QCD critical point and fluctuations

M. Stephanov

History

Cagniard de la Tour (1822): discovered continuos transition from liquid to vapour by heating alcohol, water, etc. in a gun barrel, glass tubes.

Name

Faraday (1844) – liquefying gases:

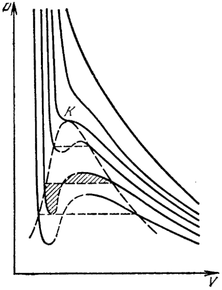
"Cagniard de la Tour made an experiment some years ago which gave me occasion to want a new word."

Mendeleev (1860) – measured vanishing of liquid-vapour surface tension: "Absolute boiling temperature".

Andrews (1869) – systematic studies of many substances established continuity of vapour-liquid phases. Coined the name "critical point".

Theory

van der Waals (1879) –
in “On the continuity of the gas and liquid state”
(PhD thesis) wrote e.o.s. with a critical point.



Smoluchowski, Einstein (1908,1910) – explained critical opalescence.

Landau – classical theory of critical phenomena

Fisher, Kadanoff, Wilson – scaling, full fluctuation theory based on RG.

Critical opalescence

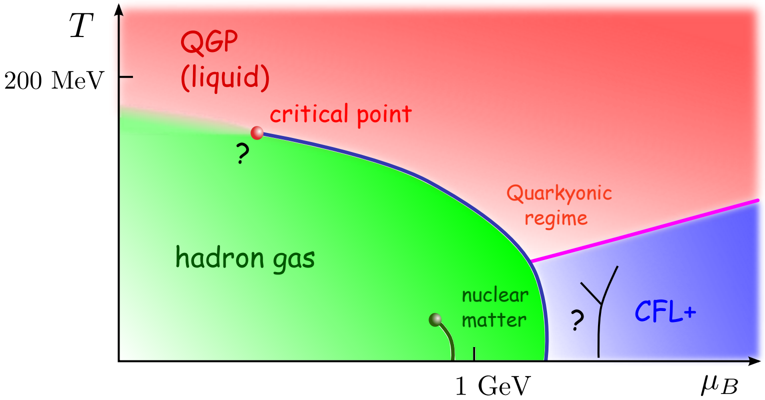
Substance[13][14]	Critical temperature	Critical pressure (absolute)
Argon	−122.4 °C (150.8 K)	48.1 atm (4,870 kPa)
Ammonia[15]	132.4 °C (405.5 K)	111.3 atm (11,280 kPa)
Bromine	310.8 °C (584.0 K)	102 atm (10,300 kPa)
Caesium	1,664.85 °C (1,938.00 K)	94 atm (9,500 kPa)
Chlorine	143.8 °C (416.9 K)	76.0 atm (7,700 kPa)
Ethanol	241 °C (514 K)	62.18 atm (6,300 kPa)
Fluorine	−128.85 °C (144.30 K)	51.5 atm (5,220 kPa)
Helium	−267.96 °C (5.19 K)	2.24 atm (227 kPa)
Hydrogen	−239.95 °C (33.20 K)	12.8 atm (1,300 kPa)
Krypton	−63.8 °C (209.3 K)	54.3 atm (5,500 kPa)
CH_4 (methane)	−82.3 °C (190.8 K)	45.79 atm (4,640 kPa)
Neon	−228.75 °C (44.40 K)	27.2 atm (2,760 kPa)
Nitrogen	−146.9 °C (126.2 K)	33.5 atm (3,390 kPa)
Oxygen	−118.6 °C (154.6 K)	49.8 atm (5,050 kPa)
CO_2	31.04 °C (304.19 K)	72.8 atm (7,380 kPa)
N_2O	36.4 °C (309.5 K)	71.5 atm (7,240 kPa)
H_2SO_4	654 °C (927 K)	45.4 atm (4,600 kPa)
Xenon	16.6 °C (289.8 K)	57.6 atm (5,840 kPa)
Lithium	2,950 °C (3,220 K)	652 atm (66,100 kPa)
Mercury	1,476.9 °C (1,750.1 K)	1,720 atm (174,000 kPa)
Sulfur	1,040.85 °C (1,314.00 K)	207 atm (21,000 kPa)
Iron	8,227 °C (8,500 K)	
Gold	6,977 °C (7,250 K)	5,000 atm (510,000 kPa)
Water[2][16]	373.946 °C (647.096 K)	217.7 atm (22.06 MPa)

Critical point is a ubiquitous phenomenon

Critical point between the QGP and hadron gas phases?

QCD is a relativistic theory of a fundamental force.

CP is a singularity of EOS, anchors the 1st order transition.



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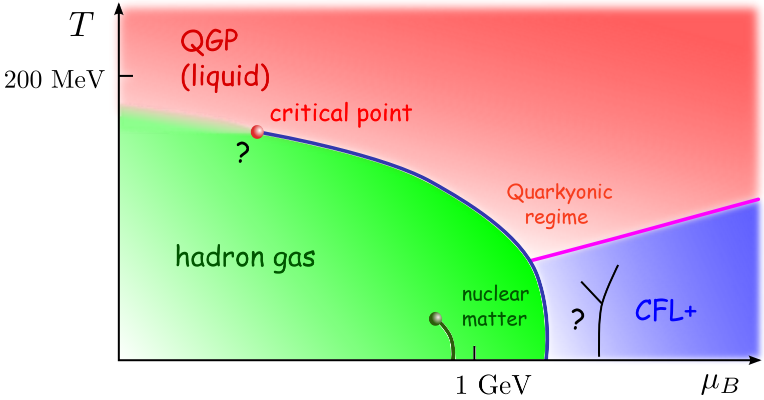


Lattice QCD at $\mu_B \lesssim 2T$ – a crossover.

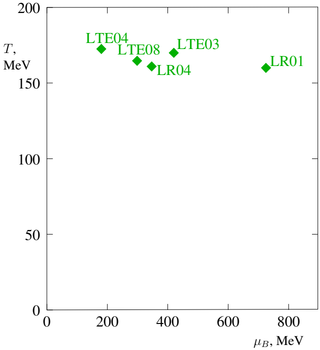
C.P. is ubiquitous in models (NJL, RM, Holog., Strong coupl. LQCD, . . .)

Essentially two approaches to discovering the QCD critical point. Each with its own challenges.

- Lattice simulations.

  The *sign problem* restricts reliable lattice calculations to $\mu_B = 0$.

  Under different assumptions one can estimate the position of the critical point, assuming it exists, by extrapolation from $\mu = 0$.



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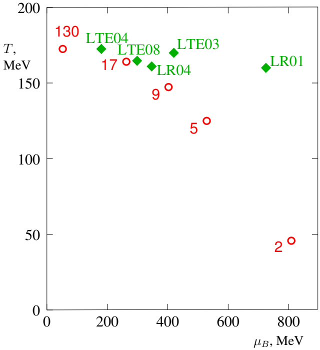

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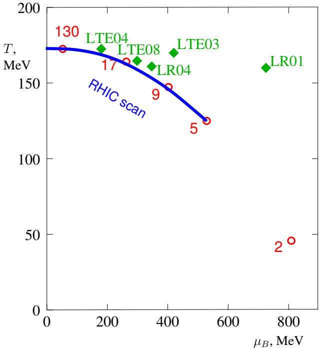

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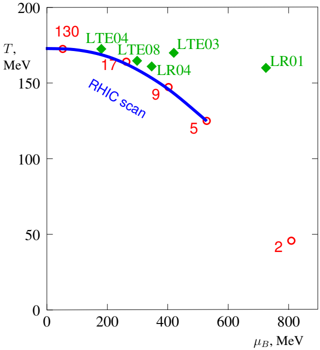


- Heavy-ion collisions. *Non-equilibrium.*

Lattice Monte Carlo

- Thermodynamics is encoded in the partition function

$$Z = \sum_{\text{quantum states}} \exp\{-\beta(\mathcal{E} - \mu N)\} = \int \mathcal{D}(\text{paths}) \exp\{-S_E\}$$

  - S_E - action on a path in imaginary time τ from 0 to β.

- Usually, S_E - real. So $\int \mathcal{D}(\text{paths})\, e^{-S_E}$ - itself is a partition function for *classical* statistical system in $3+1$ dimensions. Monte Carlo methods work.

- Not so for $\mu \neq 0$.

Sign problem

$$e^{-S_E} = e^{-S_{\text{gluons}}} \det D_{\text{quarks}}.$$

and $\det D_{\text{quarks}}$ - complex for $\mu \neq 0$.

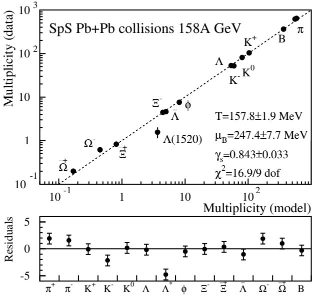
Monte Carlo translates weight e^{-S_E} into probability and fails if S_E is not real.

- Recent progress based on various techniques of circumventing the problem:
  - Reweighting (use weight at $\mu = 0$);
  - Taylor expansion;
  - Imaginary μ;
  - “Thimbles”, complex Langevin

Heavy-Ion Collisions. Thermalization.

"Little Bang"

- The final state looks thermal.
- Similar to CMB.



(Becattini et al)

- Flow – looks hydrodynamic. Initial anisotropy fluctuations are propagated to final state hydrodynamically.
- Why and when this thermalization occurs – an open question.

Outline

- Equilibrium
- Non-equilibrium

Why fluctuations are large at a critical point?

- The key equation:

$$P(X) \sim e^{S(X)} \qquad \textit{(Einstein 1910)}$$

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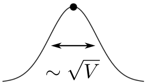

- For an extensive quantity $\langle X \rangle \sim V$:

$$\langle (\delta X)^2 \rangle_c = -\left(S''\right)^{-1} = VT\chi$$

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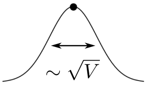

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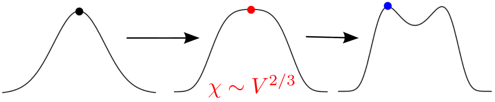

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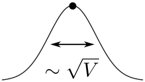

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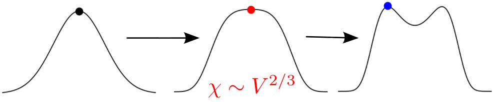
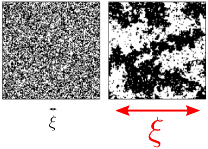




CLT? X is not a sum of ∞ many *uncorrelated* contributions: $\xi \to \infty$

Fluctuations of order parameter and ξ

- Fluctuations at CP – conformal field theory.

  Parameter-free $\to$ universality. Only one scale $\xi = m_\sigma^{-1} < \infty$,

$$P[\sigma] \sim \exp\{-\Omega[\sigma]/T\},$$

$$\Omega = \int d^3x \left[\frac{1}{2}(\boldsymbol{\nabla}\sigma)^2 + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \dots\right].$$

  - Width/shape of $P(\sigma_0 \equiv \int_{\boldsymbol{x}} \sigma)$ best expressed via cumulants:

- Higher cumulants (shape of $P(\sigma_0)$) depend stronger on ξ.

  Universal: $\boxed{\langle\sigma_0^k\rangle_c \sim V\xi^p}$, $p = k(3-[\sigma]) - 3$, $[\sigma] = \beta/\nu \approx 1/2$.

  E.g., $p \approx 2$ for $k = 2$, but $p \approx 7$ for $k = 4$.

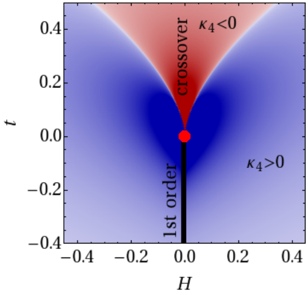
Sign

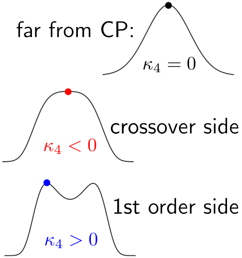
- Higher moments also depend on which side of the CP we are

$$\kappa_3[\sigma] = 2VT^{3/2}\,\tilde\lambda_3\,\xi^{4.5}\,; \quad \kappa_4[\sigma] = 6VT^2\,[\,2(\tilde\lambda_3)^2 - \tilde\lambda_4\,]\,\xi^7\,.$$

This dependence is also universal.

- 2 relevant directions/parameters. Using Ising model variables:





Experiments do not measure σ.

Mapping to QCD and experimental observables

Observed fluctuations are not the same as σ, but related:

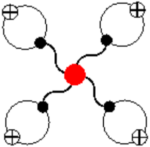
Think of a collective mode described by field σ such that $m = m(\sigma)$:

$$\delta n_{\boldsymbol{p}} = \delta n_{\boldsymbol{p}}^{\text{free}} + \frac{\partial \langle n_{\boldsymbol{p}} \rangle}{\partial \sigma} \times \delta\sigma$$

The cumulants of multiplicity $M \equiv \int_{\boldsymbol{p}} n_{\boldsymbol{p}}$:

- $\kappa_4[M] = \underbrace{\langle M \rangle}_{\text{baseline}} + \kappa_4[\sigma] \times g^4 \underbrace{\left(\bigcirc\right)^4}_{\sim M^4} + \ldots,$

  g – coupling of the critical mode ($g = dm/d\sigma$).



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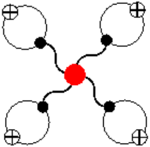

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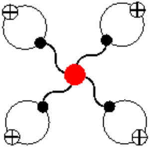

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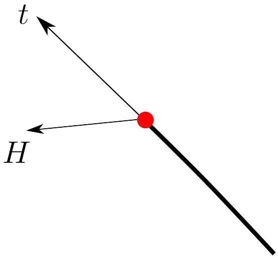
- NB: Sensitivity to M_{accepted}: $(\kappa_4)_\sigma \sim M^4$ (number of 4-tets).

Mapping Ising to QCD phase diagram

T vs μ_B:



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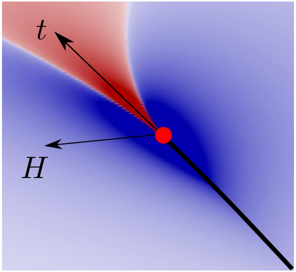

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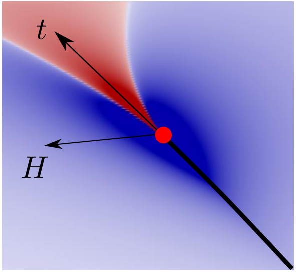
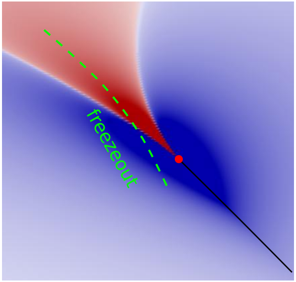
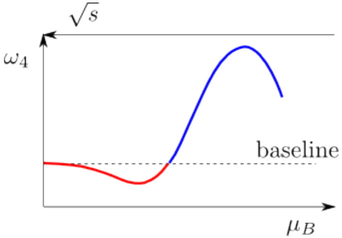


- In QCD $(t, H) \to (\mu - \mu_{\text{CP}}, T - T_{\text{CP}})$
- $\kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma))$

Beam Energy Scan





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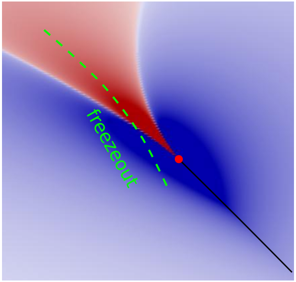

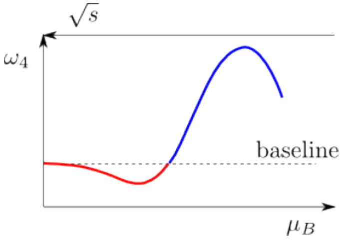

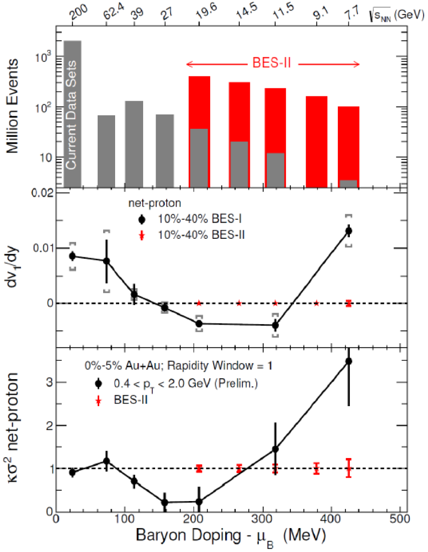

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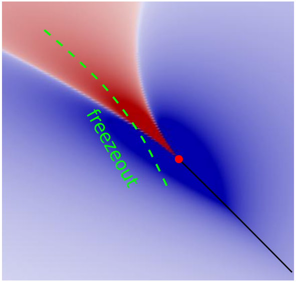

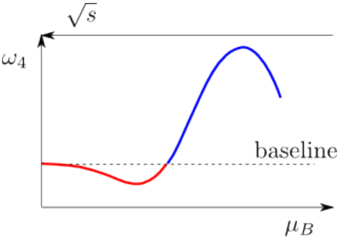

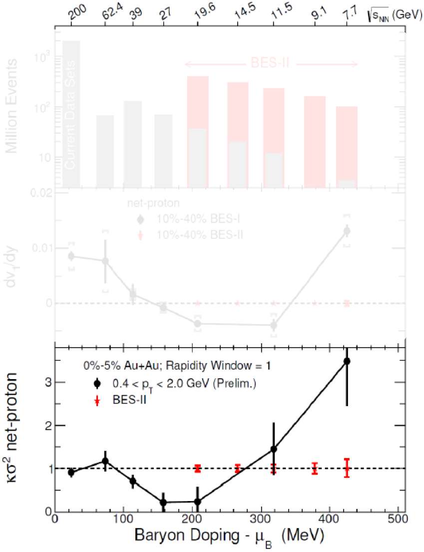

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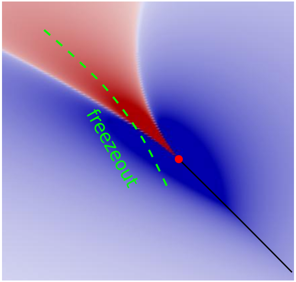
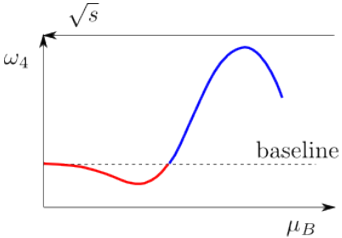
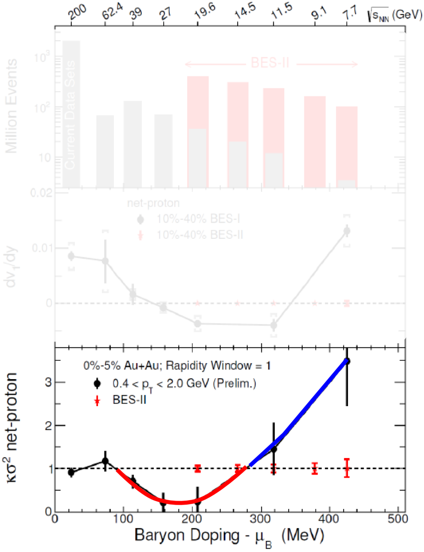


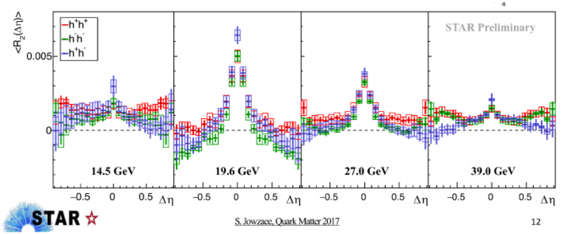


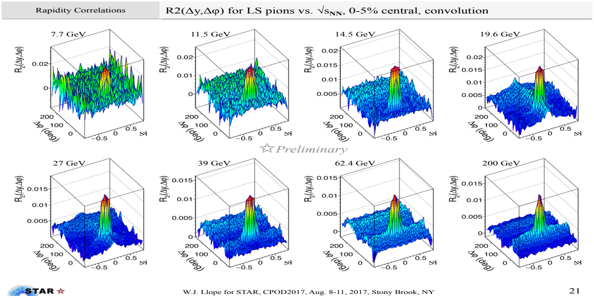


"intriguing hint" (2015 LRPNS)

Back to the two-point correlations

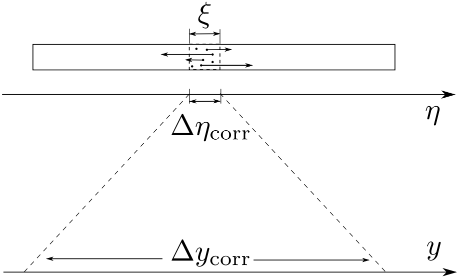
Preliminary, but very interesting:





- Non-monotonous $\sqrt{s}$ dependence with max near 19 GeV.
- Charge/isospin blind.
- $\Delta\phi$ (in)dependence is as expected from critical correlations. $C_2 \sim f(\phi_1)f(\phi_2)$.
- Width $\Delta\eta$ suggests *soft* pions – but p_T dependence need to be checked.
- But: no signal in R_2 for K or p.

Correlations – spatial vs kinematic



$\xi \sim 1 - 3$ fm

$$\Delta\eta_{\rm corr} = \frac{\xi}{\tau_{\rm f}} \sim 0.1 - 0.3$$

Particles within $\Delta\eta_{\rm corr}$ have thermal rapidity spread. Thus

$$\Delta y_{\rm corr} \sim 1 \gg \Delta\eta_{\rm corr}$$

Acceptance dependence – two regimes

How do cumulants depend on acceptance?

Let $\kappa_n(M)$ be a cumulant of M – multiplicity of *accepted*, say, protons.

- $\Delta y \gg \Delta y_{\rm corr}$ – CLT applies.

  $\kappa_n \sim M$

  or $\omega_n \equiv \dfrac{\kappa_n}{M} \to {\rm const}$ – an “intensive”, or volume indep. measure

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- $\Delta y \ll \Delta y_{\rm corr}$ – more typical in experiment.

  Subtracting trivial (uncorrelated, Poisson) contribution:

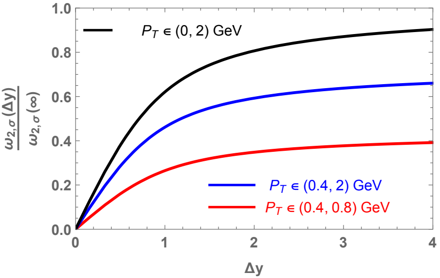
  $\kappa_n - M \sim M^n$ – proportional to number of correlated n-plets;

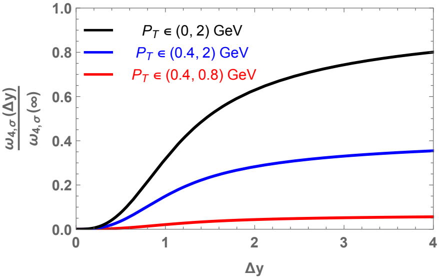
  or $\omega_n - 1 \sim M^{n-1}$.

Critical point fluctuations vs acceptance

Proton multiplicity cumulants ratio at 19.6 GeV: $\omega_{n,\sigma} \equiv \omega_n - 1$

grows as $(\Delta y)^{n-1}$ and saturates at $\Delta y \sim 1 - 2$.



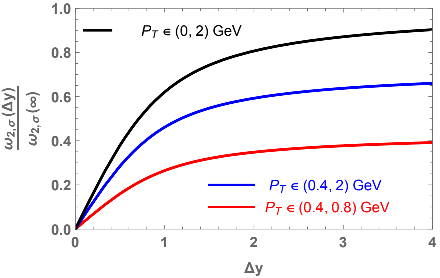


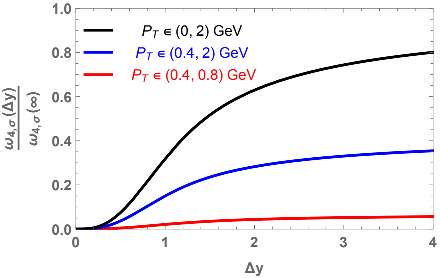
p_T and rapidity cuts have qualitatively similar effects.

Critical point fluctuations vs acceptance

Proton multiplicity cumulants ratio at 19.6 GeV: $\omega_{n,\sigma} \equiv \omega_n - 1$

grows as $(\Delta y)^{n-1}$ and saturates at $\Delta y \sim 1 - 2$.

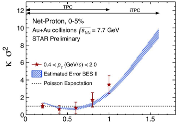




p_T and rapidity cuts have qualitatively similar effects.

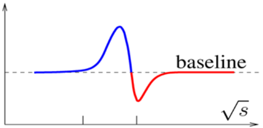
- Wider acceptance improves signal/error:

  errors grow slower than M^n.



Control Measurements for CEP Signatures

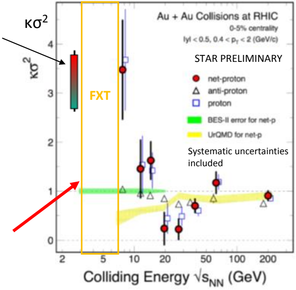
Peak behavior predicted in critical region:



M. Stephanov. J. Physics G.: Nucl. Part. Phys. ***38*** (2011) 124147

Preliminary HADES result, Quark Matter 2017

0-10%
(QM 2017)



Need data here!

→ FXT measurements needed to determine shape of $\kappa\sigma^2$ observable at lower energies

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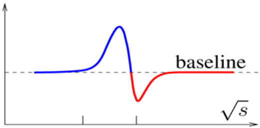

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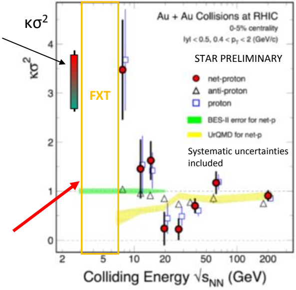

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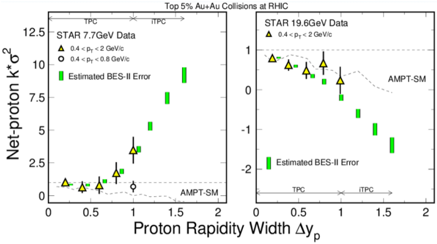
→ FXT measurements needed to determine shape of $\kappa\sigma^2$ observable at lower energies

To draw physics conclusions from this comparison, one needs to take into account rapidity acceptance Δy, different in the experiments.

Acceptance dependence

The acceptance dependence consistent with Δy^{n-1}

(Ling-MS 1512.09125; Bzdak-Koch 1607.07375)



As long as $\Delta y \ll \Delta y_{\rm corr}$ the correlators $\hat\kappa_n$ count the number of n-plets in acceptance.

Factorial cumulants

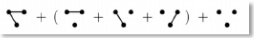
More precisely, the scaling with Δy is for *factorial* cumulants ($\hat{\kappa}_n$ or C_n).

Because they isolate irreducible n-point correlations.

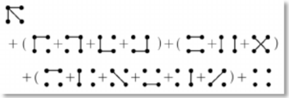
Ling & Stephanov, PRC 93, 034915 (2016)

The cumulants κ_k hold information on multi-particle correlators C_k:

$$\kappa_3 = < N > + 3C_2 + C_3$$



$$\kappa_4 = < N > + 7C_2 + 6C_3 + C_4$$



Bzdak, Koch & Strodthoff, PRC 95, 054906 (2017) ← based on STAR data (X. Luo et al., CPOD2014)

Propose C_k vs. N_{part} (& Δy) as a better approach to isolate critical fluctuations:

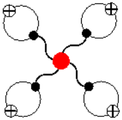
Normal cumulants ($n > 2$) are deviations from normal distribution.

Factorial cumulants – from Poisson distribution.

Physics of correlations

One can describe the correlations in the language of “clusters” Or, more physically, mean-field.

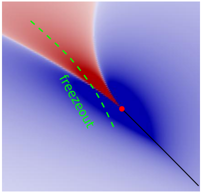
The correlations induced by critical mode have similar effect.

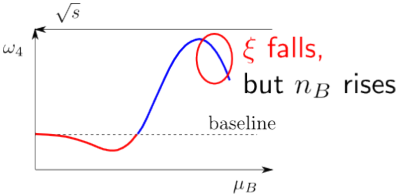


Isospin blind n-particle correlations.
Characteristic *non-monotonous* $\sqrt{s}$ dependence.

The size of the “cluster” of order number of particles within ξ^3 (qualitatively).

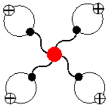
Large μ_B: n_B^4 vs ξ^7





The cumulants of multiplicity $M \equiv \int_{\boldsymbol{p}} n_{\boldsymbol{p}}$: $\quad (M_P \sim n_B \times \Delta y)$

- $$\kappa_4[M] = \underbrace{\langle M \rangle}_{\text{baseline}} + \kappa_4[\sigma] \times g^4 \underbrace{\left(\bigcirc \right)^4}_{\sim M^4} + \ldots,$$



$$\hat{\kappa}_4[M] \approx g^4 \kappa_4[\sigma] M^4 \sim \underbrace{\xi^7 \times n_B^4}_{\text{compete at large } \mu_B} \times (\Delta y)^4.$$

[Athanasiou-Rajagopal-MS]

- The ratio $\dfrac{\hat{\kappa}_4[M]}{n_B^4}$ or $\dfrac{\hat{\kappa}_4[M]}{M^4} \sim \kappa_4[\sigma] \sim \xi^7$.

Non-equilibrium physics is essential near the critical point.

The goal for BEST COLLABORATION

Why ξ is finite

System expands and is *out of equilibrium*

Kibble-Zurek mechanism:

Critical slowing down means $\tau_{\rm relax} \sim \xi^z$.
Given $\tau_{\rm relax} \lesssim \tau$ (expansion time scale):

$$\xi \lesssim \tau^{1/z},$$

$z \approx 3$ (universal).

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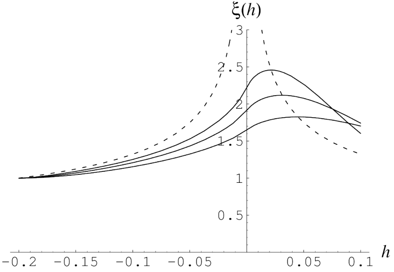
Estimates: $\xi \sim 2-3$ fm
(Berdnikov-Rajagopal)

KZ scaling for $\xi(t)$
and cumulants
(Mukherjee-Venugopalan-Yin)



Lessons

$$\kappa_n \sim \xi^p \quad \text{and} \quad \xi_{\max} \sim \tau^{1/z}$$

- Therefore, the magnitude of fluctuation signals is determined by non-equilibrium physics.
- Higher moments are more sensitive to ξ – good for detecting critical point. But harder to predict for the same reason.

Time evolution of cumulants (memory)

Mukherjee-Venugopalan-Yin

Relaxation to equilibrium

$$\frac{dP(\sigma_0)}{d\tau} = \mathcal{F}[P(\sigma_0)]$$

$$\Downarrow$$

$$\frac{d\kappa_n}{d\tau} = L[\kappa_n, \kappa_{n-1}, \ldots]$$

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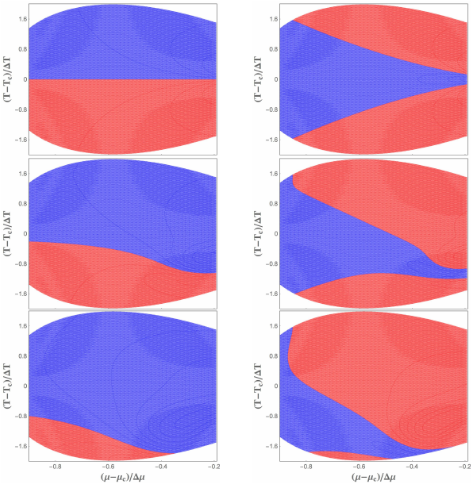

κ_3 κ_4

Signs of cumulants also depend on off-equilibrium dynamics.

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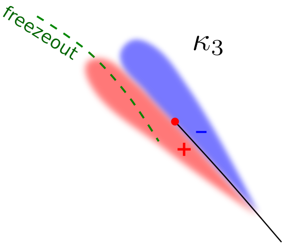

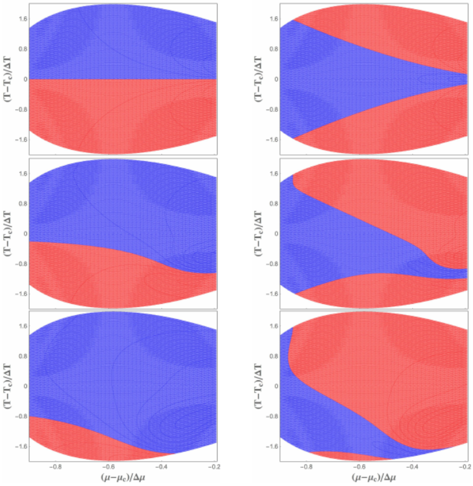

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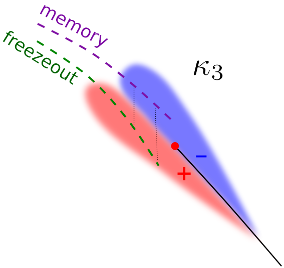
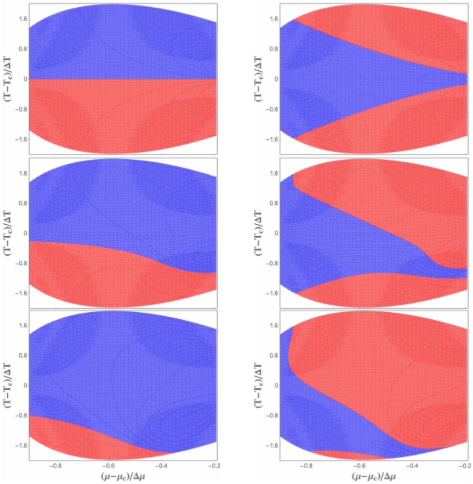




κ_3 κ_4

Signs of cumulants also depend on off-equilibrium dynamics.

Experiments do not measure σ.

Kinetics near critical point

- Soft mode couples to hadrons
- Dynamical description
  - Couple hadrons to soft mode

$$\mathcal{S} = \int_{\boldsymbol{x}} \frac{1}{2}\big(\partial_\mu\sigma\partial^\mu\sigma - U(\sigma)\big) - \int ds\ M(\sigma),$$

  - Kinetic equation

$$\frac{p^\mu}{M}\frac{\partial f}{\partial x^\mu} + \underbrace{\partial^\mu M(\sigma)}_{\text{"force" due to field grad.}} \frac{\partial f}{\partial p^\mu} = 0\,,$$

  + field equation:

$$\partial^2\sigma + dU/d\sigma + (dM/d\sigma)\int_{\boldsymbol{p}} f/\gamma = 0.$$

(M.S., PRD81:054012,2010)

Collisions, dissipation and noise

Fluctuation-dissipation requires noise: *(Fox,Uhlenbeck)*

$$\frac{p^\mu}{M}\frac{\partial f}{\partial x^\mu} + \underbrace{\partial^\mu M(\sigma)}_{\text{"force" due to field grad.}}\frac{\partial f}{\partial p^\mu} = \mathcal{C}[f] + \text{noise }(\xi)\,,$$

$$\partial^2\sigma + dU/d\sigma + (dM/d\sigma)\int_{\boldsymbol{p}} f/\gamma = -\Gamma_0\partial_0\sigma + \text{noise }(\eta).$$

$$\langle\xi(x_1,p_1)\xi(x_2,p_2)\rangle = \underbrace{(\mathcal{K}+\mathcal{K}^\dagger)}_{\text{linearized C}}\delta^3_{\boldsymbol{p}_1,\boldsymbol{p}_2}\delta^4(x_1-x_2);$$

$$\langle\eta(x_1)\eta(x_2)\rangle = 2\Gamma_0 T\delta^4(x_1-x_2);$$

$$\langle\xi(x_1,p_1)\eta(x_2)\rangle = 0.$$

Kinetic theory with critical mode

- Boltzmann equation, with collisions and noise:

$$\frac{p^\mu}{M}\frac{\partial f}{\partial x^\mu} + \partial^\mu M \frac{\partial f}{\partial p^\mu} + \mathcal{C}[f] = \xi\,,$$

(Fox-Uhlenbeck) + field equation:

$$\partial^2\sigma + dU/d\sigma + (dM/d\sigma)\int_{\boldsymbol{p}} f/\gamma + \Gamma_0\dot{\sigma} = \eta.$$

- Noise is fixed by fluctuation-dissipation relations.

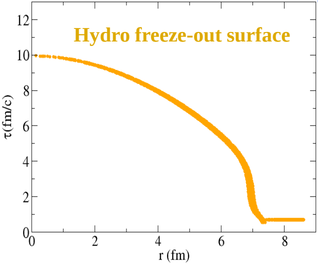
  Fluctuations in equilibrium are reproduced correctly.

  We can now study non-equilibrium evolution of fluctuations.

  E.g., memory effects can be described (PRD81:054012,2010)

Applied to realistic freezeout conditions

Freeze-out Scheme near the Critical Points



Jiang, Li & Song, PRC 2016

Particle emissions in traditional hydro

$$E\frac{dN}{d^3p}=\int_\Sigma \frac{p_\mu d\sigma^\mu}{2\pi^3} f(x,p)$$

Particle emissions near T_{cr}

$$M \longrightarrow g\sigma(x)$$

$$f(x,p) = f_0(x,p)[1-g\sigma(x)/(\gamma T)] = f_0 + \delta f$$

$$\langle \delta f_1 \delta f_2 \rangle_\sigma = f_{01} f_{02} f_{03} \left(\frac{g^2}{\gamma_1 \gamma_2} \frac{1}{T^3} \right) \langle \sigma_1 \sigma_2 \rangle_c ,$$

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$$\langle \delta f_1 \delta f_2 \delta f_3 \delta f_4 \rangle_\sigma = f_{01} f_{02} f_{03} f_{04} \left(\frac{g^4}{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \frac{1}{T^4} \right) \langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle_c .$$

What is Hydrodynamics?

Fluid left alone tends to equilibrium.

There are two time scales:

1) local thermodynamic equilibration – fast;

2) achieving same conditions throughout – slow.

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It is an *effective theory* – only operates with degrees of freedom that matter – densities of energy, momentum, charge. They are slow to change on large scales because they carry conserved quantities.

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It is an *effective theory* – only operates with degrees of freedom that matter – densities of energy, momentum, charge. They are slow to change on large scales because they carry conserved quantities.

The remaining, faster degrees of freedom are the “noise”.

Equations for stochastic hydrodynamics proposed by Landau-Lifshits in 1957. We want to study correlations in a *relativistically* expanding fireball.

Relativistic Hydrodynamics

- Equations: conservation (continuity) $\nabla_\mu T^{\mu\nu} = 0$.
- Variables: ϵ, u^μ, defined by $T^{\mu\nu} u_\nu = \epsilon u^\mu$ – fixes 4 components of $T^{\mu\nu}$.

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- Variables: ϵ, u^μ, defined by $T^{\mu\nu} u_\nu = \epsilon u^\mu$ – fixes 4 components of $T^{\mu\nu}$.
- The remaining 6 components (stress T^{ij} in l.r.f.) must be also expressed in terms of ϵ and u^μ (grad. expansion):

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \underbrace{P(\epsilon)\Delta^{\mu\nu} + \Delta T^{\mu\nu}}_{\text{stress in l.r.f}}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ and

$$\Delta T^{\mu\nu} = -\eta \Delta^\mu_\lambda \left[\nabla^\lambda u^\nu + \nabla^\nu u^\lambda - \frac{2}{3} g^{\lambda\nu} (\nabla \cdot u) \right] - \zeta \Delta^{\mu\nu} (\nabla \cdot u)$$

(velocity gradients cause stress)

Fluctuations and Noise

- The constitutive eq. is only true *on average*. Both sides fluctuate and

$$T^{\mu\nu} = \underbrace{T^{\mu\nu}_{\mathrm{id}} + \Delta T^{\mu\nu}_{\mathrm{visc}}}_{\text{function of } \epsilon,\ u^\mu} + S^{\mu\nu}.$$

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- The magnitude is determined by the condition that the equilibrium distribution is $e^{\text{Entropy}(\epsilon)}$ (Einstein 1910). Dissipation (proportional to viscosity), which damps fluctuations, must be matched by noise (Onsager):

$$\langle S^{\mu\nu}(x) S^{\alpha\beta}(y)\rangle = 2T\left[\eta\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) + \left(\zeta - \frac{2}{3}\eta\right)\Delta^{\mu\nu}\Delta^{\alpha\beta}\right]\delta^4(x-y)$$

- Now $\nabla_\mu T^{\mu\nu} = 0$ is a system of stochastic eqs. for ϵ, u^μ.

Hydrodynamics breaks down at CP

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Near CP gradient terms are dominated by $\zeta \sim \xi^3 \to \infty$
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When $k \sim \xi^{-3}$ hydrodynamics breaks down, i.e., while $k \ll \xi^{-1}$ still.

(For simplicity, measure dim-ful quantities in units of T, i.e., $k \sim T(T\xi)^{-3}$.)

Why does hydro break at so small k?

Critical slowing down and bulk viscosity

Bulk viscosity is the effect of system taking time to adjust to local equilibrium (Khalatnikov-Landau).

$$p_{\text{hydro}} = p_{\text{equilibrium}} - \zeta\, \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

$\boldsymbol{\nabla} \cdot \boldsymbol{v}$ – expansion rate

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Hydrodynamics breaks down because of large relaxation time (critical slowing down).

Similar to breakdown of an effective theory due to a low-energy mode which should not have been integrated out.

Critical slowing down and Hydro+

- There is a critically slow mode ϕ with relaxation time $\tau_\phi \sim \xi^3$.

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(MS-Yin 1704.07396, in preparation)

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  Crossover occurs when $\Gamma_{\rm hydro} \sim \Gamma_\phi$, or $k \sim \xi^{-3}$.
- Regime II: $k > \xi^{-3}$ – "Hydro+" regime.

Advantages/motivation of Hydro+

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  (Since noise $\sim \zeta$, also the noise is not large.)

Ingredients of "Hydro+"

- Nonequilibrium entropy, or quasistatic EOS:

$$s^*(\varepsilon, n, \phi)$$

Equilibrium entropy is the maximum of s^*:

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- The 6th equation (constrained by 2nd law):

$$(u \cdot \partial)\phi = -\gamma_\phi \pi - A_\phi (\partial \cdot u), \qquad \text{where } \pi = \frac{\partial s^*}{\partial \phi}$$

- Another example: relaxation of axial charge.

Linearized Hydro+

Linearized Hydro+ has 4 longitudinal modes (sound$\times 2$ + density + ϕ).

In addition to the usual c_s, D, etc. Hydro+ has two more parameters $\Delta c^2 = c_*^2 - c_s^2$ and $\Gamma = \Gamma_\phi$.

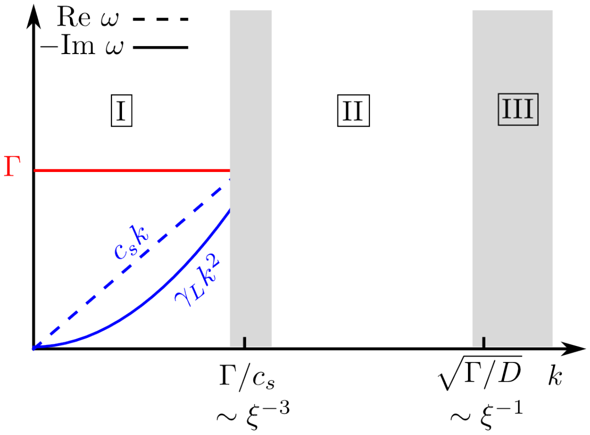
The sound velocities are different in Regime I ($c_s k \ll \Gamma$) and II:

$$c_s^2 = \left(\frac{\partial p}{\partial \varepsilon}\right)_{s/n,\pi=0} \quad \text{and} \quad c_*^2 = \left(\frac{\partial p^*}{\partial \varepsilon}\right)_{s/n,\phi}$$

The bulk viscosity receives large contribution from the slow mode given by Landau-Khalatnikov formula

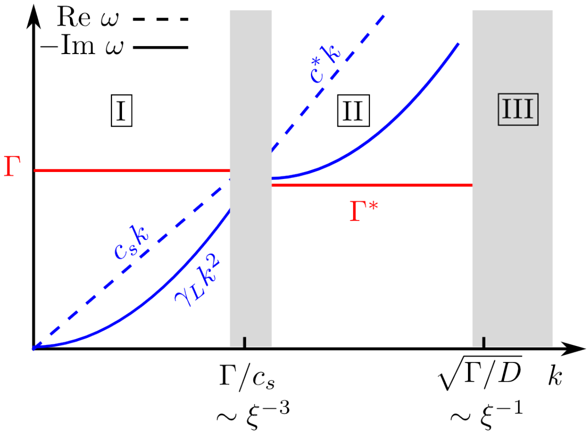
$$\Delta\zeta = w\Delta c^2/\Gamma$$

Modes



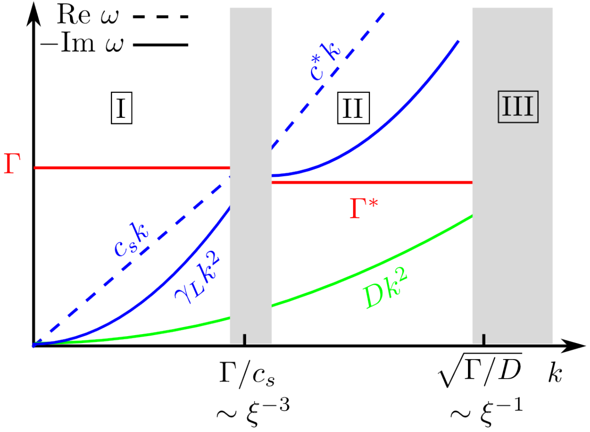
Re ω
$-$Im ω
I
II
III
Γ
$c_s k$
$\gamma_L k^2$
Γ/c_s
$\sim \xi^{-3}$
$\sqrt{\Gamma/D}$
$\sim \xi^{-1}$
k

Modes



Re ω
$-$Im ω
I
II
III
Γ
Γ^*
$c^* k$
$c_s k$
$\gamma_L k^2$
Γ/c_s
$\sim \xi^{-3}$
$\sqrt{\Gamma/D}$
$\sim \xi^{-1}$
k

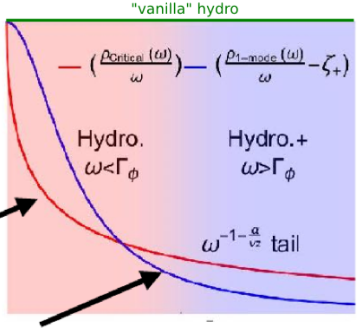
Modes



Re ω
$-$Im ω
I
II
III
Γ
Γ^*
$c_s k$
$c^* k$
$\gamma_L k^2$
Dk^2
Γ / c_s
$\sim \xi^{-3}$
$\sqrt{\Gamma / D}$
$\sim \xi^{-1}$
k

Hydro+

- "Hydro+" one mode qualitatively captures the transition from hydro regime ω< 1/ξ³ to "hydro+" regime ω> 1/ξ³ .



($\rho_{Bulk}(\omega)$ from mode H)

($\rho_{Bulk}(\omega)$ from hydro+one mode)

- One mode is not enough to fully capture the critical dynamic behavior.

- Next step: Hydro+ a spectrum of slow modes.

Microscopic origins of Hydro+

Understanding the microscopic origin of the slow mode:

The fluctuations around equilibrium are controlled by the entropy functional $P \sim e^S$.

Near the critical point convenient to "rotate" the basis of variables to "Ising"-like critical variables $\mathcal{E}$ and $\mathcal{M}$. $\mathcal{M} \sim s/n - (s/n)_{\rm CP}$.

$$\delta^2 \mathcal{S}[\delta\mathcal{E}, \delta\mathcal{M}] = \frac{1}{2}\, a_{\mathcal{M}}\ (\delta\mathcal{M})^2 + \frac{1}{2}\, a_{\mathcal{E}}\ (\delta\mathcal{E})^2 + b\, \delta\mathcal{E}\, \delta\mathcal{M}^2 + \ldots$$

Since $a_{\mathcal{M}} \ll a_{\mathcal{E}}$ fluctuations of $\mathcal{M}$ are large and are slow to equilibrate.

Their magnitude is related to the slow relaxation mode ϕ.

Slow modes near a critical point

- A general critical point: slow modes include order parameter (M), and <δMδM> (and potentially higher cumulants…).
- QCD critical point: hydro + <δMδM>.
  - M is a linear combination of ϵ, n and chiral condensate σ. σ equilibrates at microscopic time scale and the evolution of σ simply traces the evolution of ϵ, n $\Rightarrow$ ~~Eq. for M~~.

(Son-Stephanov, 04')

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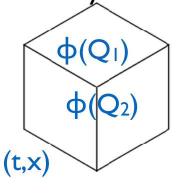
Relation between <δMδM> and $\phi(t, x\,; Q)$

- The Wigner transform of <δMδM> ⇒ $\phi(t, x\,; Q)$

$$\phi(t, x\,; Q) = \int d^3\Delta x \,\langle \delta M(t, x+\Delta x)\, \delta M(t, x-\Delta x)\rangle\, e^{-i\,Q\,\Delta x}$$

$\phi(t,x\,;Q)$ may be viewed as many local slow modes with label Q at a fluid cell (t,x).



- In equilibrium: $\phi_{eq}(Q) = 1/[(\chi_M)^{-1}+Q^2]$ ($\phi_{eq}(Q=0)=\chi_M \sim \kappa_2$).

Generalized Entropy $s_+(\epsilon,n,\phi(Q))$

- The generalized entropy $s_+(\epsilon,n, \phi(Q))$ can be *derived* following the formalism of 2PI effective action in QFT.

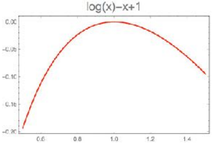
  (J. M. Cornwall, R. Jackiw, E. Tomboulis, 1974')

  - NB: 2PI effective action is a useful tool to study non-equilibrium effects. (e.g. J. Berges et al, hep-ph/0409123)

- A simple form at the leading order in "loop expansion":

$$s_+\,(\,\epsilon, n, \phi(Q)\,) = s_{\rm eq}\,(\,\epsilon, n\,) + \frac{1}{2}\int_Q \left\{ \log\left(\frac{\phi(Q)}{\phi_{eq}(Q)}\right) - \frac{\phi(Q)}{\phi_{eq}(Q)} + 1 \right\},$$

$$\pi(Q) = \frac{\delta s_+}{\delta \phi(Q)} = \phi_{\rm eq}^{-1}(Q) - \phi^{-1}(Q)$$



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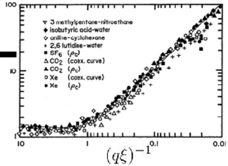
(MS-Yin, in preparation)

E.o.M for ϕ(Q)

- A Q-dependent (phenomenological) relaxation equation for ϕ:

$(u^\mu \partial_\mu)\phi = - \gamma_\phi \pi$ ⟶ $(u^\mu \partial_\mu)\, \phi(Q) = -2\,\gamma(Q)\pi(Q)$

- $\Gamma(Q)=\gamma(Q)/(\phi_{eq}(Q))^2$ is known from model H. ⟵



- $s_{(+)}(\epsilon,n,\phi(Q))$ together with $\Gamma(Q)$ successfully reproduces critical behavior of $\rho_{Bulk}(\omega)\sim \mathrm{Im} \langle T^i_i T^i_i \rangle$.

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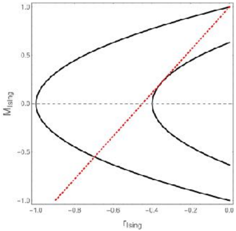
(MS-Yin, in preparation)

An example of hydro+ in an expanding QGP

Solving equation for ϕ(Q) along a trajectory



$M \sim T - T_c, \; r_{Ising} \sim \mu - \mu_c$

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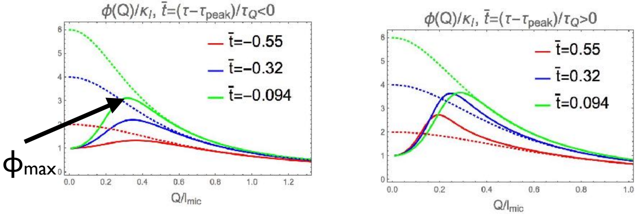
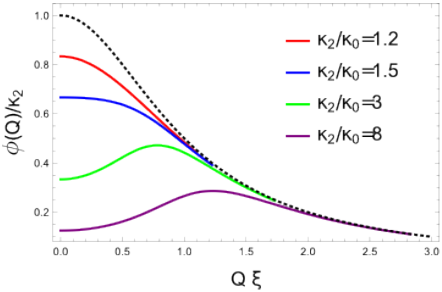
"Hydro+" describes the slow relaxation of critical fluctuations



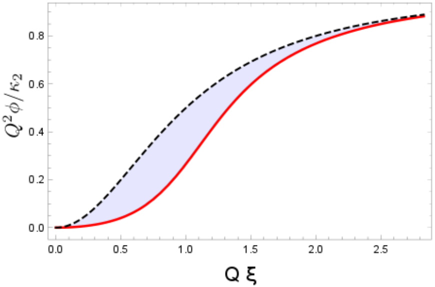
$\tau<\tau_{peak}$, fall out of equilibrium. $\tau>\tau_{peak}$, memory.

- NB: $\phi(Q)$ can be related to the baryon number balance function (if supplemented with mapping and freeze-out prescription).

Relaxation of slow mode(s).



Shaded area $\sim(\kappa_{2,eq} - \kappa_{2,ne})$



Summary

- A fundamental question for Heavy-Ion collision experiments:

  Is there a critical point on the boundary between QGP and hadron gas phases?

  Theoretical framework is needed – the goal for BEST COLLABORATION .

- Large (non-gaussian) fluctuations – universal signature of a critical point.

- In H.I.C., the magnitude of the signatures is controlled by dynamical non-equilibrium effects. The physics of the interplay of critical and dynamical phenomena can be captured by hydrodynamics with a critically slow mode(s) – Hydro+.