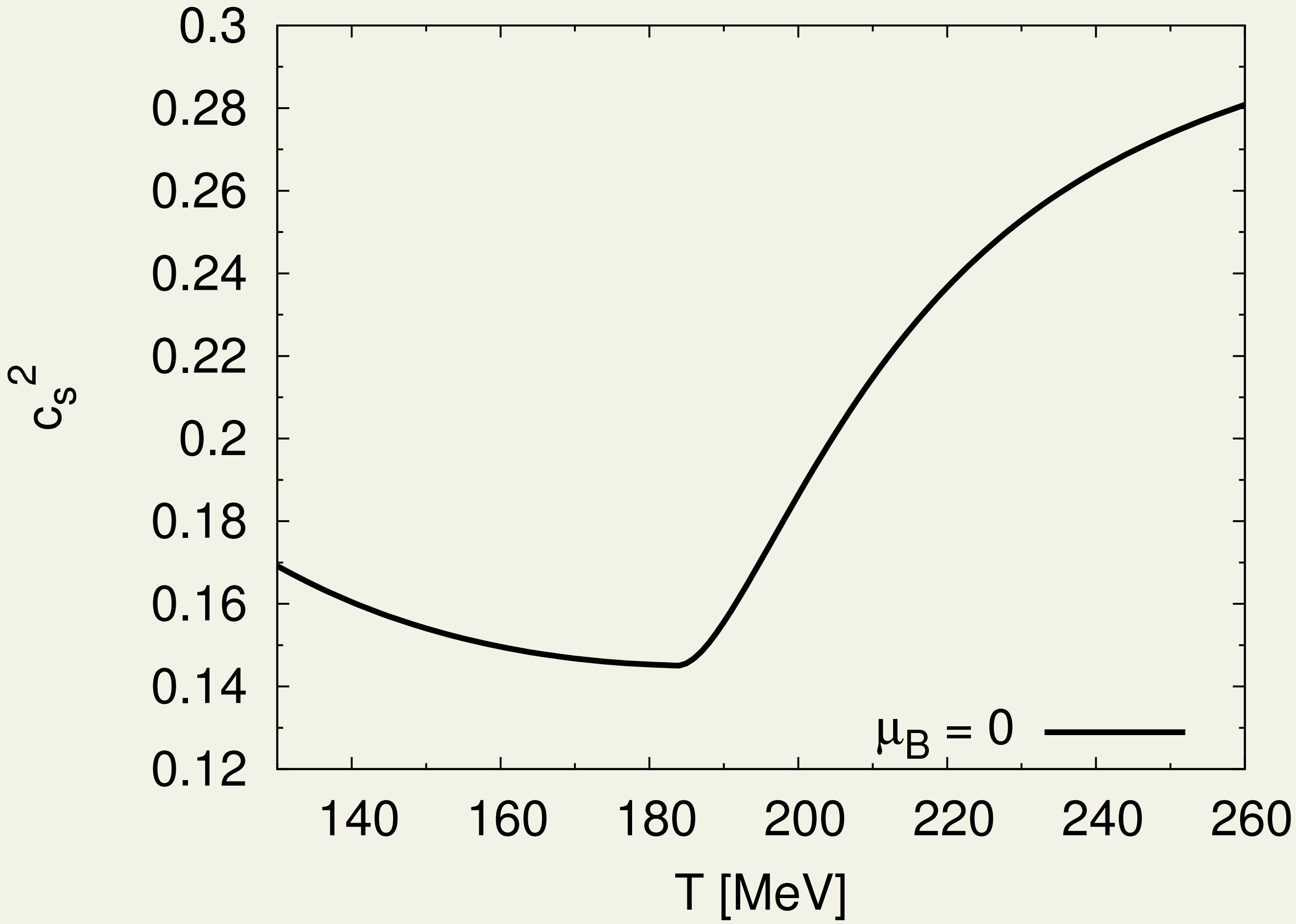


4.5
4
3.5
3
2.5
GeV/fm-3]2
1.5
1
0.5
0
400
350
300
250
200
150
100
T [MeV]
0
50
100
150
200
250
300
μB [MeV]

# $P/T^4$

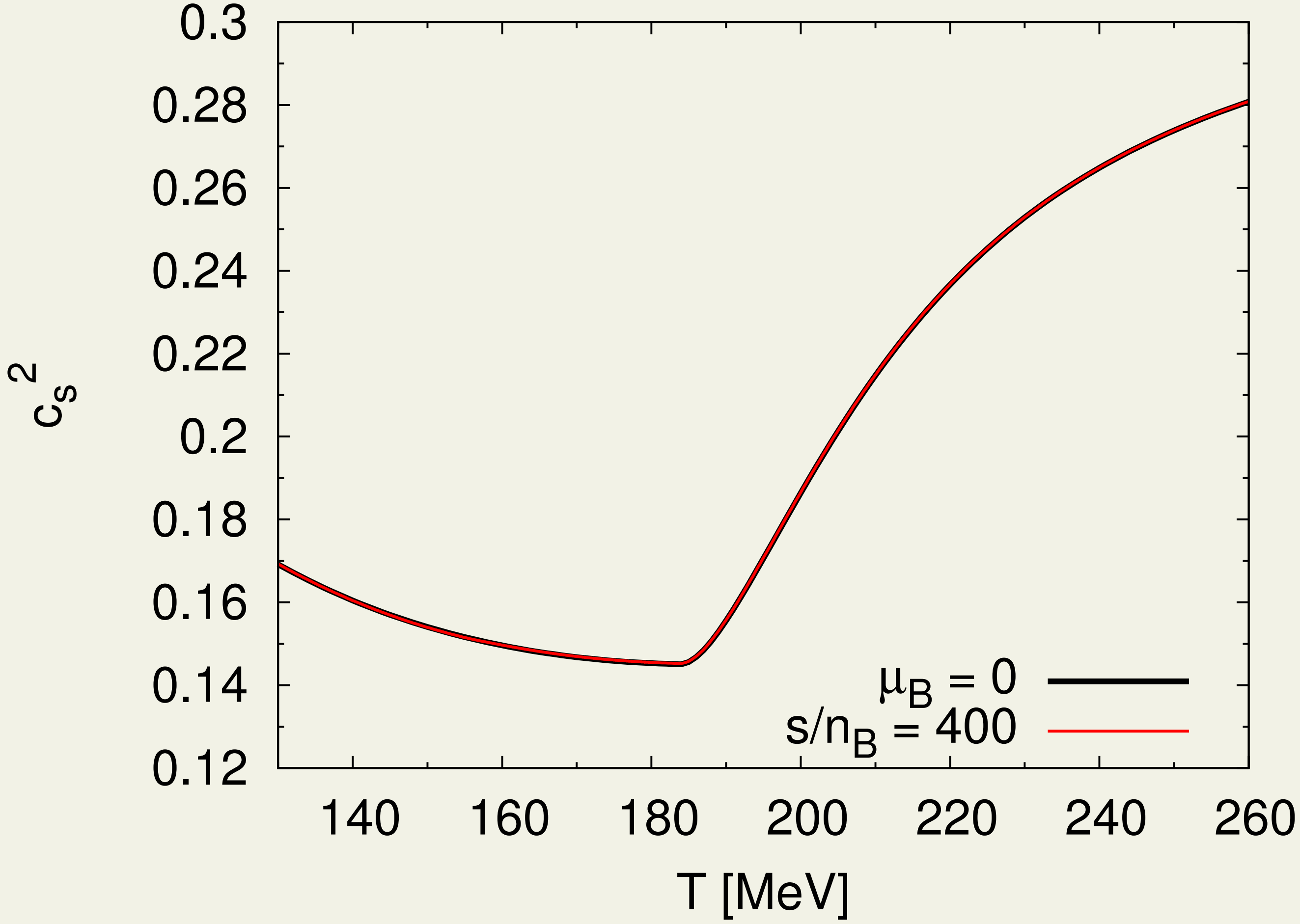

4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
eV/fm-3]
400
350
300
250
200
150
100
T [MeV]
0
50
100
150
200
250
300
μB [MeV]

# Speed of sound

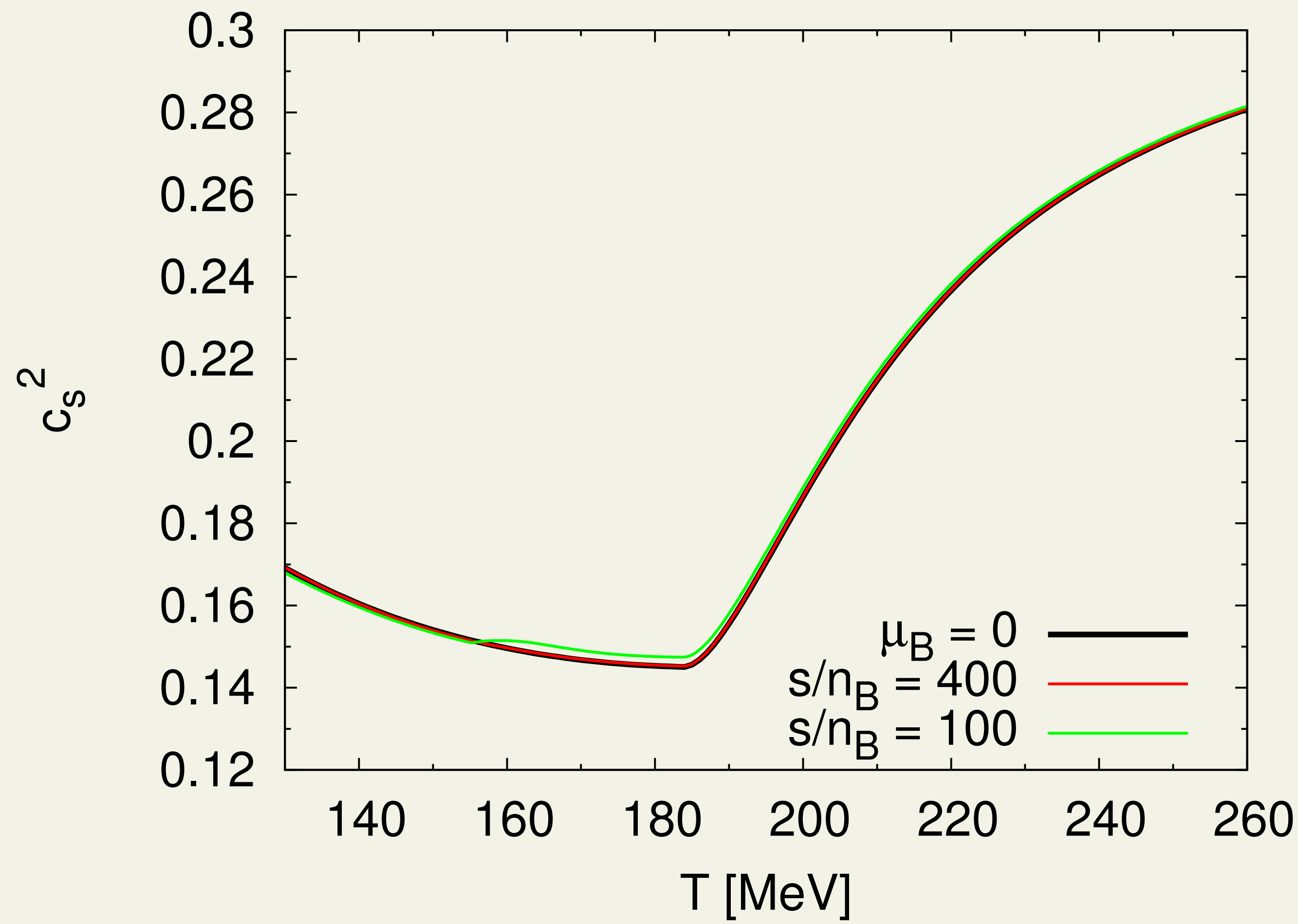



- **s95p-v1** **parametrization by P. Petreczky and P.H.**

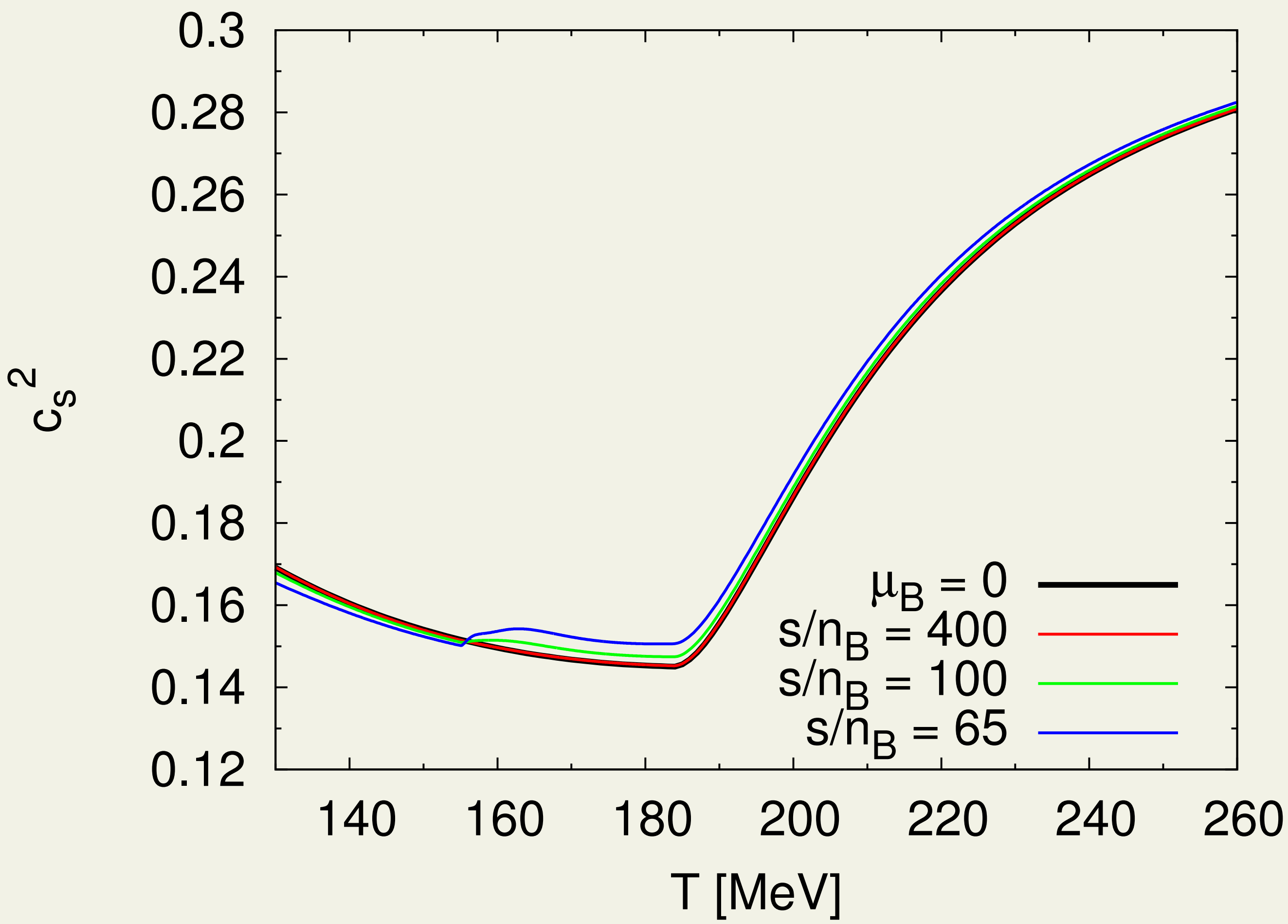
# Speed of sound



# Speed of sound



# Speed of sound



# Transverse expansion and flow

- **Define speed of sound $c_s$:**

$$c_s^2 = \left.\frac{\partial P}{\partial \epsilon}\right|_{s/n_b}$$

- **large $c_s \Rightarrow$ "stiff EoS"**
- **small $c_s \Rightarrow$ "soft EoS"**
- **For baryon-free matter in rest frame**

$$(\epsilon + P) D u^\mu = \nabla^\mu P \qquad \Longleftrightarrow \qquad \frac{\partial}{\partial \tau} u_\mu = -\frac{c_s^2}{s} \partial_\mu s$$

$\Rightarrow$ **The stiffer the EoS, the larger the acceleration**

# Blast wave

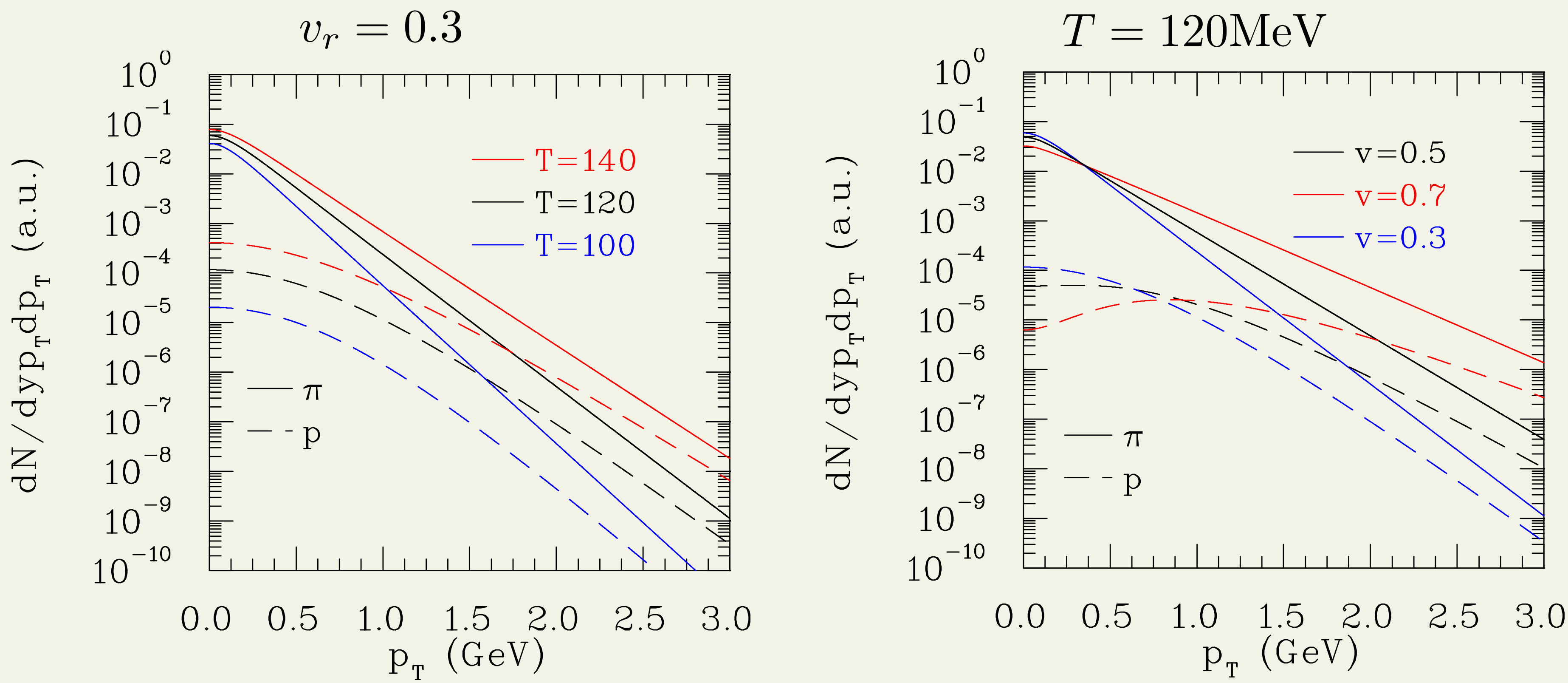
**(Siemens and Rasmussen, PRL 42, 880 (1979))**

- **Freeze-out surface a thin cylindrical shell radius $r$, thickness $\mathrm{d}r$, expansion velocity $v_r$, decoupling time $\tau_{\mathrm{fo}}$, boost invariant**

- **Cooper-Frye for Boltzmannions**

$$\frac{\mathrm{d}N}{\mathrm{d}y\, p_T\, \mathrm{d}p_T} = \frac{g}{\pi}\, \tau_{\mathrm{fo}}\, r\, m_T\, \mathrm{I}_0\!\left(\frac{v_r \gamma_r p_T}{T}\right) \mathrm{K}_1\!\left(\frac{\gamma_r m_T}{T}\right)$$

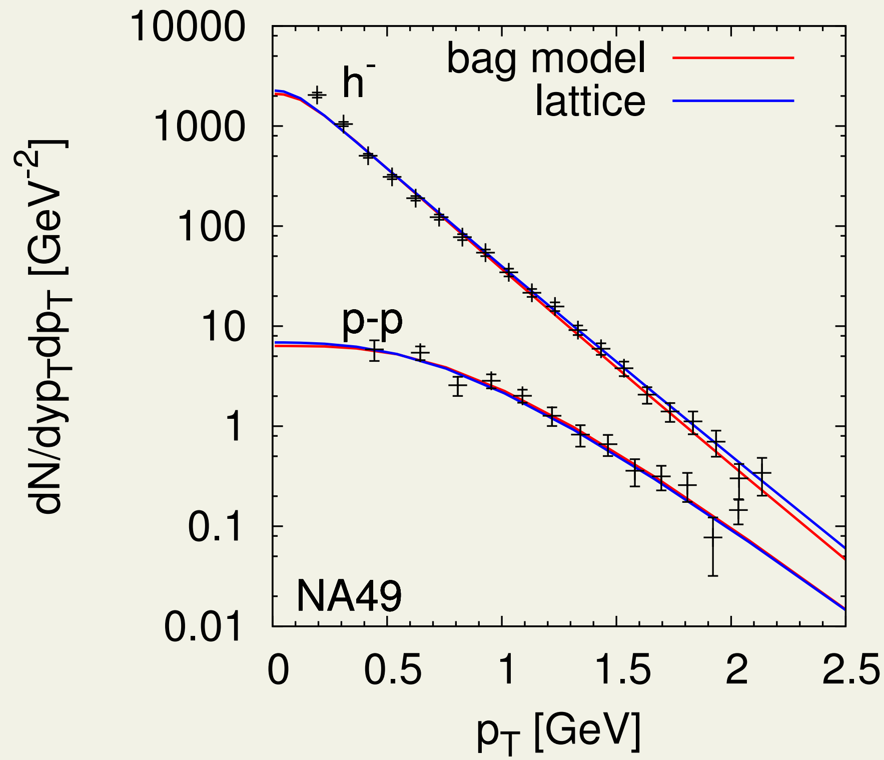
# effect of temperature and flow velocity




- **The larger the temperature, the flatter the spectra**
- **The larger the velocity, the flatter the spectra $\Rightarrow$ blueshift**
- **The heavier the particle, the more sensitive it is to flow (shape and slope)**

# EoS vs. $T_{\rm fo}$

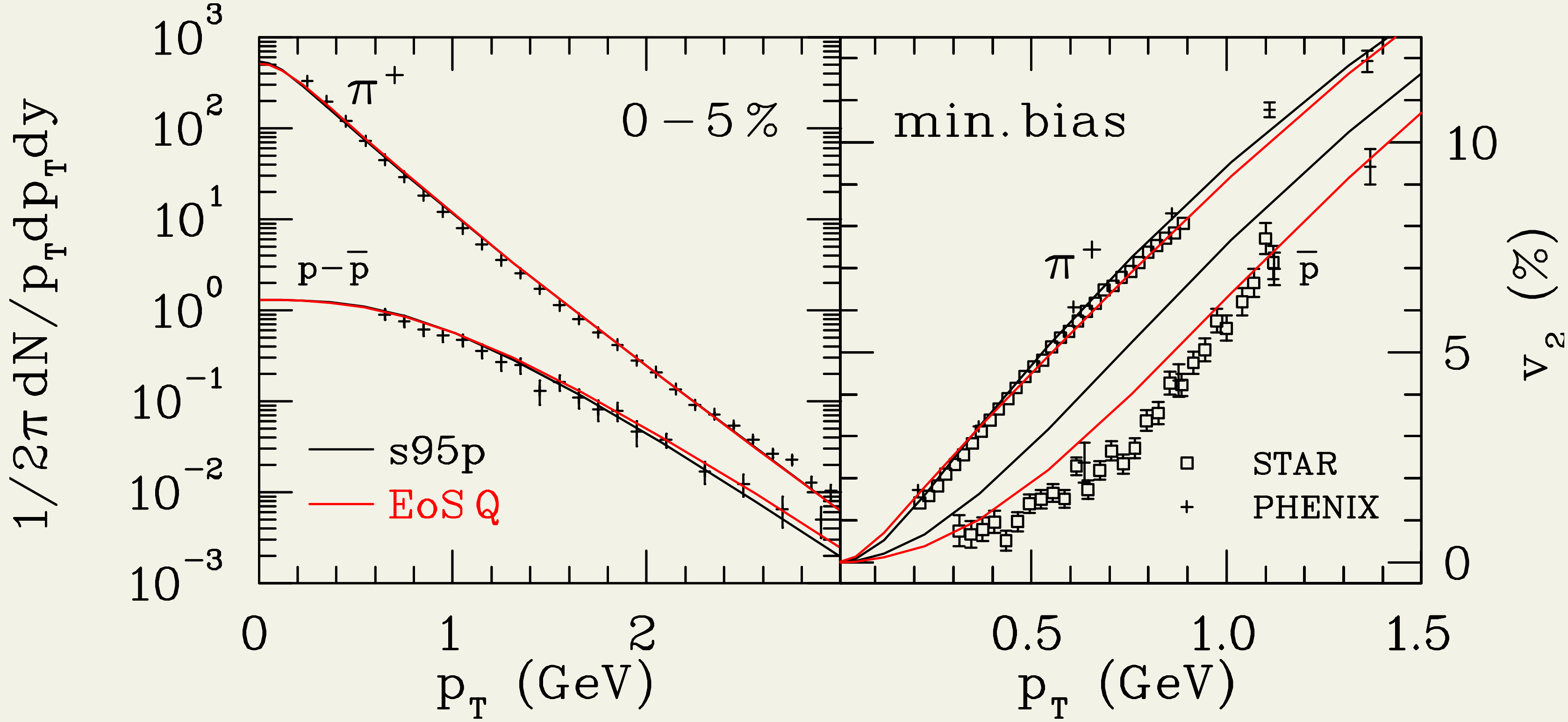
- **hard EoS** $\Leftrightarrow$ **high** $T_{\rm fo}$
- **soft EoS** $\Leftrightarrow$ **low** $T_{\rm fo}$




- $T_{\rm fo} \approx 120$ **MeV (bag),** $T_{\rm fo} \approx 130$ **MeV (lattice)**

# $v_2$ and EoS

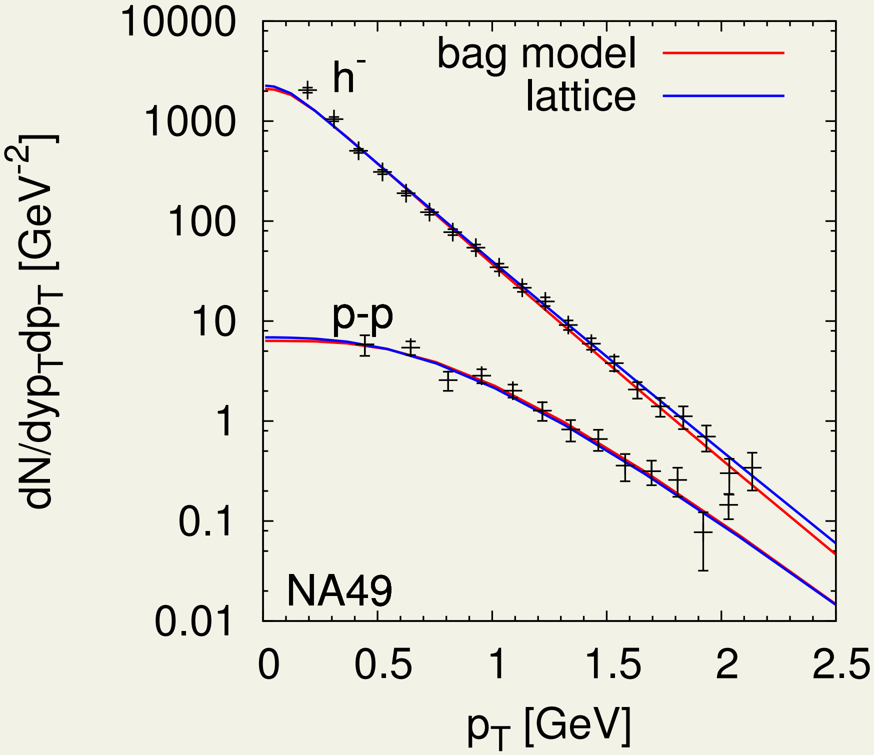
- **ideal hydro, Au+Au at $\sqrt{s_{NN}} = 200$ GeV**

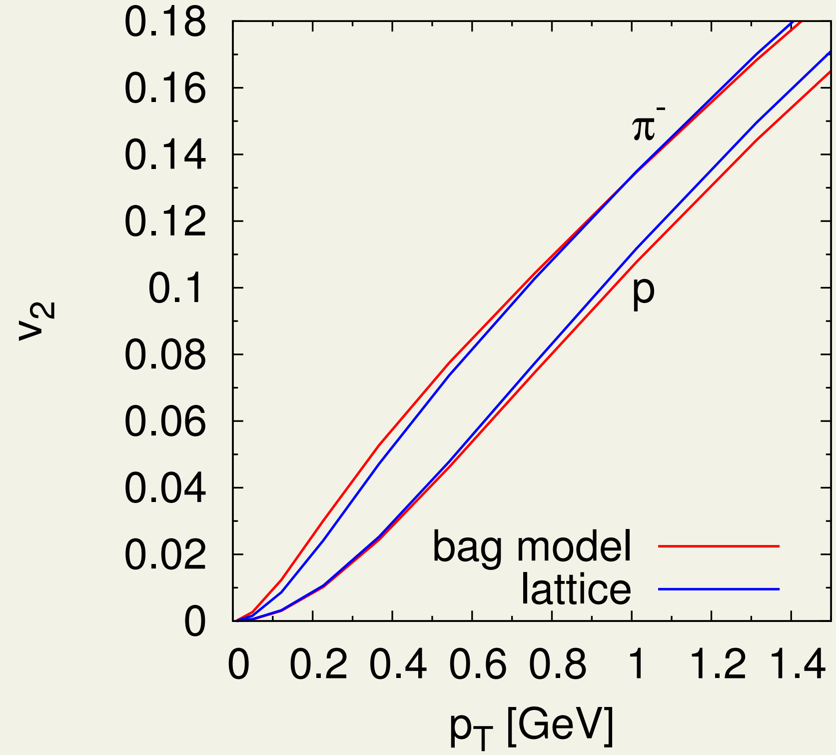



- **s95p:** $T_{dec} = 140$ **MeV**
- **EoS Q: first order phase transition at** $T_c = 170$ **MeV,** $T_{dec} = 125$ **MeV**
- $v_2(p_T)$ **of protons sensitive to phase transition!**

# $v_2$ and EoS

- **ideal hydro, Pb+Pb at** $\sqrt{s_{NN}} = 18$ **GeV**



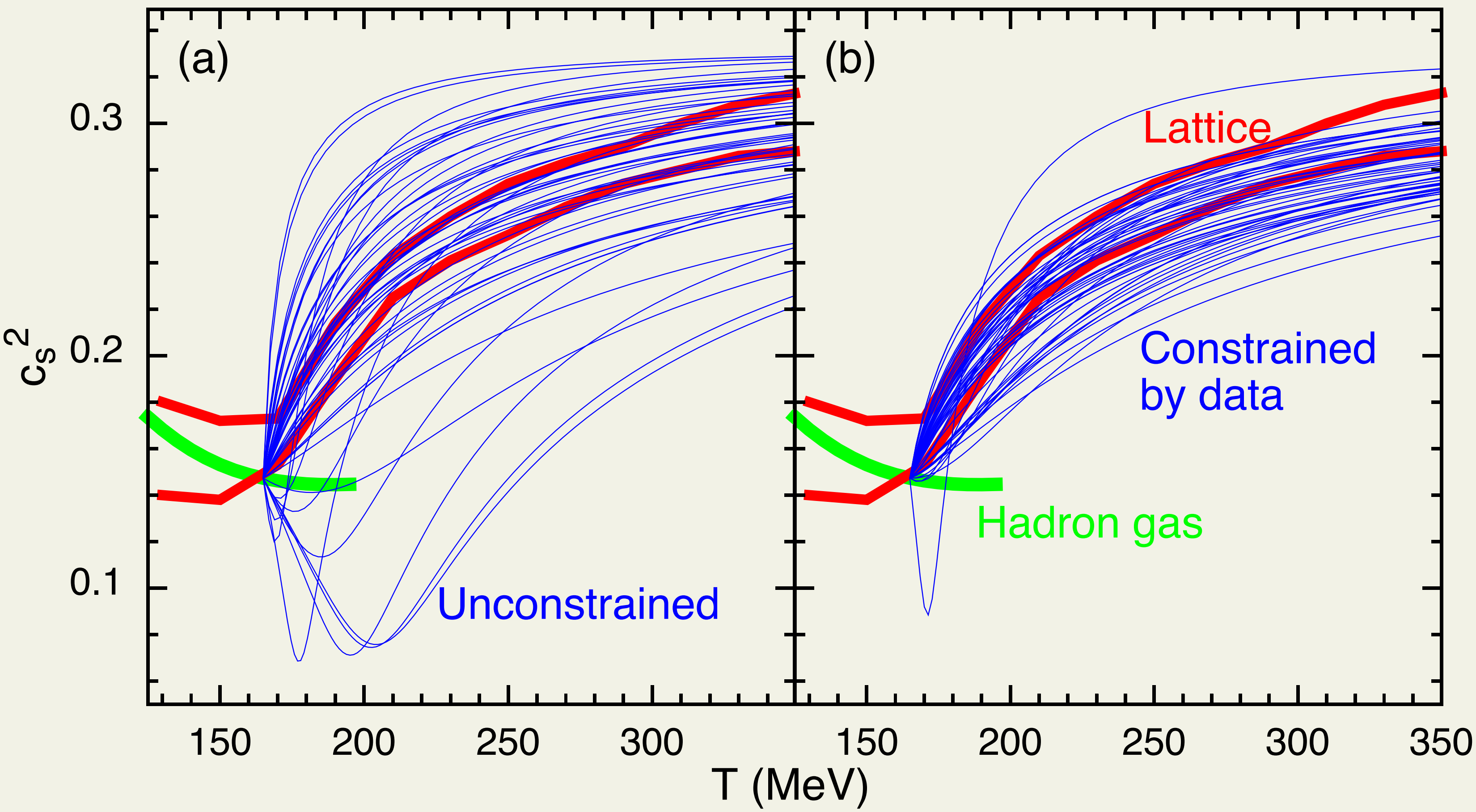




- $T_{\rm fo} \approx 120$ **MeV (bag)**
- $T_{\rm fo} \approx 130$ **MeV (lattice)**
- **protons no longer sensitive to phase transition!**

# Global analysis

- **fit to $p_T$, $v_n$, multiplicities etc.**

**Pratt et al., PRL 114, 202301 (2015)**




- **Bayesian analysis using emulators**