

Determination of electric dipole transitions in heavy quarkonia using potential non-relativistic QCD

arxiv:1701.02513, arxiv:1708.08465

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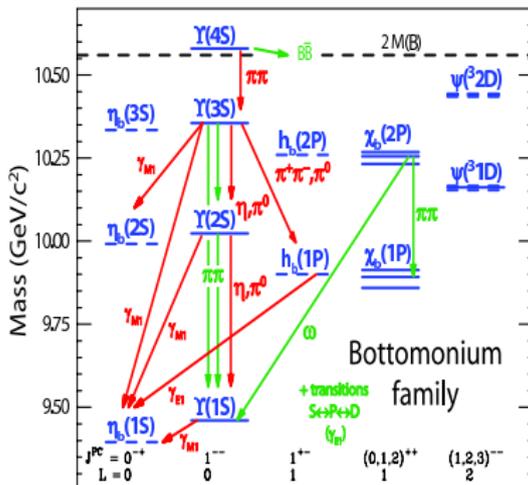
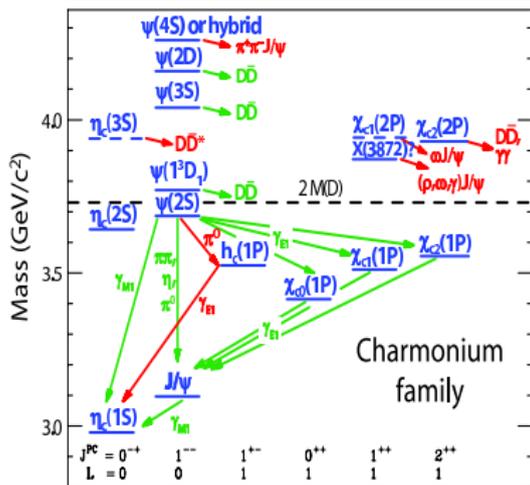
07.11.2017



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The heavy quarkonia

The Charmonium and bottomonium systems were discovered in the 1970s
Experimentally clear spectrum of narrow states below the open-flavor threshold



Eichten et al. *Rev. Mod. Phys.* 80 (2008)

- Heavy quarkonia are bound states of a heavy quark and its antiquark ($c\bar{c}$ charmonium and $b\bar{b}$ bottomonium).
- They can be classified in terms of the quantum numbers of a non-relativistic bound state \rightarrow Similar to the positronium $[(e^+e^-)$ -bound state] in QED.
- Heavy quarkonia are a very well established multi-scale system which can serve as an ideal laboratory for testing all regimes of QCD.

The nonrelativistic expansion

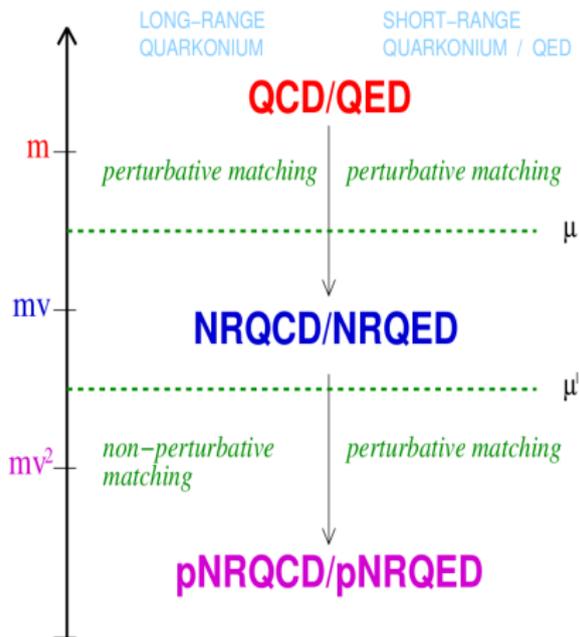
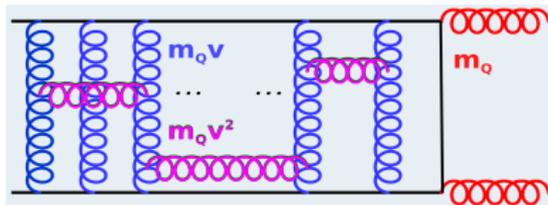
☞ Heavy quarkonium is a non-relativistic system:

$$v_c \sim 0.55, \quad v_b \sim 0.32 \quad (v_{\text{light}} = 1.0)$$

☞ Heavy quarkonium is a multi-scale system:

$$M \gg p \sim 1/r \sim Mv \gg E \sim Mv^2$$

☞ Scales are entangled in full QCD



☞ Systematic expansions in the small heavy-quark velocity v may be implemented at the Lagrangian level by constructing suitable effective field theories (EFTs):

- Expanding QCD in p/M , E/M leads to **NRQCD**.
→ Bodwin, Braaten, and Lepage. *Phys. Rev. D* 51 (1995)
- Expanding NRQCD in E/p leads to **pNRQCD**.
→ Brambilla et al. *Nucl. Phys. B* 566 (2000)

There is another scale in QCD: Λ_{QCD}

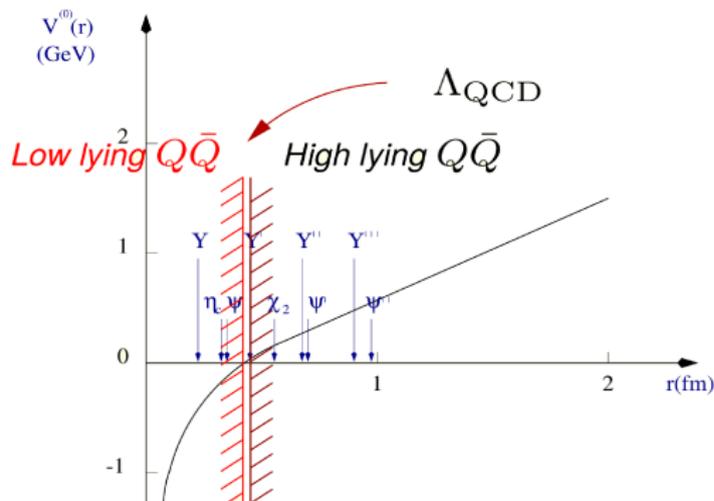
☞ The matching of QCD to NRQCD

$M \gg \Lambda_{\text{QCD}} \rightarrow$ Perturbative matching.

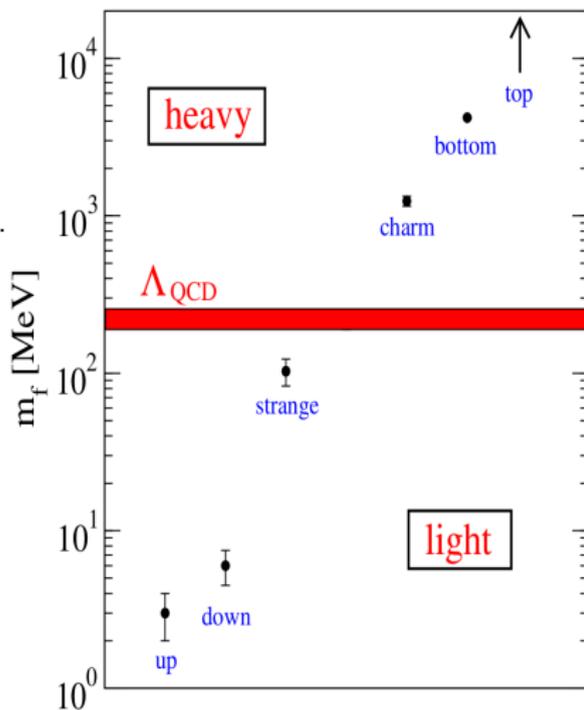
☞ The matching of NRQCD to pNRQCD

$p \sim 1/r \gg \Lambda_{\text{QCD}} \rightarrow$ Weak coupling regime.
Perturbative matching.

$p \sim 1/r \gtrsim \Lambda_{\text{QCD}} \rightarrow$ Strong coupling regime.
Nonperturbative matching.



Quarkmasses (in $\overline{\text{MS}}$ at $\mu=2$ GeV)



Potential nonrelativistic QCD at weak coupling

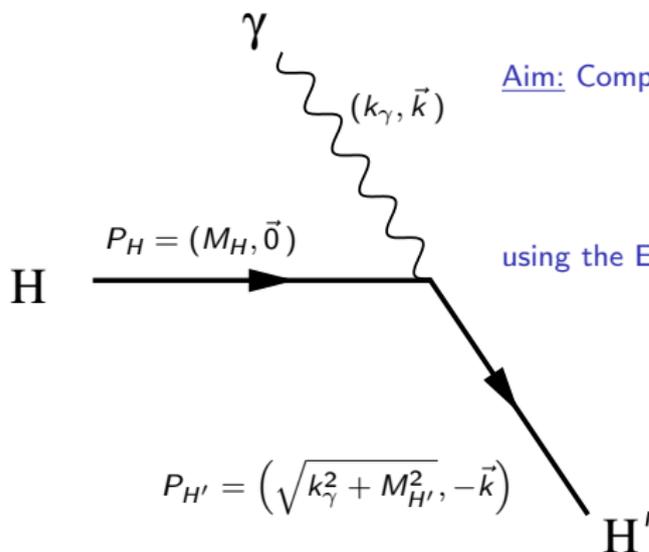
In summary...

- ☞ Provides a QM description from FT: the matching coefficients are the interaction potentials and the leading order dynamical equation is of the Schrödinger type.
- ☞ The degrees of freedom in pNRQCD (at weak coupling) are color singlet and octet fields and ultra-soft gluon fields.
- ☞ Account for non-potential terms as well. Singlet to Octet transitions via ultra-soft gluons provide loop corrections to the leading potential picture.
- ☞ The Quantum Mechanical divergences are canceled by the NRQCD matching coefficients.
- ☞ Poincaré invariance is realized via exact relations between different matching coefficients.

Potential non-relativistic QCD is the state-of-the-art tool for addressing Quarkonium bound state properties.

- ☞ Conventional meson spectrum: higher order perturbative corrections in v and α_s .
- ☞ Inclusive and semi-inclusive decays, E1 and M1 transitions, EM line-shapes.
- ☞ Doubly- and triply-heavy baryons.
- ☞ Precise extraction of Standard Model parameters: m_c , m_b , α_s , ...
- ☞ Exotic states such as gluelumps and hybrids.
- ☞ Properties of Quarkonium systems at finite temperature.

Electromagnetic transitions



Aim: Compute the electric dipole (E1) transitions:

$$\chi_{bJ}(1P) \rightarrow \gamma \Upsilon(1S) \text{ with } J = 0, 1, 2$$

$$h_b(1P) \rightarrow \gamma \eta_b(1S)$$

using the EFT called potential non-relativistic QCD.

- ☞ Electromagnetic transitions are often significant decay modes of heavy quarkonium states that are below the open-flavor threshold.
- ☞ They can be classified in a series of electric and magnetic multipoles. The electric dipole (E1) and the magnetic dipole (M1) are the most important ones.
- ☞ Large set of accurate experimental data taken by B-factories, τ -charm facilities and proton-proton colliders ask for a systematic and model-independent analysis.

Decay rate of electric dipole transitions

☞ The LO decay width, which scales as $\sim k_\gamma^3/(mv)^2$, is

$$\Gamma_{E1}^{(0)} = \frac{4}{9} \alpha_{e/m} e_Q^2 k_\gamma^3 \left[I_3^{(0)}(n1 \rightarrow n'0) \right]^2$$

☞ A probe of the internal structure of hadrons:

$$I_N^{(k)}(n\ell \rightarrow n'\ell') = \int_0^\infty dr r^N R_{n'\ell'}^*(r) \left[\frac{d^k}{dr^k} R_{n\ell}(r) \right]$$

☞ Up to order k_γ^3/m^2 , the expressions we use for the decay rates under study are:

$$\begin{aligned} \Gamma(n^3P_J \rightarrow n' ^3S_1 + \gamma) &= \Gamma_{E1}^{(0)} \left\{ 1 + R^{S=1}(J) - \frac{k_\gamma}{6m} - \frac{k_\gamma^2}{60} \frac{I_5^{(0)}(n1 \rightarrow n'0)}{I_3^{(0)}(n1 \rightarrow n'0)} \right. \\ &+ \left. \left[\frac{J(J+1)}{2} - 2 \right] \left[- \left(1 + \kappa_Q^{e/m} \right) \frac{k_\gamma}{2m} + \frac{1}{m^2} \left(1 + 2\kappa_Q^{e/m} \right) \frac{I_2^{(1)}(n1 \rightarrow n'0) + 2I_1^{(0)}(n1 \rightarrow n'0)}{I_3^{(0)}(n1 \rightarrow n'0)} \right] \right\} \\ \Gamma(n^1P_1 \rightarrow n' ^1S_0 + \gamma) &= \Gamma_{E1}^{(0)} \left\{ 1 + R^{S=0} - \frac{k_\gamma}{6m} - \frac{k_\gamma^2}{60} \frac{I_5^{(0)}(n1 \rightarrow n'0)}{I_3^{(0)}(n1 \rightarrow n'0)} \right\} \end{aligned}$$

- $R^{S=1}(J)$ and $R^{S=0}$ include the initial and final state corrections due to higher order potentials and higher order Fock states.
- The remaining corrections within the brackets are the result of taking into account $\mathcal{O}(v^2)$ -suppressed electromagnetic interaction terms in the Lagrangian.
- The terms proportional to the anomalous magnetic moment, κ_Q^{em} , go beyond our accuracy and are therefore not considered in the numerical analysis.

Coulomb-like potential in the LO Hamiltonian

☞ The states are solutions of the Schrödinger equation:

$$H^{(0)}\psi_{nlm}^{(0)}(\vec{r}) = E_n^{(0)}\psi_{nlm}^{(0)}(\vec{r})$$

☞ Only the Coulomb-like term of the static potential is exactly included in the LO Hamiltonian ($C_F = 4/3$):

$$H^{(0)} = -\frac{\vec{\nabla}^2}{2m_r} + V_s^{(0)}(r) = -\frac{\vec{\nabla}^2}{2m_r} - \frac{C_F\alpha_s}{r}$$

☞ Therefore, $\psi_{nlm}^{(0)}(\vec{r})$ and $E_n^{(0)}$ can be written in the hydrogen-like form:

$$\psi_{nlm}^{(0)}(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\Omega_r) = N_{n\ell}\rho_n^\ell e^{-\frac{\rho_n}{2}} L_{n-\ell-1}^{2\ell+1}(\rho_n)Y_{\ell m}(\Omega_r)$$

$$E_n^{(0)} = -\frac{m_r C_F^2 \alpha_s^2}{2n^2}$$

$$\rho_n = 2r/na, \quad a = 1/m_r C_F \alpha_s, \quad N_{n\ell} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]}}$$

These states are not eigenstates of the complete Hamiltonian due to higher order potentials and the presence of ultra-soft gluons that lead to singlet-to-octet transitions.

⇓

One has to consider corrections to the wave function which can contribute to the decay rate at the required order of precision!

Corrections due to higher order potentials (I)

☞ To account for $\mathcal{O}(v^2)$ -corrections to the decay width, we need to consider the complete Hamiltonian:

$$H = -\frac{\vec{\nabla}^2}{2m_r} + V_s(r) + \delta H$$

☞ The static potential is:

$$V_s(r) = V_s^{(0)}(r) \left[1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n a_n(\nu, r) \right]$$

☞ The known $\mathcal{O}(\alpha_s^n)$ radiative corrections to the LO static potential are:

$$a_1(\nu, r) = a_1 + 2\beta_0 \ln(\nu e^{\gamma_E} r)$$

$$a_2(\nu, r) = a_2 + \frac{\pi^2}{3} \beta_0^2 + (4a_1\beta_0 + 2\beta_1) \ln(\nu e^{\gamma_E} r) + 4\beta_0^2 \ln^2(\nu e^{\gamma_E} r)$$

$$\begin{aligned} a_3(\nu, r) = & a_3 + a_1\beta_0^2\pi^2 + \frac{5\pi^2}{6}\beta_0\beta_1 + 16\zeta_3\beta_0^3 \\ & + \left(2\pi^2\beta_0^3 + 6a_2\beta_0 + 4a_1\beta_1 + 2\beta_2 + \frac{16}{3}C_A^3\pi^2 \right) \ln(\nu e^{\gamma_E} r) \\ & + \left(12a_1\beta_0^2 + 10\beta_0\beta_1 \right) \ln^2(\nu e^{\gamma_E} r) + 8\beta_0^3 \ln^3(\nu e^{\gamma_E} r) \\ & + \delta a_3^{\text{US}}(\nu, \nu_{\text{US}}). \end{aligned}$$

Corrections due to higher order potentials (II)

☞ The term δH encodes the relativistic corrections to the static potential and to the non-relativistic kinetic operator:

$$\delta H = -\frac{\vec{\nabla}^4}{4m^3} + \frac{V^{(1)}}{m} + \frac{V_{\text{SI}}^{(2)}}{m^2} + \frac{V_{\text{SD}}^{(2)}}{m^2}$$

☞ At order $1/m^2$, we can split the contributions into spin-independent (SI) and spin-dependent (SD) terms:

$$V_{\text{SI}}^{(2)}(r) = V_r^{(2)}(r) + \frac{1}{2}\{V_{\rho^2}^{(2)}(r), -\vec{\nabla}^2\} + V_{L^2}^{(2)}(r) \vec{L}^2$$

$$V_{\text{SD}}^{(2)}(r) = V_{LS}^{(2)}(r) \vec{L} \cdot \vec{S} + V_{S^2}^{(2)}(r) \vec{S}^2 + V_{S_{12}}^{(2)}(r) S_{12}$$

☞ In the weak-coupling case, the above potentials read at leading (non-vanishing) order in perturbation theory:

$$V^{(1)}(r) = -\frac{C_F C_A \alpha_s^2}{2r^2}, \quad V_r^{(2)}(r) = \pi C_F \alpha_s \delta^{(3)}(\vec{r}),$$

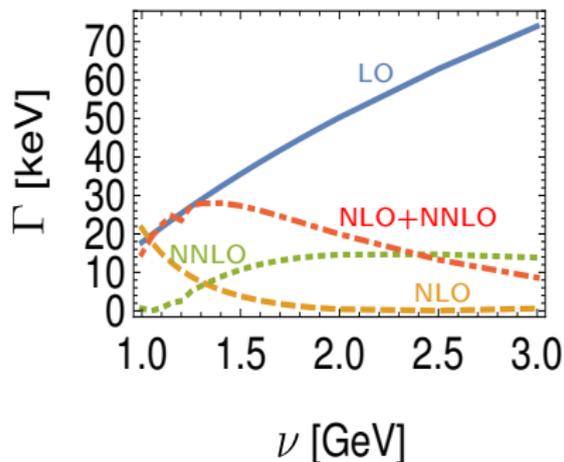
$$V_{\rho^2}^{(2)}(r) = -\frac{C_F \alpha_s}{r}, \quad V_{L^2}^{(2)}(r) = \frac{C_F \alpha_s}{2r^3},$$

$$V_{LS}^{(2)}(r) = \frac{3C_F \alpha_s}{2r^3}, \quad V_{S^2}^{(2)}(r) = \frac{4\pi C_F \alpha_s}{3} \delta^{(3)}(\vec{r}),$$

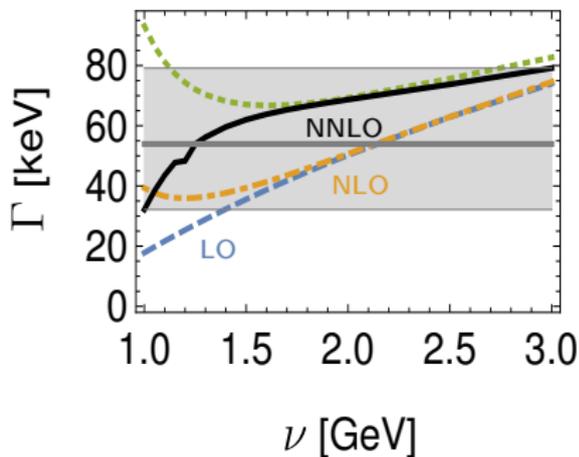
$$V_{S_{12}}^{(2)}(r) = \frac{C_F \alpha_s}{4r^3}.$$

Total matrix elements and decay widths [$\chi_{b1}(1P) \rightarrow \gamma \Upsilon(1S)$]

Individual contributions



Combined effect



Observations:

- LO, NLO and NNLO contributions depend strongly on the renormalization scale ν : running of $\alpha_s(\nu)$ and radiative terms ($\propto \log$ s) of the static potential.
- Subleading contributions are of the same order of magnitude than the leading order term.
- The renormalization scale dependence of the decay width is slightly reduced as the NLO and NNLO corrections are included.
- Setting the terms proportional to $a_1(\nu, r)$ and $a_2(\nu, r)$ to zero, the decay width exhibits a different ν -dependence (dotted green curve in the right panel).

Static potential exactly included in the LO Hamiltonian

Improving convergence: *The perturbative expansion in pNRQCD can be rearranged in such a way that the static potential is exactly included in the LO Hamiltonian.*

☞ The new expansion scheme was applied for:

- studying inclusive EM decays of heavy quarkonium (Kiyoyama, Pineda, and Signer. *Nucl. Phys.* B841 (2010)).
- computing magnetic dipole transitions between low-lying heavy quarkonia (Pineda and Segovia. *Phys. Rev.* D87 (2013)).

☞ The exact treatment of the soft logarithms of the static potential made the factorization scale dependence much smaller.

☞ We proceed herein to apply the same formalism to the electric dipole transitions under study

$$H_{\text{new}}^{(0)} = -\frac{\vec{\nabla}^2}{2m_r} + V_s^{(N)}(r),$$

where the static potential is again approximated by a polynomial of order $N + 1$ in powers of α_s

$$V_s^{(N)}(r) = -\frac{C_F \alpha_s(\nu)}{r} \left[1 + \sum_{k=1}^N \left(\frac{\alpha_s}{4\pi} \right)^k a_k(\nu, r) \right]$$

Subtraction of renormalon effects

☞ One first makes the substitution

$$(m, V_s(r)) = (m_X + \delta m_X, V_{s,X}(r) - 2\delta m_X),$$

where

$$\delta m_X^{(N)}(\nu_f) = \nu_f \sum_{k=0}^N \delta m_X^{(k)} \left(\frac{\nu_f}{\nu} \right) \alpha_s^{k+1}(\nu)$$

represents a residual mass that encodes the pole mass renormalon contribution, and X stands for the specific renormalon subtraction scheme.

☞ We mainly use the RS' scheme:

$$\delta m_{\text{RS}'}^{(0)} = 0$$

$$\delta m_{\text{RS}'}^{(1)} \left(\frac{\nu_f}{\nu} \right) = N_m \frac{\beta_0}{2\pi} S(1, b)$$

$$\delta m_{\text{RS}'}^{(2)} \left(\frac{\nu_f}{\nu} \right) = N_m \left(\frac{\beta_0}{2\pi} \right) \left[S(1, b) \frac{2d_0(\nu, \nu_f)}{\pi} + \left(\frac{\beta_0}{2\pi} \right) S(2, b) \right]$$

$$\delta m_{\text{RS}'}^{(3)} \left(\frac{\nu_f}{\nu} \right) = N_m \left(\frac{\beta_0}{2\pi} \right) \left[S(1, b) \frac{3d_0^2(\nu, \nu_f) + 2d_1(\nu, \nu_f)}{\pi^2} + \left(\frac{\beta_0}{2\pi} \right) S(2, b) \frac{3d_0(\nu, \nu_f)}{\pi} + \left(\frac{\beta_0}{2\pi} \right)^2 S(3, b) \right]$$

where

$$d_k(\nu, \nu_f) = \beta_k / 2^{1+2k} \ln \frac{\nu}{\nu_f}, \quad S(n, b) = \sum_{k=0}^n c_k \frac{\Gamma(n+1+b-k)}{\Gamma(1+b-k)},$$

and b , c_0 , c_1 and c_2 are coefficients that only depend on the β 's, and N_m is a constant.

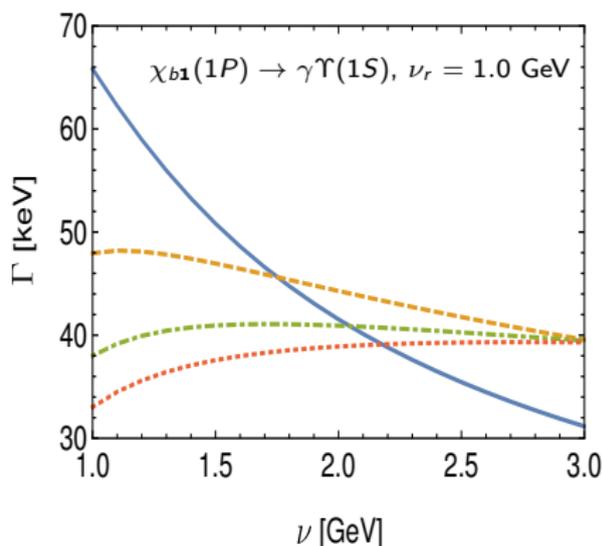
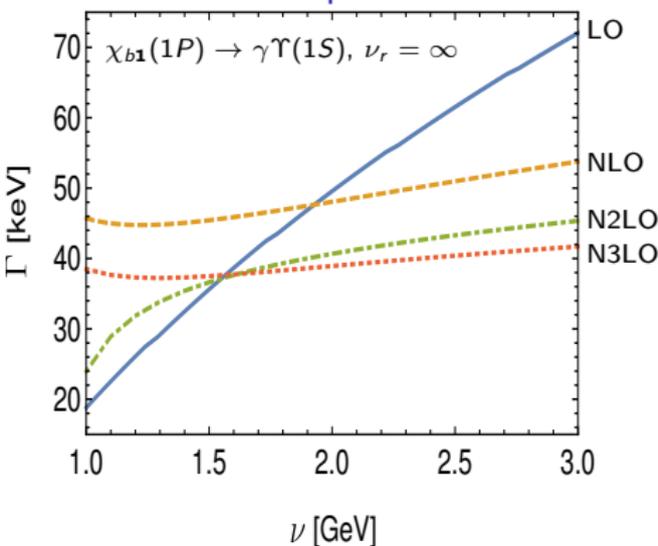
Renormalization group improvement

The last improvement to the static potential that we consider herein is the absorption of the large logarithms into the running coupling constant.

$$V_{s,RS'}^{(N)}(\nu, \nu_f, \nu_r, r) = \begin{cases} V_s^{(N)} + 2\delta m_{RS'}^{(N)}|_{\nu=\nu} \equiv \sum_{k=0}^N V_{s,RS'}^{(k)} \alpha_s^{k+1}(\nu) & \text{if } r > \nu_r^{-1}, \\ V_s^{(N)} + 2\delta m_{RS'}^{(N)}|_{\nu=1/r} \equiv \sum_{k=0}^N V_{s,RS'}^{(k)} \alpha_s^{k+1}(1/r) & \text{if } r < \nu_r^{-1}, \end{cases}$$

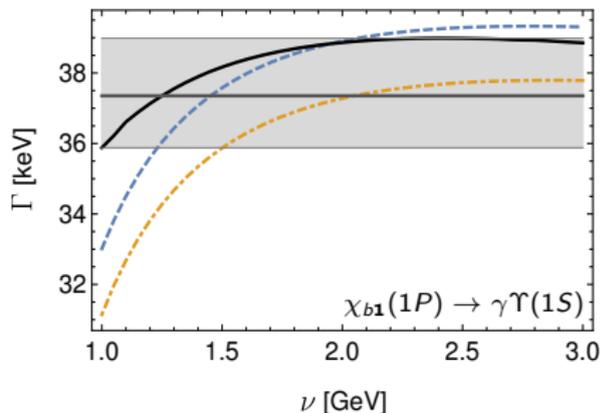
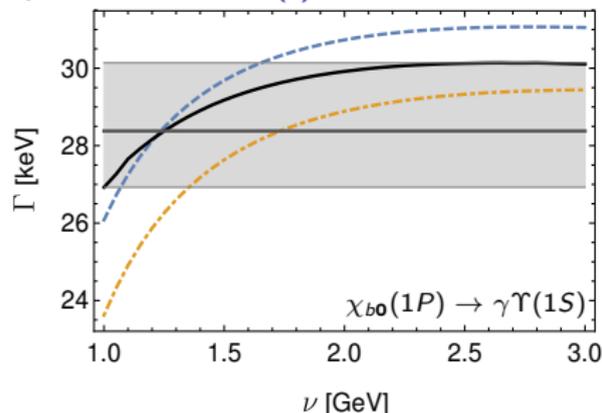
- ☞ This has to be done carefully in order not to destroy the renormalon cancellation achieved order by order in α_s .
- ☞ Large values of ν_f imply a large infrared cutoff. This makes the scheme become closer to a \overline{MS} -like scheme. Such schemes still achieve renormalon cancellation but jeopardize the power counting.
- ☞ Low values of ν_f are preferred with the constraint that one should still obtain the renormalon cancellation.
- ☞ By also taking a low value of $\nu_r = \nu_f$ we find that the convergence is accelerated and the scale dependence is significantly reduced.

Effects due to improvements



- One gets a strong dependence on the factorization scale when solving the Schrödinger equation with only the Coulomb-like term of the static potential.
- The ν -scale dependence becomes mild as NLO, NNLO and NNNLO terms of the static potential are included in the Schrödinger equation.
- The convergence of the perturbative series has improved considerably with respect the purely analytical case. This feature is clearly seen at large values of ν .
- The convergence is seen in the whole range of ν once the correct logarithmically modulated short distance behavior of the static potential is included properly.

Improved results (I)



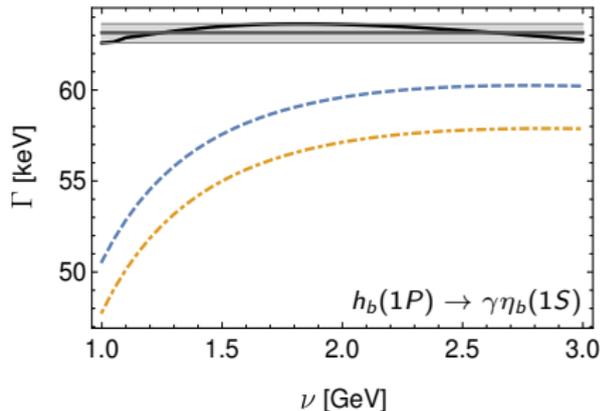
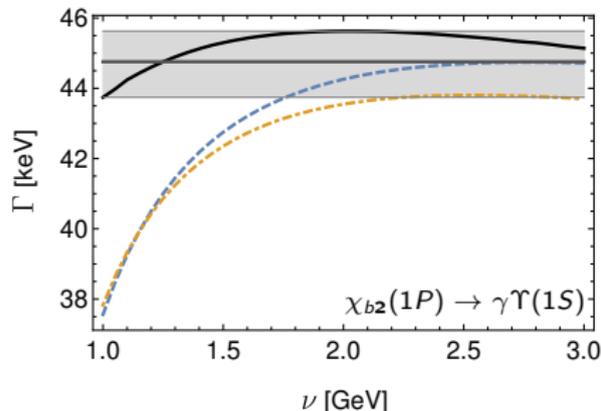
Legend:

- *Dashed blue curve*: leading order non-relativistic decay rate.
- *Dot-dashed orange curve*: taking into account the relativistic contributions stemming from higher order electromagnetic operators.
- *Solid black*: including the relativistic corrections to the wave function of the initial and final states.

Observations:

- The leading order decay width depends weakly on ν , this feature is translated to the cases in which relativistic corrections are included.
- Both relativistic contributions to the leading order decay rate are much more under control than in the purely analytical case.
- Higher order EM operators tend to diminish the LO decay rate whereas the opposite effect is found when corrections to the wave functions are incorporated.

Improved results (II)



Legend:

- Dashed blue curve: leading order non-relativistic decay rate.
- Dot-dashed orange curve: taking into account the relativistic contributions stemming from higher order electromagnetic operators.
- Solid black: including the relativistic corrections to the wave function of the initial and final states.

Observations:

- Similar conclusions to the ones already mentioned in the former slide apply here.
- The relativistic corrections stemming from higher order EM operators are clearly smaller, whereas the effect due to corrected wave functions is 2-3 times bigger.
- The correction to the decay rate due to the modification of the initial and final state wave functions is much larger for the $h_b(1P) \rightarrow \gamma\Upsilon(1S)$ transition.

Improved results (III)

Final values for the decay widths:

$$\Gamma(\chi_{b0}(1P) \rightarrow \gamma\Upsilon(1S)) = 28_{-1}^{+2} \text{ keV},$$

$$\Gamma(\chi_{b1}(1P) \rightarrow \gamma\Upsilon(1S)) = 37_{-1}^{+2} \text{ keV},$$

$$\Gamma(\chi_{b2}(1P) \rightarrow \gamma\Upsilon(1S)) = 45_{-1}^{+1} \text{ keV},$$

$$\Gamma(h_b(1P) \rightarrow \gamma\eta_b(1S)) = 63_{-1}^{+1} \text{ keV}.$$

Comparison with several other theoretical approaches:

Mode	LO	NNLO	CQM	R	GI	BT	LFQM	SNR _{0/1}
$\chi_{b0}(1P) \rightarrow \gamma\Upsilon(1S)$	28.47	28.38	28.07	29.9	23.8	25.7	-	-
$\chi_{b1}(1P) \rightarrow \gamma\Upsilon(1S)$	36.04	37.35	35.66	36.6	29.5	29.8	-	-
$\chi_{b2}(1P) \rightarrow \gamma\Upsilon(1S)$	41.00	44.76	39.15	40.2	32.8	33.0	-	-
$h_b(1P) \rightarrow \gamma\eta_b(1S)$	55.22	63.15	43.7	52.6	35.7	-	37.5	55.8/36.3

Predicted total decay widths:

Mode	$\mathcal{B} = \Gamma_i/\Gamma$ [%]	Γ_i	Γ
$\chi_{b0}(1P) \rightarrow \gamma\Upsilon(1S)$	$(1.76 \pm 0.35)\%$	28_{-1}^{+2} keV	$1.6_{-0.3}^{+0.3}$ MeV
$\chi_{b1}(1P) \rightarrow \gamma\Upsilon(1S)$	$(33.9 \pm 2.2)\%$	37_{-1}^{+2} keV	110_{-8}^{+9} keV
$\chi_{b2}(1P) \rightarrow \gamma\Upsilon(1S)$	$(19.1 \pm 1.2)\%$	45_{-1}^{+1} keV	234_{-16}^{+15} keV
$h_b(1P) \rightarrow \gamma\eta_b(1S)$	$(52_{-5}^{+6})\%$	63_{-1}^{+1} keV	121_{-12}^{+14} keV

Epilogue

We have computed the electric dipole transitions $\chi_{bJ}(1P) \rightarrow \gamma\Upsilon(1S)$ with $J = 0, 1, 2$ and $h_b(1P) \rightarrow \gamma\eta_b(1S)$ within the weak coupling version of a low-energy effective field theory called potential non-relativistic QCD.

☞ Only the Coulomb term of the static potential is included in the LO Hamiltonian:

- The decay width reveals a severe dependence on the factorization scale ν : running of $\alpha_s(\nu)$ and radiative terms ($\propto \log s$) of the static potential.
- Most of the corrections to the decay rate induced by $1/m$ and $1/m^2$ potentials are relatively small and behave well with respect the renormalization scale ν .
- The general convergence of the perturbative series for all the studied electric dipole transitions is not as good as expected.

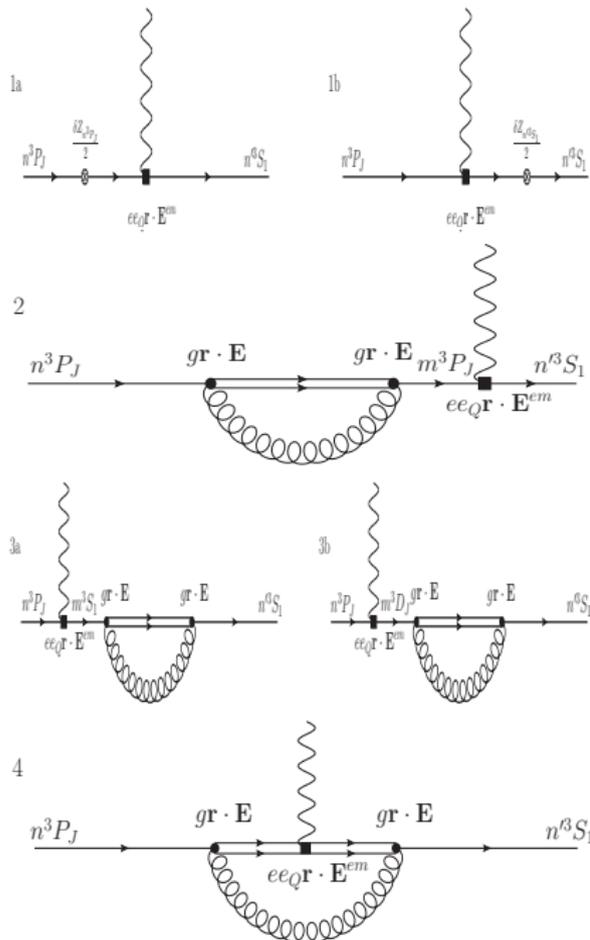
☞ The static potential is exactly included in the leading order Hamiltonian:

- The LO decay rate depends weakly on the factorization scale and this is translated to the result in which relativistic corrections are included.
- Relativistic corrections induced by higher order electromagnetic operators and wave functions are much more under control.
- Our results are in fair agreement with the ones reported by other theoretical approaches [$\Gamma(h_b(1P) \rightarrow \gamma\eta_b(1S))$ is slightly higher].
- Total decay widths of the $\chi_{bJ}(1P)$ -family and the $h_b(1P)$ meson are predicted.

BACKUP SLIDES

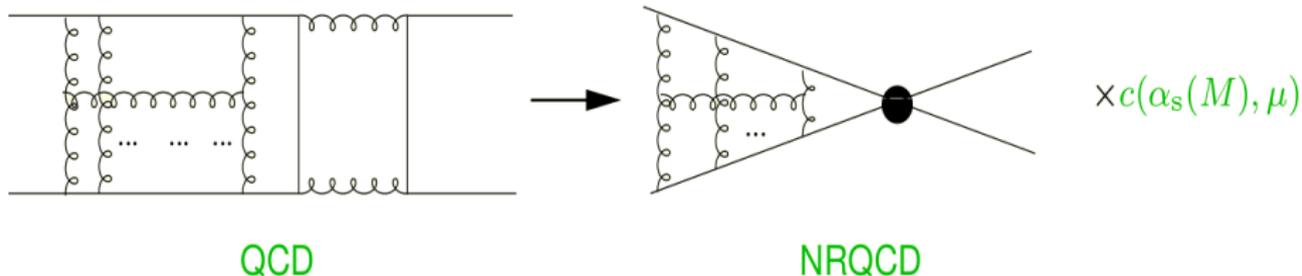
Corrections due to higher order Fock states

- Diagrams 1a and 1b correspond to the renormalization of the initial and final wave functions.
- Diagrams 2, 3a and 3b account for the correction of the initial and final wave functions due to the presence of octet states.
- Diagram 4 represents an electric dipole transition mediated by the intermediate octet state.
- The first two diagrams contribute to relative order $\Lambda_{\text{QCD}}^2/(mv)^2$ whereas the remaining ones scales as $\Lambda_{\text{QCD}}^3/(m^3v^4)$.
- We do not consider these contributions herein because in the (strict) weak-coupling regime, $E \sim mv^2 \gg \Lambda_{\text{QCD}}$, one can argue that they should be negligible.



Nonrelativistic QCD

Physics at the scale M : Quarkonium annihilation and production.



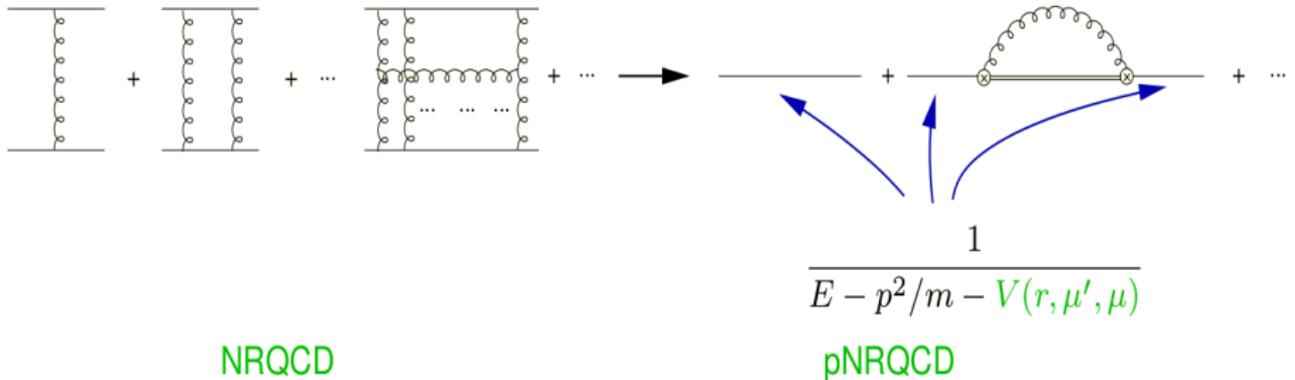
The effective Lagrangian is organized as an expansion in $1/M$ and $\alpha_s(M)$:

$$\mathcal{L}_{\text{NRQCD}} = \sum_n \frac{c_n(\alpha_s(M), \mu)}{M^n} \times \mathcal{O}_n(\mu, Mv, Mv^2, \dots)$$

- $\mathcal{L}_{\text{NRQCD}}$ is made of all low-energy operators \mathcal{O}_n that may be built from the effective degrees of freedom and are consistent with the symmetries of \mathcal{L}_{QCD} .
- The Wilson coefficients c_n encode the high energy physics. They are calculated by imposing that $\mathcal{L}_{\text{NRQCD}}$ and \mathcal{L}_{QCD} describe the same physics at $\mu = M$.

Potential nonrelativistic QCD at weak coupling

☞ Physics at the scale Mv : Quarkonium formation.



☞ The effective Lagrangian is organized as an expansion in $1/M$, $\alpha_s(M)$ and $1/p \sim r$:

$$\mathcal{L}_{\text{pNRQCD}} = \int d^3r \sum_n \sum_k \frac{c_n(\alpha_s(M), \mu)}{M^n} \times V_{n,k}(r, \mu', \mu) r^k \times \mathcal{O}_k(\mu', Mv^2, \dots)$$

where a multipole expansion of the gluon field has been performed.

☞ The Wilson coefficients of pNRQCD depends on the distance r (and scales μ, μ'):

- $V_{n,0}$ are the potentials in the Schrödinger equation.
- $V_{n,k \neq 0}$ are the couplings with the low-energy degrees of freedom, which provide corrections to the potential picture.

Standard quantum mechanics perturbation theory

First order correction to the wave function ($|n\ell\rangle^{(0)} \equiv |n\ell\rangle$ and $E_n^{(0)} \equiv E_n$):

$$|n\ell\rangle^{(1)} = \sum_{n' \neq n} \frac{\langle n'\ell | V | n\ell \rangle}{E_n - E_{n'}} |n'\ell\rangle$$

Second order correction to the wave function:

$$|n\ell\rangle^{(2)} = \sum_{k_1 \neq n} \left[\sum_{k_2 \neq n} \frac{\langle k_1\ell | V | k_2\ell \rangle \langle k_2\ell | V | n\ell \rangle}{(E_n - E_{k_1})(E_n - E_{k_2})} - \frac{\langle k_1\ell | V | n\ell \rangle \langle n\ell | V | n\ell \rangle}{(E_n - E_{k_1})^2} \right] |k_1\ell\rangle \\ - \frac{1}{2} \sum_{k_2 \neq n} \frac{|\langle k_2\ell | V | n\ell \rangle|^2}{(E_n - E_{k_2})^2} |n\ell\rangle$$

Identification with the Coulomb Green function:

$$\sum_{n'} \frac{|n'\ell\rangle \langle n'\ell|}{E_n - E_{n'}} - \sum_{n'=n} \frac{|n'\ell\rangle \langle n'\ell|}{E_n - E_{n'}} = (-1) \times \lim_{E \rightarrow E_n} \left(\frac{1}{E - H} - \frac{P_{n\ell}}{E - E_n} \right)$$

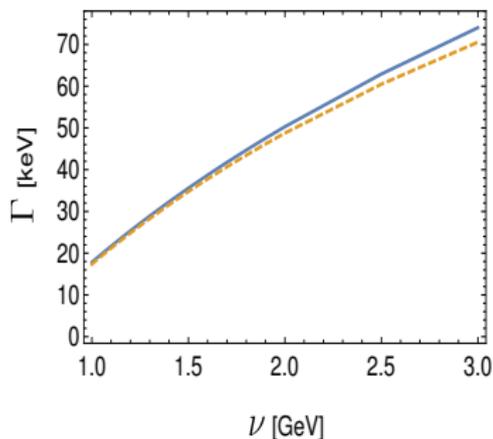
Definition of the Coulomb Green function:

$$G(\vec{r}_1, \vec{r}_2; E) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_{\ell}(\hat{r}_1 \cdot \hat{r}_2) G_{\ell}(r_1, r_2; E) \\ G_{\ell}(r_1, r_2; E) = \sum_{\nu=\ell+1}^{\infty} m_r a^2 \left(\frac{\nu^4}{\lambda} \right) \frac{R_{\nu\ell}(\rho_{\lambda,1}) R_{\nu\ell}(\rho_{\lambda,2})}{\nu - \lambda}$$

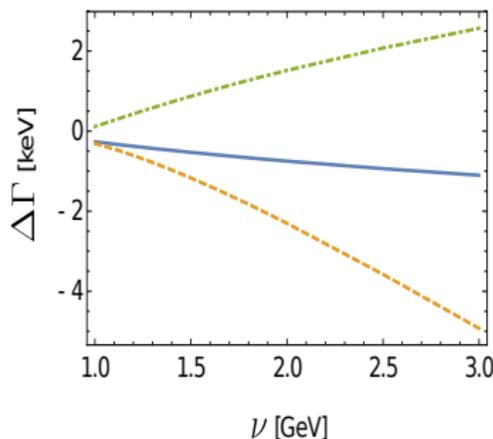
Higher order electromagnetic operators [$\chi_{b1}(1P) \rightarrow \gamma \Upsilon(1S)$]

$$\Gamma_{n^3P_J \rightarrow n'^3S_1} = \Gamma_{E_1}^{(0)} \left[1 + R^{S=1}(J) - \frac{k_\gamma}{6m} - \frac{k_\gamma^2}{60} \frac{I_5^{(0)}(n1 \rightarrow n'0)}{I_3^{(0)}(n1 \rightarrow n'0)} \right. \\ \left. + \left(\frac{J(J+1)}{2} - 2 \right) \left(- \left(1 + \cancel{\kappa \frac{e/m}{Q}} \right) \frac{k_\gamma}{2m} + \frac{1}{m^2} \left(1 + \cancel{2\kappa \frac{e/m}{Q}} \right) \frac{I_2^{(1)}(n1 \rightarrow n'0) + 2I_1^{(0)}(n1 \rightarrow n'0)}{I_3^{(0)}(n1 \rightarrow n'0)} \right) \right]$$

Combined effect



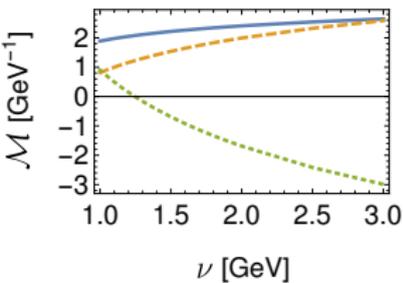
Individual contributions



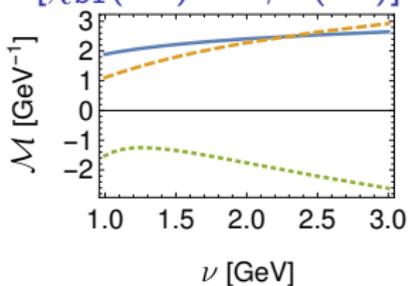
Observations:

- The leading order decay width depends strongly on the typical scale ν of the heavy quarkonia involved in the reactions.
- The relativistic effects from higher order electromagnetic terms are small, as expected. This can be understood analyzing each contribution separately.

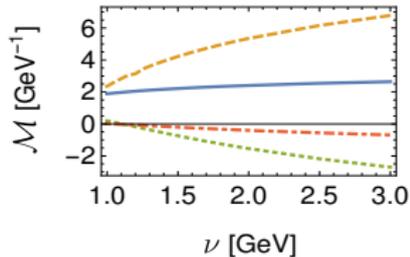
Wave function corrections [$\chi_{b1}(1P) \rightarrow \gamma \Upsilon(1S)$]



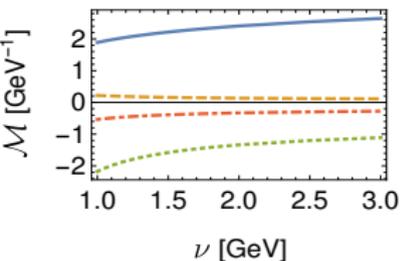
- \mathcal{M}_{LO}
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(V_0^{(1)})$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(V_0^{(1)})$



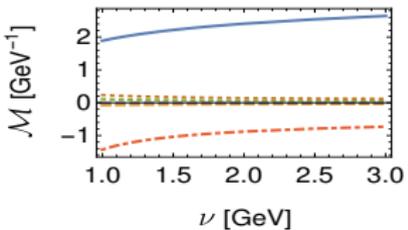
- \mathcal{M}_{LO}
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(V_0^{(2)})$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(V_0^{(2)})$



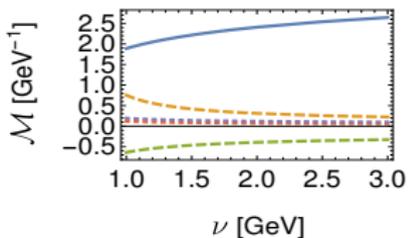
- \mathcal{M}_{LO}
- $\mathcal{M}_{\text{NNLO}}^{\text{ini,ini}}(V_0^{(1)})$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin,fin}}(V_0^{(1)})$
- $\mathcal{M}_{\text{NNLO}}^{\text{ini,fin}}(V_0^{(1)})$



- \mathcal{M}_{LO}
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(V_r^{(1)})$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(V_r^{(1)})$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(V_r^{(2)})$



- \mathcal{M}_{LO}
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(L^2)$
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(LS)$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(S^2)$
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(S_{12})$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(S_{12})$



- \mathcal{M}_{LO}
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(p^2)$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(p^2)$
- $\mathcal{M}_{\text{NNLO}}^{\text{ini}}(p^4)$
- $\mathcal{M}_{\text{NNLO}}^{\text{fin}}(p^4)$