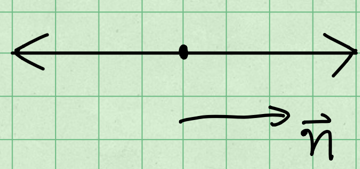
$$\sigma_{tot} = \int d\tau \, \frac{d\sigma}{d\tau}$$

if we can compute $\frac{d\sigma}{d\tau}$, we can integrate to get σ_{tot}. For two or three particle final state, τ is very simple.

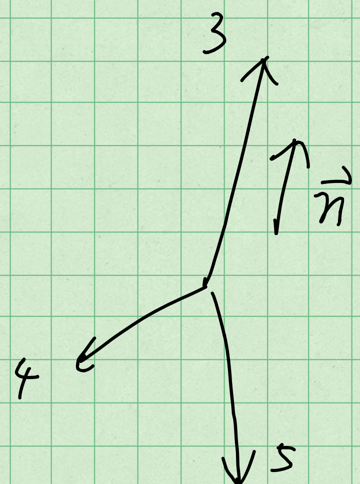
2 particle : $\tau = 0$



3 particle :

if $S_{45} < S_{34}$

& $S_{45} < S_{35}$



$$\tau = 1 - \frac{|\hat{k}_3 \cdot \vec{k}_3| + |\hat{k}_3 \cdot \vec{k}_4| + |\vec{k}_3 \cdot \vec{k}_5|}{Q}$$

$$= 1 - x_3$$

taking into account other kinematical configuration, we have

$$\tau = \min\left(1 - x_3,\ 1 - x_4,\ 1 - x_5\right)$$

exercise : Check this !

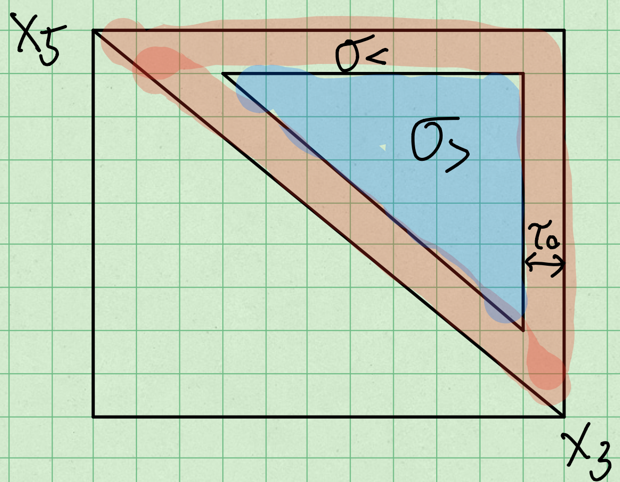
Therefore, for 3 particle final state,

$$0 \le \tau \le \frac{1}{3}$$

we can divide the total cross section into 2 parts,

$$\sigma_{tot} = \underbrace{\int_0^{\tau_0} \frac{d\sigma}{d\tau} d\tau}_{\sigma_<} + \underbrace{\int_{\tau_0}^{\frac{1}{3}} \frac{d\sigma}{d\tau} d\tau}_{\sigma_>}, \quad 0 < \tau_0 < \frac{1}{3}$$

in Dalitz plot



let $\hat{s}_{34} = \frac{s_{34}}{Q^2}$, $\hat{s}_{45} = \frac{\hat{s}_{45}}{Q^2}$, $\tilde{s}_{35} = \frac{s_{35}}{Q^2}$,

then $\tau = \min(\hat{s}_{34}, \hat{s}_{45}, \hat{s}_{35})$

the real corrections can be rewrite as

$$\sigma_R = \sigma_0 \frac{\alpha_s}{2\pi} C_F \int_0^1 d\hat{s}_{34} \int_0^{1-\hat{s}_{34}} d\hat{s}_{45} \frac{(1-\hat{s}_{34})^2 + (1-\hat{s}_{45})^2}{\hat{s}_{34}\,\hat{s}_{45}}$$

when $\hat{s}_{34}$ is the minimal,

$$\frac{d\sigma}{d\tau}\Big|_{\tau=\hat{s}_{34}} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \int_0^1 d\hat{s}_{34} \int_0^{1-\hat{s}_{34}} d\hat{s}_{45} \frac{(1-\hat{s}_{34})^2 + (1-\hat{s}_{45})^2}{\hat{s}_{34}\,\hat{s}_{45}}$$

$$\times\, \delta(\tau - \hat{s}_{34})\, \theta(\hat{s}_{45} - \hat{s}_{34})\, \theta(1 - 2\hat{s}_{34} - \hat{s}_{45})$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \int_{\tau}^{1-2\tau} d\hat{s}_{45} \frac{(1-\tau)^2 + (1-\hat{s}_{45})^2}{\tau\, \hat{s}_{45}}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left[\frac{(2-2\tau+\tau^2)\ln\left(\frac{1-2\tau}{\tau}\right)}{\tau} - \frac{3}{2\tau} + 4 + \frac{3}{2}\tau \right]$$

by symmetry, when $\hat{s}_{45}$ is the minimal, the result is the same.

when $\hat{s}_{35}$ is the minimal,

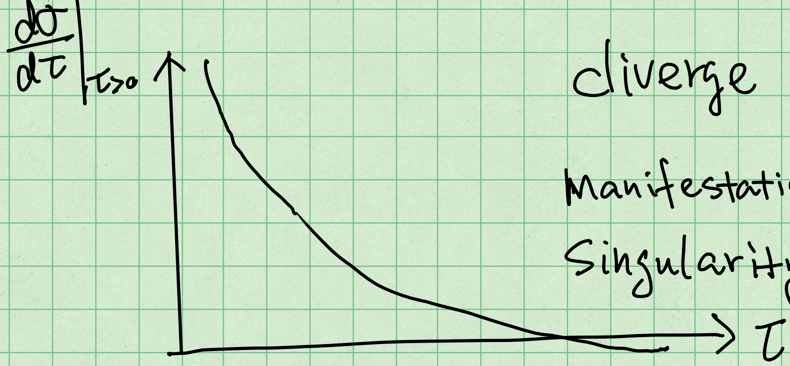
$$\frac{d\sigma}{d\tau}\Big|_{\tau=\hat{s}_{35}} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \int_{\tau}^{1-2\tau} d\hat{s}_{34} \frac{(1-\hat{s}_{34})^2 + (\hat{s}_{34}+\tau)^2}{s\cdot(1-s-\tau)}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left[\frac{2(1+\tau^2)\ln\frac{1-2\tau}{\tau}}{1-\tau} - 2 + 6\tau \right]$$

$$\therefore \frac{d\sigma}{d\tau}\Big|_{\tau>0} = 2\frac{d\sigma}{d\tau}\Big|_{\tau=\hat{s}_{34}} + \frac{d\sigma}{d\tau}\Big|_{\tau=\hat{s}_{35}}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \left[\frac{2\cdot(3\tau^2-3\tau+2)}{\tau\cdot(1-\tau)} \ln\left(\frac{1-2\tau}{\tau}\right) - 3\cdot(1-3\tau)\cdot\frac{(1+\tau)}{\tau} \right]$$

$$\lim_{\tau\to 0} \frac{d\sigma}{d\tau}\Big|_{\tau>0} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \left[-\frac{4\ln\tau}{\tau} - \frac{3}{\tau} + O(\tau^0) \right]$$



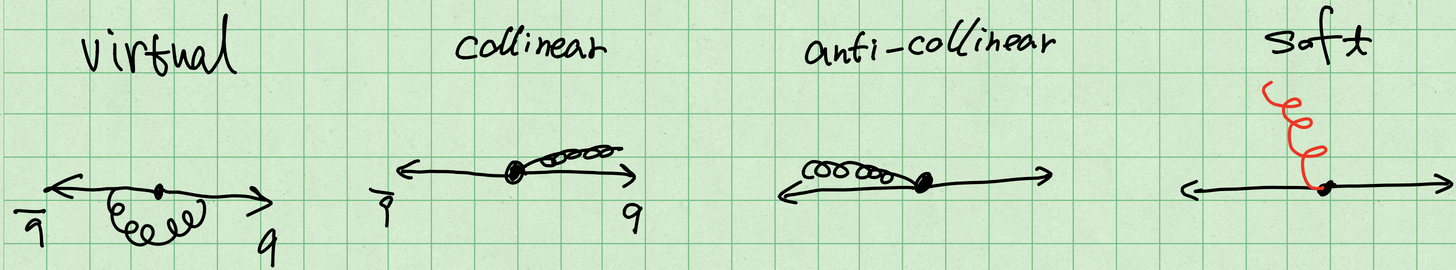
diverge at small τ!

Manifestation of soft/collinear singularity of QCD!

$$\sigma_>(\tau_0) = \int_{\tau_0}^{1/3} d\tau \left.\frac{d\sigma}{d\tau}\right|_{\tau>0}$$

$$\lim_{\tau_0\to 0} \sigma_>(\tau_0) = \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \left[2\ln^2\tau_0 + 3\ln\tau_0 + \frac{5}{2} - \frac{\pi^2}{3} + \mathcal{O}(\tau_0) \right]$$

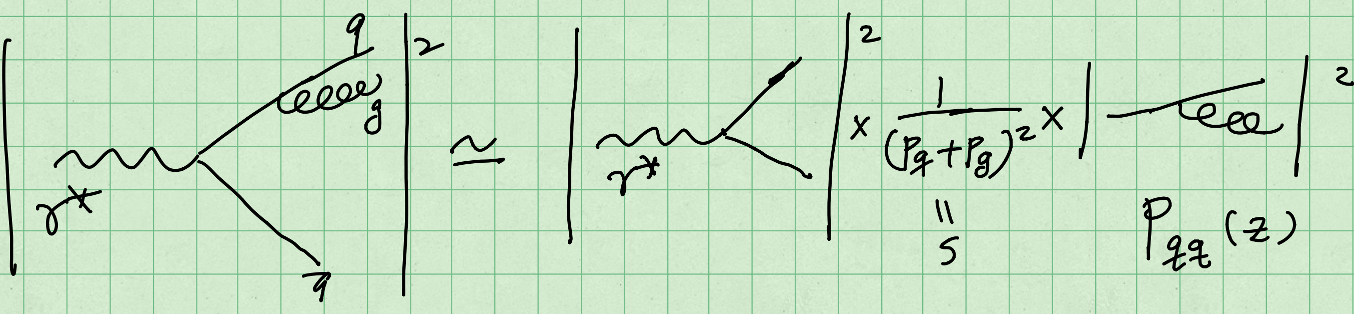
When τ is very small, 4 possibilities:



The virtual contribution is trivial.

$$\frac{d\sigma^V}{d\tau} = \delta(\tau)\,\sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \cdot \frac{(4\pi)^{\epsilon}}{e^{\epsilon\gamma_E}} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2 \right)$$

For the collinear part, we have $\tau = \frac{1}{Q^2}(p_g + p_q)^2$. Thanks to collinear factorization, we can factorize the calculation into two parts:



$$\frac{d\sigma^c}{d\tau} = \sigma_0 \cdot \int d\bar{\Phi}_2^c(s,z)\ \frac{2g_s^2}{s} \cdot P_{qq}(z,\epsilon)\ \delta\left(\tau - \frac{s}{Q^2}\right)$$

Collinear 2-body phase space measure

$$d\Phi_2^c(s,z) = \mu^{2\epsilon}\ ds\,dz\ \frac{[z\cdot(1-z)\,s]^{-\epsilon}}{(4\pi)^{2-\epsilon}\ \Gamma(1-\epsilon)}$$

where z is the longitudinal momentum fraction of quark,

$$P \simeq (p_q + p_g)\,, \quad p_q \simeq z\cdot P\,, \quad p_g \simeq (1-z)P\,.$$

$P_{qq}(z,\epsilon)$ is the splitting function in $4-2\epsilon$ dimension

$$P_{qq}(z,\epsilon) = C_F\cdot\left(\frac{1+z^2}{1-z} - \epsilon(1-z)\right)$$

$$\therefore\ \frac{d\sigma^c}{d\tau} = \sigma_0\cdot\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} g_s^2\,(4\pi)^{-2+\epsilon}\,(1-\epsilon)(4-\epsilon)\frac{\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}\ \frac{1}{\tau^{1+\epsilon}}$$

Using the formula for plus expansion

$$\frac{1}{\tau^{1+n\epsilon}} = -\frac{\delta(\tau)}{n\epsilon} + \left[\frac{1}{\tau}\right]_+ - n\epsilon\left[\frac{\ln\tau}{\tau}\right]_+ + \mathcal{O}(\epsilon^2)$$

where the plus distribution is define as

$$\int_0^1 [f(\tau)]_+\, g(\tau)\, d\tau = \int_0^1 f(\tau)\left(g(\tau)-g(0)\right) d\tau\,,$$ for a regular function $g(\tau)$.

We the have

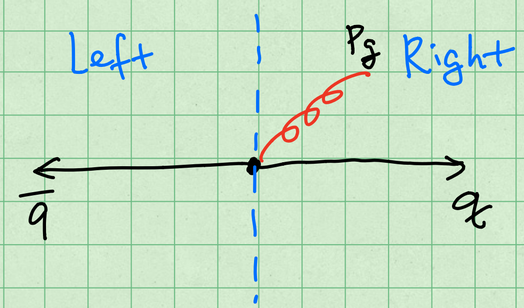
$$\frac{d\sigma^c}{d\tau} = \sigma_0\,\frac{\alpha_s}{2\pi}\,C_F\,\frac{(4\pi)^\epsilon}{e^{\epsilon\gamma_E}}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\cdot\left[\delta(\tau)\cdot\left(\frac{2}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{7}{2} - \frac{\pi^2}{2}\right)\right.$$

$$- \frac{2}{\epsilon}\left[\frac{1}{\tau}\right]_+ + 2\left[\frac{\ln\tau}{\tau}\right]_+ - \frac{3}{2}\left[\frac{1}{\tau}\right]_+ \Bigg]$$

For the anti-collinear $p_g \parallel p_{\bar q}$ contribution, we have

$$\frac{d\sigma^{\bar c}}{d\tau} = \frac{d\sigma^{c}}{d\tau}$$

Finally, there is the soft contribution, $p_g \to 0$.



Since g is soft, q and $\bar q$ are two back to back momentum with energy $\frac{Q}{2}$.

$$q^\mu = \frac{Q}{2} n^\mu, \quad \bar q^\mu = \frac{Q}{2}\bar n^\mu, \qquad q^+ = q\cdot n$$
$$q^- = q\cdot \bar n$$

$$n^\mu = (1,0,0,1) \qquad \bar n^\mu = (1,0,0,-1)$$

if g is in the right hemisphere, $\tau = \frac{2p_g\cdot p_q}{Q^2} = \frac{p_g^+}{Q}$

If g is in the left hemisphere, $\tau = \frac{2p_g\cdot p_{\bar q}}{Q^2} = \frac{p_g^-}{Q}$

We can express it in terms of a measurement function

$$M(\tau) = \delta\left(\tau - \frac{p_g^+}{Q}\right)\theta(p_g^- - p_g^+) + \delta\left(\tau - \frac{p_g^-}{Q}\right)\theta(p_g^+ - p_g^-)$$

The matrix element factorize as

$$\cdot\, g_s^2 C_F \cdot \frac{2p_q\cdot p_{\bar q}}{p_q\cdot p_g \; p_{\bar q}\cdot p_g}$$

$$\therefore \frac{d\sigma^s}{d\tau} = \sigma_0 \int d\Phi^s \cdot g_s^2 C_F \cdot \frac{4}{p_g^+ p_g^-} M(\tau)$$

where $d\Phi^s$ is the soft phase space measure

$$d\Phi^s = \frac{\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}} \int d^{4-2\epsilon} p_g \, \delta(p_g^2)\, \theta(p_g^0)$$

$$= \frac{\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}} \frac{1}{2} \int dp_g^+ \, dp_g^- \, d^{2-2\epsilon}P_{g\perp} \, \delta(p_g^+ p_g^- - \vec{P}_{g\perp}^{\,2})\, \theta(p_g^+ + p_g^-)$$

integrate out the $\perp$ component

$$= \frac{\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}} \frac{1}{4} \int_0^\infty dp_g^- \, dp_g^- \, \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$$

$$\therefore \frac{d\sigma^s}{d\tau} = \sigma_0 \frac{\alpha_s}{2\pi} \frac{(4\pi)^\epsilon}{e^{\epsilon\gamma_E}} \left(\frac{\mu^2}{Q^2}\right)^\epsilon \frac{4}{\epsilon\Gamma(1-\epsilon)} \cdot \frac{1}{\tau^{1+2\epsilon}}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} \frac{(4\pi)^\epsilon}{e^{\epsilon\gamma_E}} \left(\frac{\mu^2}{Q^2}\right)^\epsilon \frac{\alpha_s}{2\pi} C_F \left[\delta(\tau)\cdot\left(-\frac{2}{\epsilon^2} + \frac{\pi^2}{6}\right) + \frac{4}{\epsilon}\cdot\left[\frac{1}{\tau}\right]_+ - 8\left[\frac{\ln\tau}{\tau}\right]_+\right]$$

Adding the soft, collinears, and virtual contribution, we find.

$$\frac{d\sigma^{v+c+\bar{c}+s}}{d\tau} = \frac{d\sigma^v}{d\tau} + \frac{d\sigma^c}{d\tau} + \frac{d\sigma^{\bar{c}}}{d\tau} + \frac{d\sigma^s}{d\tau}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \left[\delta(\tau)\left(-1 + \frac{\pi^2}{3}\right) - 4\left[\frac{\ln\tau}{\tau}\right]_+ - 3\left[\frac{1}{\tau}\right]_+\right]$$

The large logarithmic terms agree with the full amplitude calculation.

The above results is a very good approximation when τ is small. For a small τ_0, we can use this formula to integrate to get $\sigma_<$,

$$\sigma_< \simeq \int_0^{\tau_0} d\tau \frac{d\sigma^{V+c+\bar{c}+s}}{d\tau}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \left[-2\ln^2\tau_0 - 3\ln\tau_0 - 1 + \frac{\pi^2}{3} \right] + O(\tau_0)$$

Combine it with $\sigma_>$,

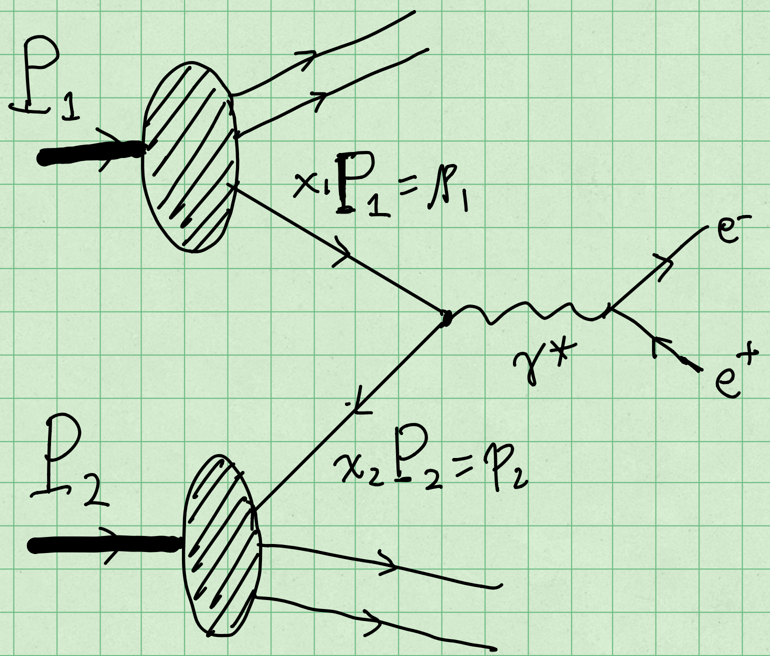
$$\lim_{\tau_0 \to 0} \sigma_>(\tau_0) = \sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot \left[2\ln^2\tau_0 + 3\ln\tau_0 + \frac{5}{2} - \frac{\pi^2}{3} + O(\tau_0) \right]$$

we obtain

$$\sigma_{tot} = \sigma_< + \sigma_> = \sigma_0 \cdot \frac{\alpha_s}{2\pi} \cdot C_F \cdot \frac{3}{2} = \sigma_0 \cdot \frac{\alpha_s}{\pi}$$

Agree with the direct calculation!

Drell-Yan Production: $PP \to \gamma^* \to e^+e^-$



High energy, $P_1^2 = P_2^2 = 0$

$$S = (P_1 + P_2)^2 = 2 P_1 \cdot P_2$$

$$M^2 = (p_{e^+} + p_{e^-})^2 \overset{LO}{=} (p_1 + p_2)^2 = x_1 x_2 S = \hat{s}$$

$$\tau = \frac{M^2}{S} \overset{LO}{=} x_1 x_2$$

Parton model:

$$\frac{d\sigma}{dM^2} = \int_0^1 dx_1 dx_2 \sum_q \left[q(x_1)\bar{q}(x_2) + q(x_2)\bar{q}(x_1) \right] \frac{d\hat{\sigma}}{dM^2}$$

$\hat{\sigma}$: partonic cross section

Recall that

$$\sigma_0(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3M^2} e_q^2 N_c$$

$$\hat{\sigma}(q\bar{q} \to e^+e^-) = \underbrace{\frac{1}{N_c^2}}_{\text{color average}} \sigma(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3M^2 N_c} e_q^2$$

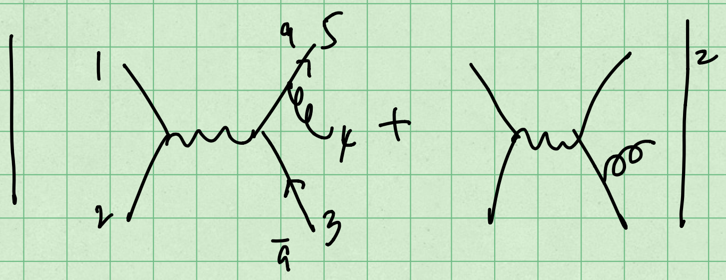
$$\frac{d\hat{\sigma}_q}{dM^2} = \sigma_0 e_q^2 \delta(M^2 - \hat{s}) \qquad \sigma_0 = \frac{4\pi\alpha^2}{3M^2 N_c}$$

$$\therefore \frac{d\sigma}{dM^2} = \sigma_0 \int_0^1 dx_1 dx_2\, \delta(x_1 x_2 S - M^2) \sum_q e_q^2 \left[q(x_1)\bar{q}(x_2) + (x_1 \leftrightarrow x_2) \right]$$

$$= \frac{\sigma_0}{S} \int_0^1 dx_1 dx_2\, \delta(x_1 x_2 - \tau) \sum_q e_q^2 \left[q(x_1)\bar{q}(x_2) + (x_1 \leftrightarrow x_2) \right]$$

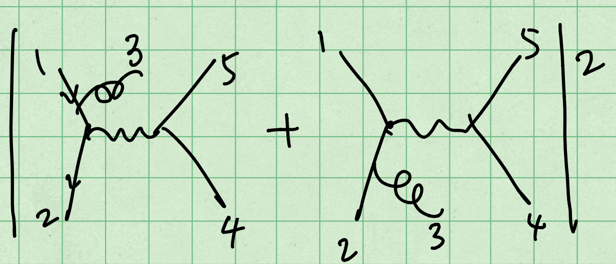
$$= \frac{\sigma_0}{S} \int_\tau^1 \frac{dx_1}{x_1} \sum_q e_q^2 \left[q(x_1)\bar{q}\left(\frac{\tau}{x_1}\right) + \bar{q}(x_1) q\left(\frac{\tau}{x_1}\right) \right]$$

NLO Corrections to DY : Real part



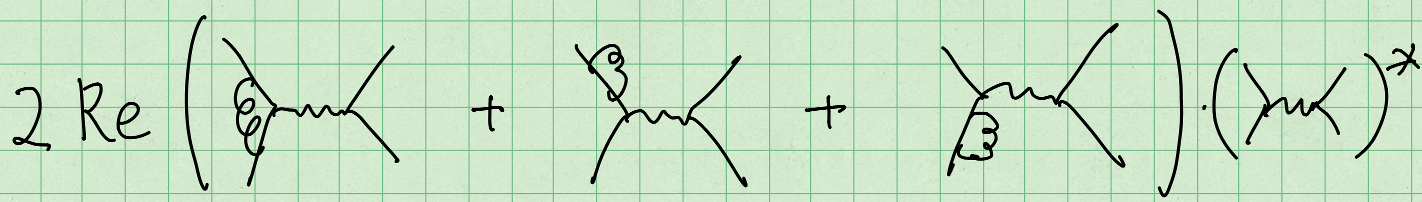
$$\propto \frac{s_{13}^2 + s_{15}^2 + s_{23}^2 + s_{25}^2}{s_{12} s_{34} s_{45}}$$

crossing : $q\bar{q}$ channel



$$\propto \frac{s_{15}^2 + s_{14}^2 + s_{25}^2 + s_{24}^2}{s_{45} s_{13} s_{23}}$$

virtual :



In reality, the amplitudes have to be calculated in $4-2\varepsilon$ dimension. But the crossing relation is the same.
When adding real and virtual corrections.
Partonic cross section is absent of soft singularity,

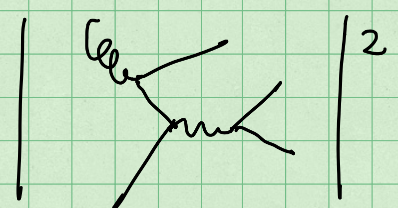
$$\frac{d\hat{\sigma}^{V}_{q\bar{q}\to e^+e^-}}{dM^2} + \frac{d\hat{\sigma}^{R}_{q\bar{q}\to e^+e^-g}}{dM^2} \propto \frac{\alpha_s(\mu_R)}{2\pi} P_{qq}(z)\left(-\frac{1}{\epsilon} + \ln\frac{M^2}{\mu_R^2}\right)$$

$$+ \cdots$$

The remaining divergence is collinear origin. Cancel by PDF counter term :

$$\delta q(x) \propto \frac{\alpha_s(\mu_R)}{2\pi} P_{qq}(z)\left(\frac{1}{\epsilon} + \ln\frac{\mu_R^2}{\mu_F^2}\right)$$

There is also a qg channel



: No soft singularity, but collinear divergent.

gluon PDF counter term

$$\delta g(x) = \frac{\alpha_s(\mu_R)}{2\pi} P_{qg}(z)\left(\frac{1}{\epsilon} + \ln\frac{\mu_R^2}{\mu_F^2}\right)$$

Final results for DY at NLO

$$\frac{d\sigma^{NLO}}{dM^2} = \frac{\sigma_0}{S}\int_0^1 dx_1 dx_2 dz\, \delta(x_1 x_2 z - \tau) \sum_q e_q^2$$

$$\times \Big[q(x_1,\mu_F)\,\bar{q}(x_2,\mu_F)\Big(\delta(1-z) + \frac{\alpha_s(\mu_R)}{2\pi} C_F D_q(z,\mu_F)\Big)$$

$$+ g(x_1,\mu_F)\big(q(x_2,\mu_F)+\bar{q}(x_2,\mu_F)\big)\frac{\alpha_s(\mu_R)}{2\pi} T_R D_g(z,\mu_F)$$

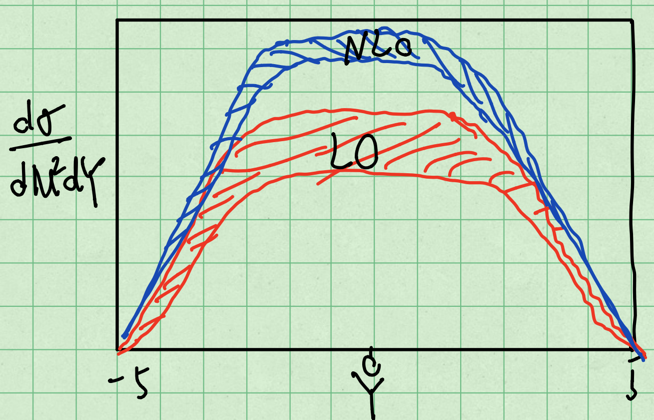
$$+ (x_1 \leftrightarrow x_2)\Big]$$

$$D_q(z,\mu_F) = 4(1+z^2)\left(\frac{\ln(1-z) + \ln\frac{M}{\mu_F}}{1-z}\right)_+$$

$$- 2\,\frac{1+z^2}{1-z}\ln z + \delta(1-z)\left(\frac{2}{3}\pi^2 - 8\right)$$

$$D_g(z,\mu_F) = \left(z^2 + (1-z)^2\right)\left[\ln\frac{(1-z)^2}{z} + 2\ln\frac{M}{\mu_F}\right] + \frac{1}{2} + 3z - \frac{7}{2}z^2$$

Numerically, Large Corrections at NLO!



At central, No overlap between LO and NLO!

Origin of Large corrections,

$$\lim_{z\to 1} D_q(z,\mu_F) = 8\left(\frac{\ln(1-z)+\ln\frac{\mu}{\mu_F}}{1-z}\right)_+ + \delta(1-z)\left(\frac{2}{3}\pi^2-8\right)$$

$$\lim_{z\to 1} D_g(z,\mu_F) = O((1-z)^0)$$

$$z = \frac{M^2}{\hat{s}} = \frac{(p_e+p_{e^+})^2}{(p_q+p_{\bar{q}})^2}$$

$$1-z = \frac{(p_e+p_{\bar{e}}+p_g)^2-(p_e+p_{\bar{e}})^2}{\hat{s}} = \frac{2p_g\cdot(p_e+p_{e^+})+O(p_g^2)}{\hat{s}} \qquad \sim 2p_g^0/M$$

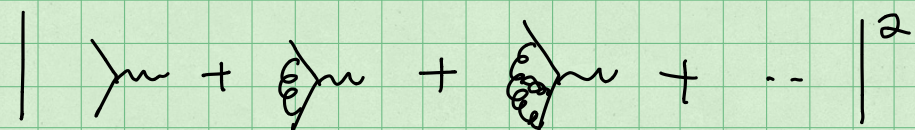
Therefore, $z\to 1$ corresponds to p_g soft

Two physical scales: $\underbrace{M}_{\text{Hard}}$ and $\underbrace{(1-z)M}_{\text{soft}}$

expect QCD dynamics factorized between these two scales:

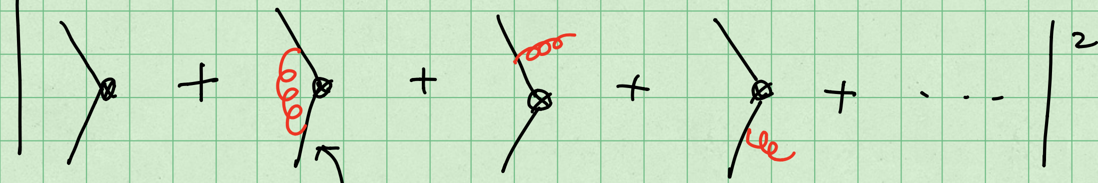
$$\hat{\sigma}(z) \sim H(M^2,\mu)\, S((1-z),\mu) + \text{power corrections in } (1-z)$$

$H(M^2,\mu)$: Hard function

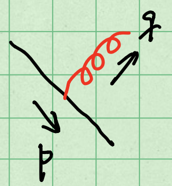


$$= 1 + \frac{\alpha_s}{2\pi} C_F\cdot\left(-8+\frac{7}{6}\pi^2\right)+\cdots$$

$S(1-z), \mu)$: soft function



eikonal

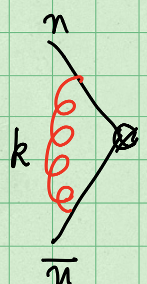


$$= i g_s t_a \cdot \frac{p^\mu}{q\cdot p} = i g_s t_a \frac{n^\mu}{q\cdot n} \qquad p^\mu = E\cdot n^\mu$$

One-loop corrections to soft function : virtual

$k\cdot n = k^+$

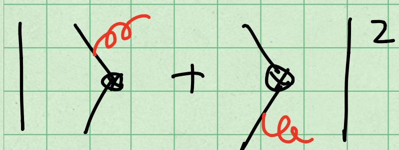
$k\cdot \bar{n} = k^-$



$$\propto \mu^{2\epsilon}\int \frac{d^d k}{(2\pi)^d} \frac{n\cdot\bar{n}}{k\cdot n\, k\cdot\bar{n}} \equiv 0 \quad : \text{scaleless integral.}$$

$$= \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$$

Real corrections



$$= \mu^{2\epsilon}\int \frac{d^d k}{(2\pi)^{3-2\epsilon}} \delta(k^2)\theta(k^0)\frac{4}{k^+ k^-}\delta\left(1-z-\frac{1}{Q}\cdot(k^+ + k^-)\right)$$

$$= \frac{\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}}\frac{1}{2}\int dk^+ dk^- d^{2-2\epsilon}k_\perp\, \delta(k^+k^- - \vec{k}_\perp^{\,2})\,\theta(k^0)$$

$$\times \frac{4}{k^+k^-}\delta\left(1-z-\frac{1}{Q}(k^+ + k^-)\right)$$

$$= \frac{\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}}\frac{1}{2}\int dk^+ dk^- |\vec{k}_\perp|\, d|\vec{k}_\perp|\, d\Omega_{1-2\epsilon}\, \delta(k^+k^- - |\vec{k}_\perp|^2)$$

$$\times \frac{4}{k^+k^-}\delta\left(1-z-\frac{1}{Q}(k^+ + k^-)\right)$$

$$\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}} \frac{1}{4} \Omega_{1-2\epsilon} \int dk^+ dk^- \frac{4}{(k^+ k^-)^{1+\epsilon}} \delta\left(1-z-\frac{1}{Q}(k^+ + k^-)\right)$$

$$= \frac{\mu^{2\epsilon}}{(2\pi)^{3-2\epsilon}} \Omega_{1-2\epsilon} Q \int_{Q\cdot(1-z)}^{\infty} dk^+ \cdot \frac{1}{(k^+)^{1+\epsilon} \cdot (Q(1-z) - k^+)^{1+\epsilon}}$$

$$\simeq \frac{\alpha_s}{2\pi} C_F \frac{2^{1+2\epsilon} \pi^\epsilon \Gamma(-\epsilon)^2}{\Gamma(1-\epsilon)\Gamma(-2\epsilon)} \left(\frac{\mu^2}{Q^2}\right)^\epsilon \frac{1}{(1-z)^{1+2\epsilon}}$$

$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^\epsilon \frac{(4\pi)^\epsilon}{e^{\epsilon\gamma_E}} \Bigg[\boxed{\frac{2\delta(1-\epsilon)}{\epsilon^2} - \frac{4}{\epsilon}\frac{1}{[1-z]_+}}$$

remove by renormalization

$$+ 8\left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{4\ln\frac{\mu^2}{Q^2}}{[1-z]_+} + \delta(1-z)\left(\ln^2\frac{\mu^2}{Q^2} - \frac{1}{2}\pi^2\right)\Bigg]$$

$\therefore$ $H(Q, \mu=Q)\, S(z, \mu=Q)$

$$= \sigma_0 \left\{ \delta(1-z) + \frac{\alpha_s}{2\pi} C_F \left(8\left[\frac{\ln(1-z)}{1-z}\right]_+ + \delta(1-z)\left(-8 + \frac{2}{3}\pi^2\right)\right) + O(\alpha_s^2) \right\}$$

which reproduce the leading behavior of $D_q(z, \mu)$.

The soft function can be define to all orders as vacuum expection value of cusped Wilson Loop (Ref. 0710.0680)

$$S = \frac{1}{N_c} \sum_{X_s} \langle 0 | T\{Y_n Y_{\bar{n}}^\dagger\} \delta\left(1-z-\frac{2\hat{P}_0}{Q}\right) | X_s \rangle \langle X_s | T\{Y_{\bar{n}} Y_n^\dagger\} | 0 \rangle$$

where $\hat{P}_0 |X_s\rangle = 2E_{X_s} |X_s\rangle$

$$Y_n(x) = P\exp\left[ig_s\int_{-\infty}^{0} ds\, n\cdot A(ns+x)\right]$$

The definition of soft function as a gauge invariant object also facilitate the resummation large logarithms by RGE. It's convenient to go to Laplace space,

$$S(\omega) = \int_{-\infty}^{1} dz \exp\left[-\frac{Q(1-z)}{e^{\gamma_E}\omega}\right] S(z)$$

To NLO

$$S(\omega) = 1 + \frac{\alpha_s}{4\pi}\left[2C_F \ln^2\frac{\mu^2}{\omega^2} + \frac{1}{3}\pi^2 C_F\right] + O(\alpha_s^2)$$

It can be shown that $S(\omega)$ satisfy RGE

$$\frac{dS(\omega,\mu)}{d\ln\mu} = \left(2\Gamma_{cusp}(\alpha_s)\ln\frac{\mu^2}{\omega^2} - 2\gamma_s(\alpha_s)\right) S(\omega,\mu)$$

where $\Gamma_{cusp}(\alpha_s) = \frac{\alpha_s}{4\pi}\cdot 4C_F + O(\alpha_s^2)$ cusp anomalous dimensi

$$\gamma_s(\alpha_s) = 0 + O(\alpha_s^2)$$

One can solve the RGE to all orders

$$S(\omega,\mu) = \exp\left[\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left(2\Gamma_{cusp}(\alpha_s(\mu'))\ln\frac{\mu'^2}{\omega^2} - 2\gamma_s(\alpha_s(\mu'))\right)\right]$$

$$\times S(\omega,\mu_0)$$

To compute the integral on the exponent, use

$$\mu \frac{d\alpha_s(\mu)}{d\mu} = \beta(\alpha_s) \Rightarrow \frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta(\alpha_s)}$$

The solution of the RGE expanded to NNLO:

$$S(\omega,\mu) = 1 + \frac{\alpha_s}{4\pi}\left[\frac{1}{2}\Gamma_0^{cusp} L_\omega^2 - \gamma_0^S L_\omega + C_1^S\right]$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{8}\left(\Gamma_0^{cusp}\right)^2 L_\omega^4 + \left(\frac{1}{6}\beta_0 \Gamma_0^{cusp} - \frac{1}{2}\gamma_0^S \Gamma_0^{cusp}\right) L_\omega^3\right.$$

$$\left. + \# L_\omega^2 \cdots \right]$$

The leading term L_ω^4, when transform back to momentum space, gives

$$L_\omega^4 \xrightarrow[\text{Laiplace}]{\text{inverse}} 64\left[\frac{\ln^3(1-z)}{1-z}\right]_+ - 32\pi^2\left[\frac{\ln(1-z)}{1-z}\right]_+ + \frac{128\,\zeta_3}{[1-z]_+}$$

So the resummation of $\log\frac{\mu}{\omega}$ is equivalent to resummation of $\left[\frac{\ln^k(1-z)}{1-z}\right]_+$.