Non-extensive Statistical Mechanics

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Outline

A summary on Entropy Statistics

With the purpose to study as a whole the major part of entropy measures, the entropy-functional is proposed in[1]

$$H_{h,v}^{\varphi_1,\varphi_2}(p_i) = h\left(\frac{\sum_i^W v_i\varphi_1(p_i)}{\sum_i^W v_i\varphi_2(p_i)}\right) \qquad (1)$$

Here I just list some of them: (1) **Shannon-1948**[2], (2) **Renyi-1961**[3], and *etc*[4, 5, 6, 1, 2, 3, 4, 5, 3, 6, 1, 2, 3].

Measure	$h(x)$	$\varphi_1(x)$	$\varphi_2(x)v_i$	
1	x	$-x\log x$	x	v
2	$(1-r)^{-1}\log x$	x^r	x	v
3	x	$-x^r\log x$	x^r	v
4	$(s-r)^{-1}\log x$	x^r	x^s	v
5	$(1/s)\arctan x$	$x^r\sin(s\log x)$	$x^r\cos(s\log x)$	v
6	$(m-r)^{-1}\log x$	x^{r-m+1}	x	v
7	$(m(m-r))^{-1}\log x$	$x^{r/m}$	x	v
8	$(1-t)^{-1}\log x$	x^{t+s-1}	x^s	v
9	$(1-s)^{-1}(x-1)$	x^s	x	v
10	$(t-1)^{-1}(x^t-1)$	$x^{1/t}$	x	v
11	$(1-s)^{-1}(e^x-1)$	$(s-1)x\log x$	x	v
12	$(1-s)^{-1}(x^{\frac{s-1}{r-1}}-1)$	x^r	x	v
13	x	$-x^r\log x$	x	v
14	$(s-r)^{-1}x$	x^r-x^s	x	v
15	$(\sin s)^{-1}x$	$-x^r\sin(s\log x)$	x	v
16	$\left(1+\frac{1}{\lambda}\right)\log(1+\lambda)-\frac{x}{\lambda}$	$(1+\lambda x)\log(1+\lambda x)$	x	v
17	x	$-x\log\left(\frac{\sin(sx)}{2\sin(s/2)}\right)$	x	v
18	x	$-\frac{\sin(xs)}{2\sin(s/2)}\log\left(\frac{\sin(sx)}{2\sin(s/2)}\right)$	x	v
19	x	$-x\log x$	x	w_i
20	x	$-\log x$	1	v_i
21	$(1-r)^{-1}\log x$	x^{r-1}	1	v_i
22	$(1-s)^{-1}(e^x-1)$	$(s-1)\log x$	1	v_i
23	$(1-s)^{-1}(x^{\frac{r-1}{s-1}}-1)$	x^{r-1}	1	v_i

Shannon's Entropy

First of all, it is well known of the Shannon entropy, which satisfies the Fadeev's postulates[4],

$$S_{Sh} = \frac{\sum_i^W (-p_i \ln p_i)}{\sum_i^W p_i} = -\sum_i^W p_i \ln p_i \tag{2}$$

whose maximum is nothing but the case that $p_i = 1/W$, namely the Boltzmann Entropy,

$$S_{BG} = -\sum_i^W \frac{1}{W} \ln \frac{1}{W} = \ln W \tag{3}$$

Renyi's Entropy

While, Shannon's Entropy is not the only one satisfying the postulates if we weaken the fourth one to $H[\mathcal{O}+\mathcal{Q}]=H[\mathcal{O}]+H[\mathcal{Q}]$.[3] In 1961, Alfred Renyi gave another one

$$S_{Re}=\frac{1}{1-q}\ln\sum_{i}^{W}p_i^q \tag{4}$$

which is also shown in the table.

P.S. Think of this: what will it be when $\sum_i^W p_i^q$ goes to 1 with $q\to 1$?

Outline

Tsallis' Entropy

Since Tsallis suggested to use the non-extensive entropy formula with the use of a quantity normally scaled in multifractals firstly,[5]

$$S_{Ts} = \frac{\sum_{i=1}^{W} p_i^q - 1}{1 - q} \tag{5}$$

the corresponding generalized statistical mechanics have been substantially developed and spread over many fields of application[6, 1, 2, 3, 4, 5].

P.S. It also has the property that its maximum is $S_{Ts}^{max} = \ln_q W$ *with* $p_i = 1/W$.

Uniqueness Theorem for Shannon Entropy

1. Shannon [2]

$$\begin{cases} S(A+B) = S(A) + S(B) & p_{ij}^{A+B} = p_i^A p_j^B \\ S(\{p_i\}) = S(p_L, p_M) + p_L S(\{p_i/p_L\}) + p_M S(\{p_i/p_M\}) & L + M = W \end{cases} \tag{6}$$

2. Khinchin [4]

$$\begin{cases} S(\{p_i\}, 0) = S(\{p_i\}) \\ S(A+B) = S(A) + S(B|A) \quad S(B|A) = \sum p_i^A S(\{p_{ij}^{A+B}/p_i^A\}) \end{cases} \tag{7}$$

Uniqueness Theorem for Tsalli Entropy

1. Santos [5]

$$\begin{cases} S(A+B) = S(A) + S(B) + (1-q)S(A)S(B) & p_{ij}^{A+B} = p_i^A p_j^B \\ S(\{p_i\}) = S(p_L, p_M) + p_L^q S(\{p_i/p_L\}) + p_M^q S(\{p_i/p_M\}) & L+M=W \end{cases} \tag{8}$$

2. Abe [6]

$$\begin{cases} S(\{p_i\}, 0) = S(\{p_i\}) \\ S(A+B) = S(A) + S(B|A) + (1-q)S(A)S(B|A) \quad S(B|A) = \frac{\sum (p_i^A)^q S(\{p_{ij}^A}{\sum (p_i^A)} \end{cases} \tag{9}$$

Thermodynamical Foundations/Applications

1. Correlated Anomalous Diffusion: generalized Fokker-Planck Equation.
2. Central Limit Theorems. [4]
3. Zeroth Law. [5]
4. Equipartition and Virial theorems. [6]
5. Second Law. [1]
6. Quantum H-theorem. [6]
7. The classical N-body problem. [2]
8. Fluctuation-Dissipation Theorem. [3]
9. . . .

q-Relation

Considering the thermal equilibrium of two systems, one with energy E_1 (subsystem) and the other with $E - E_1$ (reservoir), for a maximal entropy state,

$$L(S(E)) = L(S(E_1)) + L(S(E - E_1)) = max \tag{10}$$

where $L(S)$ is the additive form of the entropy.[6]

Under the Universal Thermal Independence (UTI) Principle, we have

$$\frac{L''(S)}{L'(S)} = -\frac{S''(E)}{S'(E)^2} = \frac{1}{C} \tag{11}$$

Then we have, $L(S) = C(e^{S/C} - 1)$, with finite constant heat capacity C.

q-Relation

Substituting the Renyi Entropy into it,

$$L(S) = C(e^{\frac{1}{C}\frac{1}{1-q}\ln\sum_i^W p_i^q} - 1) = \frac{\sum_i^W p_i^q - 1}{1-q} \tag{12}$$

where we have $q = 1 - 1/C$.

More deeply, to solve the negative heat capacity problem, we get the more generalized result

$$q = 1 - \frac{1}{C} + \frac{\Delta T^2}{T^2} \tag{13}$$

more details can be seen in [1] and its followings.

Canonical Ensemble

Under the Optimal Lagrange Multipliers(OLM)-Tsallis technique, with the normalized ($\sum_i^W p_i = 1$) and energy constraints ($U_q = \sum_i^W P_i\epsilon_i = \sum_i^W \frac{p_i^q}{\sum_j^W p_j^q}\epsilon_i$) respectively,

$$p_i = \frac{1}{Z_q}[1-(1-q)\frac{\beta}{\sum_j^W p_j^q}\epsilon_i]^{\frac{1}{1-q}} := \frac{1}{Z_q}e_q^{-\beta'\epsilon_i} \tag{14}$$

More discussions are given out in paper[3] and book[4].

q-Thermodynamics

Re-writing the q-expectation of energy, U_q,

$$U_q = \sum_i^W \frac{p_i^q}{\sum_j^W p_j^q}\epsilon_i = \sum_i^W \frac{1}{\sum_j^W p_j^q}\frac{1}{Z_q^q}(e_q^{-\beta'\epsilon_i})^q\epsilon_i$$
$$= -\frac{\partial}{\partial\beta}\ln_q Z_q \quad (15)$$

Similarly, the generalized force is

$$Y_q = -\frac{1}{\beta}\frac{\partial}{\partial y}\ln_q Z_q \quad (16)$$

where y is the generalized coordinates.

q-Thermodynamics

All the above lead to

$$\begin{aligned}\beta(dU_q - Y_q dy) &= \beta d(-\frac{\partial}{\partial\beta}\ln_q Z_q) - \beta(-\frac{1}{\beta}\frac{\partial}{\partial y}\ln_q Z_q)dy \\ &= d(\ln_q Z_q - \beta\frac{\partial}{\partial\beta}\ln_q Z_q) \end{aligned} \tag{17}$$

Comparing with the q-thermodynamical relation

$$dS_q = \frac{1}{T}(dU_q - Y_q dy) \tag{18}$$

which tells us nothing but

$$S_q = \ln_q Z_q - \beta\frac{\partial}{\partial\beta}\ln_q Z_q \tag{19}$$

$$\beta = \frac{1}{T} \tag{20}$$

Grand Canonical Ensemble

Consider a non-interacting quantum gas composed of N particles in heat and particle baths. With $\sum p_R = 1$, $\sum p_R^q E_R = \bar{E}$ and $\sum p_R^q N_R = \bar{N}$,

$$p_R = \frac{1}{Z_q} e_q^{-\beta(E_R - \mu N_R)} \tag{21}$$

The factorization approximation (FA) was proposed by Buyukkilic *et al.* in 1995[5]. Then the generalized distribution functions are

$$\bar{n}_k = \frac{1}{e_{2-q}^{\beta(\epsilon_k - \mu)} \pm 1} \tag{22}$$

where $\bar{n}_k = \sum p_{n_k} n_k$ and $N_R = \sum n_k$.

q-FDD and BED and Non-extensive quantum H-Theorem

Let us start by presenting the specific functional form for entropy[6]

$$S_Q = -\sum n_k^q \ln_q n_k \mp (1 \pm n_k)^q \ln_q (1 \pm n_k) \tag{23}$$

With $C_q(n_k) = \frac{dn_k}{dt}$ denotes the quantum q-collisional term, it is proved that

$$\frac{dS_Q}{dt} \geq 0 \tag{24}$$

which is the quantum H_q-theorem.

Furthermore, considering the equilibrium case, namely, $dS_Q/dt = 0$,

$$n_k = \frac{1}{e_q^{\alpha+\beta\epsilon_k} \pm 1} \tag{25}$$

Escort Distributions

Meantime, C. Beck generalized Hagedorn's theory[1, 2] to re-consider the grand partition function in non-extensive statistics. [3]

$$P_{i_1,i_2,\cdots,i_N} = \frac{1}{Z}\prod_{j=1}^{N}[1-(1-q)\beta\varepsilon_{i_j}]^{q/(1-q)} = \frac{1}{Z}\prod_{j=1}^{N}(e_q^{-\beta\varepsilon_{i_j}})^q \qquad (26)$$

where $1-(1-q)\beta H = \prod_{j=1}^{N}(1-(1-q)\beta\varepsilon_{i_j})$. So we have

$$H = \sum_j \varepsilon_{i_j} + (q-1)\beta\sum_{j,k}\varepsilon_{i_j}\varepsilon_{i_k} + \cdots \qquad (27)$$

Thus the non-extensive average number turns to be

$$\bar{n}_i = \frac{1}{(e_{2-q}^{\beta\varepsilon_{i_j}})^q \pm 1} \qquad (28)$$

Outline

the $(1-q)$ expansion

We've known that the non-extensive statistics recovers the classical Boltzmann case when $q \to 1$. Moreover, in most cases there are no need to calculate the exact values or forms based on q-BED (q-FDD). Here I list some of the approximation methods[1]:

1. Factorization Approximation. (FA)[4]

$$\frac{1}{e_{2-q}^{x}-1} \approx \frac{1}{e^{x}-1}\left(1-\frac{1-q}{2}\frac{x^{2}e^{x}}{e^{x}-1}\right) \tag{29}$$

2. Asymptotic Approximation. (AA)[5]

$$\begin{aligned}\langle n\rangle_{q} &\approx \frac{1}{e^{x}-1}(1-e^{-x})^{q-1}\left[1+(1-q)x\left(\frac{e^{x}+1}{e^{x}-1}-\frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^{2}}\right)\right] \\ &\approx \frac{1}{e^{x}-1}\left[1+(1-q)x\left(\frac{e^{x}+1}{e^{x}-1}-\frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^{2}}-\ln(1-e^{-x})\right)\right]\end{aligned} \tag{30}$$

3. see below.

the $(1-q)$ expansion

1. Exact Approximation. (EA)[6, 2]

$$\frac{1}{e_{2-q}^{x}-1}=\frac{\Gamma(\frac{1}{1-q})}{\int dv v^{\frac{1}{1-q}-1}e^{-v}(e^{v(1-q)x}-1)} \tag{31}$$

where $\Gamma(a)=\int dt t^{a-1}e^{-t}$ for any $a>0$.

Blackbody Radiation

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A unified description of the thermostatistic properties of a class of Bose systems

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Abstract

The generalized statistic distribution derived from Tsallis' entropy is used to study the thermostatistic properties of a q-generalized Bose system trapped in a generic power-law potential. General expressions for some important parameters of the system such as the total number of particles, critical temperature that Bose–Einstein condensation occurs, internal energy, and heat capacity at constant volume are derived and the characteristics of Bose–Einstein condensation are discussed in detail. It is found that the thermostatistic properties depend closely on the nonextensivity parameter q, space dimensionality, kinematic characteristics of particles, shapes of external potentials and generalized thermal wavelength. Moreover, it is shown that the results obtained here can describe the thermostatistic properties of a class of Bose systems in a unified way so that many important conclusions in literature are included in this Letter.

PACS: 05.30.Jp; 05.40.-a; 05.70.-a; 05.20.-y

Keywords: q-generalized statistics; Bose system; Tsallis' entropy; Generic power-law potential; Thermostatistic property; Unified description

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Generalization of the Planck radiation law and application to the cosmic microwave background radiation

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Within the framework of the recently introduced nonextensive statistical mechanics, we generalize the Planck law for the blackbody radiation. The generalized thermal spectrum contains an additional parameter q, for which $q = 1$ in the Planck limit. For the cosmic microwave background radiation, we find a 95% confidence limit of $|q-1| < 3.6\times10^{-5}$ from the data of Mather *et al.* [Astrophys. J. **420**, 439 (1994)] that were obtained with the *Cosmic Background Explorer* satellite, under the assumption that the internal reference of the apparatus has a Planck spectrum.

PACS number(s): 05.20.−y, 05.30.Jp, 04.80.−y, 95.30.−k

Analysis of residual spectra and the monopole spectrum for 3 K blackbody radiation by means of non-extensive thermostatistics

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Abstract

We analyze residual spectra of 3 K blackbody radiation (CMB) using non-extensive thermostatistics with a parameter $q-1$. The limits of $|q-1| < 1.2\times10^{-5}$ and the temperature fluctuation $|\delta T| < (1.6 \sim 4.3)\times10^{-5}$ are smaller than those by Tsallis et al. Moreover, analyzing the monopole spectrum by a formula including the chemical potential μ, we obtain the limits $|q-1| < 2.3\times10^{-5}$ and $|\mu| < 1.6\times10^{-4}$. $|q-1|$ is comparable with the Sunyaev-Zeldovich effect y.

Keywords: 3K CMB; non-extensive thermostatistics; dimensionless chemical potential; Sunyaev-Zeldovich effect

A New Blackbody Radiation Law Based on Fractional Calculus and its Application to NASA COBE Data

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Abstract

By applying fractional calculus to the equation proposed by M. Planck in 1900, we obtain a new blackbody radiation law described by a Mittag–Leffler (ML) function. We have analyzed NASA COBE data by means of a non-extensive formula with a parameter $(q-1)$, a formula proposed by Ertik et al. with a fractional parameter $(\alpha-1)$, and our new formula including a parameter $(p-1)$, as well as the Bose–Einstein distribution with a dimensionless chemical potential μ. It can be said that one role of the fractional parameter $(p-1)$ is almost the same as that of chemical potential (μ) as well as that of the parameter $(q-1)$ in the non-extensive approach.

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Nonextensive black-body distribution function and Einstein's coefficients A and B

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Abstract

The nonextensive statistics of a boson gas within the dilute gas approximation is applied to black-body radiation. Einstein's spontaneous emission coefficient A_{21}, stimulated emission coefficient B_{21}, and absorption coefficient B_{12} are studied in connection with this new statistical mechanics. Relations including nonextensivity between these coefficients are obtained.

Blackbody radiation in a nonextensive scenario

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Abstract

An exact analysis of the N-dimensional blackbody radiation process in a nonextensive à la Tsallis scenario is performed for values of the nonextensive's index in the range $(0 < q < 1)$. The recently advanced "Optimal Lagrange Multipliers" (OLM) technique has been employed. The results are consistent with those of the extensive, $q = 1$ case. The generalization of the celebrated laws of Planck, Stefan-Boltzmann, and Wien are investigated.

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EDP Sciences

Exact and approximate results of non-extensive quantum statistics

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Abstract. We develop an analytical technique to derive explicit forms of thermodynamical quantities within the asymptotic approach to non-extensive quantum distribution functions. Using it, we find an expression for the number of particles in a boson system which we compare with other approximate scheme (*i.e.* factorization approach), and with the recently obtained exact result. To do this, we investigate the predictions on Bose-Einstein condensation and the blackbody radiation. We find that both approximation techniques give results similar to (up to $\mathcal{O}(q-1)$) the exact ones, making them a useful tool for computations. Because of the simplicity of the factorization approach formulae, it appears that this is the easiest way to handle with physical systems which might exhibit slight deviations from extensivity.

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PHYSICS LETTERS A
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Some bounds upon the nonextensivity parameter using the approximate generalized distribution functions

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Abstract

In this study, the approximate generalized quantal distribution functions and their applications which appeared in the literature so far, have been summarized. Making use of the generalized Planck radiation law, which has been obtained by us [Physica A 240 (1997) 657], some alternative bounds for the nonextensivity parameter q have been estimated. It has been shown that these results are similar to those obtained by Tsallis et al. [Phys. Rev. E 52 (1995) 1447] and by Plastino et al. [Phys. Lett. A 207 (1995) 42].

BEC

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A Nonextensive Approach to Bose-Einstein Condensation of Trapped Interacting Boson Gas

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Abstract In the Bose-Einstein condensation of interacting atoms or molecules such as ^{87}Rb, ^{23}Na and ^{7}Li, the theoretical understanding of the transition temperature is not always obvious due to the interactions or zero point energy which cannot be exactly taken into account. The S-wave collision model fails sometimes to account for the condensation temperatures. In this work, we look at the problem within the nonextensive statistics which is considered as a possible theory describing interacting systems. The generalized energy U_q and the particle number N_q of boson gas are given in terms of the nonextensive parameter q. $q > 1$ ($q < 1$) implies repulsive (attractive) interaction with respect to the perfect gas. The generalized condensation temperature T_{cq} is derived versus T_c given by the perfect gas theory. Thanks to the observed condensation temperatures, we find $q \approx 0.1$ for ^{87}Rb atomic gas, $q \approx 0.95$ for ^{7}Li and $q \approx 0.62$ for ^{23}Na. It is concluded that the effective interactions are essentially attractive for the three considered atoms, which is consistent with the observed temperatures higher than those predicted by the conventional theory.

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Generalized thermostatistics and Bose–Einstein condensation

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Abstract

Analytical expressions for Bose–Einstein condensation of an ideal Bose gas analyzed within the strictures of nonextensive, generalized thermostatistics are here obtained.

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Note on BEC in Nonextensive Statistical Mechanics

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The generalized Bose-Einstein distribution, within the dilute gas assumption, in the nonextensive Tsallis statistics is worked without approximation for the Bose-Einstein condensation (BEC). The results obtained are compared with the recent results presented in Int. J. Mod. Phys. B **14**, 405 (2000) by L. Salasnich. Furthermore, in order to promote a complete analysis for the BEC in the nonextensive scenario we also find exact expression within the normalized constraints in a harmonic trap.

BOSE-EINSTEIN CORRELATIONS IN CASCADE PROCESSES AND NON-EXTENSIVE STATISTICS

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We discuss the effect of nonextensivity of the emitting source on the Bose-Einstein correlations (BEC). This is done numerically by comparing cascade hadronization model (CAS), which is known to exhibit fractal structure in both space-time and phase-space, with its equivalent obtained from the information theory approach (MaxEnt), in which hadronization proceeds uniformly in the phase-space. To this

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More accurate theory for Bose–Einstein condensation fraction

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Abstract

Bose–Einstein statistics is derived in the thermodynamic limit when the ratio of system size to thermal de Broglie wavelength goes to infinity. However, according to the experimental setup of Bose–Einstein condensation of harmonically trapped Bose gas of alkali atoms, the ratio near the condensation temperature (T_0) is 30–50. And, at ultralow temperatures well below T_0, this ratio becomes comparable to 1. We argue that finite

Non-extensive Bose-Einstein Condensation Model

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Abstract

The imperfect Boson gas supplemented with a gentle repulsive interaction, is completely solved. In particular it is proved that it has non-extensive Bose-Einstein condensation, i.e. there is condensa

Outline

Basic assumptions

$$S_R = -\int d\Omega p^q \ln_q p \tag{32}$$

Similarly we have $p(x,v) = f(x,v)/Z_q$, where

$$f = [1-(1-q)\beta(H-\langle H\rangle_q)]^{1/(1-q)} := e_q^{-\beta(H-\langle H\rangle_q)} \tag{33}$$

Easy to get

$$\int d\Omega f \equiv \int d\Omega f^q \tag{34}$$

$$\langle A\rangle_q := \frac{\int d\Omega f^q A}{\int d\Omega f^q} = \frac{1}{Z_q}\int d\Omega f^q A \tag{35}$$

Relativistic kinetic theory

The corresponding particle four-flow and energy-momentum flow are as

$$N^{\mu}(x) = \frac{1}{Z_q}\int \frac{d^3p}{p^0} p^{\mu} f \tag{36}$$

$$T^{\mu\nu}(x) = \frac{1}{Z_q}\int \frac{d^3p}{p^0} p^{\mu} p^{\nu} f^q \tag{37}$$

Thus can we have the probability density $n = N^{\mu}U_{\mu}$, the energy density $\epsilon = T^{\mu\nu}U_{\mu}U_{\nu}$ and the equilibrium pressure $P = -\frac{1}{3}T^{\mu\nu}\Delta_{\mu\nu}$. Thus we also obtain that for massless system,

$$\epsilon = 3P \tag{38}$$

theoretical work

Eur. Phys. J. A **40**, 325–340 (2009)
DOI 10.1140/epja/i2009-10806-6
THE EUROPEAN PHYSICAL JOURNAL A
Regular Article – Theoretical Physics

Non-extensive approach to quark matter*

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Communicated by U.-G. Meißner

Abstract. We review the idea of generating non-extensive stationary distributions based on abstract composition rules of the subsystem energies, in particular the parton cascade method, using a Boltzmann equation with relativistic kinematics and modified two-body energy composition rules. The thermodynamical behavior of such model systems is investigated. As an application hadronic spectra with power law tails are analyzed in the framework of a quark coalescence model.

PACS. 21.65.Qr Quark matter – 25.75.Ag Global features in relativistic heavy-ion collisions – 05.20.Dd Kinetic theory

Acta Phys. Hung. A 27/2–3 (2006) 367–371
DOI: 10.1556/APH.27.2006.2–3.42
HEAVY ION PHYSICS

Numerical Simulation of Non-Extensive Boltzmann Equation

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Received 28 October 2005

Abstract. We present first results of the development of a test particle simulation for solving non-extensive extensions of the elastic two-particle Boltzmann equation. Stationary one-particle energy distributions with power-law tail are obtained.

Acta Phys. Hung. A 22/3–4 (2005) 223–229
HEAVY ION PHYSICS

Quark Matter and Non-Extensive Thermodynamics

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Received 29 April 2004

Abstract. We investigate different ways of describing the thermodynamics of prehadronic matter in high-energy heavy-ion collisions. The non-extensive thermodynamics with certain assumptions enables an agreement between two important experimental facts. That cannot be achieved using the conventional Gibbs statistics. A massless quark–gluon plasma with $E/N = 1$ GeV energy per particle and $T = 175$ MeV spectral slope temperature can be assumed.

PHYSICAL REVIEW C **85**, 024905 (2012)

Fluid dynamical equations and transport coefficients of relativistic gases with non-extensive statistics

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(Received 12 September 2011; published 17 February 2012)

We derive equations for fluid dynamics from a non-extensive Boltzmann transport equation consistent with Tsallis' non-extensive entropy formula. We evaluate transport coefficients employing the relaxation time approximation and investigate non-extensive effects in leading order dissipative phenomena at relativistic energies, like heat conductivity, shear, and bulk viscosity.

DOI: 10.1103/PhysRevC.85.024905 PACS number(s): 24.10.Nz, 25.75.−q, 47.75.+f

168 Progress of Theoretical Physics Supplement No. 174, 2008

Nonextensive/Dissipative Correspondence in Relativistic Hydrodynamics

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[2] *The Andrzej Sołtan Institute for Nuclear Studies, Hoża 69, 00681, Warsaw, Poland*

We argue that there is profound correspondence (the nonextensive/dissipative correspondence - NexDC) between the perfect nonextensive hydrodynamics and the usual dissipative hydrodynamics which leads to simple expression for dissipative entropy current.

Cent. Eur. J. Phys. • 7(3) • 2009 • 432-443
DOI: 10.2478/s11534-008-0163-5
Central European Journal of **Physics**

Nonextensive perfect hydrodynamics – a model o dissipative relativistic hydrodynamics?

Resear

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Received 14 October 2008; accepted 19 Decem

Abstract: We demonstrate that nonextensive perfect relativistic hydrodynamics (q-hydrodynamics) c model of the usual relativistic dissipative hydrodynamics (d-hydrodynamics) therefore facilita ably its applications. As an illustration, we show how using q-hydrodynamics one gets the expressions for the dissipative entropy current and the corresponding ratios of the bulk and sh to entropy density, ζ/s and η/s respectively.

phenomenological analysis

Cent. Eur. J. Phys. • 7(3) • 2009 • 395-400
DOI: 10.2478/s11534-009-0004-1

VERSITA

Central European Journal of **Physics**

Non-extensive equilibration in relativistic matter

Research Article

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Received 27 September 2008; accepted 19 November 2008

Abstract: We present a view of the non-extensive thermodynamics based on general composition rules. A formal logarithm maps these rules to the addition, which can be used to generate stationary distributions by standard techniques. We review the most commonly used rules and as an application we discuss the Tsallis-Pareto distribution of transverse momenta of energetic hadrons, which emerge from relativistic heavy-ion collisions.

Acta Phys. Hung. A 21/1 (2004) 85–94

HEAVY ION PHYSICS

What Is the Temperature in Heavy Ion Collisions?[a]

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[1] KFKI Res. Inst. Part. Nucl. Phys., H-1525 Budapest, P.O. Box 49, Hungary
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Received 27 February 2004

Abstract. We consider the Tsallis distribution as a source of the apparent slope of one-particle spectra in heavy-ion collisions and investigate the equation of state of this special kind of quark matter in the framework of non-extensive thermodynamics. We relate the energy per particle to the power-law tail of spectra at a given temperature.

Indian J. Phys. **Vol. 85**, No. 6, pp 941-946, June, 2011

IJP an international journal

Dissipative or just nonextensive hydrodynamics? - Nonextensive/dissipative correspondence

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Abstract : We argue that there is correspondence between the perfect nonextensive hydrodynamics and the usual dissipative hydrodynamics, which we call nonextensive/dissipative correspondence (NexDC). It leads to simple expression for dissipative entropy current and allows for predictions for the ratio of bulk and shear viscosities to entropy density, ζ/s and η/F .

theoretical basis

Tsallis' non-extensive statistical mechanics can be considered an appropriate basis to deal with physical phenomena where **strong dynamical correlations, long-range interactions and microscopic memory effects** take place.[5]

And we expect in the range of temperature and density considered, the presence of strange particles does not significantly affect the main conclusions regarding the relevance of non-extensive statistical effects to the nuclear EOS.[6]

In many-body long-range-interacting systems, there has been observed the emergence of long-living quasi-stationary (metastable) states characterized by non-Gaussian power-law velocity distributions.[1]

Moreover, for such systems like QGP formed in heavy-ion collisions, the size ($N \ll N_A$) needs be re-consideration whether Boltzmann statistics is still appropriate.

phase transition

By requiring the Gibbs conditions on the global conservation of baryon number and electric charge fraction, the phase transition from hadronic matter to QGP is studied in the frame of non-extensive statistics.[5, 2]

$$P_B = \frac{2}{3}\sum_{i=n,p}\int\frac{d^3k}{(2\pi)^3}\frac{k^2}{E_i*(k)}[n_i^q(k)+\bar{n}_i^q(k)]-\frac{1}{2}m_\sigma^2\sigma^2-U(\sigma)+\frac{1}{2}m_\omega^2\omega^2+\frac{1}{2}m_\rho^2\rho^2 \tag{39}$$

$$P_q = \frac{\gamma_f}{3}\sum_{f=u,d}\int\frac{d^3k}{(2\pi)^3}\frac{k^2}{e_f}[n_f^q(k)+\bar{n}_f^q(k)]-B \tag{40}$$

$$P_g = \frac{\gamma_g}{3}\int\frac{d^3k}{(2\pi)^3}kn_g^q(k) \tag{41}$$

heavy-ion collisions

R. Hagedorn[2] proposed the QCD inspired empirical formula to describe experimental hadron production data:[3]

$$E\frac{d^3\sigma}{d^3p} = C(1+\frac{p_T}{p_0})^{-n} \rightarrow \begin{cases} \exp(-\frac{np_T}{p_0}) & p_T \rightarrow 0 \\ (\frac{p_0}{p_T})^n & p_T \rightarrow \infty \end{cases} \quad (42)$$

which coincides with

$$h_q(p_T) = C_q e_q^{-\beta p_T} = C_q[1-(1-q)\frac{p_T}{T}]^{\frac{1}{1-q}} \quad (43)$$

for $n = 1/(q-1)$ and $p_0 = nT$.

heavy-ion collisions

Others[4, 5, 6, 1, 2] criticized that thermodynamical consistency leads to

$$E\frac{d^3N}{d^3p} \propto [1-(1-q)\beta p_T]^{\frac{q}{1-q}} \tag{44}$$

which comes from

$$\epsilon = g\int d\Omega E f^q \tag{45}$$

heavy-ion collisions

During the last few years, G. G. Barnafoldi *et al.*[3] proposed a "soft+hard" model for p_T spectra as well as v_2 measured in both pp and AA collisions.

$$\left.\frac{dN}{2\pi p_T dp_T dy}\right|_{y=0} = \sum_{i=soft,hard} A_i e_q^{-\beta_i[\gamma_i(m_T - v_{0,i} p_T) - m]} \quad (46)$$

where "soft" is referred to hadrons yields stemming from the QGP part but "hard" to jets.

heavy-ion collisions

IOP Publishing Journal of Physics G: Nuclear and Particle Physics
J. Phys. G: Nucl. Part. Phys. 37 (2010) 115009 (4pp) doi:10.1088/0954-3899/37/11/115009

The non-extensivity parameter of a thermodynamical model of hadronic interactions at LHC energies

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Published 11 October 2010
Online at stacks.iop.org/JPhysG/37/115009

Abstract
The LHC measurements above SPS and Tevatron energies give an opportunity to test predictions of the non-extensive thermodynamical picture of hadronic interaction to examine the measured transverse momentum distributions for a new interaction energy range. We determined the Tsallis model non-extensivity parameter for the hadronization process before short-lived particles decay and distort the initial $p_\perp$ distribution. We have shown that it follows exactly the smooth rise determined at lower energies below the present LHC record. The energy dependence of the q parameter is consistent with expectations and the evidence of the asymptotic limit may be seen.

Vol. 5 (2012) *Acta Physica Polonica B Proceedings Supplement* No 2

GENERALISED MICROCANONICAL STATISTICS AND FRAGMENTATION IN ELECTRON–POSITRON COLLISIONS*

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(Received January 16, 2012)

A statistical fragmentation model based on the microcanonical ensemble and on a Koba–Nielsen–Olesen (KNO) type scaling of the multiplicity distribution of charged hadrons in electron–positron collisions is presented.

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Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics A 774 (2006) 845–848
NUCLEAR PHYSICS A

Power-law tailed spectra from equilibrium

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We propose that power-law tailed hadron spectra may be viewed as stemming from a matter in an unconventional equilibrium state typical for non-extensive thermodynamics. A non-extensive Boltzmann equation, which is able to form such spectra as a stationary solution, is utilized as a rough model of quark matter hadronization. Basic ideas about a non-extensive simulation of the QCD equation of state on the lattice are presented.

IOP Publishing Journal of Physics G: Nuclear and Particle Physics
J. Phys. G: Nucl. Part. Phys. 39 (2012) 025006 (12pp) doi:10.1088/0954-3899/39/2/025006

The Tsallis distribution in proton–proton collisions at $\sqrt{s}$ = 0.9 TeV at the LHC

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Abstract
The Tsallis distribution has been used recently to fit the transverse momentum distributions of identified particles by the STAR (Abelev *et al* (STAR Collaboration) 2007 *Phys. Rev.* C **75** 064901) and PHENIX (Adare *et al* (PHENIX Collaboration) 2011 *Phys. Rev.* C **83** 064903) collaborations at the Relativistic Heavy Ion Collider and by the ALICE (Aamodt *et al* (ALICE Collaboration) 2011 *Eur. Phys. J.* C **71** 1655) and CMS (Khachatryan *et al* (CMS Collaboration) 2011 *J. High Energy Phys.* JHEP05(2011)064) collaborations at the Large Hadron Collider. Theoretical issues are clarified concerning the thermodynamic consistency of the Tsallis distribution in the particular case of relativistic high-energy quantum distributions. An improved form is proposed for describing the transverse momentum distribution and fits are presented together with estimates of the parameter q and the temperature T.

International Journal of Modern Physics E
Vol. 16, No. 6 (2007) 1687–1700

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NON-EXTENSIVE THERMODYNAMICS, HEAVY ION COLLISIONS AND PARTICLE PRODUCTION AT RHIC ENERGIES

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Received 16 August 2006

In the light of ideas of the nonextensive thermodynamics, we have analyzed here the transverse momentum spectra of pions and protons produced at different centralities in the interactions of $P+P$, $D+Au$ and $Au+Au$ interactions, all of them at $\sqrt{s_{NN}}$ = 200 GeV at RHIC-BNL. Comparison of the results and the comments thereon have also been made with indications of suitable hints to the physical import and implications. The overall impact and the utility of the approach along with the obtained results are discussed in detail.

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Physics Letters B
www.elsevier.com/locate/physletb

Systematic properties of the Tsallis distribution:
Energy dependence of parameters in high energy p–p collisions

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ABSTRACT

Changes in the transverse momentum distributions with beam energy are studied using the Tsallis distribution as a parameterization. The dependence of the Tsallis parameters q, T and the volume are determined as a function of beam energy. The Tsallis parameter q shows a weak but clear increase with beam energy with the highest value being approximately 1.15. The Tsallis temperature and volume are consistent with being independent of beam energy within experimental uncertainties.

heavy-ion collisions

ELSEVIER Physica A 286 (2000) 156–163 PHYSICA A www.elsevier.com/locate/physa

A nonextensive thermodynamical equilibrium approach in $e^+e^- \rightarrow hadrons$

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Received 6 July 2000

Abstract

We show that the inclusion of long distance effect, expected in strong interactions, through a nonextensive thermodynamical approach is able to explain the experimental distribution of the transverse momentum of the hadrons with respect to the jet axis (p_t) $e^+e^- \rightarrow$ *hadron* reaction. The observed deviation from the exponential behavior, predicted by the Boltzmann–Gibbs thermodynamical treatment, is automatically recovered by the nonextensive Tsallis statistics used here. We fitted the observed p_t spectrum in the range of 14–161 GeV and obtained, besides a good fit, the theoretically important fact that the temperature becomes independent of the primary energy.

Eur. Phys. J. A (2012) **48**: 161
DOI 10.1140/epja/i2012-12161-y
THE EUROPEAN PHYSICAL JOURNAL A

Review

Consequences of temperature fluctuations in observables measured in high-energy collisions

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Communicated by T. Bíró

Abstract. We review the consequences of intrinsic, nonstatistical temperature fluctuations as seen in observables measured in high-energy collisions. We do this from the point of view of nonextensive statistics and Tsallis distributions. Particular attention is paid to multiplicity fluctuations as a first consequence of temperature fluctuations, to the equivalence of temperature and volume fluctuations, to the generalized thermodynamic fluctuations relations allowing us to compare fluctuations observed in different parts of the phase space, and to the problem of the relation between Tsallis entropy and Tsallis distributions. We also discuss the possible influence of conservation laws on these distributions and provide some examples of how one can get them *without* considering temperature fluctuations.

Eur. Phys. J. A **40**, 299–312 (2009)
DOI 10.1140/epja/i2009-10803-9
THE EUROPEAN PHYSICAL JOURNAL A

Regular Article – Theoretical Physics

Power laws in elementary and heavy-ion collisions*

A story of fluctuations and nonextensivity?

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Communicated by U.-G. Meißner

Abstract. We review from the point of view of nonextensive statistics the ubiquitous presence in elementary and heavy-ion collisions of power law distributions. Special emphasis is placed on the conjecture that this is just a reflection of some intrinsic fluctuations existing in the hadronic systems considered. These systems are summarily described by a single parameter q playing the role of a nonextensivity measure in the nonextensive statistical models based on Tsallis entropy.

PACS. 89.75.-k Complex systems – 24.60.-k Statistical theory and fluctuations – 25.75.Dw Particle and resonance production – 25.75.-q Relativistic heavy-ion collisions

IOP PUBLISHING JOURNAL OF PHYSICS G: NUCLEAR AND PARTICLE PHYSICS
J. Phys. G: Nucl. Part. Phys. **37** (2010) 094042 (6pp) doi:10.1088/0954-3899/37/9/094042

A non-extensive equilibrium analysis of π^+ p_T spectra at RHIC

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Abstract
By analyzing the dynamical properties of particle production in relativistic heavy ion collisions, it is possible to characterize the final stage of the equilibration process occurring in the collision fireball. In this work, we use the Hagedorn model coupled with non-extensive statistics to evaluate the transverse momentum spectra of positive pions for various event centrality classes in Au+Au collisions at RHIC with center-of-mass energies of 62.4GeV A^{-1} and 200 GeV A^{-1}. We find that, by assuming an energy distribution that incorporates particle correlations, it is possible to explain the entire π^+ p_T spectrum as measured by RHIC. We find that spectra from central collisions, when compared to peripheral collisions, are consistent with a system that has smaller values of the non-extensivity parameter 'q' and higher values of temperature. Comparison between different beam energies also shows a variation of the 'q' parameter. The result is discussed using the interpretation that the 'q' parameter is a measure of particle correlations within the system. Under this assumption, our results show that more central collisions are consistent with a system with less particle correlations.

arXiv:0911.1040v1 [nucl-th] 5 Nov 2009

Analyzing Non-Extensivity of η-spectra in Relativistic Heavy Ion Collisions at $\sqrt{s_{NN}} = 200$ GeV

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Abstract

The transverse momentum spectra of secondary η particles produced in $P+P$, $D+Au$ and $Au+Au$ interactions at $\sqrt{s_{NN}} = 200$ GeV at different centralities have been studied in the light of a non-extensive thermodynamical approach. The results and the possible thermodynamical insights, thus obtained, about the hadronizing process have also been discussed in detail.

PHYSICAL REVIEW D **87**, 114007 (2013)

Tsallis fits to p_T spectra and multiple hard scattering in pp collisions at the LHC

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(Received 12 May 2013; published 5 June 2013)

Phenomenological Tsallis fits to the CMS, ATLAS, and ALICE transverse momentum spectra of hadrons for pp collisions at LHC were recently found to extend over a large range of the transverse momentum. We investigate whether the few degrees of freedom in the Tsallis parametrization may arise from the relativistic parton-parton hard-scattering and related processes. The effects of the multiple hard-scattering and parton showering processes on the power law are discussed. We find empirically that whereas the transverse spectra of both hadrons and jets exhibit power-law behavior of $1/p_T^n$ at high p_T, the power indices n for hadrons are systematically greater than those for jets, for which $n \sim 4$–5.

DOI: 10.1103/PhysRevD.87.114007 PACS numbers: 25.75.Bh, 24.10.Jv, 24.85.+p, 25.40.Ep

heavy-ion collisions

ELSEVIER

Physica A 286 (2000) 164–180

PHYSICA A

www.elsevier.com/locate/physa

Non-extensive statistical mechanics and particle spectra in elementary interactions

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Received 2 March 2000; received in revised form 31 March 2000

Abstract

We generalize Hagedorn's statistical theory of momentum spectra of particles produced in high-energy collisions using Tsallis' formalism of non-extensive statistical mechanics. Suitable non-extensive grand canonical partition functions are introduced for both fermions and bosons. Average occupation numbers and moments of transverse momenta are evaluated in an analytic way. We analyse the energy dependence of the non-extensivity parameter q as well as the q-dependence of the Hagedorn temperature. We also take into account the multiplicity. As a final result we obtain formulas for differential cross sections that are in very good agreement with e^+e^- annihilation experiments.

PHYSICAL REVIEW D **89**, 054014 (2014)

Particle production in relativistic $pp(\bar{p})$ and AA collisions at RHIC and LHC energies with Tsallis statistics using the two-cylindrical multisource thermal model

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(Received 19 December 2013; published 14 March 2014)

An improved Tsallis statistics is implemented in a multisource thermal model to describe systematically pseudorapidity spectra of charged particles produced in relativistic nucleon-nucleon (pp or $p\bar{p}$) collisions at various collision energies and in relativistic nucleus-nucleus (AA) collisions at different energies with different centralities. The results with Tsallis statistics using the two-cylindrical multisource thermal model are in good agreement with the experimental data measured at RHIC and LHC energies. It is found that the rapidity shifts of longitudinal sources increase linearly with collision energies and centralities in the framework. According to the laws, we also give a prediction of the pseudorapidity distributions in $pp(\bar{p})$ collisions at higher energies.

Outline

κ-distributions

To get rid of the KMS problem[4] and others Tsallis' q-exponential meets, G. Kaniadakis[3, 4, 5, 6] proposed another form of distributions which lead to the κ-deformed statistical mechanics.

$$S_{Ka} = \int d\Omega \frac{f^{1-\kappa} - f^{1+\kappa}}{2\kappa} := -\int d\Omega f \ln_\kappa f \tag{47}$$

where $\ln_\kappa x = \frac{x^\kappa - x^{-\kappa}}{2\kappa}$ is the κ-logarithm. With the OLM and MEM can we get its distribution:

$$f = e_\kappa^{-\beta(U-\mu)} \tag{48}$$

where the κ-exponential is introduced,

$$e_\kappa^x = (\sqrt{1+\kappa^2 x^2} + \kappa x)^{1/\kappa} = \exp(\frac{1}{\kappa} \operatorname{arcsin} \kappa x) \tag{49}$$

κ-distributions of a QGP[1]

Using the κ-deformed statistics to describe the QGP, the single particle distribution functions of quarks/anti-quarks and gluons respectively,

$$\bar{n}_{q/\bar{q}} = \frac{1}{\sqrt{1+\kappa^2\beta^2(k\mp\mu_q)^2}+\kappa\beta(k\mp\mu_q)+1} := \frac{1}{e_\kappa(\beta(k\mp\mu_q))+1} \tag{50}$$

$$\bar{n}_g = \frac{1}{\sqrt{1+\kappa^2\beta^2k^2}+\kappa\beta k-1} := \frac{1}{e_\kappa(\beta k)-1} \tag{51}$$

Thus can we study the phase transition with the similar steps. The same phase diagram as in the Tsallis case is obtained, since both of them are fractal in nature.

Thank You!!!

Backup Slides

Outline

Fadeev's postulates

The amount of uncertainty of the distribution $\Omega = (p_1, p_2, \cdots, p_n)$, that is, the amount of uncertainty concerning the outcome of an experiment, the possible results of which have the probabilities $p_1, p_2, \cdots, p_n$, is called the **Entropy** of the distribution Ω. In 1957, Fadeev proposed that the simplest such set of postulates are as follows.

1. It is a symmetric function of its variables for n.
2. It is a continuous function of p_i.
3. $H(1/2, 1/2) = 1$.
4. $H(tp_1, (1-t)p_1, p_2, \cdots, p_n) = H(p_i) + p_1 H(t, (1-t))$.

q-Algorithm

From now on, Tsallis Entropy is re-written as

$$S_q = \frac{\sum_{i=1}^{W} p_i^q - 1}{1-q} := \sum_i^W p_i \ln_q\left(\frac{1}{p_i}\right) = -\sum_i^W p_i^q \ln_q p_i \tag{52}$$

where the q-logarithm is introduced,

$$\ln_q(x) := \frac{x^{1-q} - 1}{1-q} \tag{53}$$

with its inverse function, q-exponential

$$e_q^x := [1 + (1-q)x]^{\frac{1}{1-q}} \tag{54}$$

q-Algorithm

Consistently, non-linear generalized algebraic forms emerge,

q-sum

$$x \oplus_q y := x + y + (1 - q)xy \tag{55}$$

q-product

$$x \otimes_q y := (x^{1-q} + y^{1-q} - 1)^{1/(1-q)} \tag{56}$$

q-substraction

$$x \ominus_q y := \frac{x - y}{1 + (1 - q)y} \tag{57}$$

q-division

$$x \oslash y := (x^{1-q} - y^{1-q} + 1)^{1/(1-q)} \tag{58}$$

More are seen in [2, 1].

Conditional Probability

Consider two cases A and B with probabilities $P(A)$ and $P(B)$. The conditional probability of A under B is:

$$P(A|B) = \frac{P(AB)}{P(B)} \tag{59}$$

If all A_i are independent(mutually incompatible), $P(B) > 0$, then,

$$P(\sum A_i|B) = \sum P(A_i|B) \tag{60}$$

Moreover, for the case that $B \subset \bigcup A_i$,

$$P(B) = \sum P(A_i)P(B|A_i) \tag{61}$$

which is the total probability formula.

Bayes Formula

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum P(A_j)P(B|A_j)} \tag{62}$$

Grand Canonical Ensemble***

Think about the constraints with the escort distribution $P_i = p_i^q / \sum p_j^q$, similarly we have

$$p_i = \frac{1}{\Xi_q} e_q^{-\beta'(\epsilon_i - \mu N_i)} \tag{63}$$

Easy to prove the q-thermodynamics above,

$$N_q = \sum P_i N_i = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln_q \Xi_q \tag{64}$$

$$U_q = -\frac{\partial}{\partial \beta} \ln_q \Xi_q \tag{65}$$

Grand Canonical Ensemble***

Consider each distinct microstate i in energy levels l $(l = 1, 2, \cdots)$, that is, $\epsilon_i = \sum_l \varepsilon_l n_l$ and $N_i = \sum_l n_l$. So the q-grand partition function turns to be

$$\Xi_q = \sum_{\{n_l\}} e_q^{-\beta' \sum_l (n_l \varepsilon_l - \mu n_l)} = \sum_{\{n_l\}} \prod_l^q e_q^{-\beta'(\varepsilon_l - \mu) n_l}$$
$$= \prod_l^q \sum_{n_l} e_q^{-\beta'(\varepsilon_l - \mu) n_l} = \prod_l^q Z_q(l) \tag{66}$$

where $Z_q(l) = \sum_{n_l} e_q^{-\beta'(\varepsilon_l - \mu) n_l}$.

Grand Canonical Ensemble***

1. For Fermions,

$$Z_q(l) = 1 + e_q^{-\beta'(\varepsilon_l - \mu)} \tag{67}$$

so

$$\bar{n}_l = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln_q Z_q(l) = \frac{1}{\sum p_m^q}\left(\frac{1}{e_{2-q}^{\beta'(\varepsilon_l-\mu)}+1}\right)^q \rightarrow \left(\frac{1}{e_{2-q}^{\beta'(\varepsilon_l-\mu)}+1}\right)^q \tag{68}$$

2. For Bosons, similarly,

$$\bar{n}_l = \frac{1}{\sum p_m^q}\left(\frac{1}{e_{2-q}^{\beta'(\varepsilon_l-\mu)}-1}\right)^q \rightarrow \left(\frac{1}{e_{2-q}^{\beta'(\varepsilon_l-\mu)}-1}\right)^q \tag{69}$$

Asymptotic Approximation

$$\rho = e_q^{-\beta\mathcal{H}}/Z_q \tag{70}$$

$$Z_q = Tr\exp(\frac{1}{1-q}\ln[1-(1-q)\beta\mathcal{H}]) \approx Tr\exp(-\beta\mathcal{H} - \frac{1}{2}(1-q)\beta^2\mathcal{H}^2)$$
$$\approx Tr\exp(-\beta\mathcal{H})[1-\frac{1}{2}(1-q)\beta^2\mathcal{H}^2] = Z_{BG}[1-\frac{1}{2}(1-q)\beta^2\langle\mathcal{H}^2\rangle_{BG}] \tag{71}$$

Asymptotic Approximation

$$
\begin{aligned}
\langle \mathcal{O} \rangle_q &= Tr\rho^q \mathcal{O} = \langle \rho^{q-1} \mathcal{O} \rangle_1 = Z_q^{1-q} \langle \frac{\mathcal{O}}{1-(1-q)\beta\mathcal{H}} \rangle_1 \\
&= Z_q^{1-q} Z_q^{-1} Tr([1-(1-q)\beta\mathcal{H}]^{1/(1-q)} \frac{\mathcal{O}}{1-(1-q)\beta\mathcal{H}}) \\
&\approx Z_q^{1-q} Z_{BG}^{-1} [1+\frac{1}{2}(1-q)\beta^2 \langle \mathcal{H}^2 \rangle_{BG}] Tr\{\mathcal{O} \exp(\frac{q}{1-q} \ln[1-(1-q)\beta\mathcal{H}])\} \\
&\approx Z_q^{1-q} Z_{BG}^{-1} [1+\frac{1}{2} a\beta^2 \langle \mathcal{H}^2 \rangle_{BG}] Tr\{\mathcal{O} \exp(-(1-a)\beta\mathcal{H} + \frac{a}{2}(1-a)\beta^2\mathcal{H}^2)\} \\
&\approx Z_q^{1-q} Z_{BG}^{-1} [1+\frac{1}{2} a\beta^2 \langle \mathcal{H}^2 \rangle_{BG}] Tr\{\mathcal{O} \exp(-\beta\mathcal{H})[1 + a\beta\mathcal{H} - \frac{a}{2}\beta^2\mathcal{H}^2]\} \\
&= Z_{BG}^{1-q} [1+\frac{1}{2}(1-q)\beta^2 \langle \mathcal{H}^2 \rangle_{BG}] \\
&\quad \{\langle \mathcal{O} \rangle_{BG} + (1-q)\beta \langle \mathcal{O}\mathcal{H} \rangle_{BG} - \frac{1-q}{2}\beta^2 \langle \mathcal{O}\mathcal{H}^2 \rangle_{BG}\} \qquad (72)
\end{aligned}
$$

Asymptotic Approximation

Consider the system $\mathcal{H} = nh\nu$, with $\mathcal{O} = n$,

$$\langle n \rangle_q \approx \langle n \rangle_{BG} Z_{BG}^{1-q} \{1 + (1-q)x[\frac{\langle n^2 \rangle_{BG}}{\langle n \rangle_{BG}} + x(\langle n^2 \rangle_{BG} - \frac{\langle n^3 \rangle_{BG}}{\langle n \rangle_{BG}})]\} \tag{73}$$

deformed differential

From the generalized substraction rules we can easily have,

$$d_q x = (x + dx) \ominus_q x = \frac{dx}{1 + (1 - q)x} \tag{74}$$

$$d_\kappa x = (x + dx) \ominus_\kappa x = \frac{dx}{\sqrt{1 + \kappa^2 x^2}} \tag{75}$$

Easy to see $\frac{d}{d_q x} e_q^x = e_q^x$ and $\frac{d}{d_\kappa x} e_\kappa^x = e_\kappa^x$.

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