Singularity and Stability in a Periodical System

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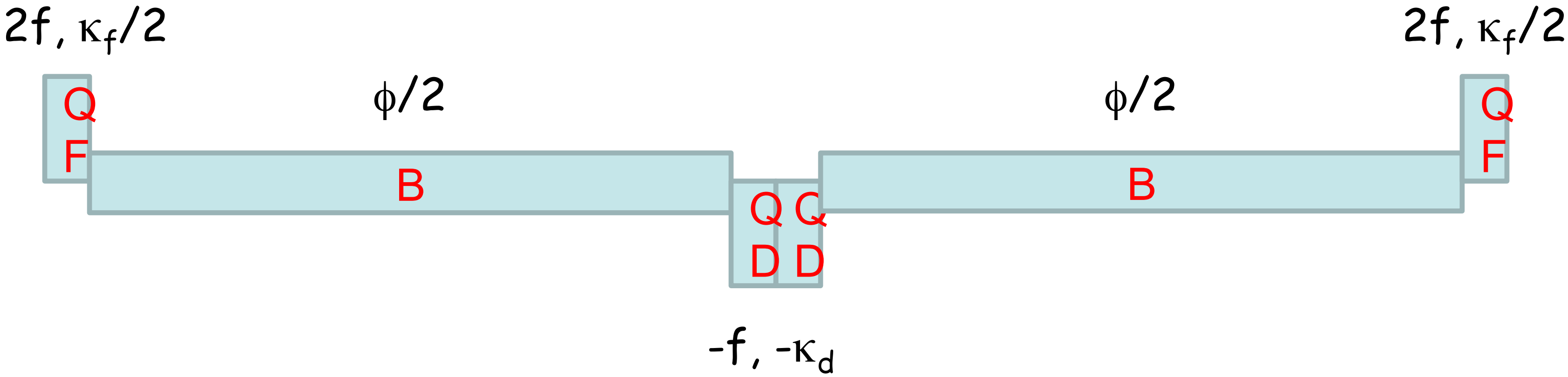
Periodic Cell: FODO



How to compute the Courant-Synder parameters and dispersions? For simplicity, we can use thin-lens approximation for quadrupoles, and small angle approximation for dipoles, and no gaps between any magnets.

What's the problem if we use these FODO cells to build entire ring? Why do we need to introduce sextupole magnets? How they work?

Can we understand nonlinear beam dynamics?

Hamiltonian and Transfer Map for a Sector Bend Magnet

Use s as the independent variable, Hamiltonian in the paraxial approximation is given by [1,2]

$$H_{Sbend} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x}{\rho}\delta.$$

Solving the Hamiltonian equations, we obtain the transfer map of a sector bend:

$$M_1 = x + \frac{L}{1+\delta}(p_x + \frac{\theta\delta}{2}),$$

$$M_2 = p_x + \theta\delta,$$

$$M_3 = y + \frac{Lp_y}{1+\delta},$$

$$M_4 = p_y,$$

$$M_5 = \delta,$$

$$M_6 = \ell + \theta x + \frac{L}{2(1+\delta)^2}[p_x^2 + p_y^2 + \theta(1+2\delta)(p_x + \frac{\theta\delta}{3})],$$

where L is the length and θ = L/ρ the bending angle of the dipole.

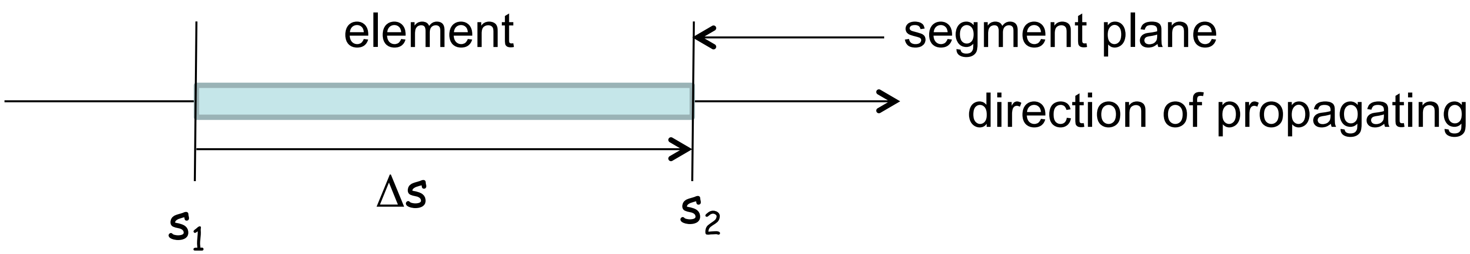
Transfer Map
for Thin Quadrupole and Sextupole

Transfer map is given by a kick:

$$
\begin{aligned}
M_1 &= x, \\
M_2 &= p_x - \frac{x}{f} - \frac{\kappa}{2}(x^2 - y^2), \\
M_3 &= y, \\
M_4 &= p_y + \frac{y}{f} + \kappa x y, \\
M_5 &= \delta,, \\
M_6 &= \ell,
\end{aligned}
$$

where f is the focusing (in horizontal plane) length of quadrupole and κ is the integrated strength of sextupole.

Concept of Transfer Map



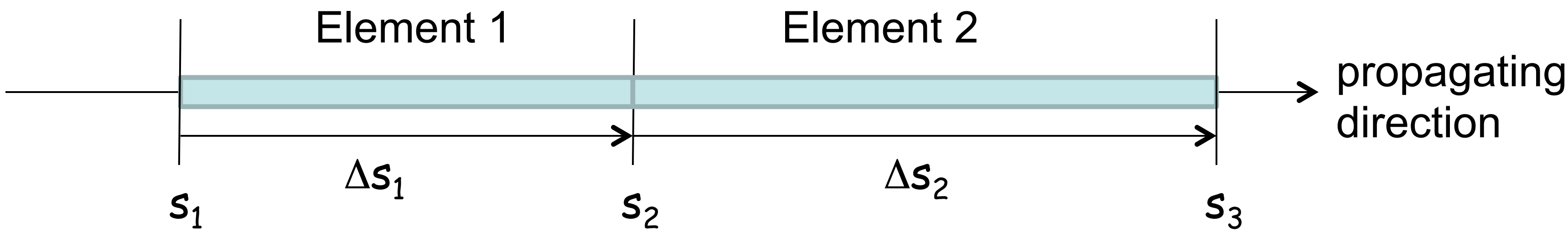
$$z(s_1) = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_1} \quad M_{1\to 2} \quad \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_2} = z(s_2)$$

$$z(s_2) = M_{1\to 2}(z(s_1)).$$

abbreviated map notation

A set (six) of functions of canonical coordinates. It's called symplectic if its Jacob is symplectic.

Concatenation of Maps



If we have the transfer map for each individual elements:

$$z(s_2) = M_{1\to 2}(z(s_1)),$$

$$z(s_3) = M_{2\to 3}(z(s_2)).$$

Then the transfer map for the combined elements is given by

$$z(s_3) = M_{1\to 2} \circ M_{2\to 3}(z(s_1)) \equiv M_{2\to 3}(M_{1\to 2}z(s_1)),$$

$M_{1\to 3}$

nested functions

Property of Symplectic Maps

Jacobian of a map:

$$J(M)=\begin{pmatrix} \frac{\partial M_1}{\partial x} & \frac{\partial M_1}{\partial p_x} & \frac{\partial M_1}{\partial y} & \frac{\partial M_1}{\partial p_y} & \frac{\partial M_1}{\partial \delta} & \frac{\partial M_1}{\partial \ell} \\ \frac{\partial M_2}{\partial x} & \frac{\partial M_2}{\partial p_x} & \frac{\partial M_2}{\partial y} & \frac{\partial M_2}{\partial p_y} & \frac{\partial M_2}{\partial \delta} & \frac{\partial M_2}{\partial \ell} \\ \frac{\partial M_3}{\partial x} & \frac{\partial M_3}{\partial p_x} & \frac{\partial M_3}{\partial y} & \frac{\partial M_3}{\partial p_y} & \frac{\partial M_3}{\partial \delta} & \frac{\partial M_3}{\partial \ell} \\ \frac{\partial M_4}{\partial x} & \frac{\partial M_4}{\partial p_x} & \frac{\partial M_4}{\partial y} & \frac{\partial M_4}{\partial p_y} & \frac{\partial M_4}{\partial \delta} & \frac{\partial M_4}{\partial \ell} \\ \frac{\partial M_5}{\partial x} & \frac{\partial M_5}{\partial p_x} & \frac{\partial M_5}{\partial y} & \frac{\partial M_5}{\partial p_y} & \frac{\partial M_5}{\partial \delta} & \frac{\partial M_5}{\partial \ell} \\ \frac{\partial M_6}{\partial x} & \frac{\partial M_6}{\partial p_x} & \frac{\partial M_6}{\partial y} & \frac{\partial M_6}{\partial p_y} & \frac{\partial M_6}{\partial \delta} & \frac{\partial M_6}{\partial \ell} \end{pmatrix}$$

constant J matrix:

$$J=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Symplectic condition:

$$J(M)\bullet J\bullet J(M)^T=J$$

Specifically, R-matrix [3] is given by $J(M)|_{x=px=y=py=d=l=0.}$ So it is sympletic as well.

Courant-Snyder Parameters

Matrix of periodic system: [1]

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Rotation matrix:

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

We have:

$$M = ARA^{-1}$$

where A^{-1} is a transformations from physical to normalized coordinates:

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}, A = \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

A is an "ascript" and is not unique. Since two-dimensional rotational group is commutative AR(q) is also an ascript. Courant and Synder choose to have A_{12}=0.

Linear Optics

Using the transfer map of the cell and the R-matrix, we find that the betatron phase advances in both planes are the same $\mu_x=\mu_y=\mu$ and given by,

$$\sin\frac{\mu}{2} = \frac{L}{4f},$$

where L is the cell length. The beta functions at the beginning:

$$\beta_x = \frac{L(1+\sin\frac{\mu}{2})}{\sin\mu}, \beta_y = \frac{L(1-\sin\frac{\mu}{2})}{\sin\mu},$$

and the periodical dispersion:

$$\eta_0 = \frac{L\phi(1+\frac{1}{2}\sin\frac{\mu}{2})}{4\sin^2\frac{\mu}{2}}.$$

No surprises. They agree with the well-known results.

To the first-order of δ

Make a similar transformation to obtain the feed-down effects from the dispersive orbit,

$$M_{\eta 0} = A_{\eta 0} \circ M_{cell} \circ A^{-1}_{\eta 0},$$

where the dispersive map is given by,

$$A_1 = x + \eta_0 \delta,$$
$$A_2 = p_x,$$
$$A_3 = y,$$
$$A_4 = p_y,$$
$$A_5 = \delta,$$
$$A_6 = \ell - \eta_0 p_x,$$

Introducing a Jacobian operator, we find the matrix with dependence of δ:

$$R_{\eta 0}(\delta) = J[M_{\eta 0}] \equiv J(M_{\eta 0})\,|_{x=px=y=py=l=0}$$

Like the R-matrix, it is symplectic.

Linear Chromaticity

Betatron phase advances up to the first-order of δ:

$$\mu_x(\delta) = \mu - \tan\frac{\mu}{2}\left[2 - \frac{1}{4\sin\frac{\mu}{2}}\left(\frac{1}{2} + \frac{1}{\sin^2\frac{\mu}{2}}\right)(\kappa_f - \kappa_d) fL\phi - \frac{3}{8\sin^2\frac{\mu}{2}}(\kappa_f + \kappa_d) fL\phi\right]\delta,$$

$$\mu_y(\delta) = \mu - \tan\frac{\mu}{2}\left[2 - \frac{1}{4\sin\frac{\mu}{2}}\left(\frac{1}{2} - \frac{1}{\sin^2\frac{\mu}{2}}\right)(\kappa_f - \kappa_d) fL\phi - \frac{1}{8\sin^2\frac{\mu}{2}}(\kappa_f + \kappa_d) fL\phi\right]\delta,$$

where κ_f, κ_d are the integrated strengths of the sextupoles. We can set their values:

$$\kappa_f = \frac{4\sin^2\frac{\mu}{2}}{fL\phi\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}, \kappa_d = \frac{4\sin^2\frac{\mu}{2}}{fL\phi\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)},$$

to cancel the linear chromaticities in both planes. The settings are expected for the local compensation to the chromatic errors by quadrupoles.

To the second-order of δ

Make a similar transformation to obtain the feed-down effects from the dispersive orbit,

$$M_{\eta 1} = A_{\eta 1} \circ A_{\eta 0} \circ M_{cell} \circ A^{-1}_{\eta 0} \circ A^{-1}_{\eta 1},$$

where the new dispersive map is given by,

$$A_1 = x + \eta_1 \frac{\delta^2}{2},$$
$$A_2 = p_x,$$
$$A_3 = y,$$
$$A_4 = p_y,$$
$$A_5 = \delta,$$
$$A_6 = \ell - \eta_1 p_x \delta,$$

where the first-order dispersion is found in the same way as the zeroth-order one, we have

$$\eta_1 = -\frac{f\phi}{2}.$$

Using the Jacobian operator, we find the matrix with dependence of d:

$$R_{\eta 1}(\delta) = \mathrm{J}\ [\mathrm{M}\ _{\eta 1}]$$

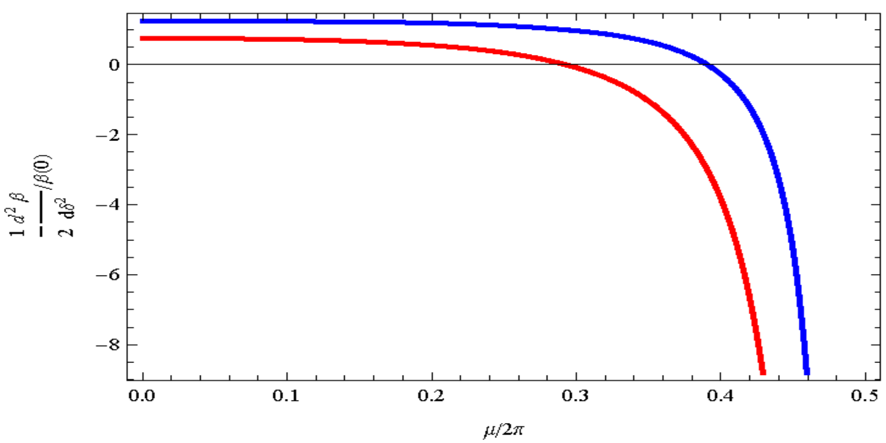
Like the R-matrix, it is symplectic.

Second-Order Beta Beating

The beta functions at the beginning of the cell, up to the second-order of δ:

$$\beta_x(\delta) = \beta_x[1 - \delta + \frac{10 - 13\sin^2\frac{\mu}{2} - \sin^3\frac{\mu}{2} + 3\sin^4\frac{\mu}{2}}{2(4 - 5\sin^2\frac{\mu}{2} + \sin^4\frac{\mu}{2})}\delta^2],$$

$$\beta_y(\delta) = \beta_y[1 - \delta + \frac{3(2 - 3\sin^2\frac{\mu}{2} - \sin^3\frac{\mu}{2} + \sin^4\frac{\mu}{2})}{2(4 - 5\sin^2\frac{\mu}{2} + \sin^4\frac{\mu}{2})}\delta^2].$$



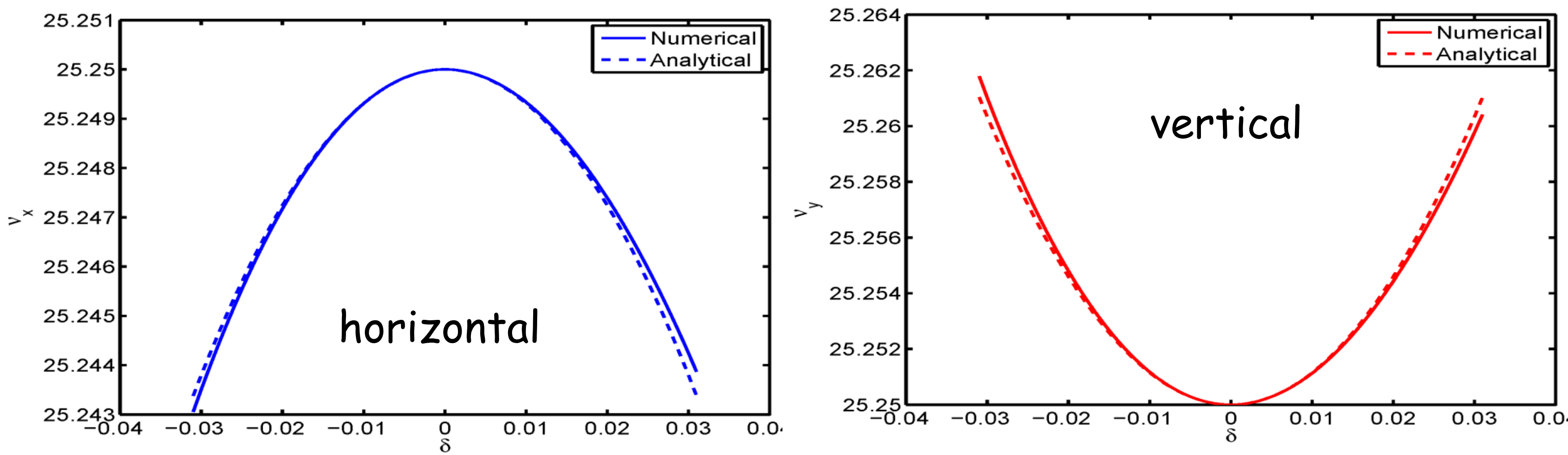
- half integer resonance seen
- not good if

  m >135⁰

Second-Order Chromatic Effects

The betatron phase advances up to the second-order of δ:

$$\mu_x(\delta) = \mu - \frac{\tan\frac{\mu}{2}\left(1 - \frac{1}{2}\sin^2\frac{\mu}{2}\right)}{2\left(1 - \frac{1}{4}\sin^2\frac{\mu}{2}\right)}\delta^2, \mu_y(\delta) = \mu + \frac{\tan\frac{\mu}{2}\left(1 + \frac{1}{2}\sin^2\frac{\mu}{2}\right)}{2\left(1 - \frac{1}{4}\sin^2\frac{\mu}{2}\right)}\delta^2.$$



Comparison to a numerical simulation in LEGO in a ring that consists of 101 90^0 cells.

Poisson Bracket

Given coordinate q_i, and its conjugate momentum p_i, the Poisson bracket is defined as,

$$[f,g] = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Fundamental brackets:

$$[q_i, q_j] = 0$$

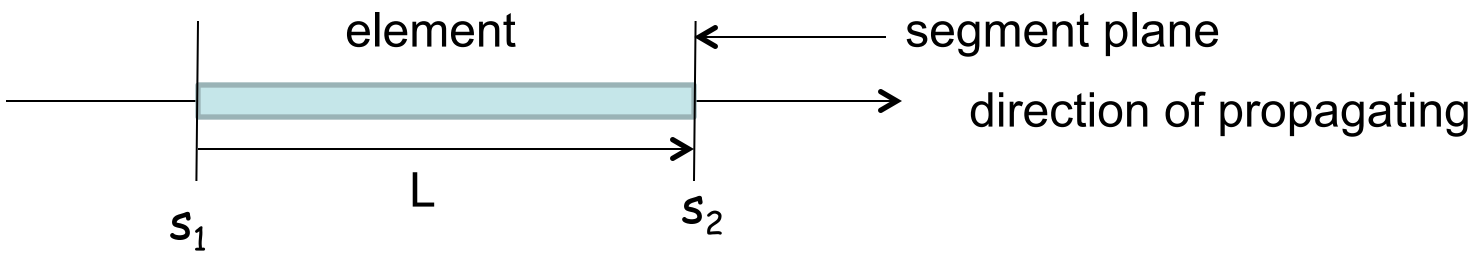
$$[p_i, p_j] = 0$$

$$[q_i, p_j] = \delta_{ij}$$

It is closely resemble the commutator in Quantum mechanics. It acts like a derivative with respect to its conjugate, for example,

$$[q_1, g] = \frac{\partial g}{\partial p_1}$$

Taylor Series and Exponential Lie Operator



For any function f(s), we have the Taylor expansion

$$f(s_2) = \sum_{n=0}^{\infty} \frac{L^n}{n!} \frac{d^n f}{ds^n}\Big|_{s_1} \equiv e^{L\frac{d}{dx}} f(s)\Big|_{s_1}$$ ← a symbolic notation

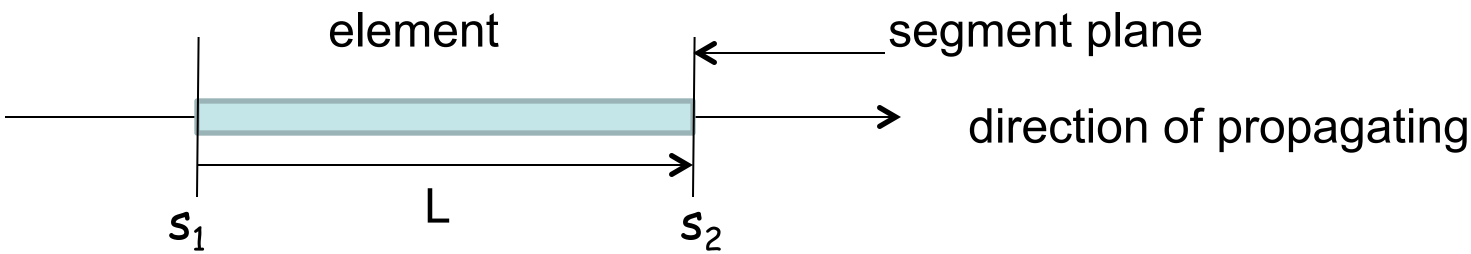
In particular, if there is no explicit dependent of s in the function f(s), namely f(s) = f(x(s),p_x(s),…), we have

$$\frac{df}{ds} = -[H,f] \equiv -:H:f,$$ ← another symbolic notation

Used Hamiltonian equation and the definition of the Poisson bracket. Combining these symbolic notations, we have the exponential Lie operator

$$f(s_2) = e^{-L:H:} f(s)\Big|_{s_1}$$

Lie Operator as a Transfer Map



In the previous slide, we have shown that

$$f(s_2) = e^{-L:H:} f(s)\big|_{s_1}.$$

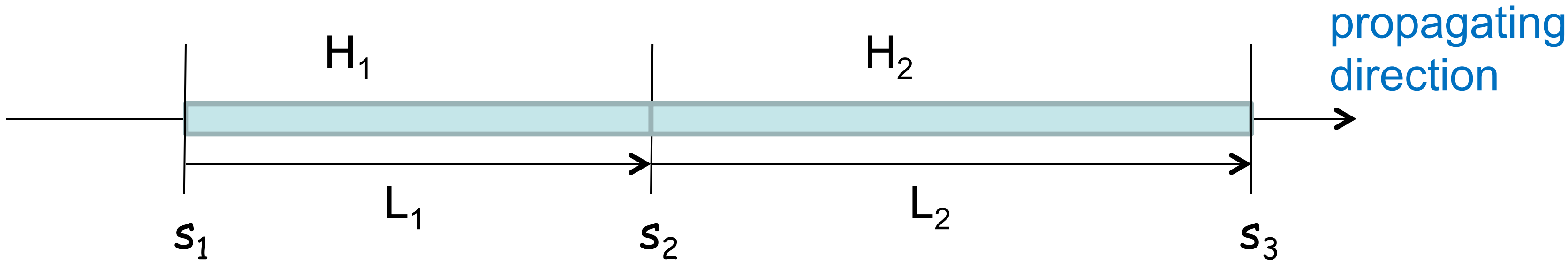
If we apply this formula to a particular function: z=x, or p_x, or y, or p_y, or δ or l, and then we have

$$z(s_2) = e^{-L:H:} z(s_1).$$

Therefore, this exponential Lie operator is a transfer map. We have

$$\mathcal{M}_{1\to 2} = e^{-L:H:}$$

Lie Operators and Map Concatenation



It is obvious that

$$f(s+L) = e^{-:L:H:} f(x, p_x, \ldots) = f(e^{-:L:H:} x, e^{-:L:H:} p_x, \ldots) = f(x(s+L), p_x(s+L), \ldots)$$

just shown

just shown

obviously true

The Lie operator acts only on the arguments of function . This precisely the definition of the map concatenation we introduced early. So we have

$$\mathcal{M}_{1\to3} = \mathcal{M}_{1\to2} \circ \mathcal{M}_{2\to3} = e^{-L_1:H_1:} e^{-L_2:H_2:}.$$

The dot is removed because Lie operator automatically has the property.

Similarity Transformation

$$e^{:A:}e^{:B:}e^{-:A:} = e^{:e^{:A:}B:}$$

Here is a proof. Set f=e$^{-:A:}$g, so we have

$$\begin{aligned}
e^{:A:}e^{:B:}f &= e^{:A:}\sum\frac{1}{n!}[B,[B,\ldots[B,f]\ldots]]] \\
&= \sum\frac{1}{n!}e^{:A:}[B,[B,\ldots[B,f]\ldots]] \\
&= \sum\frac{1}{n!}[e^{:A:}B,e^{:A:}[B,\ldots[B,f]\ldots]] \\
&= \sum\frac{1}{n!}[e^{:A:}B,[e^{:A:}B,\ldots[e^{:A:}B,e^{:A:}f]\ldots]] \\
&= \sum\frac{1}{n!}[e^{:A:}B,[e^{:A:}B,\ldots[e^{:A:}B,g]\ldots]] \\
&= e^{:e^{:A:}B:}g
\end{aligned}$$

We used $e^{:A:}[f_1,f_2] = [e^{:A:}f_1,e^{:A:}f_2]$ ($e^{:A:}[x,p_x] = [e^{:A:}x,e^{:A:}p_x]$)

The Cambell-Baker-Hausdorf (CBH) Theorem

To combine two exponential Lie operators, we have

$$e^{:A:}e^{:B:} = e^{:A+B+\frac{1}{2}[A,B]+\dots:}$$

The bracket notes the Poisson bracket. This theorem can be shown easily using the definition of the exponential Lie operator and the Jacob identity for the Poisson brackets:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

In general, it should be considered as a part of perturbation theory. It is good when A and B are small.

If [A, B]=0, then

$$e^{:A:}e^{:B:} = e^{:A+B:}$$

In particular, $e^{-:A:}$ is the inverse of $e^{:A:}$.

CBH Theorem, in Second Form

To combine two exponential Lie operators, we have[6]

$$e^{:A:}e^{:B:} = \exp[:A + (\frac{:A:}{1-e^{-:A:}})B + O(B^2):]$$

It should be used when A is large and B is small, for example,

$$A = f_2 = -\mu J$$

and B = $f_{3,}$ the third-order polynomial.

Dragt-Finn Factorization

Given a nonlinear Taylor map M, we

$$\mathcal{M}_1^{-1} \circ \mathcal{M} = I_2$$

Here M_1 is the linear part of M. It is clear that I_2 is a second order of nonlinear map near identity. It's lowest perturbation is the second order, indicated with its subscript. Now, we would like to write I_2 as a Lie operator , namely

$$\mathcal{M}_1^{-1}\,\mathcal{M} = I_2 = \exp[:f_3:]$$

Once we have f_3, then we can compute the next of by

$$\mathrm{e}^{-:f_3:}\,\mathcal{M}_1^{-1}\,\mathcal{M} = I_3$$

I_3 is a third of order nonlinear map near identity. Similar process can be continued to the next order. Finally, this procedure leads to the Dragt-Finn factorization,

$$\mathcal{M} = \mathcal{M}_1\,\mathrm{e}^{:f_3:}\mathrm{e}^{:f_4:}\ldots\mathrm{e}^{:f_{n+1}:}$$

Here n is the truncation order of the Taylor map M.

Extraction of a First Order Lie Factor

To solve the equation,

$$[f_{n+1}, z] = I_n$$

Here z is the vector in the phase space in the Poisson bracket. Its solution is given by

$$f_{n+1} = \frac{1}{n+1}\sum_{k=1}^{3}[z_{2k-1}(I_n - I)_{2k} - z_{2k}(I_n - I)_{2k-1}]$$

It is valid only if the map is symplectic.

f_3: Geometric Aberration

The third-order Lie factor is given by,

$$f_3^{(g)} = F \times \{-\frac{3[7(J_x+2J_y)+(J_x+2J_y)\cos\mu+2(J_x+6J_y)\sin\frac{\mu}{2}]}{(\cos\frac{\mu}{4}+\sin\frac{\mu}{4})^2}\cos(\psi_x-\frac{\mu}{2})$$

$$-\frac{J_x(-2+9\cos\mu+\cos 2\mu-14\sin\frac{\mu}{2})}{(\cos\frac{\mu}{4}+\sin\frac{\mu}{4})^2}\cos(3\psi_x-\frac{3\mu}{2})$$

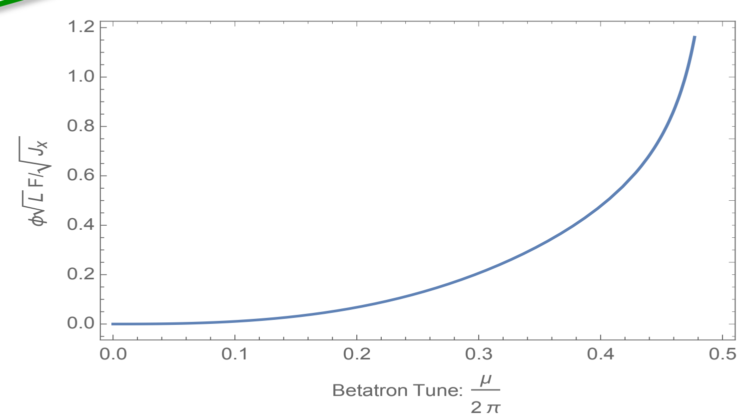
$$+6J_y(2-6\cos\mu-10\sin\frac{\mu}{2}+\sin\frac{3\mu}{2})\cos(\psi_x+2\psi_y-\frac{3\mu}{2})$$

$$+6J_y(-4+\sin\frac{\mu}{2})\cos(\psi_x-2\psi_y-\frac{\mu}{2})\}$$

resonance driving terms

where

$$F = \frac{4\sin^3\frac{\mu}{2}\sqrt{2J_x\csc\mu(1+\sin\frac{\mu}{2})}}{2\phi\sqrt{L}(7+\cos\mu)}$$



Effective Hamiltonian

Given a single-Lie factor written in terms of the action-angle variables,

$$f = -\sum_{m,n}[C_{m,n}(J_x,J_y)\cos(m\psi_x + n\psi_y) + S_{m,n}(J_x,J_y)\sin(m\psi_x + n\psi_y)]$$

combine it with the linear map in the normalized coordinates, at the 1st-order Perturbation. The effective Hamiltonian is given by,

$$H_{Eff} = 2\pi(\nu_x J_x + \nu_y J_y) + \sum_{m,n}\frac{\pi(m\nu_x + n\nu_y)}{\sin[\pi(m\nu_x + n\nu_y)]}\{C_{m,n}(J_x,J_y)$$
$$\cos[m(\psi_x + \pi\nu_x) + n(\psi_y + \pi\nu_y)] + S_{m,n}(J_x,J_y)\sin[m(\psi_x + \pi\nu_x) + n(\psi_y + \pi\nu_y)]\}$$

where ν_x, ν_y are the betatron tunes. Noting that at the resonance condition,

$$n\nu_x + m\nu_y = p$$

for an integer p, the effective Hamiltonian becomes singular. And near the resonance, we have the problem of a small denominator.

Third-Order Effective Hamiltonian

The effective Hamiltonian ($e^{-:H:} = M_1 e^{:f_3:}$) is given by,

$$H^{(g)} = \mu(J_x + J_y) - F \times \{-\frac{3\mu[7(J_x+2J_y)+(J_x+2J_y)\cos\mu+2(J_x+6J_y)\sin\frac{\mu}{2}]}{2\sin\frac{\mu}{2}(\cos\frac{\mu}{4}+\sin\frac{\mu}{4})^2}\cos\psi_x$$

$$-\frac{3\mu J_x(-2+9\cos\mu+\cos2\mu-14\sin\frac{\mu}{2})}{2\sin\frac{3\mu}{2}(\cos\frac{\mu}{4}+\sin\frac{\mu}{4})^2}\cos3\psi_x$$

$$+\frac{9\mu J_y(2-6\cos\mu-10\sin\frac{\mu}{2}+\sin\frac{3\mu}{2})}{\sin\frac{3\mu}{2}}\cos(\psi_x+2\psi_y)$$

$$+\frac{3\mu J_y(-4+\sin\frac{\mu}{2})}{\sin\frac{\mu}{2}}\cos(\psi_x-2\psi_y)\}$$

Only see the problem of the small denominators in the sum resonances: $3\nu_x$ and $\nu_x+2\nu_y$.

Single Resonance in Horizontal Motion, ν=(1/3)+Δν

The effective Hamiltonian ($e^{-:3H:} = \left(M_1 e^{:f_3:}\right)^3$) is given by,

$$H = 2\pi\Delta\nu J_x - \frac{3\pi\Delta\nu F J_x(-2 + 9\cos\mu + \cos 2\mu - 14\sin\frac{\mu}{2})}{\sin(3\pi\Delta\nu)(\cos\frac{\mu}{4} + \sin\frac{\mu}{4})^2}\cos 3\psi_x$$

$$= \pi\Delta\nu(\bar{x}^2 + \bar{p}_x^2) + k\bar{x}(\bar{x}^2 - 3\bar{p}_x^2)$$

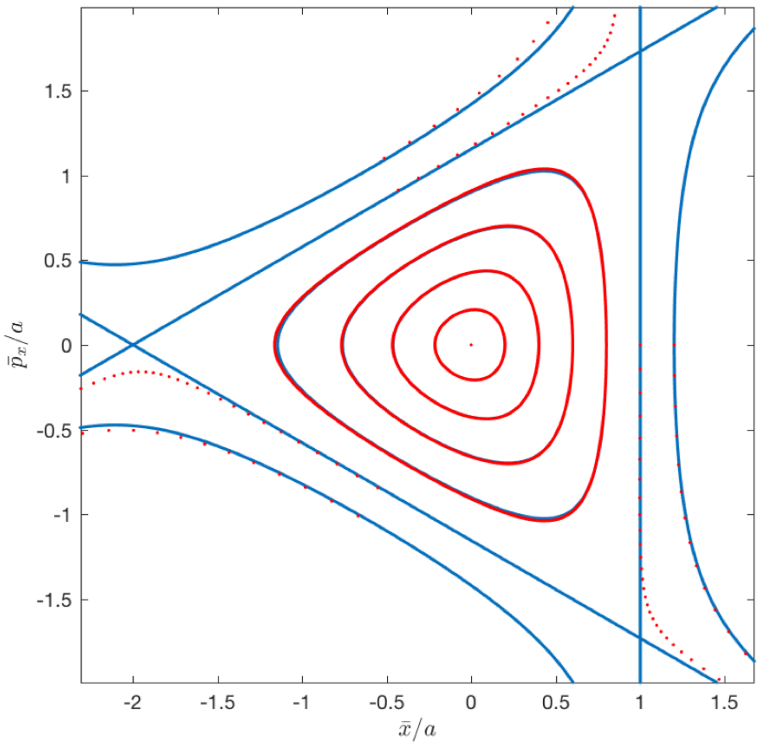
where

$$k = -\frac{2\pi\Delta\nu\sin^3\frac{\mu}{2}\sqrt{\csc\mu(1+\sin\frac{\mu}{2})}(-2 + 9\cos\mu + \cos 2\mu - 14\sin\frac{\mu}{2})}{\phi\sqrt{L}\sin(3\pi\Delta\nu)(7+\cos\mu)(\cos\frac{\mu}{4} + \sin\frac{\mu}{4})^2}$$

and $\bar{x}$ and $\bar{p}_x$ are the coordinates in the normalized phase space.

Single Resonance, $\Delta\nu$=0.005

Scale: $a=\pi\Delta\nu/3k$, defined by singularity or separatrix.



Tracking in comparison to the effective Hamiltonian:

$$H = \pi\Delta\nu(\bar{x}^2 + \bar{p}_x^2) + k\bar{x}(\bar{x}^2 - 3\bar{p}_x^2)$$

None Resonance in Horizontal Motion

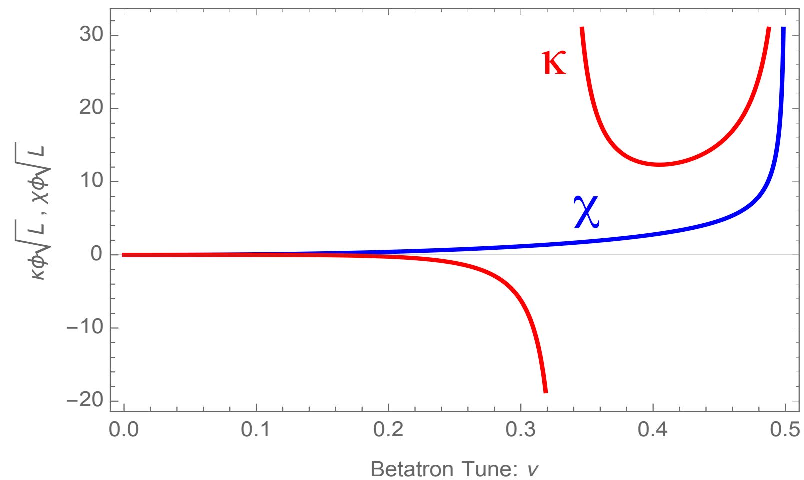
The effective Hamiltonian is given by,

$$H = \pi\nu(\bar{x}^2 + \bar{p}_x^2) + \kappa\bar{x}(\bar{x}^2 - 3\bar{p}_x^2) + \chi\bar{x}(\bar{x}^2 + \bar{p}_x^2)$$

where

$$\kappa = \frac{\mu\sin^3\frac{\mu}{2}\sqrt{\csc\mu(1+\sin\frac{\mu}{2})}(-2+9\cos\mu+\cos2\mu-14\sin\frac{\mu}{2})}{\phi\sqrt{L}\sin\frac{3\mu}{2}(7+\cos\mu)(\cos\frac{\mu}{4}+\sin\frac{\mu}{4})^2}$$

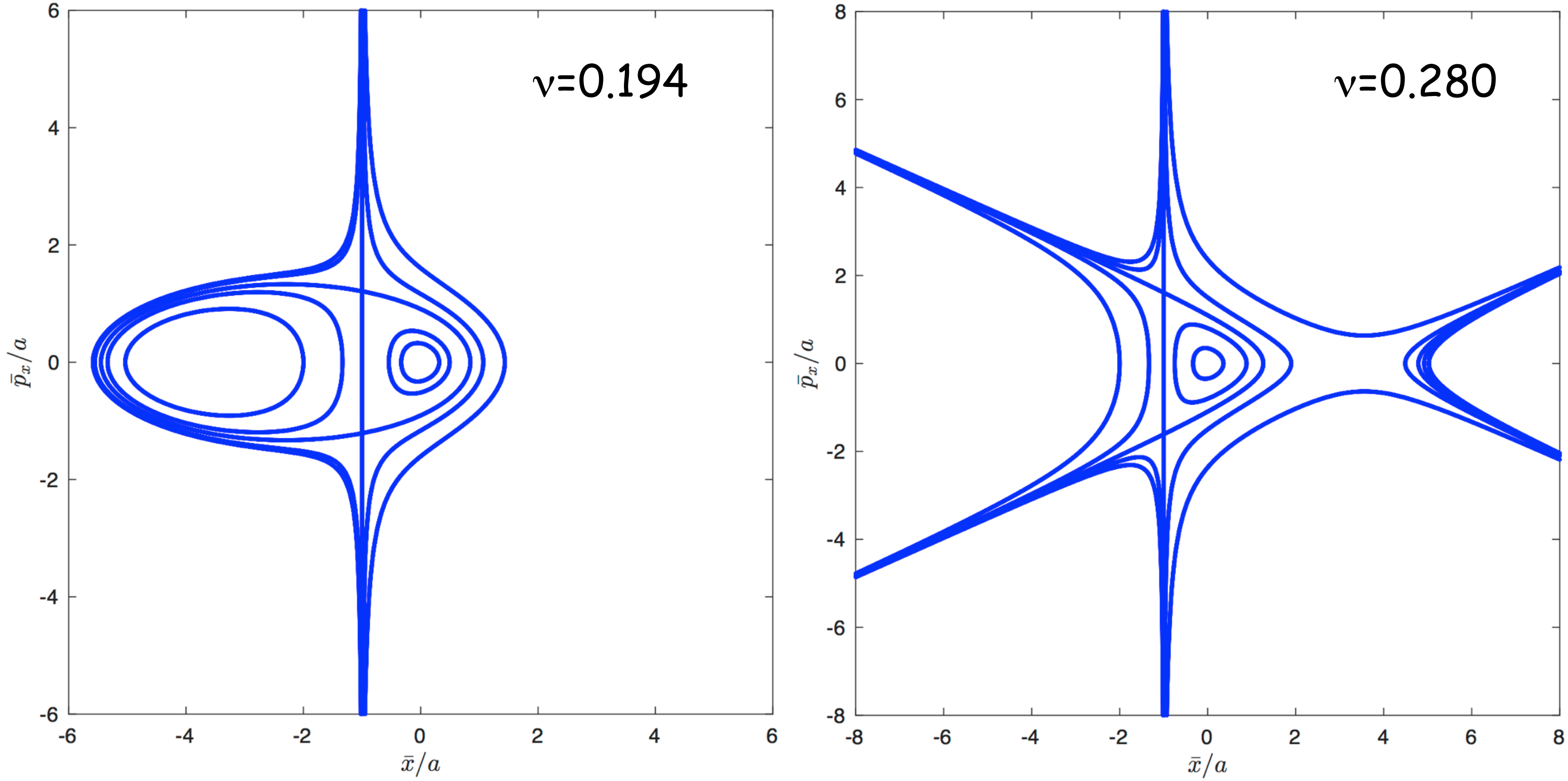
$$\chi = \frac{\mu\sin^2\frac{\mu}{2}\sqrt{\csc\mu(1+\sin\frac{\mu}{2})}(7+\cos\mu+2\sin\frac{\mu}{2})}{\phi\sqrt{L}(7+\cos\mu)(\cos\frac{\mu}{4}+\sin\frac{\mu}{4})^2}$$



The third integer driving term ($3\nu_x$) dominates.

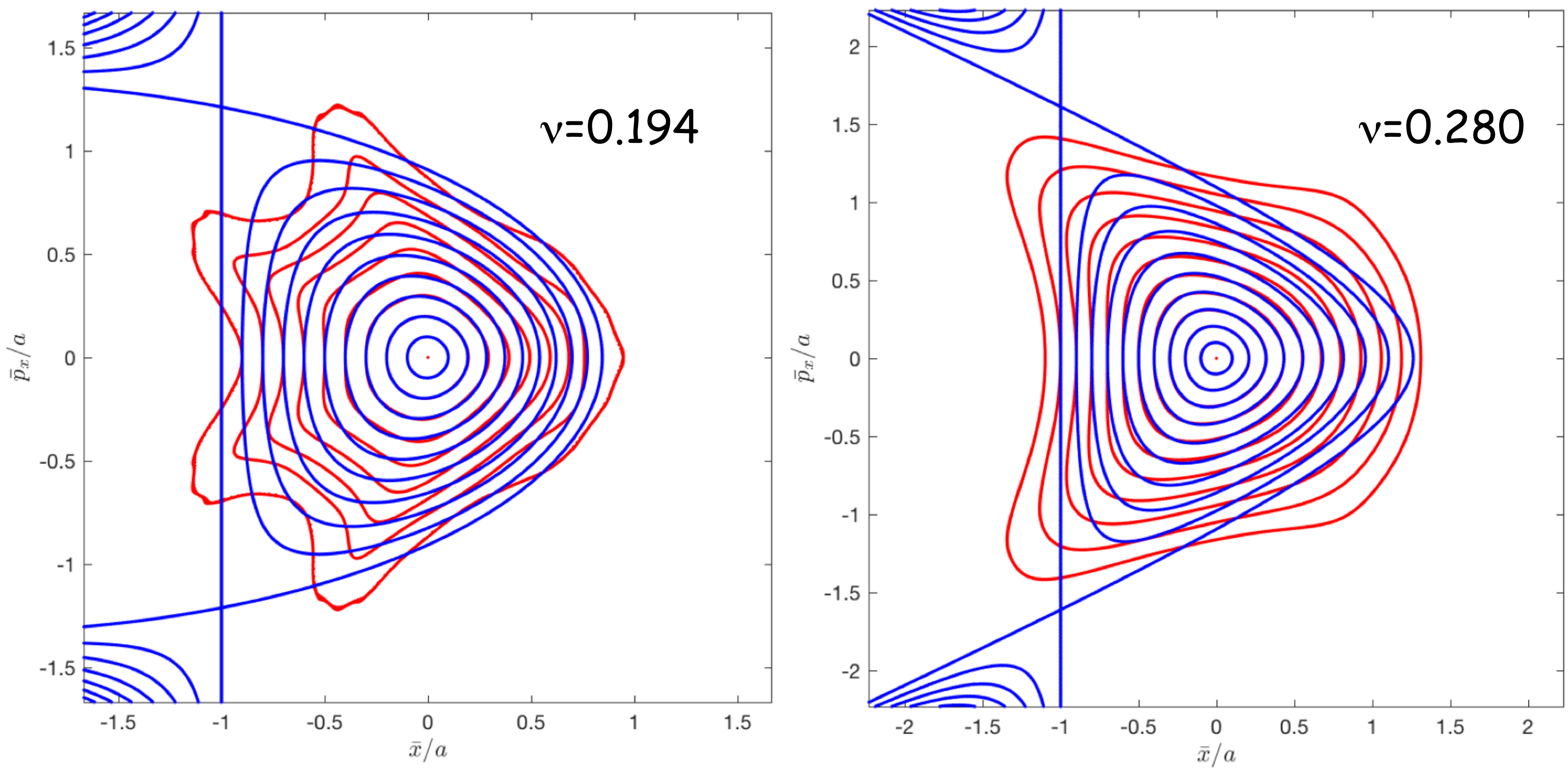
Topology of Phase Space

Scale: a=$\pi\nu/(\chi-3\kappa)$, again defined by singularity or separatrix.



Three points on the horizontal axis: $\bar{x} = -a, \dfrac{2(\kappa+\sqrt{-\kappa\chi})}{(\kappa+\chi)}a, \dfrac{2(\kappa-\sqrt{-\kappa\chi})}{(\kappa+\chi)}a$

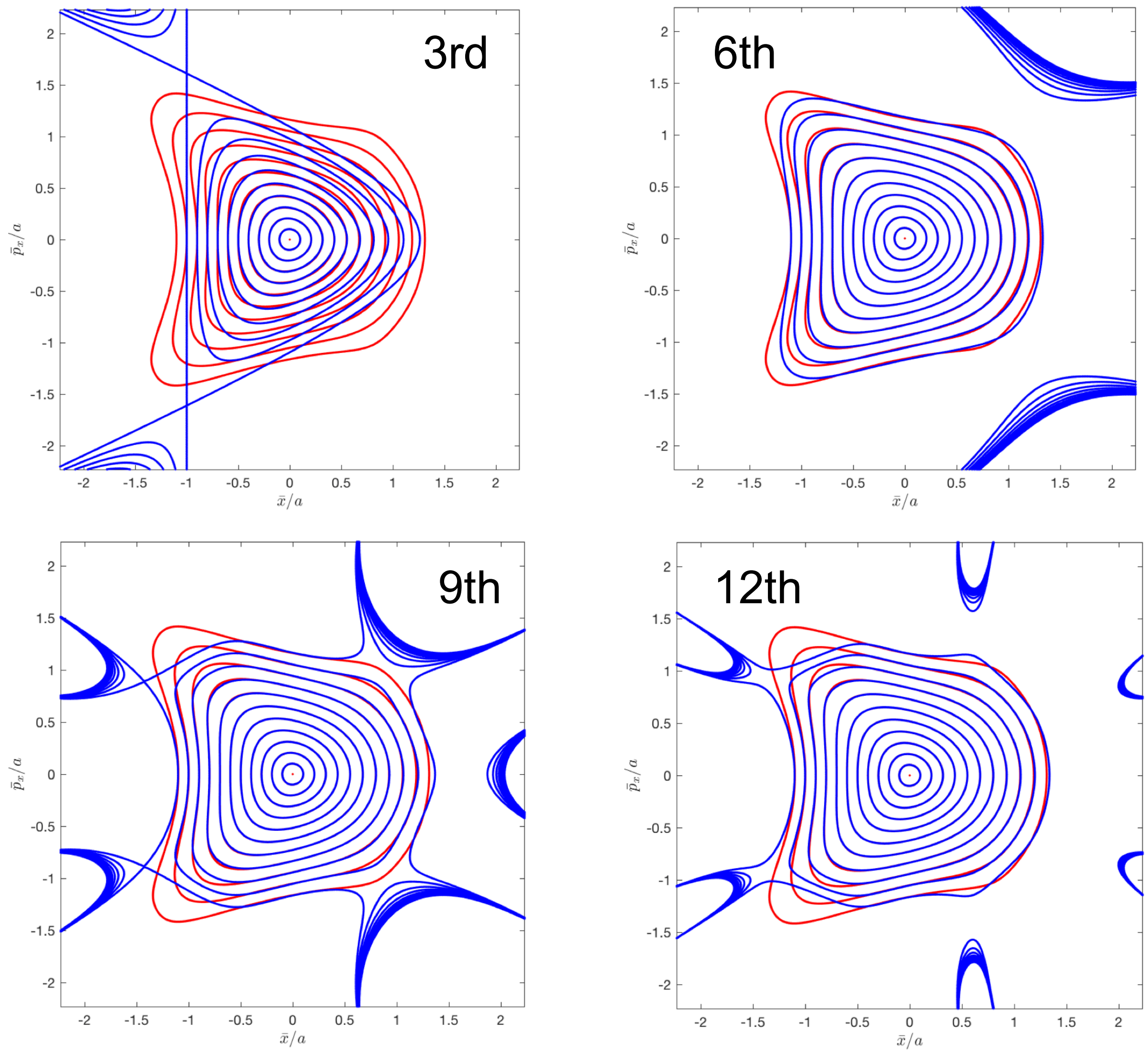
Persistence



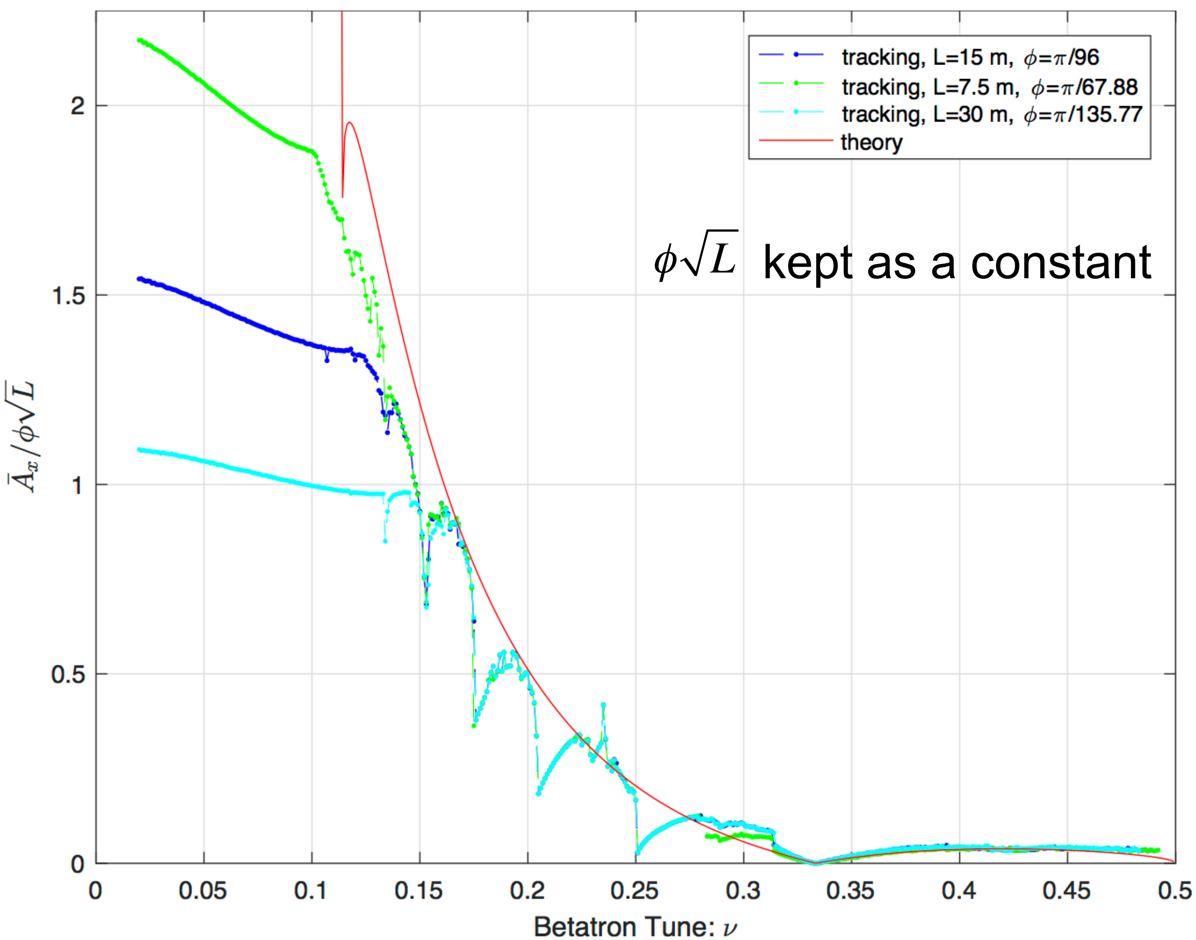
The higher order perturbation distorted the contours but not break them when the tune is sufficiently away from the resonances. Tracking with L=15 m and $\phi=\pi/96$.

Convergence of Perturbation Series

ν=0.28



Dynamic Aperture in Horizontal Plane



Degenerate Resonance, $\nu=(1/3)+\Delta\nu$

The effective Hamiltonian ($e^{-:3H:} = \left(M_1 e^{:f_3:}\right)^3$) is given by,

$$H = \pi\Delta\nu[(\bar{x}^2 + \bar{p}_x^2) + (\bar{y}^2 + \bar{p}_y^2)] + k\bar{x}(\bar{x}^2 - 3\bar{p}_x^2) + q[\bar{x}(\bar{y}^2 - \bar{p}_y^2) - 2\bar{y}\bar{p}_x\bar{p}_y]$$

where

$$k = -\frac{2\pi\Delta\nu\sin^3\frac{\mu}{2}\sqrt{\csc\mu(1+\sin\frac{\mu}{2})}(-2+9\cos\mu+\cos2\mu-14\sin\frac{\mu}{2})}{\phi\sqrt{L}\sin(3\pi\Delta\nu)(7+\cos\mu)(\cos\frac{\mu}{4}+\sin\frac{\mu}{4})^2}$$

$$q = \frac{12\pi\Delta\nu\sin^3\frac{\mu}{2}\sqrt{\csc\mu(1+\sin\frac{\mu}{2})}(2-6\cos\mu-10\sin\frac{\mu}{2}+\sin\frac{3\mu}{2})}{\phi\sqrt{L}\sin(3\pi\Delta\nu)(7+\cos\mu)}$$

and $\bar{x}, \bar{p}_x, \bar{y}, \bar{p}_y$ are the coordinates in the normalized phase space.

A Special Solution

We find a reduced Hamiltonian,

$$H = \pi\Delta\nu(\bar{x}^2 + \bar{p}_x^2) - \frac{2q}{3}\bar{x}(\bar{x}^2 - 3\bar{p}_x^2)$$

under the condition of,

$$\bar{p}_y = c_1\bar{x}$$

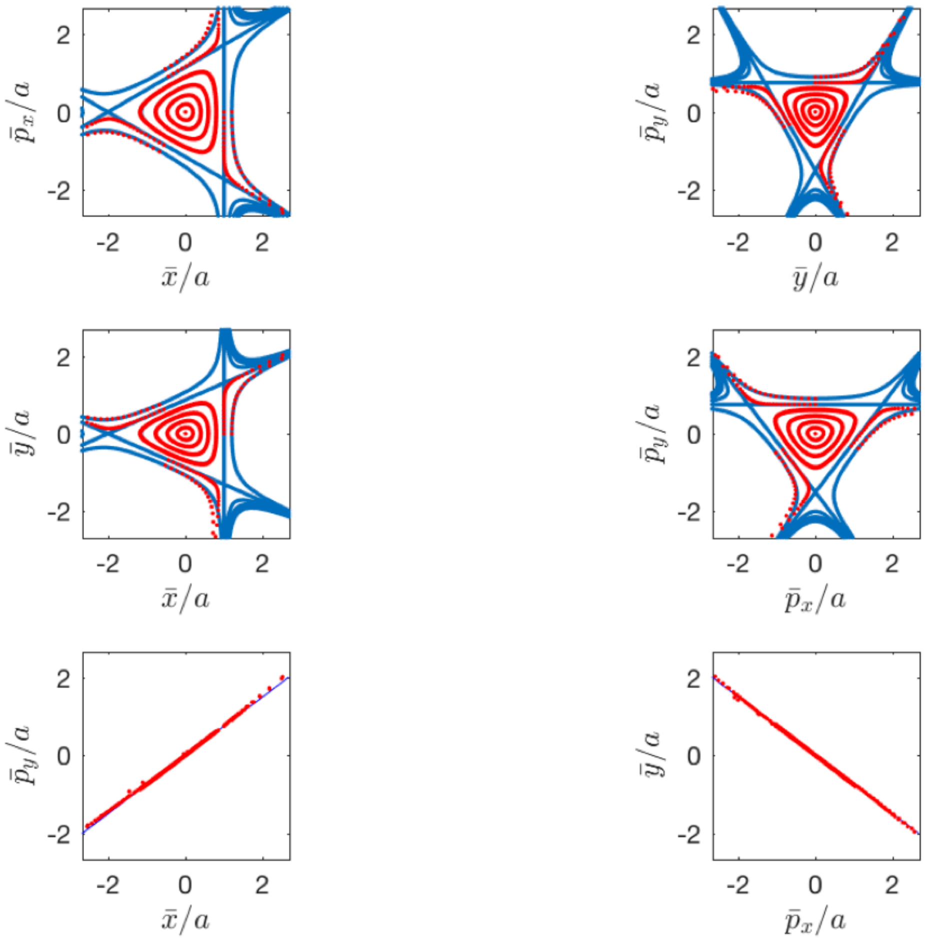
$$\bar{y} = -c_1\bar{p}_x$$

with a constant,

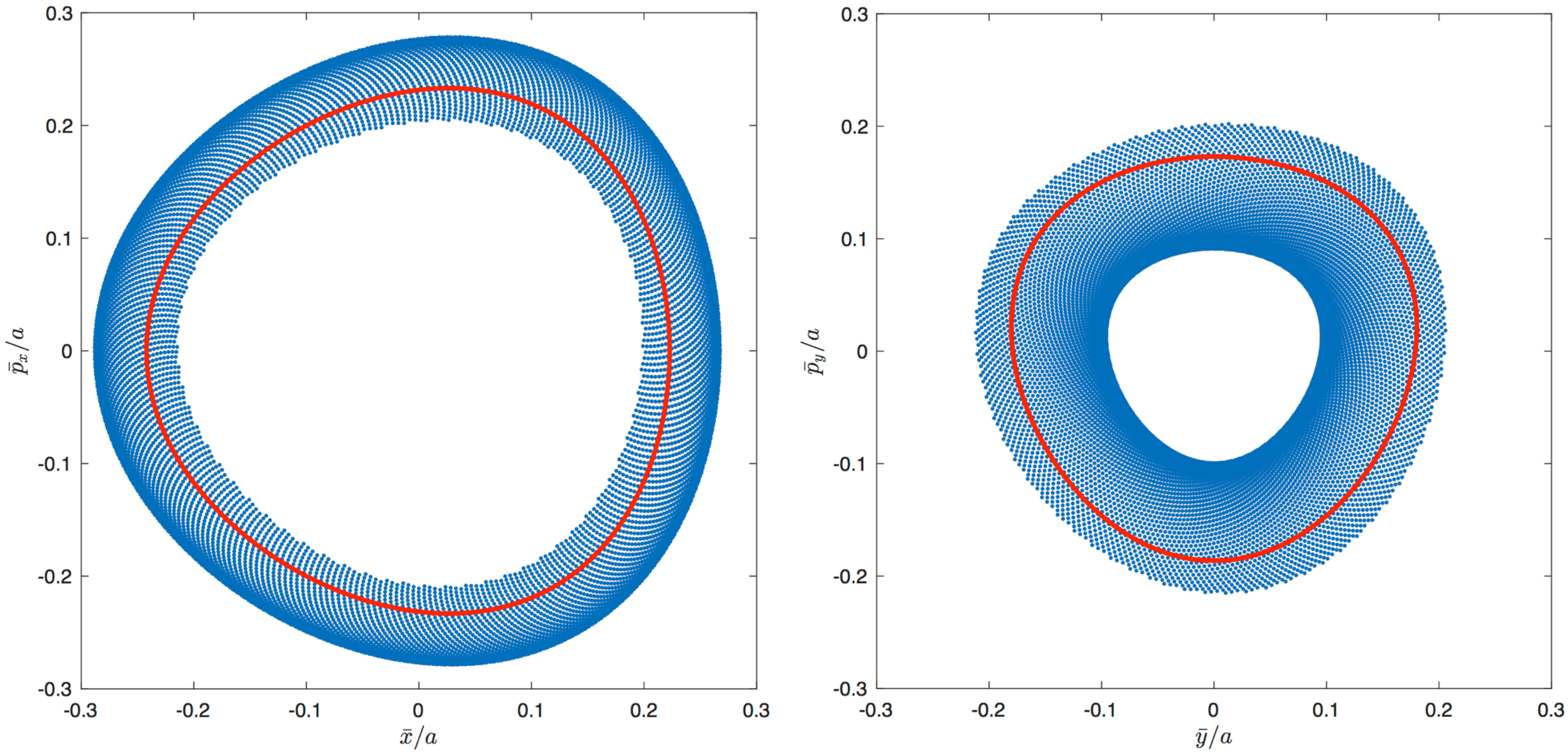
$$c_1 = \pm\sqrt{\frac{2q+3k}{q}}$$

Invariant Surfaces, $\Delta\nu$=0.005

Scale: $a=-\pi\Delta\nu/2q$, again defined by singularity or separatrix.

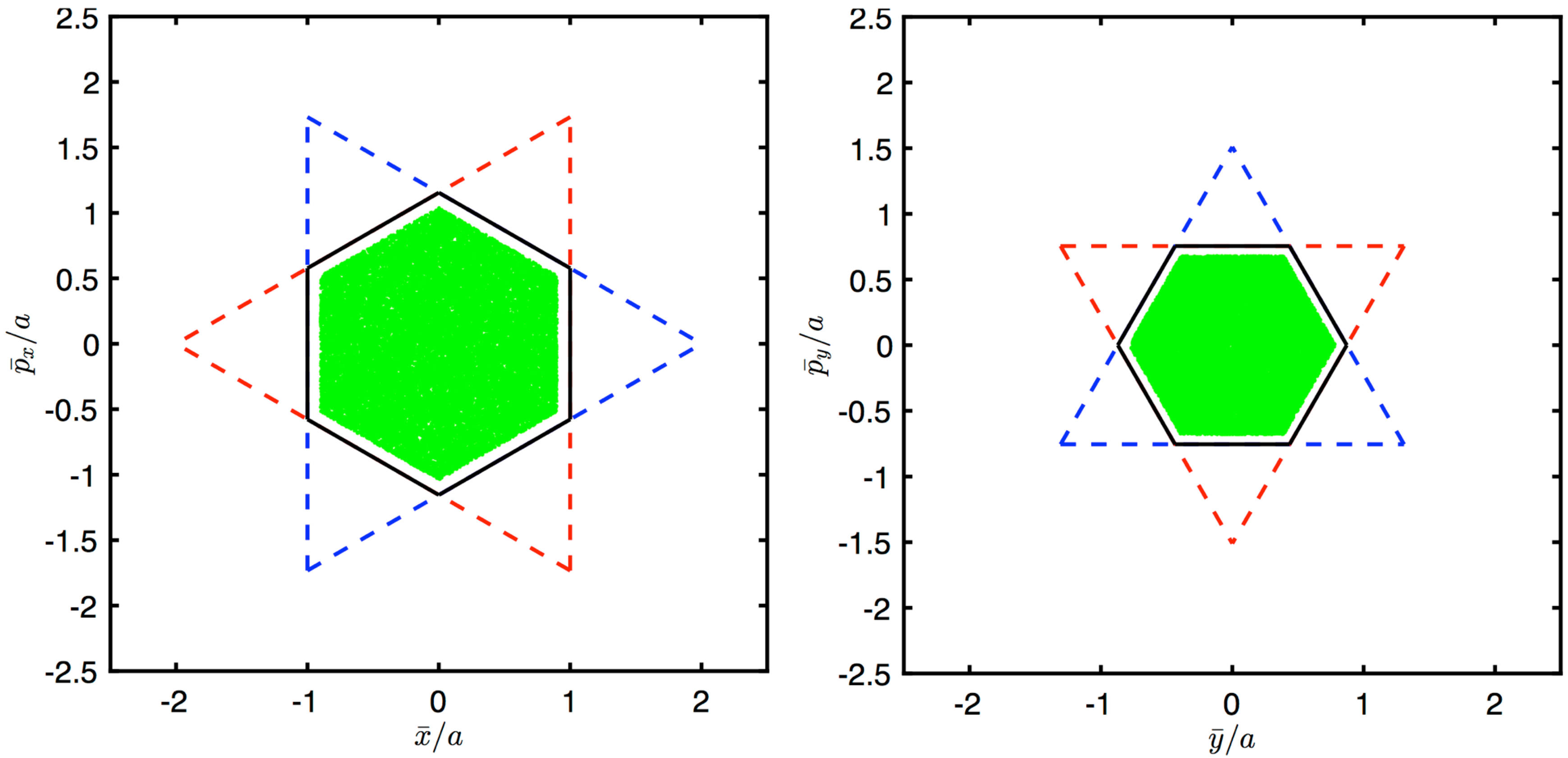


Comparison to a General Solution with same energy



Volume of Stability

the common region covered by the largest surface from each kind.



Two Degree of Freedom

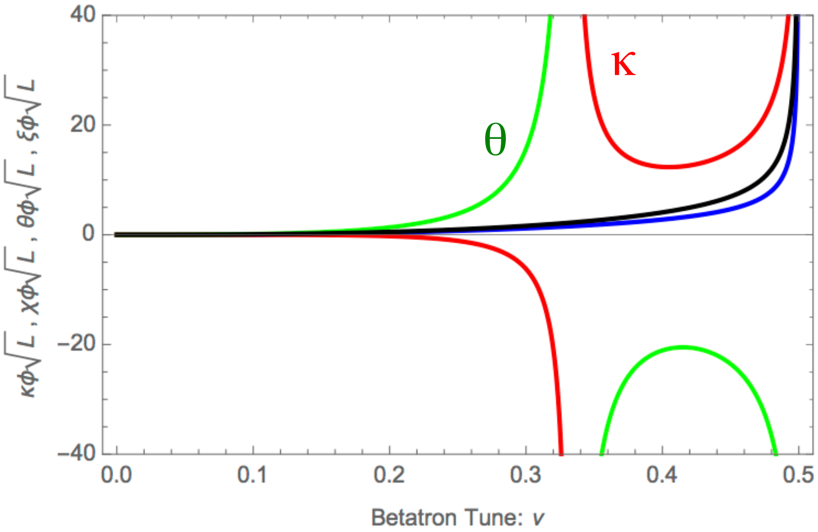
The effective Hamiltonian is given by,

$$H = \pi\nu[(\bar{x}^2 + \bar{p}_x^2) + (\bar{y}^2 + \bar{p}_y^2)] + \kappa\bar{x}(\bar{x}^2 - 3\bar{p}_x^2) + \chi\bar{x}(\bar{x}^2 + \bar{p}_x^2)$$
$$+\theta[\bar{x}(\bar{y}^2 - \bar{p}_y^2) - 2\bar{y}\bar{p}_x\bar{p}_y] + \xi[\bar{x}(3\bar{y}^2 + \bar{p}_y^2) + 2\bar{y}\bar{p}_x\bar{p}_y]$$

where

$$\theta = -\frac{6\mu\sin^3\frac{\mu}{2}\sqrt{\csc\mu(1+\sin\frac{\mu}{2})(2-6\cos\mu-10\sin\frac{\mu}{2}+\sin\frac{3\mu}{2})}}{\phi\sqrt{L}\sin\frac{3\mu}{2}(7+\cos\mu)}$$

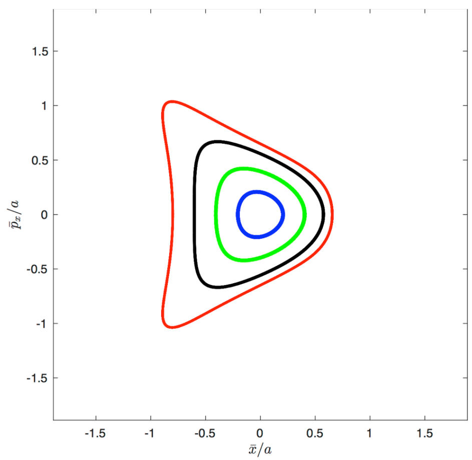
$$\xi = -\frac{2\mu\sin^2\frac{\mu}{2}\sqrt{\csc\mu(1+\sin\frac{\mu}{2})(-4+\sin\frac{\mu}{2})}}{\phi\sqrt{L}(7+\cos\mu)}$$



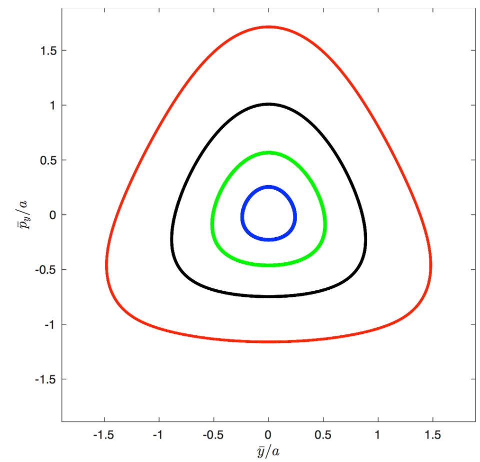
The sum resonances driving terms: $3\nu_x$, $\nu_x+2\nu_y$, dominate.

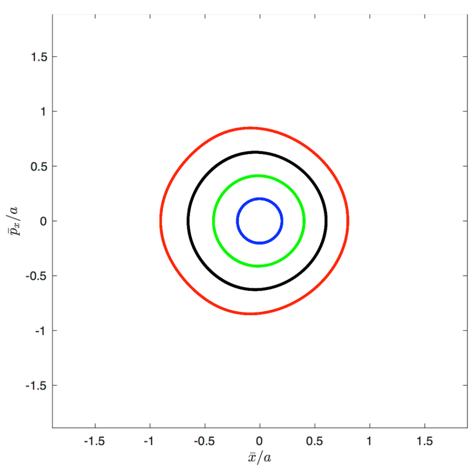
Quasi-Invariant Surfaces

ν=0.28

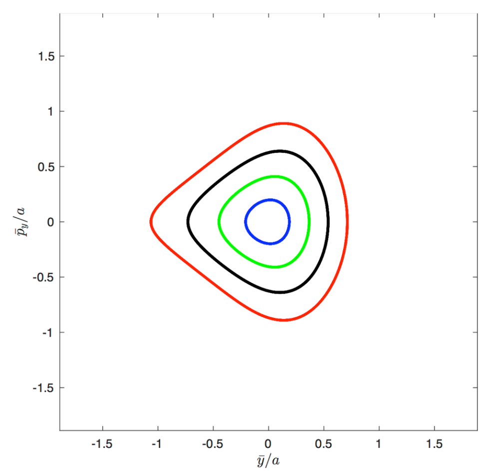


First kind

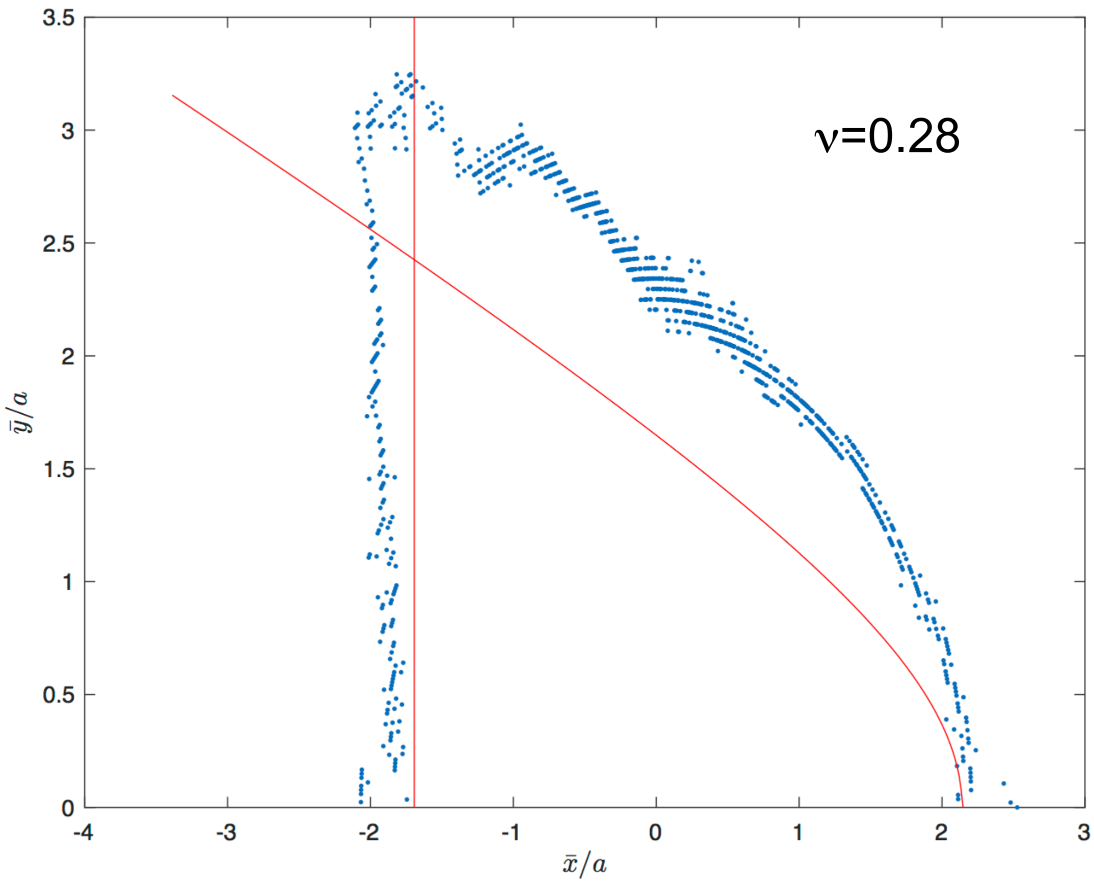




Second kind



Dynamic Aperture



Two lines are given by, $\bar{x} = \dfrac{\pi\nu}{3\kappa - \chi}$ and $\bar{y} = \sqrt{\dfrac{-4\pi^2\kappa\nu^2 + 4\pi\kappa\nu(-3\kappa+\chi)\bar{x} - (\kappa+\chi)(-3\kappa+\chi)^2\bar{x}^2}{(\theta+3\xi)(-3\kappa+\chi)^2}}$

Conclusion

1. Hamiltonian and symplectic maps are fundamental for the beam dynamics in storage rings, including the linear and chromatic optics.
2. Chromatic optics can be computed order-by-order analytically. And linear chromaticity can be corrected by sextupoles.
3. The first-order perturbation of the sextupoles, or more precisely the third-order effective Hamiltonian, largely determines the dynamics away from the other major resonances: 1/4, 1/5, 1/6, and 1/7.
4. Dynamic aperture in the normalized phase space is given by,

$$\bar{A} \propto \phi\sqrt{L}$$

in FODO cell, where ϕ is the bending angle and L length of the cell.

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