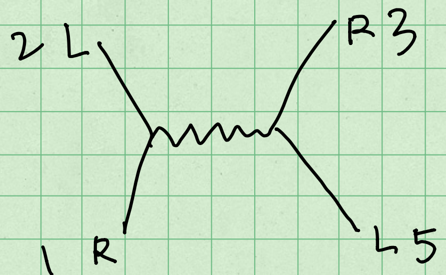
sum over spin and color, we obtain the square amplitude for $e^+e^- \to q(p_3)\,\bar{q}(p_5)\,g(p_4)$.

$$\sum |M|^2 = 16 e^2 e_q^2 g_s^2\, tr[t^a t^a] \cdot \frac{s_{13}^2 + s_{15}^2 + s_{23}^2 + s_{25}^2}{s_{12}\, s_{34}\, s_{45}}$$

$$= 48 e^2 e_q^2 g_s^2\, C_F \cdot \frac{s_{13}^2 + s_{15}^2 + s_{23}^2 + s_{25}^2}{s_{12}\, s_{34}\, s_{45}}$$

where $s_{ij} = 2 k_i \cdot k_j$

Soft / collinear limit of amplitude.



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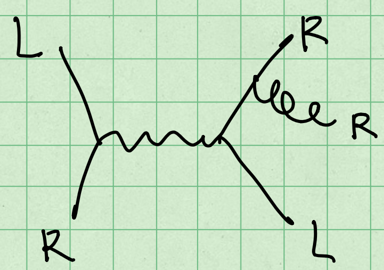
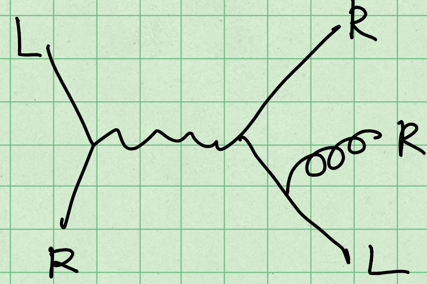




$$A_4 = \frac{\langle 25\rangle^2}{\langle 12\rangle\langle 35\rangle} \qquad A_5 = \frac{-\sqrt{2}\,\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle}$$

soft gluon emission: $k_4^0 \to 0$

$$A_5 = \frac{-\sqrt{2}\,\langle 35\rangle}{\langle 34\rangle\langle 45\rangle} \cdot A_4$$

$$= \underbrace{S(RRL)}_{\text{eikonal amplitude.}}\, A_4$$

$$S(RLL) = \frac{-\sqrt{2}\,[35]}{[34][45]}$$

collinear behavior

consider $k_3 // k_4$ limit.

$$k_3 = zK, \quad k_4 = (1-z)K, \quad K = k_3 + k_4 \quad K^2 \to 0$$

$$|3\rangle \sim \sqrt{z}\,|K\rangle, \quad |4\rangle \sim \sqrt{1-z}\,|K\rangle$$

$$A_5(RLRRL) = \frac{-\sqrt{2}\,\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle}$$

$$\approx -\sqrt{2}\,\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle K5\rangle}\cdot\frac{1}{\sqrt{1-z}}$$

$$= -\sqrt{2}\,\frac{1}{\sqrt{1-z}\,\langle 34\rangle}\cdot\frac{\langle 25\rangle^2}{\langle 12\rangle\langle K5\rangle}$$

$$= \underbrace{\frac{-\sqrt{2}}{\sqrt{1-z}\,\langle 34\rangle}}_{\text{split amplitude.}} A_4(1_R 2_L K_R 5_L)$$

$$\equiv Sp_L(3^q_R, 4^g_R)\, A_4(1_R 2_L K_R 5_L)$$

$$Sp_L(a^q_R, b^g_R) = \frac{-\sqrt{2}}{\sqrt{1-z}\,\langle ab\rangle}$$

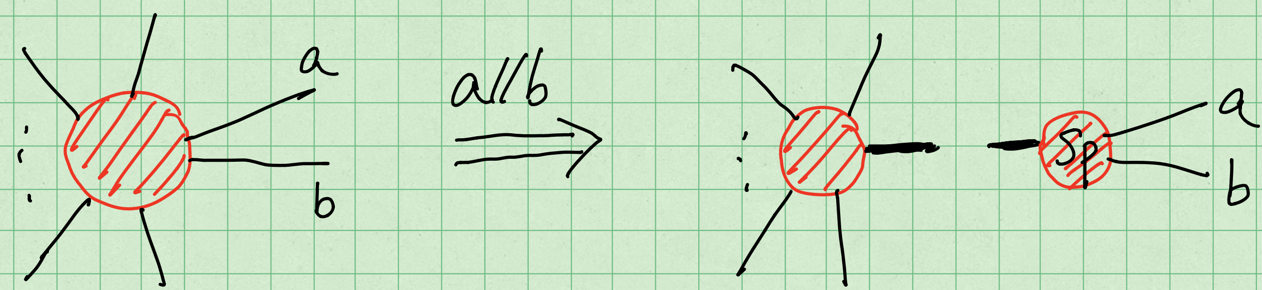
from $k_4 // k_5$ limit, we find

$$Sp_R(a^g_R, b^q_L) = \frac{-\sqrt{2}\,(1-z)}{\sqrt{z}\,\langle ab\rangle}$$

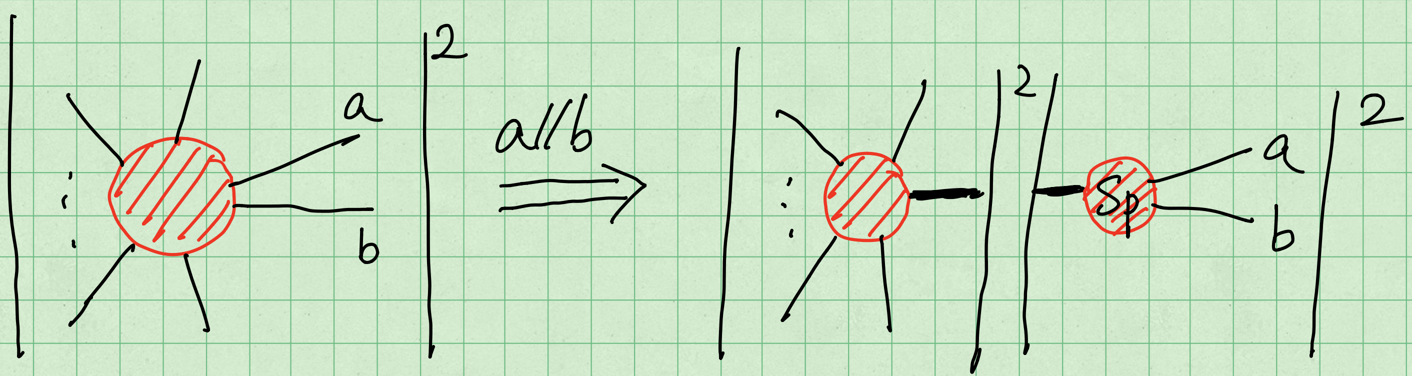
Applying C and P

$$Sp_L(a^q_R, b^g_L) = \frac{\sqrt{2}\,z}{\sqrt{1-z}\,[ab]}$$

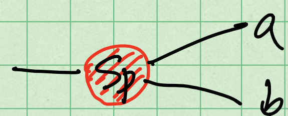
Splitting Amplitude works for any tree-level amplitude



it also holds at squared amplitude level:



Splitting function

$$P_{ab}(z) = \sum_{\text{pol.}} \Bigg|$$  $$\Bigg|^2$$

E.g.,

$$\underbrace{P_{qq}(z)}_{\text{time-like}} = C_F\left(\left| \frac{1}{\sqrt{1-z}} \frac{1}{\langle ab\rangle} \right|^2 + \left| \frac{z}{\sqrt{1-z}} \frac{1}{\langle ab\rangle} \right|^2 \right)$$

$$= C_F \frac{1+z^2}{1-z}$$ * unregularized splitting function.

can be regularized by plus prescription:

$$\hat{P}_{qq}(z) = C_F\left[\frac{1+z^2}{[1-z]_+} + \lambda\, \delta(1-z) \right]$$

fix λ by quark number conservation sum rule.

$$\int_0^1 dz\, \hat{P}_{qq}(z) = C_F \int_0^1 \frac{1+z^2-2}{1-z} dz + \lambda C_F \equiv 0$$

$$\Rightarrow \quad \lambda = \frac{3}{2}$$

$$\therefore \quad P_{qq}(z) = C_F \left(\frac{1+z^2}{[1-z]_+} + \frac{3}{2}\delta(1-z) \right)$$

Physical consquence of universality in timelike and spacelike splitting: Foundation of most pQCD Application at the LHC (parton shower, resummation. fixed order calculations, amplitude bootstrap, jet substructure, ------)

However, collinear factorization only holds universally at tree level. See also Prof. Ma's talk.

Propertites of soft and collinear also impose restriction on which observables are perturbatively calculable.

IR Safe Observable: Observables which are not sensitive to soft or collinear emission $O_n(k_1, k_2, \cdots, k_n)$.

$$O_{n+1}(\cdots, k_s, \cdots) \to O_n(\cdots, \cancel{k_s}, \cdots) \qquad k_s \to 0$$

$$O_{n+1}(\cdots, k_a, k_b, \cdots) \to O_n(\cdots, k_p, \cdots) \qquad k_a \parallel k_b, \; k_a + k_b \simeq k_P$$

Examples:

- jet cross section
- thrust $\quad T = \max\limits_{\vec{n}} \dfrac{\sum_{j=1}^{N} |\vec{n}\cdot\vec{k}_j|}{\sum_{j=1}^{N} |\vec{k}_j|}$
- inclusive cross section
- W/Z/H rapidity / pT distribution

NLO QCD corrections to $e^+e^- \to q\bar{q}$.

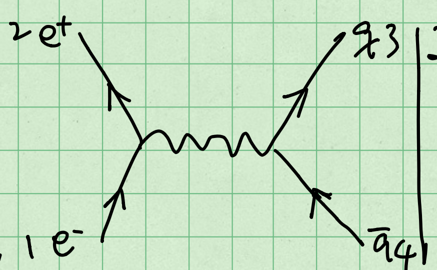
$k_1 = E\cdot(1,0,0,1)$

$k_2 = E\cdot(1,0,0,-1)$

$k_3 = E\cdot(1,0,\sin\theta,\cos\theta)$

$k_4 = E(1,0,-\sin\theta,-\cos\theta)$

LO: $\displaystyle\int \Bigg|$



$\Bigg|^2$ Phase Space

$s = (k_1+k_2)^2 = 4E^2$

$t = (k_1-k_4)^2 = -2E^2(1+\cos\theta)$

$u = (k_1-k_3)^2 = -2E^2(1-\cos\theta)$

recall that

$$\frac{1}{4}\sum_{\text{pol}} |M_4^{(0)}|^2 = \frac{4}{4}e^2 e_q^2 N_c \cdot \frac{2}{s^2}\cdot(t^2+u^2)$$

$$= \frac{8}{4} e^2 e_q^2 N_c \cdot \frac{1}{16E^4}\cdot\left(4E^4\cdot(1+\cos\theta)^2 + 4E^4(1-\cos\theta)^2\right)$$

$$= e^2 e_q^2 N_c\cdot(1+\cos^2\theta)$$

flux factor

$$\sigma_0 = \frac{1}{2s}\cdot\int\frac{d^3k_3}{(2\pi)^3 2k_3^0}\cdot\int\frac{d^3k_4}{(2\pi)^3 2k_4^0}(2\pi)^4\cdot\delta^4(k_1+k_2-k_3-k_4)$$

$$\times\, e^2 e_q^2 N_c\cdot(1+\cos^2\theta)$$

$$= \frac{4\pi}{3s} Q_q^2 N_c \alpha^2$$

$e^2 = g_e^2, \quad e_q^2 = Q_q^2 g_e^2$

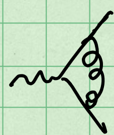
$\alpha = \dfrac{g_e^2}{4\pi}$

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q Q_q^2$$

Early evidence for color!

NLO Virtual corrections

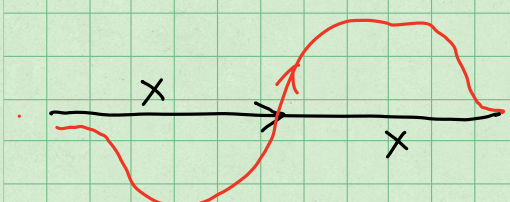
$\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}}$, vanish in dim-reg.

The one-loop integral  $\sim \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+i0} \cdot \frac{1}{(\ell-k_3)^2+i0} \cdot \frac{1}{(\ell+k_4)^2+i0}$

is divergent in $D=4$ dimension and require regularization.

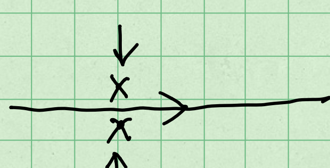
By simple power counting, no UV divergence. IR divergence can arise when propagator become on-shell. However, vanish of propagator doesn't necessarily leads to singularity.

complex integral.



not pinched

singularity can be avoid by contour deformation



pinched

end point singularity

Singularity can not be avoided

Structure of IR singular integral surface captured by Landau Equation.

For a generic L loop integral,

$$\mathcal{I}(\{p_i\},\{m_i\}) = \int \prod_{k=1}^{L} \frac{d^4 l_k}{(2\pi)^4} \frac{N(\{l_i\},\{p_i\})}{\prod_{i=1}^{N} D_i}$$

$$D_i = q_i^2 - m_i^2, \quad \forall i \in N$$

Landau Equation :

each choice of the $\{D_i\}$ in the 1st Equations specify a cut configuration.

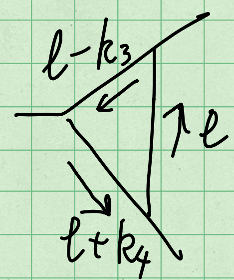
$$\begin{cases} D_i = q_i^2 - m_i^2 = 0, \ (i = 1, 2, \cdots, r \leq 4L) \\ \sum_{i \subset \text{loop } j} \alpha_i^{(j)} q_i = 0 \quad \text{for every loop } j \end{cases}$$

cutted momentum

Singularity corresponds to Non-trivial solution of Landau eq.

Apply Landau Equation to the one-loop vertex integral

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 \ (l-k_3)^2 \ (l+k_4)^2}$$



Case 1 :
3 particle cut

$$\begin{cases} l^2 = 0, \quad (l-k_3)^2 = 0, \quad (l+k_4)^2 = 0 \\ \alpha_1 l^\mu + \alpha_2 (l-k_3)^\mu + \alpha_3 (l+k_4)^\mu = 0 \end{cases}$$

Contracting the second line by l, k_3, k_4,

we get $\alpha_2 = \alpha_3 = 0$, $\alpha_1 \neq 0$, and $l^\mu \to 0$

physically corresponds to l soft, $l \sim Q(\lambda^2, \lambda^2, \lambda^2, \lambda^2)$

Case 2:
two particle cut

$$\begin{cases} l^2 = 0, \quad (l-k_3)^2 = 0 \\ \alpha_1 l^\mu + \alpha_2 (l-k_3)^\mu = 0 \end{cases}$$

contracting the second line by l, k_3, k_4,
the only non-trivial equation is

$$(\alpha_1+\alpha_2)\, l\cdot k_4 - \alpha_2\, k_3\cdot k_4 = 0$$

$$\therefore \quad \frac{l\cdot k_4}{k_3\cdot k_4} = \frac{\alpha_2}{\alpha_1+\alpha_2}$$

physically $l \,//\, k_3$, and $l^2=0$, collinear singularity.

similarily, cutting l^2 and $(l+k_4)^2$ gives the $l \,//\, k_4$ collinear singularity.

exercise: Check that the remaining cut configuration do not give rise to IR singularities for the scalar vertex integral.

Therefore, the IR singularities in the vertex corrections come from soft & collinear region!

We use dimensional Regularization to regulate both UV and IR singularities.

$$D = 4-2\epsilon,$$

$$\int \frac{d^4 l}{(2\pi)^4} \to \mu^{2\epsilon} \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}}$$

$$\sigma_V = \mathrm{Re}\left[\sigma_0 \cdot \frac{\alpha_s}{\pi} C_F \cdot \frac{(4\pi)^\epsilon}{e^{\epsilon\gamma_E}} \cdot \left(\frac{\mu^2}{-s-i0^+}\right)^\epsilon \cdot \left[-\frac{1}{\epsilon_{IR}^2} - \frac{\frac{3}{2}}{\epsilon_{IR}} - \frac{7}{2} - \frac{\delta_R}{2} + \frac{\pi^2}{12}\right]\right]$$

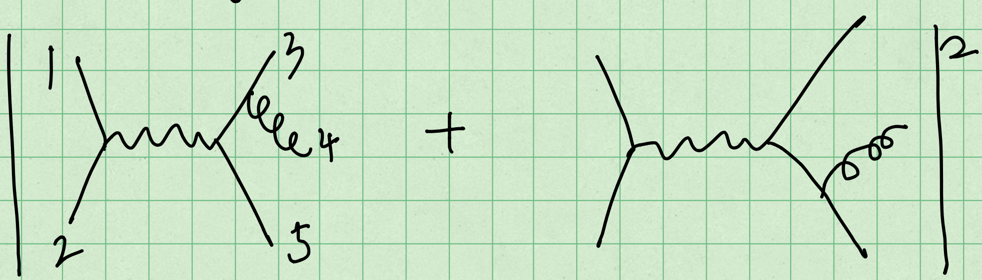
δ_R control the scheme dependence

	loop/phase space	Dirac alg.	$g^{\mu\nu}$	# of gluon pol.	δ_R
CDR	$4-2\epsilon$	$4-2\epsilon$	$4-2\epsilon$	$2-2\epsilon$	1
FDH	$4-2\epsilon$	4	4	2	0

Advantage of FDH scheme: Can use spinor-helicity method.
Disvantage: NNLO not completely understood.

reference: Kilgore, 1205.4015
We will use CDR in this lecture.

NLO Real Corrections:



recall that the squared amplitude is

$$|M_5|^2 = 16 N_c e^2 e_q^2 g_s^2 C_F \frac{s_{13}^2 + s_{15}^2 + s_{23}^2 + s_{25}^2}{s_{12} s_{34} s_{45}}$$

some kinematics: $\quad Q^\mu = k_1^\mu + k_2^\mu$

let $k_1 = \frac{Q}{2}(1,0,0,1)$, $\quad k_2 = \frac{Q}{2}(1,0,0,-1)$

$s_{12} = Q^2$, $\qquad s_{34} + s_{45} + s_{35} = Q^2$

$$\therefore \quad s_{34} = Q^2 - s_{45} - s_{35} = Q^2 - 2k_5\cdot(k_3+k_4+k_5)$$

$$= Q^2 - 2k_5 \cdot Q \;\hat{=}\; Q^2 - 2E_5 Q$$

$$S_{35} = Q^2 - 2E_4 \cdot Q$$

$$S_{45} = Q^2 - 2E_3 \cdot Q$$

let $x_3 = \frac{2E_3}{Q}$, $x_4 = \frac{2E_4}{Q}$, $x_5 = \frac{2E_5}{Q}$,

then

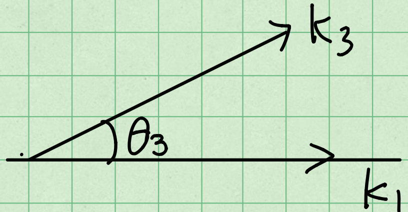
$$\boxed{\begin{aligned} S_{34} &= Q^2 \cdot (1 - x_5) \\ S_{35} &= Q^2 (1 - x_4) \\ S_{45} &= Q^2 (1 - x_3) \end{aligned}}$$

3-body phase space

$$d\Phi_3 = \int \frac{d^3 k_3}{(2\pi)^3 2E_3} \int \frac{d^3 k_4}{(2\pi)^3 2E_4} \int \frac{d^3 k_5}{(2\pi)^3 2E_5} (2\pi)^4 \delta^{(4)}(Q - k_3 - k_4 - k_5)$$

$$= \frac{1}{8(2\pi)^5} \int_0^{\frac{Q}{2}} dE_3 \int_0^{\frac{Q}{2} - E_3} dE_4 \underbrace{\int_{-1}^{1} d\cos\theta_3 \int_0^{2\pi} d\varphi_3 \int_0^{2\pi} d\varphi_{53}}_{\text{Euler angle}}$$

where in the center of mass frame,



θ_3 is the polar angle,

φ_3 is the azimuthal angle around k_1

φ_{53} is the azimuthal angle around k_3

$$S_{13}^2 = E_3^2 Q^2 \cdot (1 - \cos\theta_3)^2, \quad S_{15}^2 = E_5^2 Q^2 (1 - \cos\theta_5)^2$$

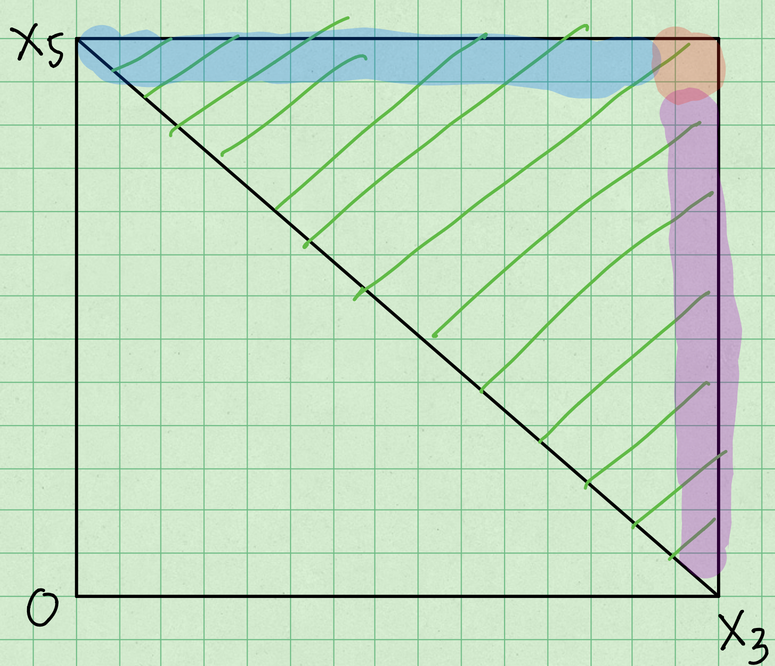
$$S_{23}^2 = E_3^2 Q^2 (1 + \cos\theta_3)^2, \quad S_{25}^2 = E_5^2 Q^2 (1 + \cos\theta_5)^2$$

The Euler angle can be trivially integrated out.

$$\sigma_R = \underbrace{\frac{1}{2Q^2}}_{\text{flux}} \; \underbrace{\frac{1}{4}}_{\text{spin ave.}} \int d\Phi_3 \; |M_5|^2$$

$$= \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \int_0^1 dx_3 \int_{1-x_3}^{1} dx_5 \frac{x_3^2 + x_5^2}{(1-x_3)(1-x_5)}$$

Dalitz plot



$\boxed{x_3 \to 1 : \; k_4 // k_5}$

$\boxed{x_5 \to 1 : \; k_3 // k_3}$

$\boxed{x_3 \;\&\; x_5 \to 1 : \; k_4 \to 0}$

σ_R IR singular, requires regularization. (Dim-reg)

$$\int \frac{d^3p}{(2\pi)^3 2p^0} = \int \frac{d^4p}{(2\pi)^3} \delta(p^2)\theta(p^0) \to \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \delta(p^2)\theta(p^0)$$

In the following, I will give two approach to compute σ_R.

Approach 1, standart textbook treatment.

$$\sigma_R = \sigma_0 \frac{\alpha_s}{2\pi} C_F \frac{\left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon}{\Gamma(2-\epsilon)} \int_0^1 dx_3 \int_{1-x_3}^1 dx_5 \frac{(1-x_3)^{-\epsilon}(1-x_5)^{-\epsilon}}{(x_3+x_5-1)^\epsilon}$$

$$\cdot \frac{(1-\epsilon)\left(x_3^2+x_5^2+\epsilon(x_3+x_5-1)^2\right)}{(1-x_3)(1-x_5)}$$

exercise:

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^\epsilon \frac{(4\pi)^\epsilon}{e^{\epsilon\gamma_E}} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{7}{6}\pi^2\right)$$

$$\sigma_V + \sigma_R = \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \Big(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2$$

$$+ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{7}{6}\pi^2\Big)$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \frac{3}{2}$$

$$= \sigma_0 \frac{\alpha_s}{\pi}$$

Approach 2: Compute an IR safe observable first, then integrate.

thrust: $$T = \max_{\vec{n}} \frac{\sum_{j=1}^{N} |\vec{n}\cdot\vec{k}_j|}{\sum_{j=1}^{N} |\vec{k}_j|}, \qquad \vec{n}^2 = 1$$

also define $\tau = 1 - T$