JP. Ma's Note, 3.42

3. 43

An interesting observation:

$$x^- \sim \frac{1}{|q^+|} \sim \frac{\sqrt{2}}{x_B m} \gtrsim \frac{1}{\Lambda} \sim R$$

If x_B is small enough, $x^- > R$ or $x^- \gg R$.

At least, one interaction point is not located insider the hadron. How can the interaction happens outside the hadron?

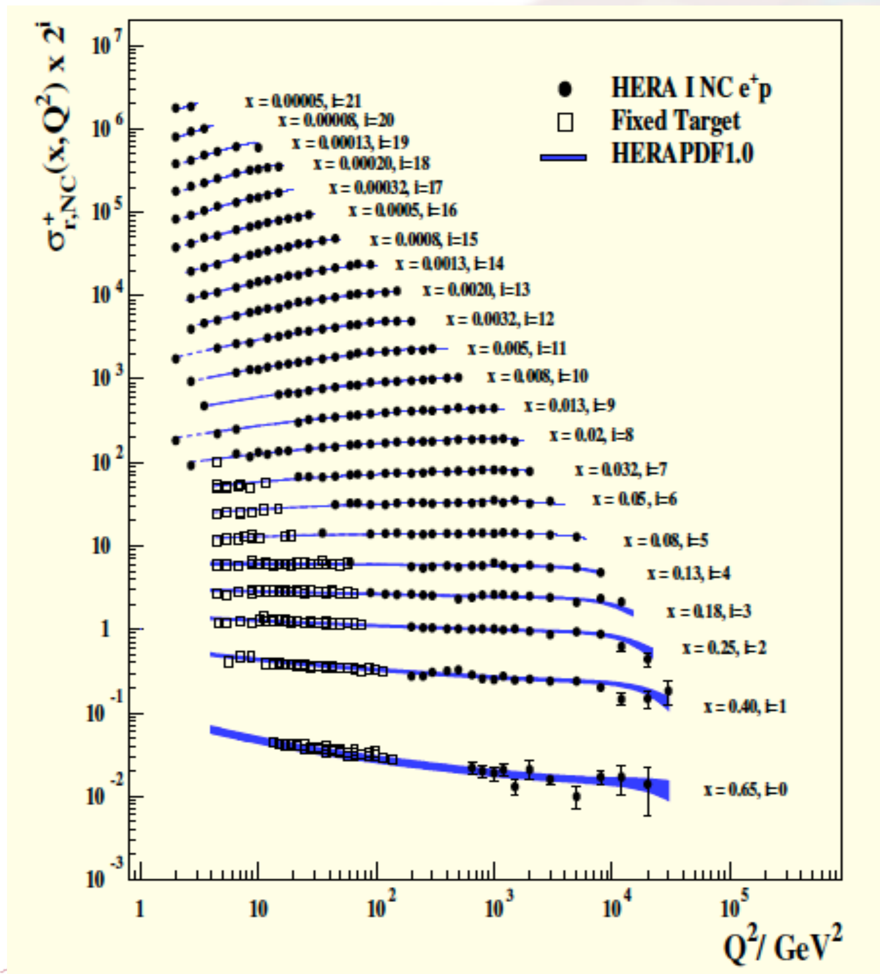
Hot topic !!

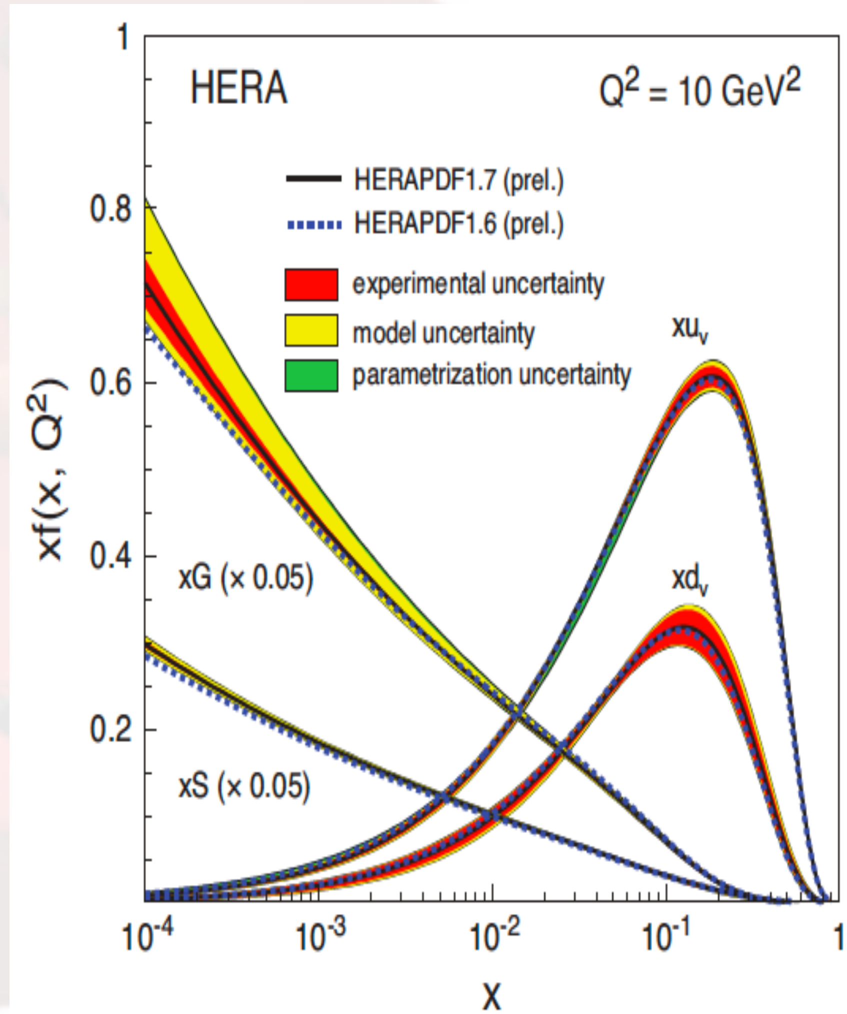
Small-x physics

Parton Physics at Small-x

- Goal is to introduce some basic ideas about small-x physics
  - Why is it interesting, how relevant
  - Current theoretical approach
- Connections to the TMDs
- References
  - Al Mueller, arXiv: hep-ph/9911289, hep-ph/0111244
  - Dominguez, Marquet, Xiao, Yuan, 1101.0715

Gluon saturation inevitable at small-x





- QCD evolution drives the gluon distribution rising at small-x

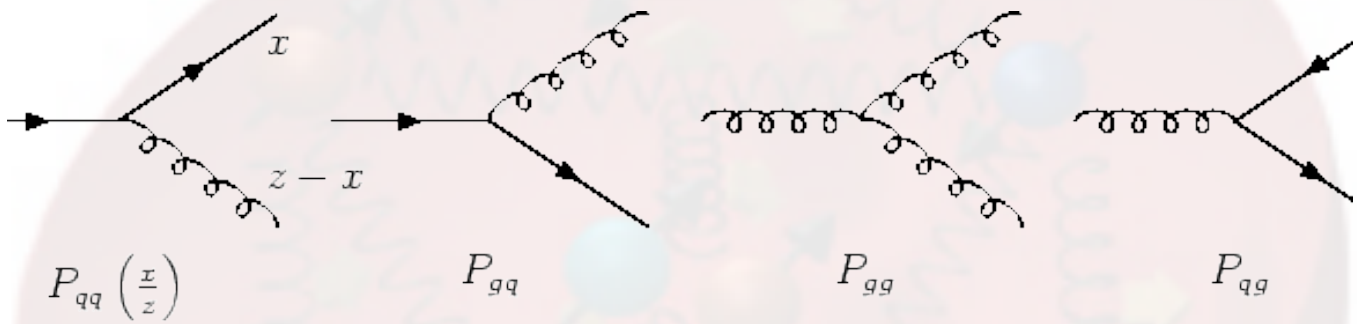


Figure 1.1: *The processes related to the lowest order QCD splitting functions. Each splitting function $P_{p'p}(x/z)$ gives the probability that a parton of type p converts into a parton of type p′, carrying fraction x/z of the momentum of parton p*

$$\mu\frac{d}{d\mu}f_{j/h}(x,\mu) = \sum_k \int_x^1 \frac{dz}{z} P_{jk}(z,\alpha_s(\mu))\, f_{k/h}(x/z,\mu)$$

$$\mathcal{P}_{gg}(x) = \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) + \delta(x-1)\beta_0$$

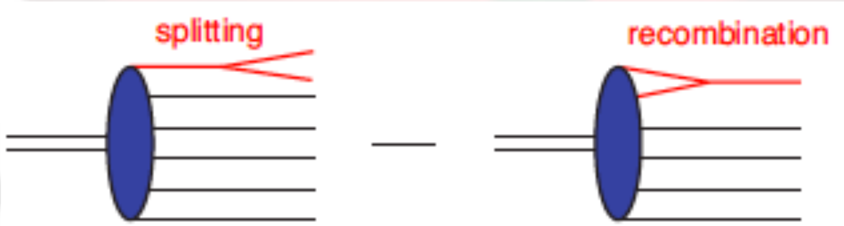
BFKL evolution becomes relevant at small-x

- Balitsky-Fadin-Lipatov-Kuraev, 1977-78

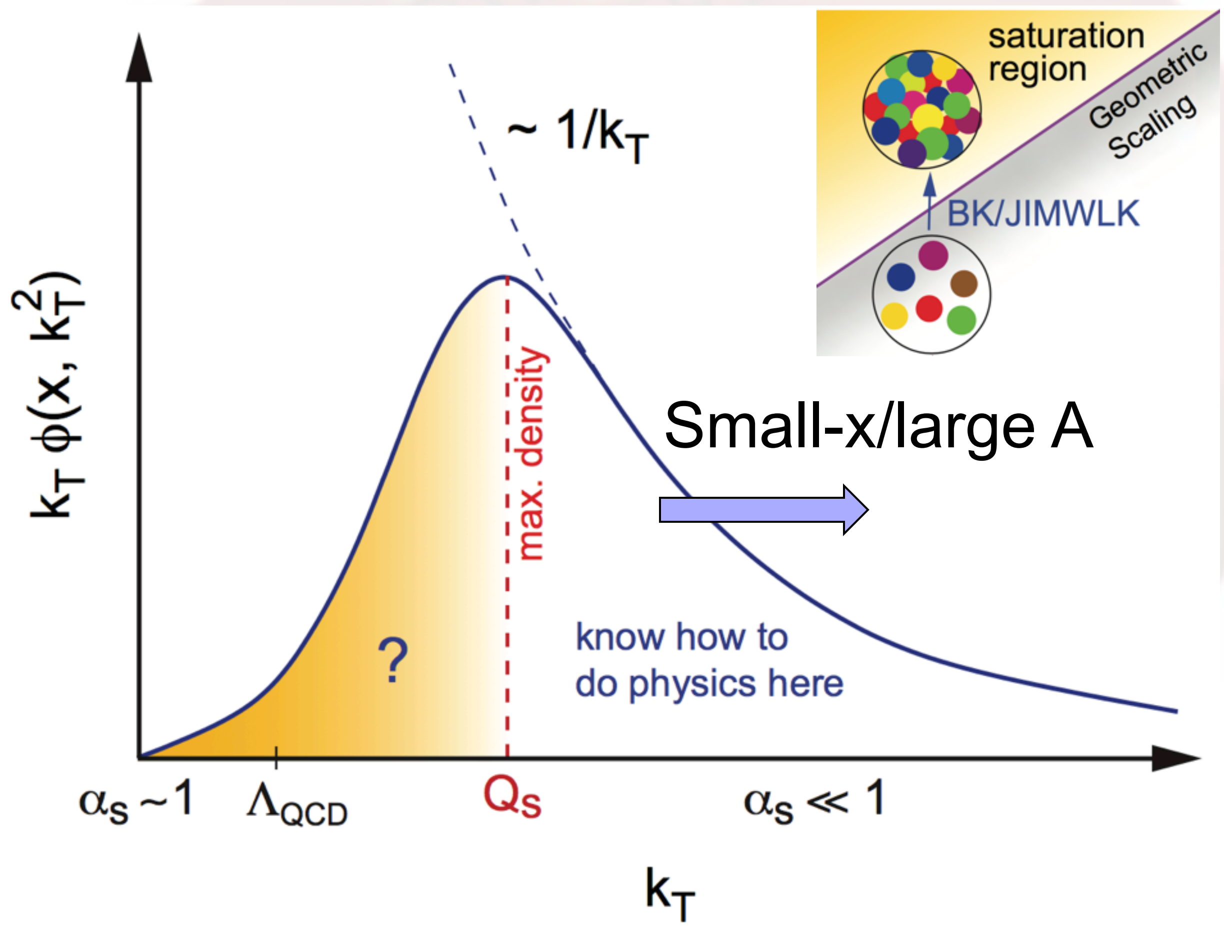
$$\frac{\partial N(x, r_T)}{\partial \ln(1/x)} = \alpha_s K_{\mathrm{BFKL}} \otimes N(x, r_T)$$

- Balitsky-Kovchegov: Non-linear term, 98

$$\frac{\partial N(x, r_T)}{\partial \ln(1/x)} = \alpha_s K_{\mathrm{BFKL}} \otimes N(x, r_T) - \alpha_s \left[N(x, r_T)\right]^2.$$



Saturation at small-x/large A



Small-x approximation

- Take the leading contribution of high energy scattering (eikonal approx)
- Take the small-x limit whenever applicable, and neglect all higher order terms
  - There have been some recent developments to deal with sub-leading contributions, however, very subtle and complicated

Light-cone decomposition

$$k^{\pm} = \left(k^0 \pm k^z\right)/\sqrt{2}$$

- Nucleon/nucleus moving in +z direction, the probe in –z direction

$$p^+ \gg p^-, \quad q^- \gg q^+$$

- Useful Fourier transform

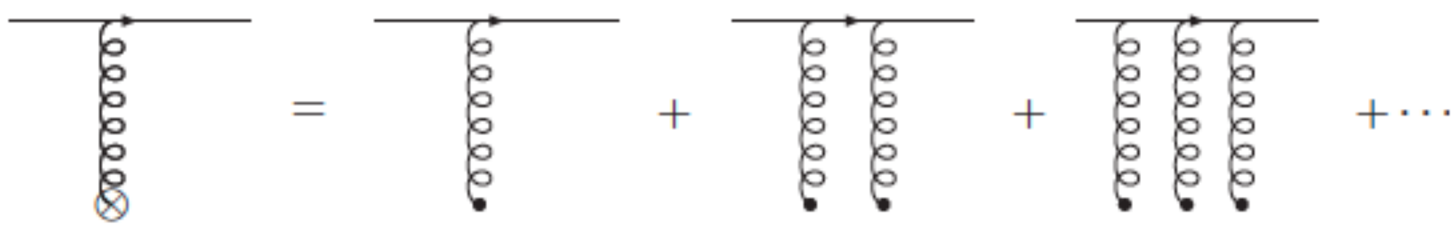
$$\int_{-\infty}^{+\infty} dx^- \Theta(-x^-) e^{-ik^+x^-} = \frac{i}{k^+ + i\epsilon} \quad \Rightarrow \quad \Theta(-x^-) = \int_{-\infty}^{+\infty} dk^+ e^{ik^+x^-} \frac{i}{k^+ + i\epsilon}$$

$$\int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp \cdot x_\perp} \frac{k_\perp^\alpha}{k_\perp^2} = \frac{1}{2\pi} \frac{ix_\perp^\alpha}{x_\perp^2}$$

Small-x factorization

Mueller, 1994

- eikonal approximation in high energy scattering

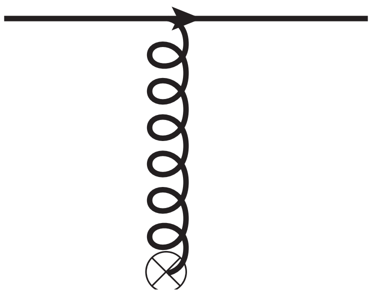


$$\int_{-\infty}^{+\infty} dx^- A^+(x^-, x_\perp) + \int_{-\infty}^{+\infty} dx_1^- dx_2^- \Theta(x_1^- - x_2^-) A^+(x_1^-, x_\perp) A^+(x_2^-, x_\perp)$$

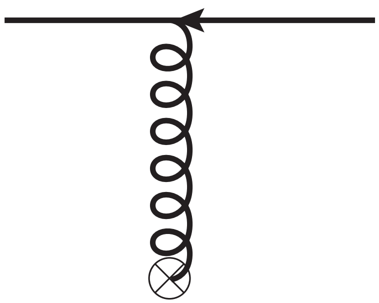
$$\Longrightarrow \int d^2x_\perp e^{ik_{g\perp}\cdot x_\perp}\left(U(x_\perp)-1\right)$$

$$U(x_\perp) = \mathcal{P}exp\left(-ig\int_{-\infty}^{+\infty} dx^- A^+(x^-, x_\perp)\right)$$

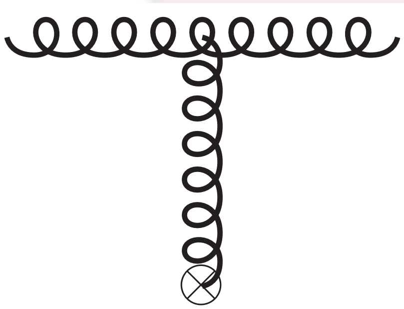
Basic rules



$$(U(x_\perp) - 1)$$

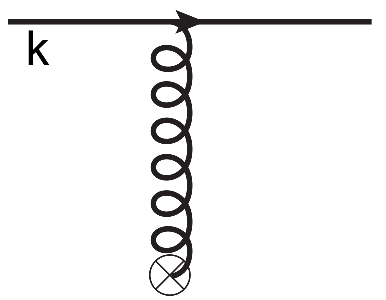


$$(U^\dagger(x_\perp) - 1)$$



$$(W(x_\perp) - 1)_{ab} = \frac{1}{T_F}\mathrm{Tr}[U(x_\perp)T^a U^\dagger(x_\perp)T^b - T^a T^b]$$

Example #1: qA→q+X



$$\mathcal{A} \propto \bar{u}_i(k)\gamma^\mu u_j(k')p_\mu \int d^2x_\perp e^{ik_{g\perp}\cdot x_\perp} \left(U(x_\perp) - 1\right)_{ij}$$

$$\frac{d\sigma_{\mathrm{LO}}^{pA\to qX}}{d^2k_\perp dy} = \sum_f xq_f(x) \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} \frac{1}{N_c} \left\langle \mathrm{Tr} U(x_\perp) U^\dagger(y_\perp) \right\rangle_Y$$

Dipole amplitude

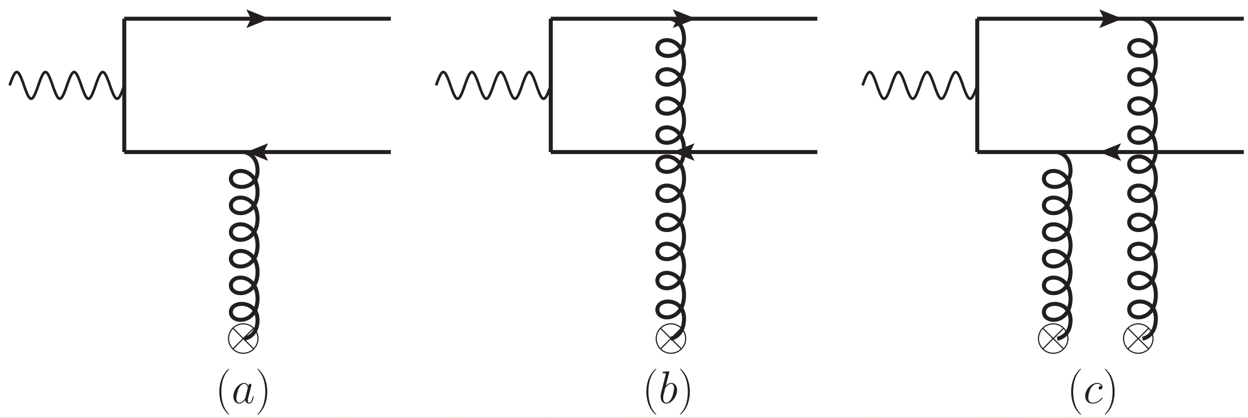
- S-matrix describes quark-antiquark dipole scattering on nucleon/nucleus

$$S_Y^{(2)}(x_\perp, y_\perp) = \frac{1}{N_c}\left\langle \mathrm{Tr} U(x_\perp) U^\dagger(y_\perp) \right\rangle_Y$$

- Also referred as the un-integrated gluon distribution in heavy ion community

$$\mathcal{F}(k_\perp) = \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} S_Y^{(2)}(x_\perp, y_\perp)$$

Example #2: DIS



$$\bar{u}(k_1)\, \not{p} \frac{i}{\not{q} - \not{k}_2 - m} \gamma^\mu v(k_2) \left(U(x_\perp) - 1\right)$$

$$\bar{u}(k_1)\gamma^\mu \frac{i}{\not{k}_1 - \not{q} - m} \not{p} v(k_2) \left(U^\dagger(x_\perp) - 1\right)$$

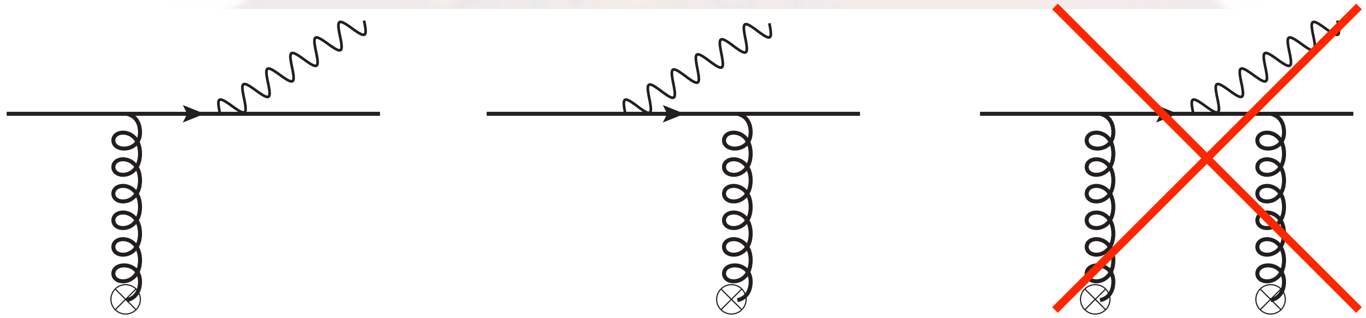
$$\bar{u}(k_1)\, \not{p} \frac{i}{\not{k}_1 - \not{k}_{g1} - m} \gamma^\mu \frac{i}{\not{k}_{g2} - \not{k}_2 - m} \not{p} v(k_2) \left(U(x_\perp) - 1\right) \left(U^\dagger(x_\perp) - 1\right)$$

- The amplitude (transverse photon) proportional to

$$A \propto \int d^2x_{1\perp} d^2x_{2\perp} e^{ik_{g1\perp}\cdot(x_{1\perp}-x_{2\perp})} e^{ik_{g\perp}\cdot x_{2\perp}} \times \frac{k_{1\perp}^{\alpha} - k_{g1\perp}^{\alpha}}{(k_{1\perp}-k_{g1\perp})^2 + z(1-z)Q^2} \left(U(x_1)U^{\dagger}(x_2) - 1\right)$$

$$F_2(x,Q^2) = \sum_f e_f^2 \frac{Q^2}{4\pi^2\alpha_{em}} \int_0^1 dz \int d^2x_\perp d^2y_\perp \left[|\psi_T(z,r_\perp,Q)|^2 + |\psi_L(z,r_\perp,Q)|^2\right] \times [1 - S(r_\perp)], \quad \text{with} \quad r_\perp = x_\perp - y_\perp .$$

Example #3: Drell-Yan



$$\bar{u}(k_1)\gamma^\mu \frac{i}{\not{k}_1 + \not{k}_2 + i\epsilon} \not{p} u(p_2)\left(U(x_\perp) - 1\right)$$

$$\bar{u}(k_1)\not{p} \frac{i}{\not{p}_2 - \not{k}_2 + i\epsilon}\gamma^\mu u(p_2)\left(U(x_\perp) - 1\right)$$

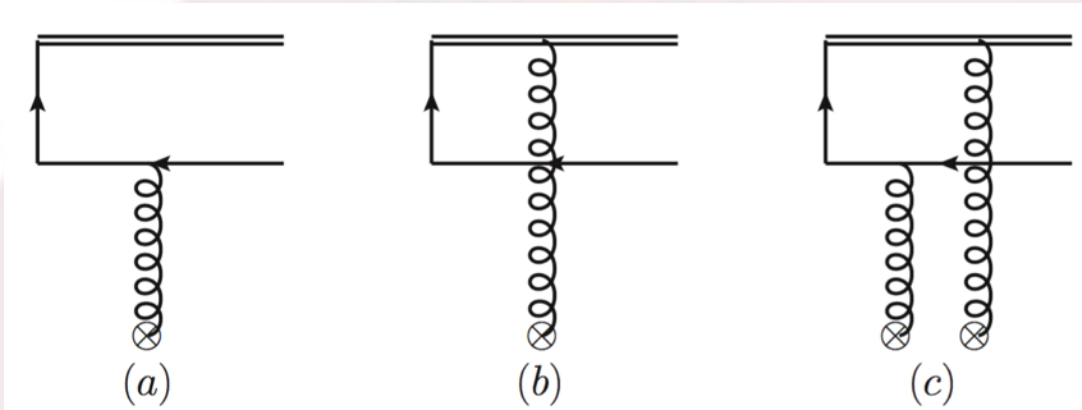
$$\frac{i}{\not{p}_2 + \not{k}_{g1} + i\epsilon}\gamma^\mu \frac{i}{\not{k}_1 - \not{k}_{g2}} \to 0$$

- Amplitude squared proportional to

$$|\mathcal{A}|^2 \propto \frac{(k_{1\perp}+k_{2\perp})^2}{k_{2\perp}^2(k_{2\perp}-z_2k_{g\perp})^2}\int d^2x_\perp d^2y_\perp e^{ik_{g\perp}\cdot(x_\perp-y_\perp)}\langle U(x_\perp)U^\dagger(y_\perp)\rangle$$
$$= \frac{1}{k_{2\perp}^2(k_{2\perp}-z_2k_{g\perp})^2}k_{g\perp}^2\mathcal{F}(k_{g\perp})$$

- Directly probe the dipole gluon distribution

Example #4: quark distribution at small-x

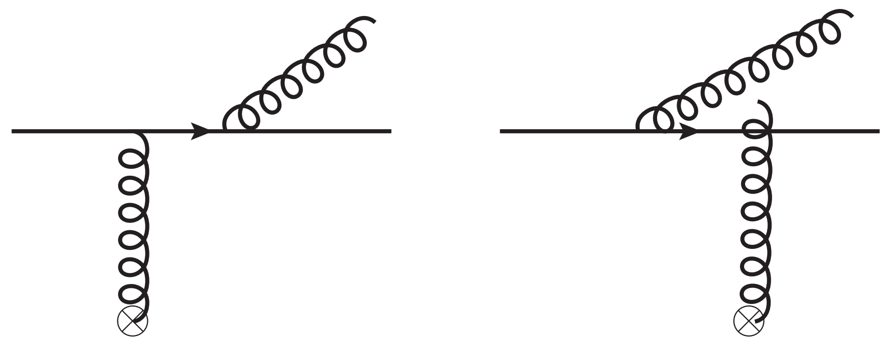


$$xq^{(DY)}(x,k_\perp)=\frac{N_c}{4\pi^4}\int d^2k_{g\perp}F_x(k_{g\perp})\left(1-\frac{k_\perp\cdot(k_\perp-k_{g\perp})}{k_\perp^2-(k_\perp-k_{g\perp})^2}\ln\frac{k_\perp^2}{(k_\perp-k_{g\perp})^2}\right)$$

- It can be shown that the DIS quark is the same as the DY quark, although the diagrams are not

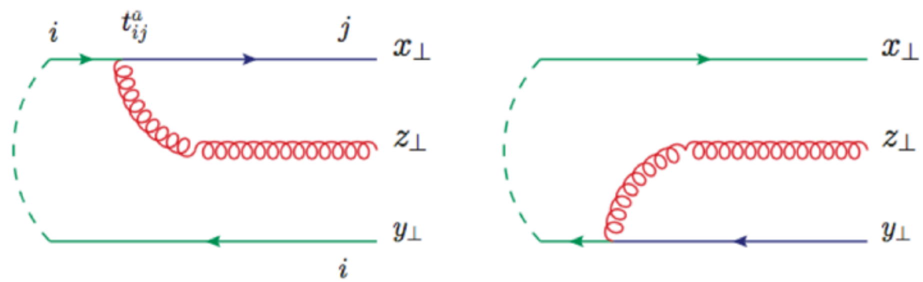
Example #4: one gluon radiation BK evolution

- Soft gluon limit

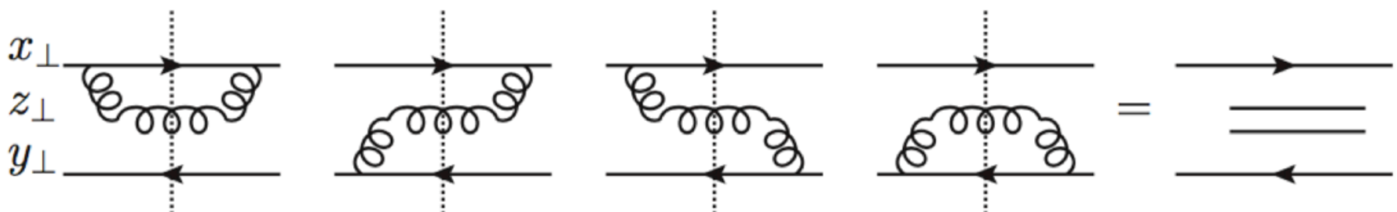


$$A \propto \int d^2x_{1\perp} d^2x_{2\perp} e^{ik_{g1\perp}\cdot(x_{1\perp}-x_{2\perp})} e^{ik_{g\perp}\cdot x_{2\perp}}$$
$$\times \left(\frac{k_{1\perp}^\alpha - k_{g1\perp}^\alpha}{(k_{1\perp}-k_{g1\perp})^2} - \frac{k_{1\perp}^\alpha - k_{g\perp}^\alpha}{(k_{1\perp}-k_{g\perp})^2} \right) \left(U(x_1)U^\dagger(x_2)T^aU(x_2)\right)$$

BK: Real+Virtual



$$\mathcal{M}(x_\perp, z_\perp, y_\perp) = 4\pi g T^a \left[\frac{\epsilon_\perp \cdot (x_\perp - z_\perp)}{(x_\perp - z_\perp)^2} - \frac{\epsilon_\perp \cdot (y_\perp - z_\perp)}{(y_\perp - z_\perp)^2} \right] \Rightarrow$$



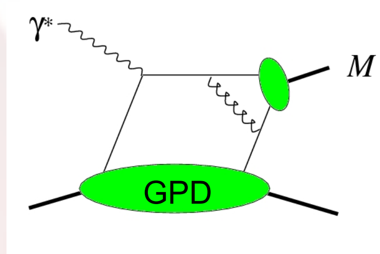
$$= \frac{\alpha_s N_c}{2\pi^2} \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (y_\perp - z_\perp)^2}$$

$$\frac{\partial}{\partial Y} S_Y^{(2)}(x_\perp, y_\perp) = -\frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 b_\perp \, (x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2 (y_\perp - b_\perp)^2} \left[S_Y^{(2)}(x_\perp, y_\perp) - S_Y^{(4)}(x_\perp, b_\perp, y_\perp) \right]$$

Wigner Distribution
$W(x,r,k_t)$

d^3r ↓ Transverse Momentum Dependent PDF $f(x,k_t)$

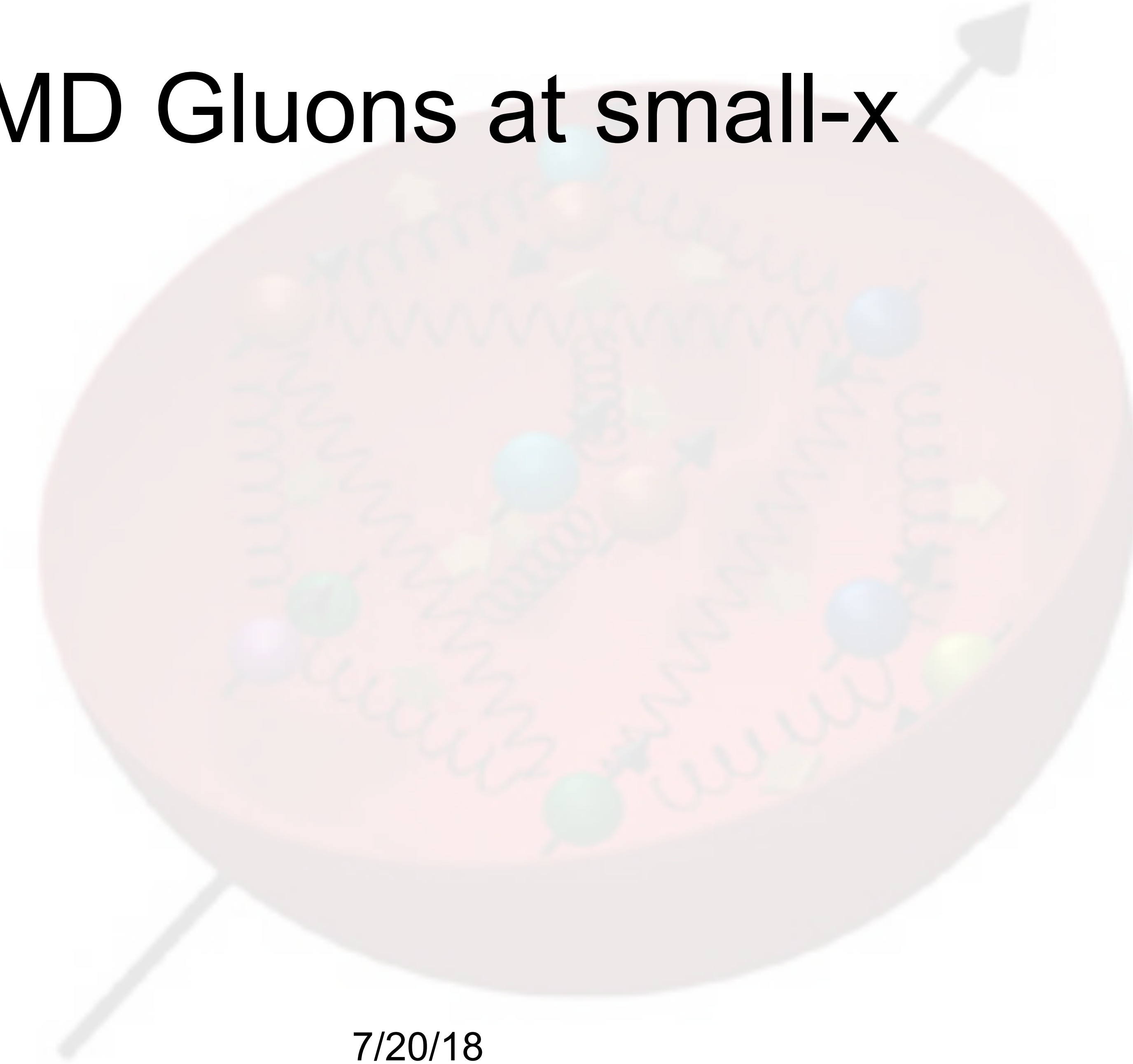
$d^2k_t dz$ F.T. ↓ Generalized Parton Distr. $H(x,\xi,t)$



d^2k_t ↓ PDF $f(x)$

dx ↓ Form Factors $F_1(Q),F_2(Q)$

TMD Gluons at small-x



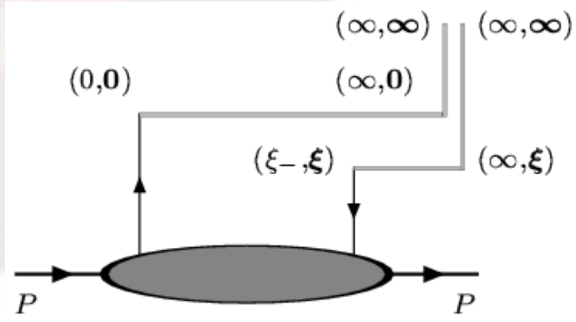
Conventional gluon distribution

- Collins-Soper, 1981

$$xG^{(1)}(x,k_\perp) = \int \frac{d\xi^- d^2\xi_\perp}{(2\pi)^3 P^+} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp} \times \langle P|F^{+i}(\xi^-,\xi_\perp)\mathcal{L}_\xi^\dagger \mathcal{L}_0 F^{+i}(0)|P\rangle$$

  - Gauge link in the adjoint representation

$$\mathcal{L}_\xi = \mathcal{P}\exp\{-ig\int_{\xi^-}^{\infty} d\zeta^- A^+(\zeta,\xi_\perp)\}$$
$$\mathcal{P}\exp\{-ig\int_{\xi_\perp}^{\infty} d\zeta_\perp\cdot A_\perp(\zeta^-=\infty,\zeta_\perp)\}$$



Physical interpretation

- Choosing light-cone gauge, with certain boundary condition (either one, but not the principal value) $A_\perp(\zeta^- = \infty) = 0$
- Gauge link contributions can be dropped
- Number density interpretation, and can be calculated from the wave functions of nucleus
  - McLerran-Venugopalan
  - Kovchegov-Mueller

Classic YM theory: WW-gluon

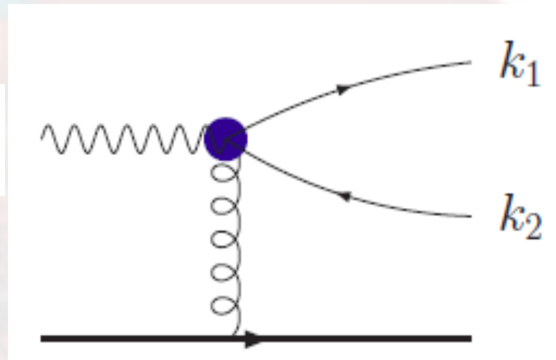
- McLerran-Venugopalan

$$xG^{(1)}(x,k_\perp) = \frac{S_\perp}{\pi^2\alpha_s}\frac{N_c^2-1}{N_c}\int \frac{d^2r_\perp}{(2\pi)^2}\frac{e^{-ik_\perp\cdot r_\perp}}{r_\perp^2}\left(1-e^{-\frac{r_\perp^2 Q_s^2}{4}}\right)$$

  - See also, Kovchegov-Mueller
  - Weizsacker-Williams gluon distribution is the conventional one

DIS dijet probes **WW** gluons

$$\gamma_T^* A \rightarrow q(k_1) + \bar{q}(k_2) + X$$



- Hard interaction includes the gluon attachments to both quark and antiquark
- The q_t dependence is the gluon distribution w/o gauge link contribution at this order

Fundamental representation

$$xG^{(1)}(x,k_\perp) = 2\int \frac{d\xi^- d\xi_\perp}{(2\pi)^3 P^+} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp}$$
$$\times \langle P|\mathrm{Tr}\left[F^{+i}(\xi^-,\xi_\perp)\mathcal{U}^{[+]\dagger}F^{+i}(0)\mathcal{U}^{[+]}\right]|P\rangle$$

$$\mathcal{U}_\xi^{[+]} = U^n\left[0,+\infty;0\right]U^n\left[+\infty,\xi^-;\xi_\perp\right]$$

- Apply the following identity

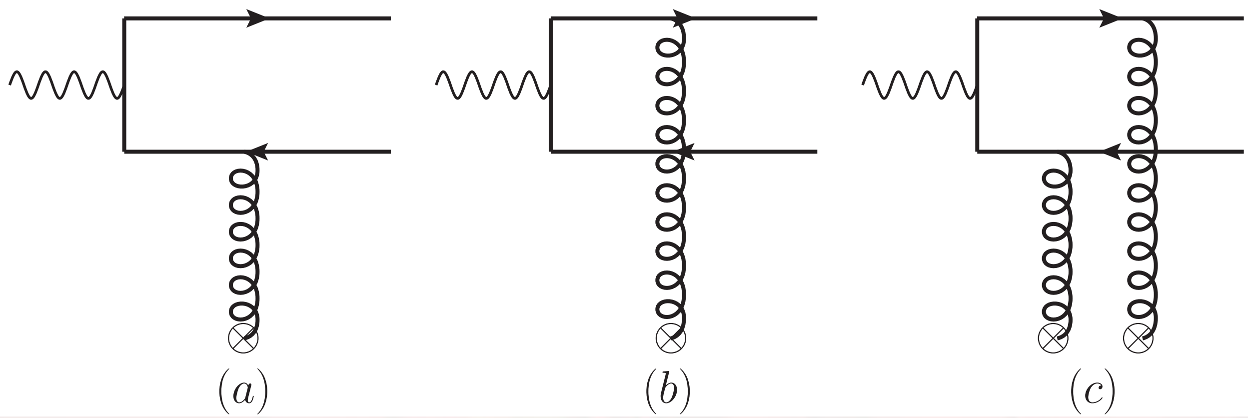
$$\partial_i U(v) = ig_S \int_{-\infty}^{\infty} \mathrm{d}v^+\, U[-\infty, v^+; v]\,\left(\partial_i A^-(v^+,v)\right)\, U[v^+,\infty;v].$$

$$\Downarrow$$

$$-\left\langle \mathrm{Tr}\left[\partial_i U(v)\right]U^\dagger(v')\left[\partial_j U(v')\right]U^\dagger(v)\right\rangle_{x_g} =$$

$$g_S^2 \int_{-\infty}^{\infty} \mathrm{d}v^+ \mathrm{d}v'^+ \left\langle \mathrm{Tr}\left[F^{i-}(\vec{v})\mathcal{U}^{[+]\dagger}F^{j-}(\vec{v}')\mathcal{U}^{[+]}\right]\right\rangle_{x_g}$$

Dipole calculation



$$A \propto \int d^2x_{1\perp} d^2x_{2\perp} e^{ik_{g1\perp}\cdot(x_{1\perp}-x_{2\perp})} e^{ik_{g\perp}\cdot x_{2\perp}}$$

$$\times \frac{k_{1\perp}^{\alpha} - k_{g1\perp}^{\alpha}}{(k_{1\perp} - k_{g1\perp})^2 + z(1-z)Q^2} \left(U(x_1)U^{\dagger}(x_2) - 1\right)$$

Expansion in the correlation limit, $q_t << P_t$

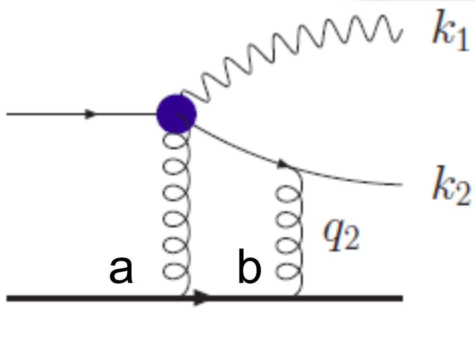
- There is cancellation between two-point and four-point functions
- final result

$$\frac{d\sigma^{\gamma_T^* A\to q\bar{q}X}}{d\mathcal{P}.\mathcal{S}.} = \alpha_{em} e_q^2 \alpha_s \delta\left(x_{\gamma^*}-1\right) z(1-z)\left(z^2+(1-z)^2\right)\frac{P_\perp^4+\epsilon_f^4}{(P_\perp^2+\epsilon_f^2)^4}$$

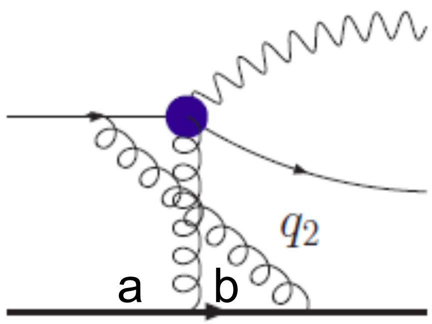
$$\times(16\pi^3)\int\frac{d^3v d^3v'}{(2\pi)^6}e^{-iq_\perp\cdot(v-v')}2\left\langle \mathrm{Tr}F^{i+}(v)\mathcal{U}^{[+]\dagger}F^{i+}(v')\mathcal{U}^{[+]}\right\rangle_{x_g}$$

- Agrees with the TMD result

Photon-jet correlation probes the dipole gluon distribution



(e)



(f)

$$\frac{i}{-q_2^+ + i\epsilon} T^b \Gamma^a$$

$$\frac{i}{q_2^+ + i\epsilon} \Gamma^a T^b$$

$$(-ig)\left(\frac{i}{-q_2^+ + i\epsilon} T^b \Gamma^a + \frac{i}{q_2^+ + i\epsilon} \Gamma^a T^b\right)$$

There is no color structure corresponding to this, We have to express the gluon Distribution in the Fundamental representation

- Dipole gluon distribution

$$xG^{(2)}(x,k_\perp) = 2\int \frac{d\xi^- d\xi_\perp}{(2\pi)^3 P^+} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp}$$

$$\langle P|\mathrm{Tr}\left[F^{+i}(\xi^-,\xi_\perp)\mathcal{U}^{[-]\dagger}F^{+i}(0)\mathcal{U}^{[+]}\right]|P\rangle$$

- This is the dipole gluon distribution, also called unintegrated gluon distribution

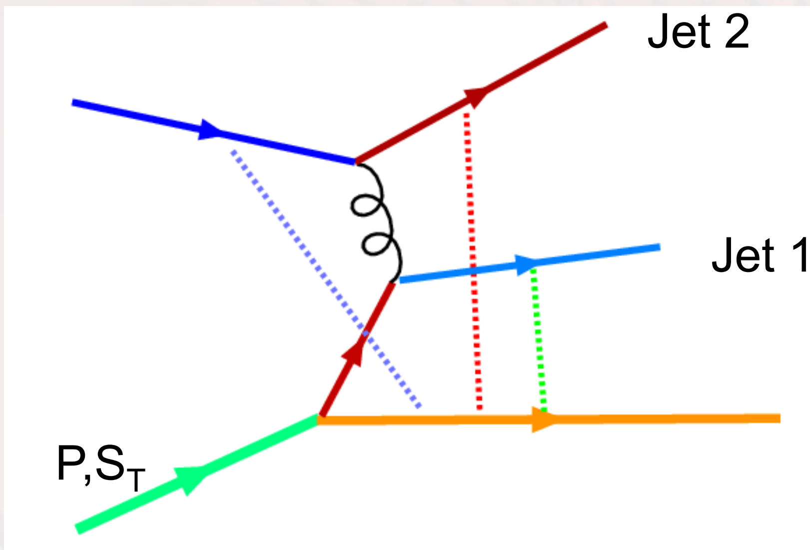
$$xG^{(2)}(x,q_\perp) \simeq \frac{q_\perp^2 N_c}{2\pi^2\alpha_s} S_\perp \int \frac{d^2 r_\perp}{(2\pi)^2} e^{-iq_\perp\cdot r_\perp} S_{x_g}^{(2)}(0,r_\perp)$$

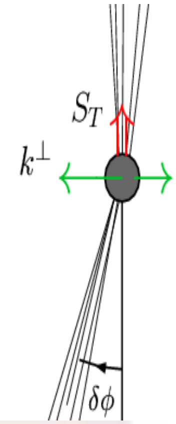
Intuitive explanations

- Final state interactions in DIS can be eliminated by choosing the light-cone gauge → number density interpretation
- Photon-jet correlation have both initial/final state interactions, can not be eliminated by choosing LC gauge → there is no number density interpretation → dipole gluon distribution

Dijet-correlation at RHIC

- Initial state and/or final state interactions





Boer-Vogelsang 03

Standard (naïve) Factorization breaks!

Becchetta-Bomhof-Mulders-Pijlman, 04-06
Collins-Qiu 08; Vogelsang-Yuan 08
Rogers-Mulders 10; Xiao-Yuan, 10

Modified factorization

- Dilute system on a dense target, in the large Nc limit,

$$\frac{d\sigma^{(pA\to \text{Dijet}+X)}}{d\mathcal{P}.\mathcal{S}.} = \sum_q x_1 q(x_1)\frac{\alpha_s^2}{\hat{s}^2}\left[\mathcal{F}_{qg}^{(1)}H_{qg\to qg}^{(1)} + \mathcal{F}_{qg}^{(2)}H_{qg\to qg}^{(2)}\right] + x_1 g(x_1)\frac{\alpha_s^2}{\hat{s}^2}\left[\mathcal{F}_{gg}^{(1)}\left(H_{gg\to q\bar{q}}^{(1)} + H_{gg\to gg}^{(1)}\right) + \mathcal{F}_{gg}^{(2)}\left(H_{gg\to q\bar{q}}^{(2)} + H_{gg\to gg}^{(2)}\right) + \mathcal{F}_{gg}^{(3)}H_{gg\to gg}^{(3)}\right],$$

- Hard partonic cross section

$$H^{(1)}_{qg\to qg} = \frac{\hat{u}^2\left(\hat{s}^2+\hat{u}^2\right)}{-2\hat{s}\hat{u}\hat{t}^2}, \quad H^{(2)}_{qg\to qg} = \frac{\hat{s}^2\left(\hat{s}^2+\hat{u}^2\right)}{-2\hat{s}\hat{u}\hat{t}^2}$$

$$H^{(1)}_{gg\to q\bar{q}} = \frac{1}{4N_c}\frac{2\left(\hat{t}^2+\hat{u}^2\right)^2}{\hat{s}^2\hat{u}\hat{t}}, \quad H^{(2)}_{gg\to q\bar{q}} = \frac{1}{4N_c}\frac{4\left(\hat{t}^2+\hat{u}^2\right)}{\hat{s}^2}$$

$$H^{(1)}_{gg\to gg} = \frac{2\left(\hat{t}^2+\hat{u}^2\right)\left(\hat{s}^2-\hat{t}\hat{u}\right)^2}{\hat{u}^2\hat{t}^2\hat{s}^2}, \quad H^{(2)}_{gg\to gg} = \frac{4\left(\hat{s}^2-\hat{t}\hat{u}\right)^2}{\hat{u}\hat{t}\hat{s}^2}$$

$$H^{(3)}_{gg\to gg} = \frac{2\left(\hat{s}^2-\hat{t}\hat{u}\right)^2}{\hat{u}^2\hat{t}^2},$$

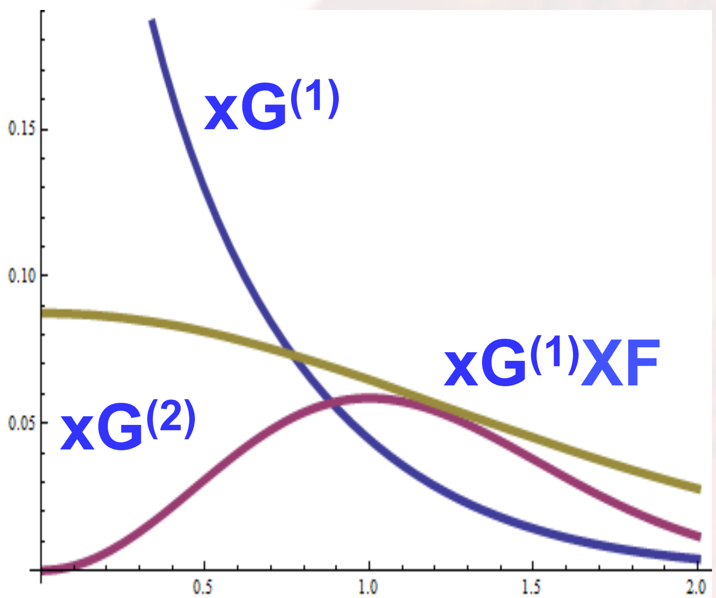
- Kt-dependent gluon distributions

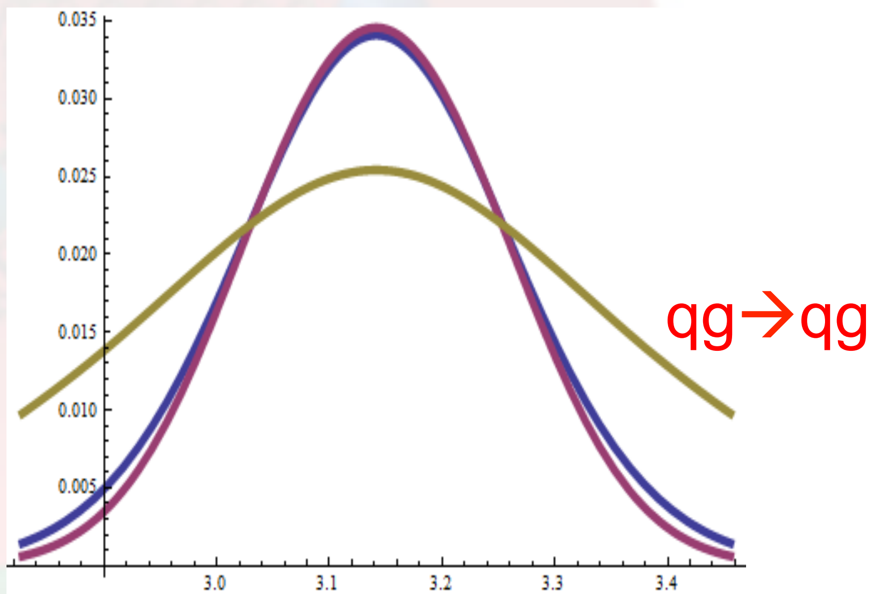
$$\mathcal{F}_{qg}^{(1)} = xG^{(2)}(x, q_\perp), \quad \mathcal{F}_{qg}^{(2)} = \int xG^{(1)}(q_1) \otimes F(q_2) \ ,$$

$$\mathcal{F}_{gg}^{(1)} = \int xG^{(2)}(q_1) \otimes F(q_2), \quad \mathcal{F}_{gg}^{(2)} = \int \frac{q_{1\perp} \cdot q_{2\perp}}{q_{1\perp}^2} xG^{(2)}(q_1) \otimes F(q_2)$$

$$\mathcal{F}_{gg}^{(3)} = \int xG^{(1)}(q_1) \otimes F(q_2) \otimes F(q_3) \ ,$$

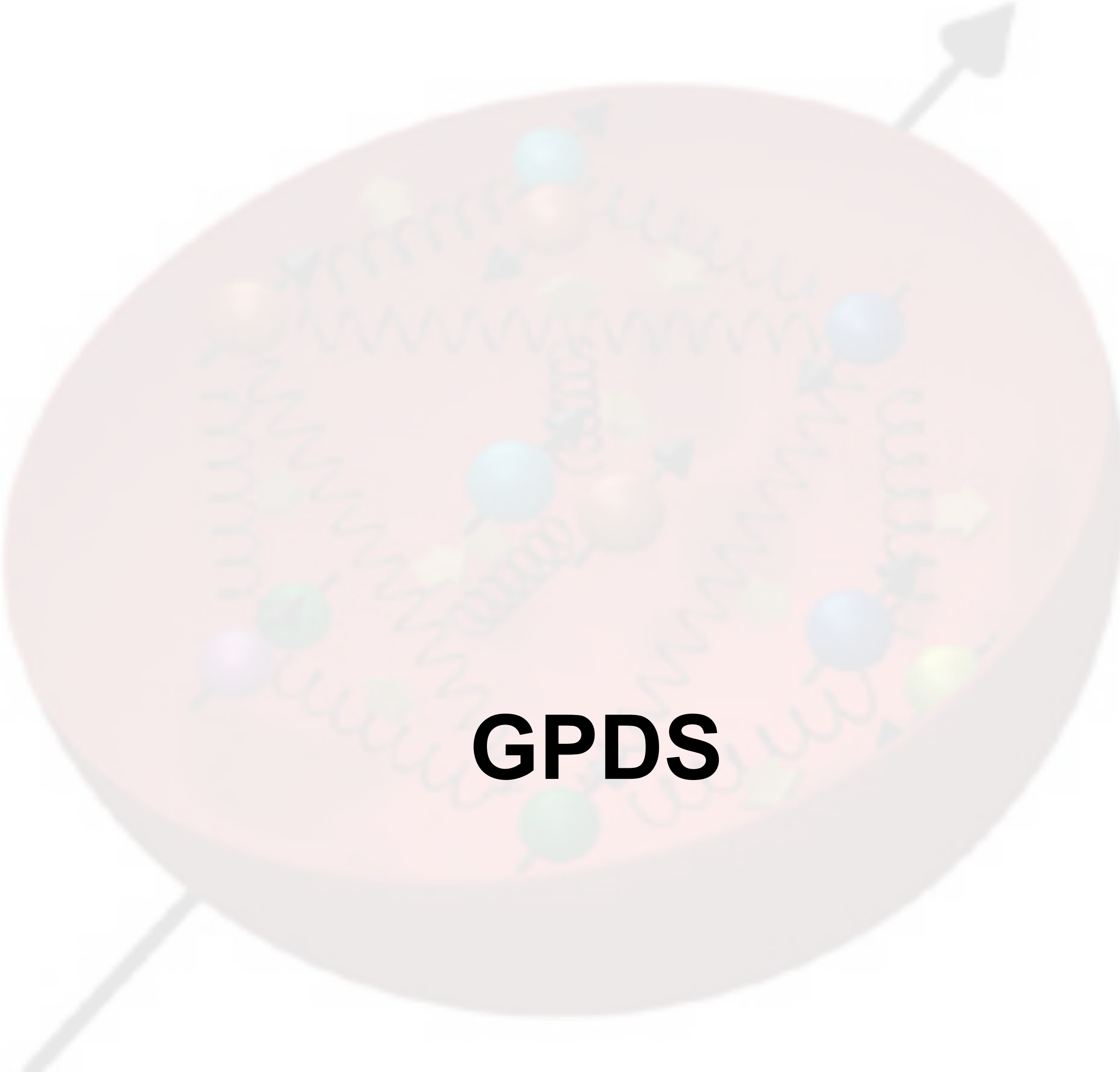
Violation effects





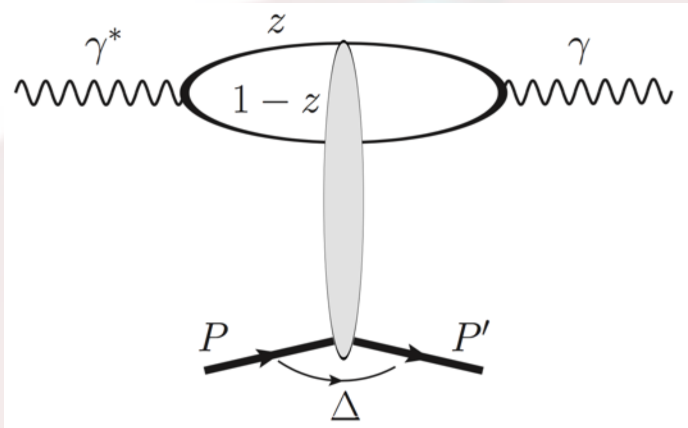
Further developments

- Sudakov resummation for small-x TMDs
  - Mueller-Xiao-Yuan, PRL110, 082301 (2013); Xiao-Yuan-Zhou, NPB921, 104 (2017)
  - Balitsky-Tarasov, JHEP1510,017 (2015)
- Transverse spin-dependent TMD gluon at small-x
  - Related to the spin-dependent odderon, Boer-Echevarria-Mulders-Zhou, PRL 2016



GPDS

DVCS and GPDs at small-x

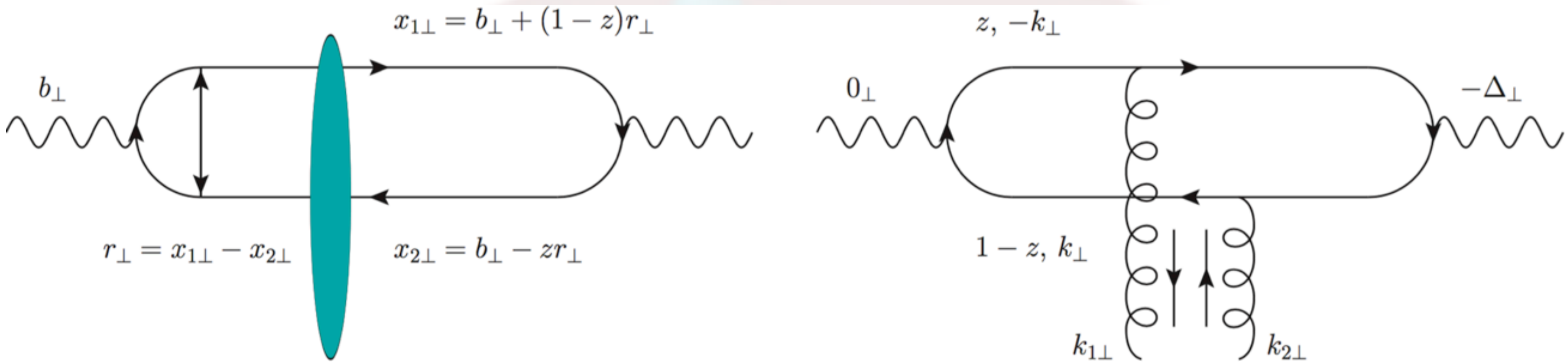


$$\frac{1}{P^+}\int\frac{d\zeta^-}{2\pi}e^{ixP^+\zeta^-}\langle p'|F^{+i}(-\zeta/2)F^{+j}(\zeta/2)|p\rangle$$
$$=\frac{\delta^{ij}}{2}xH_g(x,\Delta_\perp)+\frac{xE_{Tg}(x,\Delta_\perp)}{2M^2}\left(\Delta_\perp^i\Delta_\perp^j-\frac{\delta^{ij}\Delta_\perp^2}{2}\right)+$$

Hoodbhoy-Ji 98
Diehl 01

- All other GPDs suppressed at small-x

Dipole formalism



$$F_x(q_\perp, \Delta_\perp) = \int \frac{d^2 r_\perp d^2 b_\perp}{(2\pi)^4} e^{ib_\perp \cdot \Delta_\perp + i r_\perp \cdot q_\perp} S_x\left(b_\perp + \frac{r_\perp}{2}, b_\perp - \frac{r_\perp}{2}\right)$$

- Elliptic gluon distribution (Hatta-Xiao-Yuan 16)

$$F_x(q_\perp, \Delta_\perp) = F_0(|q_\perp|, |\Delta_\perp|) + 2\cos 2(\phi_{q_\perp} - \phi_{\Delta_\perp}) F_\epsilon(|q_\perp|, |\Delta_\perp|)$$

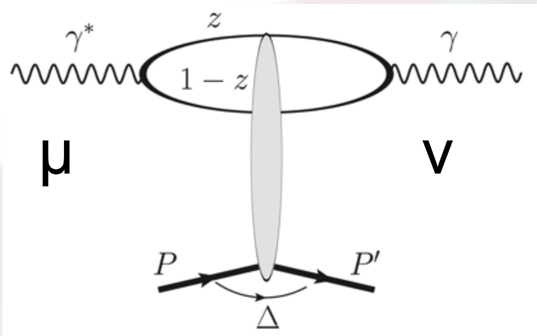
GPDs and dipole

$$xH_g(x,\Delta_\perp) = \frac{2N_c}{\alpha_s}\int d^2q_\perp q_\perp^2 F_0\,,$$

$$xE_{Tg}(x,\Delta_\perp) = \frac{4N_cM^2}{\alpha_s\Delta_\perp^2}\int d^2q_\perp q_\perp^2 F_\epsilon$$

Elliptic gluon distribution

DVCS: Helicity-conserved Amp.



$$g_\perp^{\mu\nu}\mathcal{A}_0(\Delta_\perp) + h_\perp^{\mu\nu}\mathcal{A}_2(\Delta_\perp)$$

$$h_\perp^{\mu\nu} = \frac{2\Delta_\perp^\mu \Delta_\perp^\nu}{\Delta_\perp^2} - g_\perp^{\mu\nu}$$

$$\int dz d^2q_\perp d^2k_\perp \frac{(z^2+(1-z)^2)k_\perp\cdot(k_\perp+q_\perp)}{(k_\perp+q_\perp)^2(k_\perp^2+\epsilon_q^2)} F_x(q_\perp,\Delta_\perp)$$

$$\epsilon_q^2 = z(1-z)Q^2$$

- Dominant contributions from z~1 or 0,

$$\int \frac{d^2k'_\perp}{(2\pi)^2}\frac{1}{k'^2_\perp}\int d^2q_\perp q_\perp^2 F_x(q_\perp,\Delta_\perp) \Rightarrow \int \frac{d^2k'_\perp}{(2\pi)^2}\frac{1}{k'^2_\perp} xH_g(x)$$

Hatta-Xiao-Yuan 1703.02085

Helicity-flip amplitude

$$\int dz d^2q_\perp d^2q_{1\perp} \frac{z(1-z)\left[2q_{1\perp}\cdot\Delta_\perp k_\perp\cdot\Delta_\perp - q_{1\perp}\cdot k_\perp \Delta_\perp^2\right]}{q_{1\perp}^2 (k_\perp^2+\epsilon_q^2)\Delta_\perp^2} F_x(q_\perp,\Delta_\perp)$$

- In the DVCS limit, Q>>Δ

$$\mathcal{A}_2 = -\sum_q \frac{e_q^2 N_c}{Q^2}\int d^2q_\perp q_\perp^2 F_\epsilon(q_\perp,\Delta_\perp)$$

$$= -\frac{e_q^2\alpha_s\Delta_\perp^2}{4Q^2M^2} E_{Tg}(x,\Delta_\perp)$$

Hatta-Xiao-Yuan 1703.02085

DVCS: Collinear factorization

$$T^{\mu\nu} = i\int d^4 z e^{-iq\cdot z}\langle P'|j^\mu(z/2)j^\nu(-z/2)|P\rangle \equiv g_\perp^{\mu\nu}T_0 + h_\perp^{\mu\nu}T_2$$

$$T_0 = -\sum_q e_q^2 \int dx\, \alpha(x) H_q(x,\xi,\Delta_\perp^2)\ ,$$

$$T_2 = \sum_q e_q^2 \frac{\alpha_s}{4\pi}\frac{\Delta_\perp^2}{4M^2}\int dx\, \alpha(x) E_{Tg}(x,\xi,\Delta_\perp^2)$$

$$\alpha(x) = \frac{1}{x-\xi+i\epsilon} + \frac{1}{x+\xi-i\epsilon}$$

Hoodbhoy-Ji 98

- Imaginary part at xi=x

$$\mathrm{Im}\, T_0 = \frac{\pi}{\xi}\sum_q e_q^2 \left[\xi H_q(\xi,\xi,\Delta_\perp^2) + \xi H_{\bar q}(\xi,\xi,\Delta_\perp^2)\right]$$

$$\mathrm{Im}\, T_2 = -\frac{\pi}{\xi}\frac{\alpha_s}{2\pi}\frac{\Delta_\perp^2}{4M^2}\sum_q e_q^2 \xi E_{Tg}(\xi,\xi,\Delta_\perp^2)\ ,$$

Quark/GPD quark at small-x

- DGLAP splitting dominated by gluon distribution/GPD gluon

$$xq(x) = \frac{\alpha_s}{2\pi}\frac{1}{2}\int_x^1 d\zeta\big(\zeta^2+(1-\zeta)^2\big)x'G(x')\int\frac{dk_\perp^2}{k_\perp^2} \approx xG(x)\frac{\alpha_s}{2\pi}\frac{1}{2}\cdot\frac{2}{3}\int\frac{dk_\perp^2}{k_\perp^2}$$

$$xH_q(x,\xi,\Delta_\perp^2) = \frac{\alpha_s}{2\pi}\frac{1}{2}\int_x^1 d\zeta\frac{\zeta^2+(1-\zeta)^2-\frac{\xi^2}{x^2}\zeta^2}{(1-\frac{\xi^2}{x^2}\zeta^2)^2}x'H_g(x',\xi,\Delta_\perp^2)\int\frac{dk_\perp^2}{k_\perp^2}$$

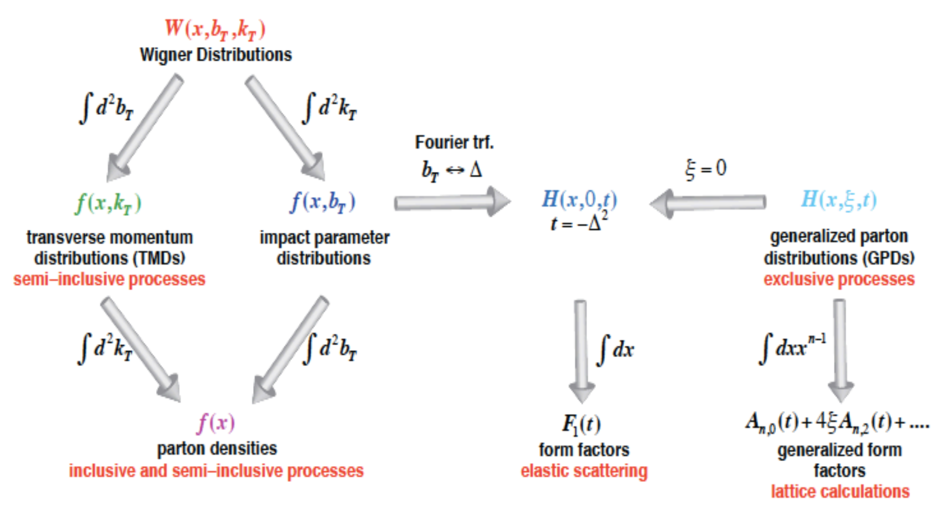
GPD quark distribution

$$\approx \xi H_g(\xi,\xi)\frac{\alpha_s}{2\pi}\frac{1}{2}\cdot 1\int\frac{dk_\perp^2}{k_\perp^2}$$

- We have established the consistency between the small-x dipole formalism and the collinear GPD framework
- The cos(2phi) asymmetry in DVCS will provide information on the elliptic gluon distribution at small-x
- Extension to polarized parton distributions/ OAMs is anticipated, but much more involved
  - Kovchegov et al, 1511.06737,1610.06188
  - Hatta et al, 1612.02445

Grand Jewels of Hadron Physics

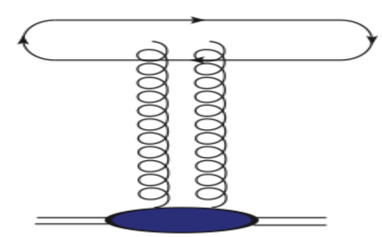
- **Wigner distributions (Belitsky, Ji, Yuan)**



Small-x

Dipole scattering amplitudes

$$\frac{1}{N_c}\left\langle \mathrm{Tr}\left[U(R_\perp)\,U^\dagger(R'_\perp)\right]\right\rangle_x$$

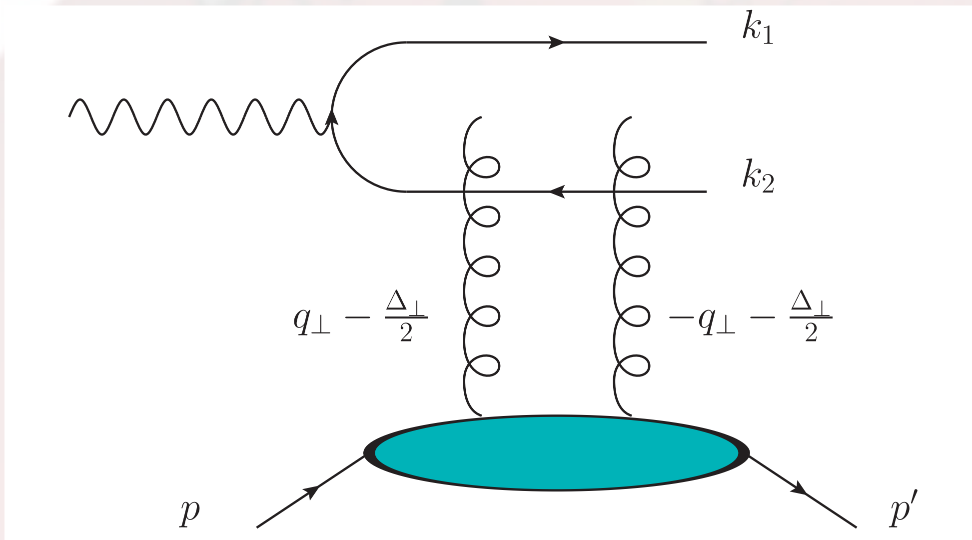


Hatta-Xiao-Yuan,1601.01585
earlier: Mueller, NPB 1999

Probing 3D Tomography of Protons at Small-x at EIC

Diffractive back-to-back dijet productions at EIC:

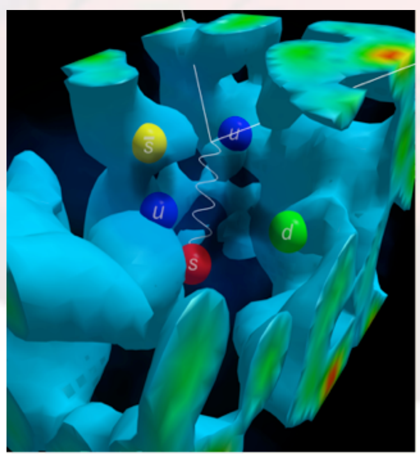
Hatta-Xiao-Yuan,1601.01585



- In the Breit frame, by measuring the recoil of final state proton, one can access Δ_T. By measuring jets momenta, one can approximately access q_T.
- The diffractive dijet cross section is proportional to the square of the Wigner distribution.

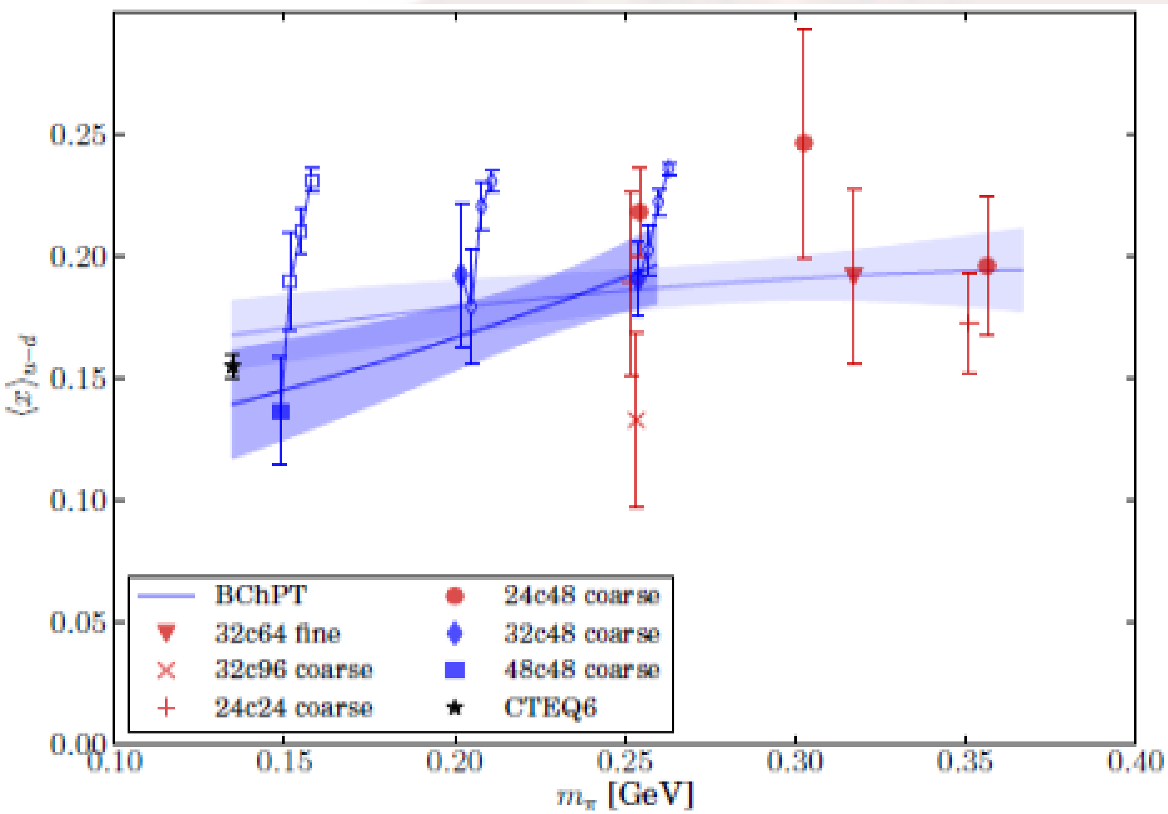
Parton Physics: Lattice QCD

- The only known rigorous framework for *ab-initio* calculation of the structure of protons and neutrons with controllable errors.
- After decades of effort, one can finally calculate nucleon properties with dynamical fermions at physical pion mass!



Nucleon Structure from Lattice QCD

J.R. Green et al, 2012 & 2014

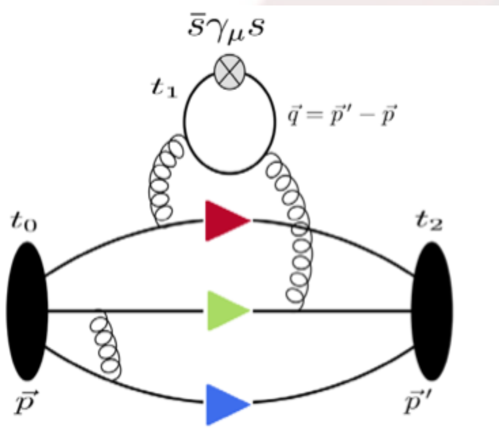


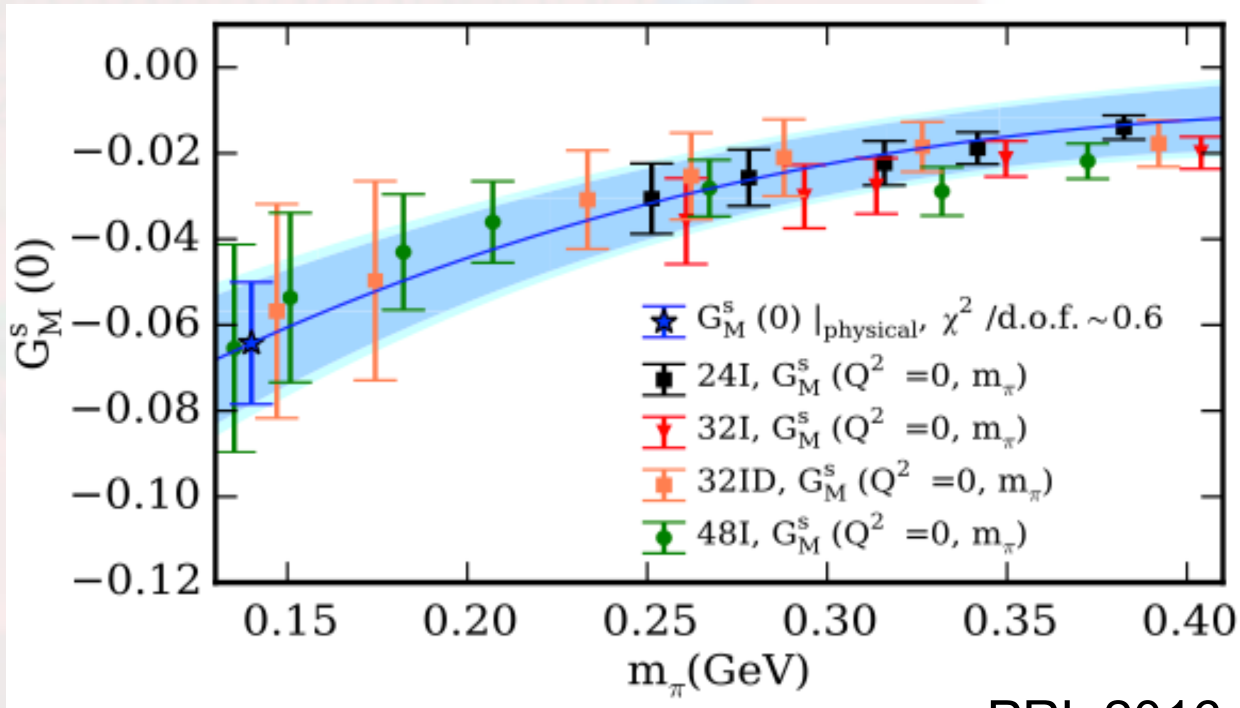
Nearly physical pion mass m_π=149MeV

Quark momentum fraction $\langle x \rangle_{u-d} = \int dx\, x\,(u + \bar{u} - d - \bar{d})$

Strange Quark Magnetic Moment

- R.S. Sufian et al, (2+1) flavor of overlap domain wall fermions at physical pion mass

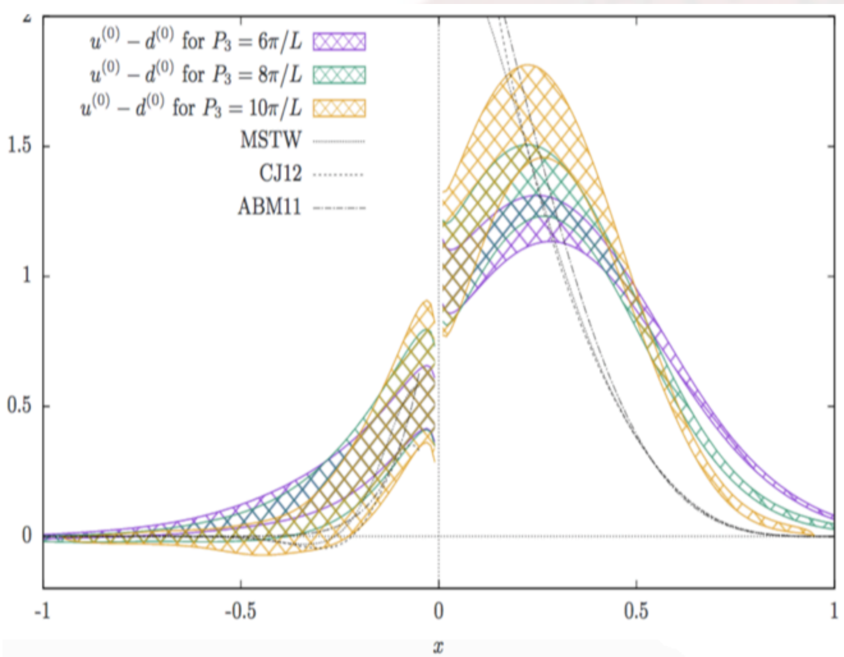


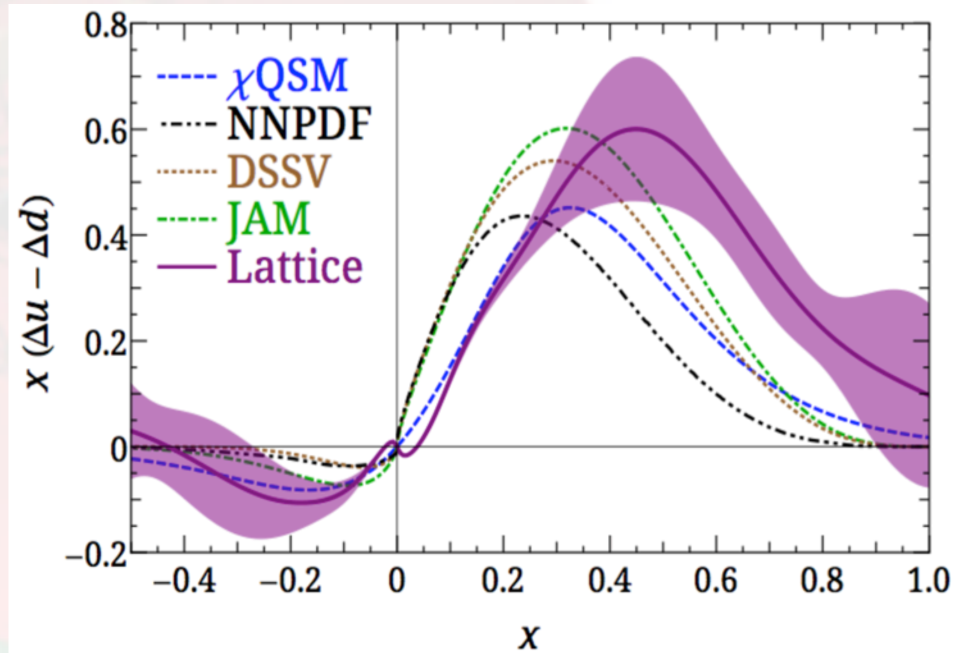


PRL 2016

Directly compute PDFs from lattice QCD

Ji, PRL, 2013

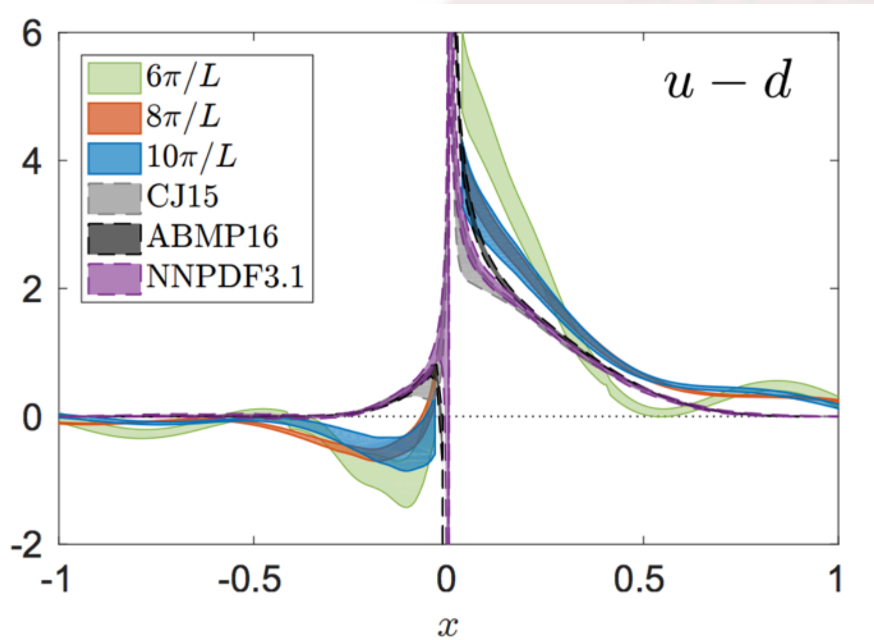




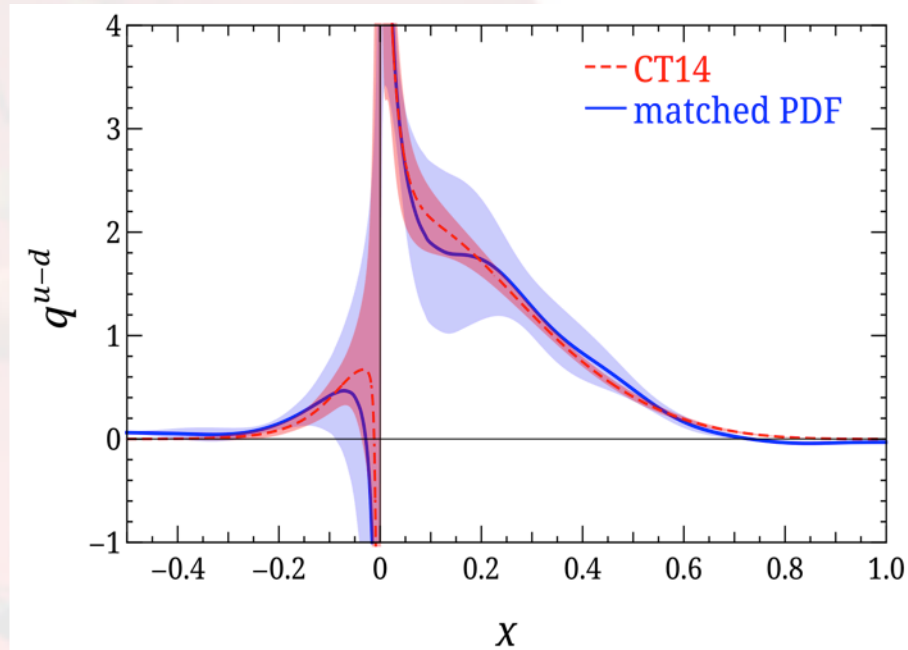
Alexsandrou et al., 2016

Chen et al., 2016

Directly compute PDFs from lattice QCD at physical pion mass

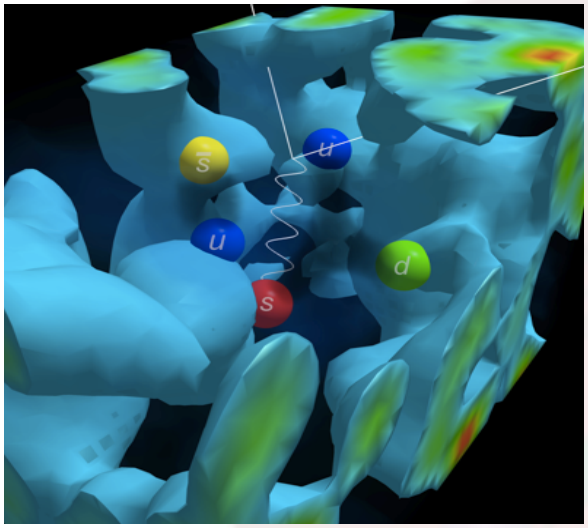


Alexsandrou et al., 2018



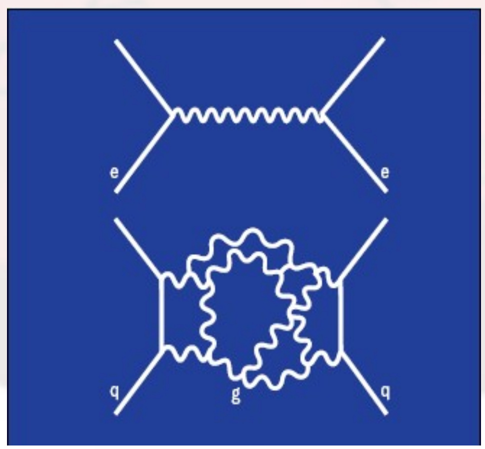
Chen et al., 2018

Fundamental Understanding of the Nucleon Structure in QCD



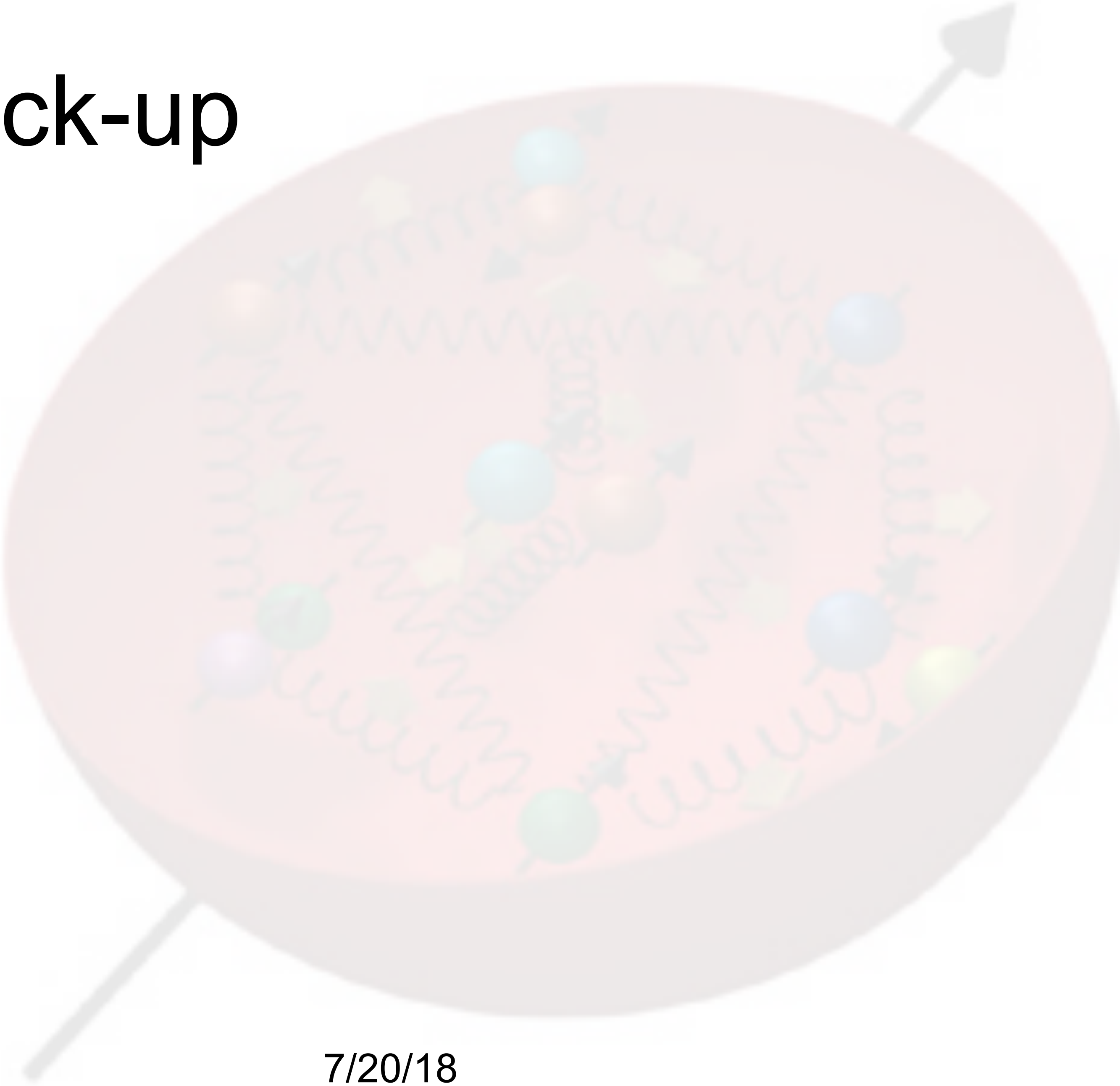
Lattice QCD

EXP. Measurements

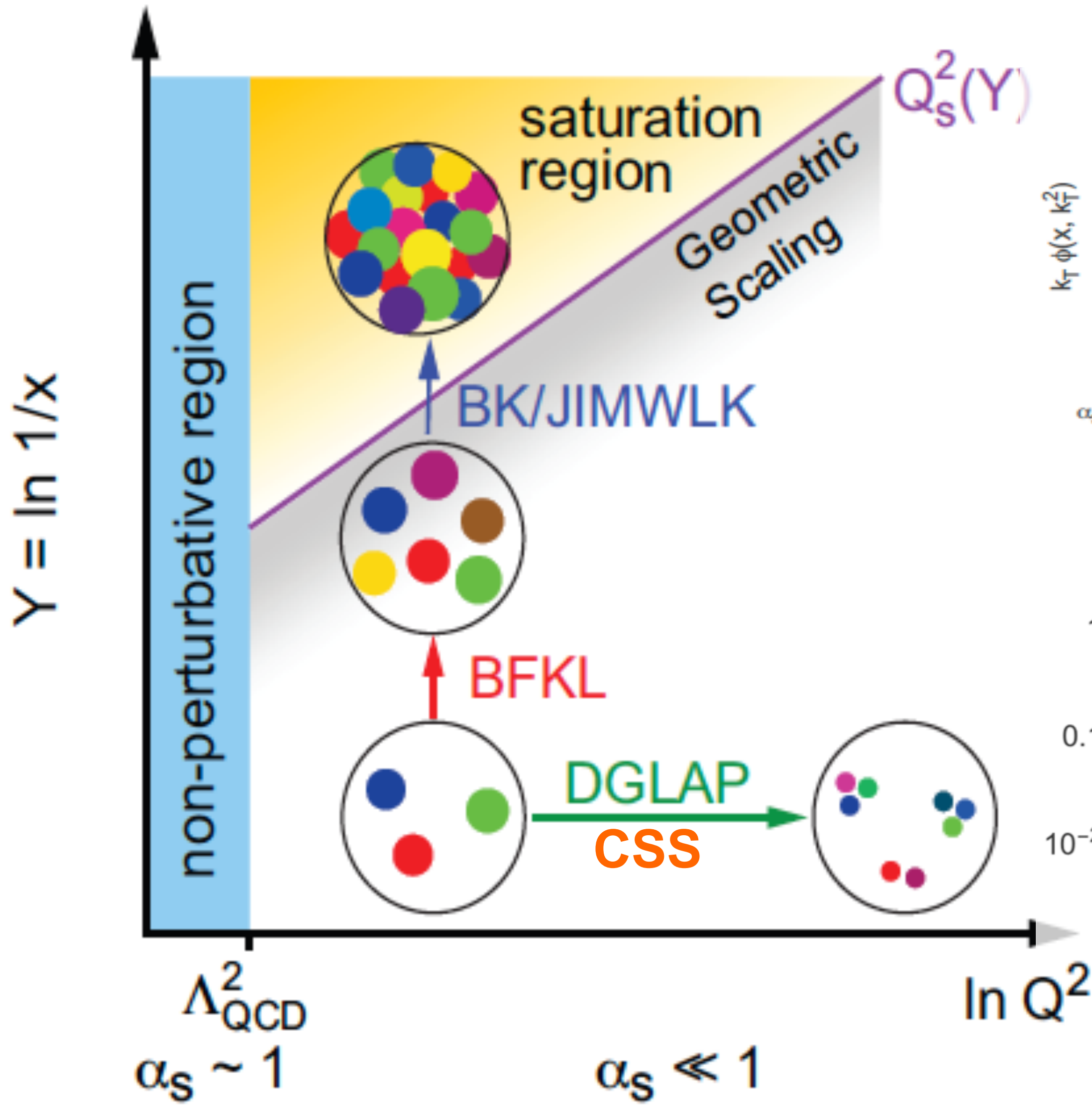


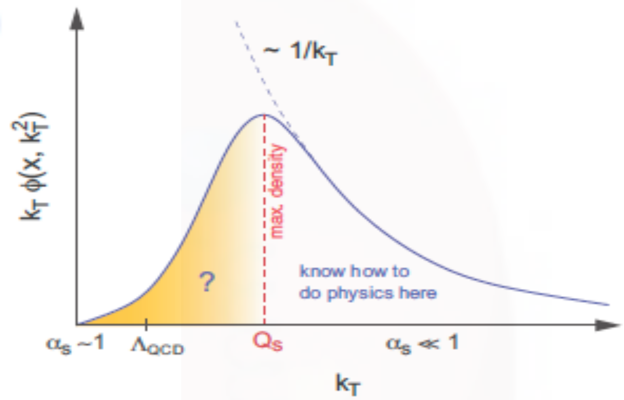
Theory/ Phenomenology

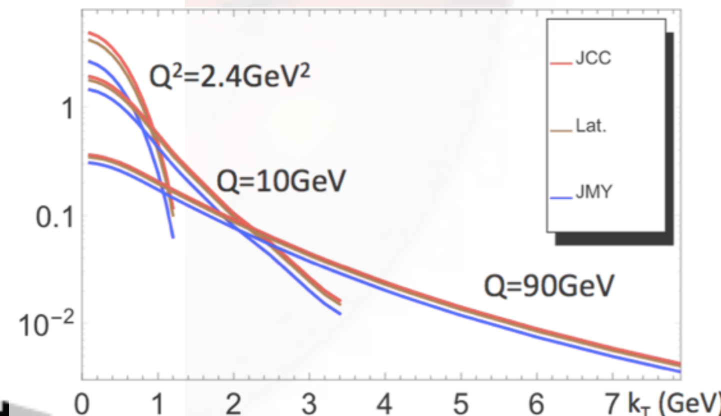
Back-up



Transverse momentum distributions: A unified picture







Prokudin-Sun-Yuan 15

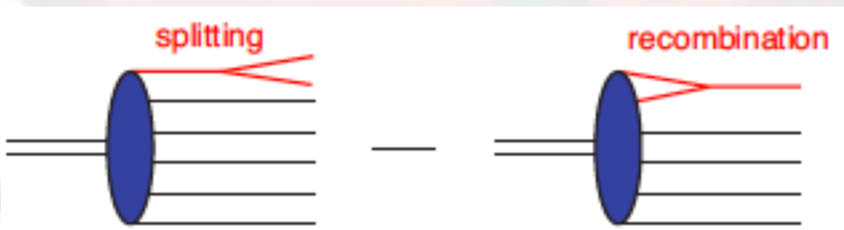
Small-x evolution: Non-linear term at high density

- Balitsky-Fadin-Lipatov-Kuraev, 1977-78

$$\frac{\partial N(x, r_T)}{\partial \ln(1/x)} = \alpha_s K_{\mathrm{BFKL}} \otimes N(x, r_T)$$

- Balitsky-Kovchegov: Non-linear term, 98

$$\frac{\partial N(x, r_T)}{\partial \ln(1/x)} = \alpha_s K_{\mathrm{BFKL}} \otimes N(x, r_T) - \alpha_s [N(x, r_T)]^2.$$

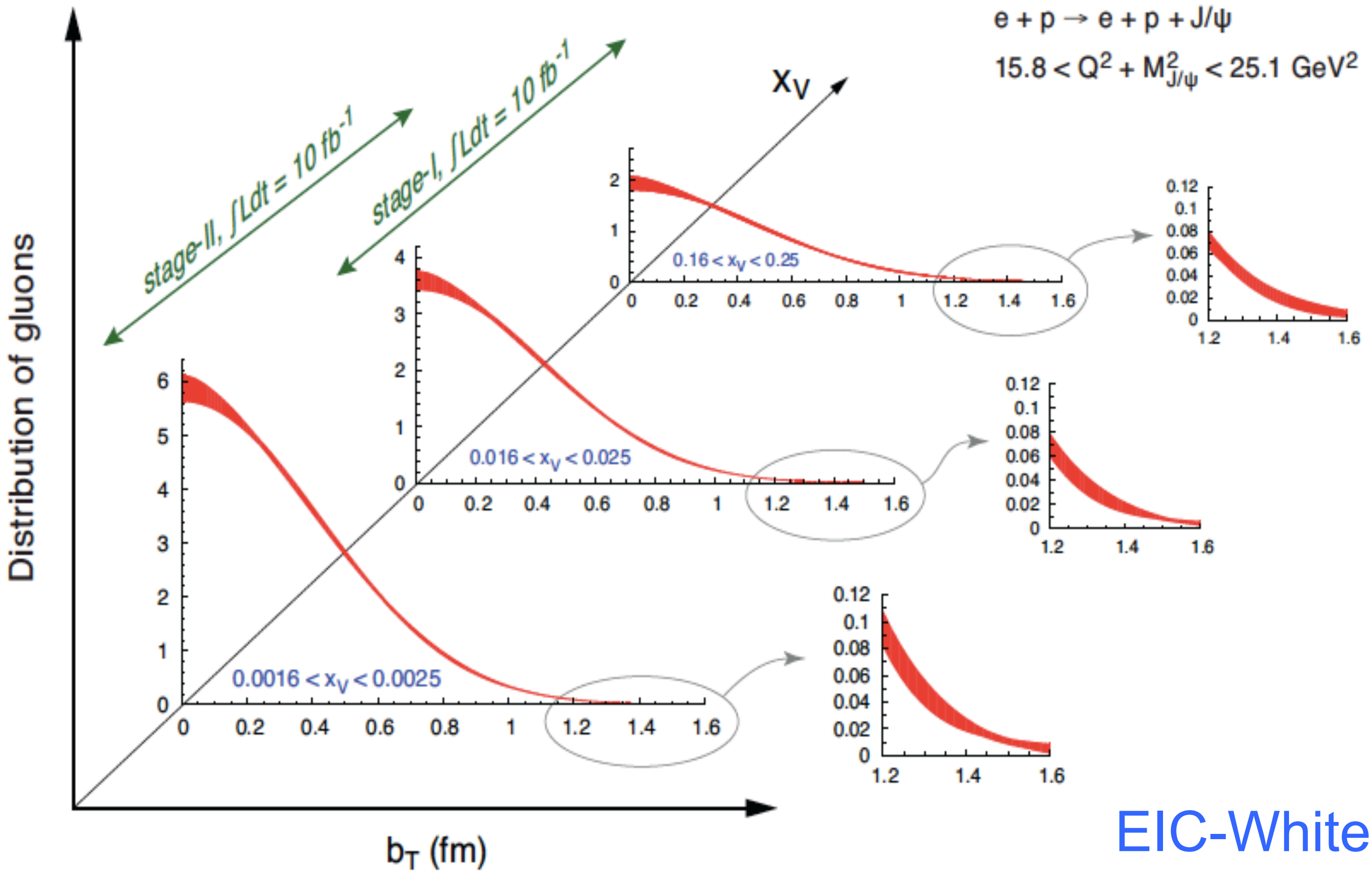


Therefore

- x-dependence of the TMDs at small-x, in principle, can be calculated from the QCD evolution (BK-JIMWLK)
- How about Q^2
  - Sudakov double log resummation (which controls Q-evolution) can be performed consistently in the small-x formalism

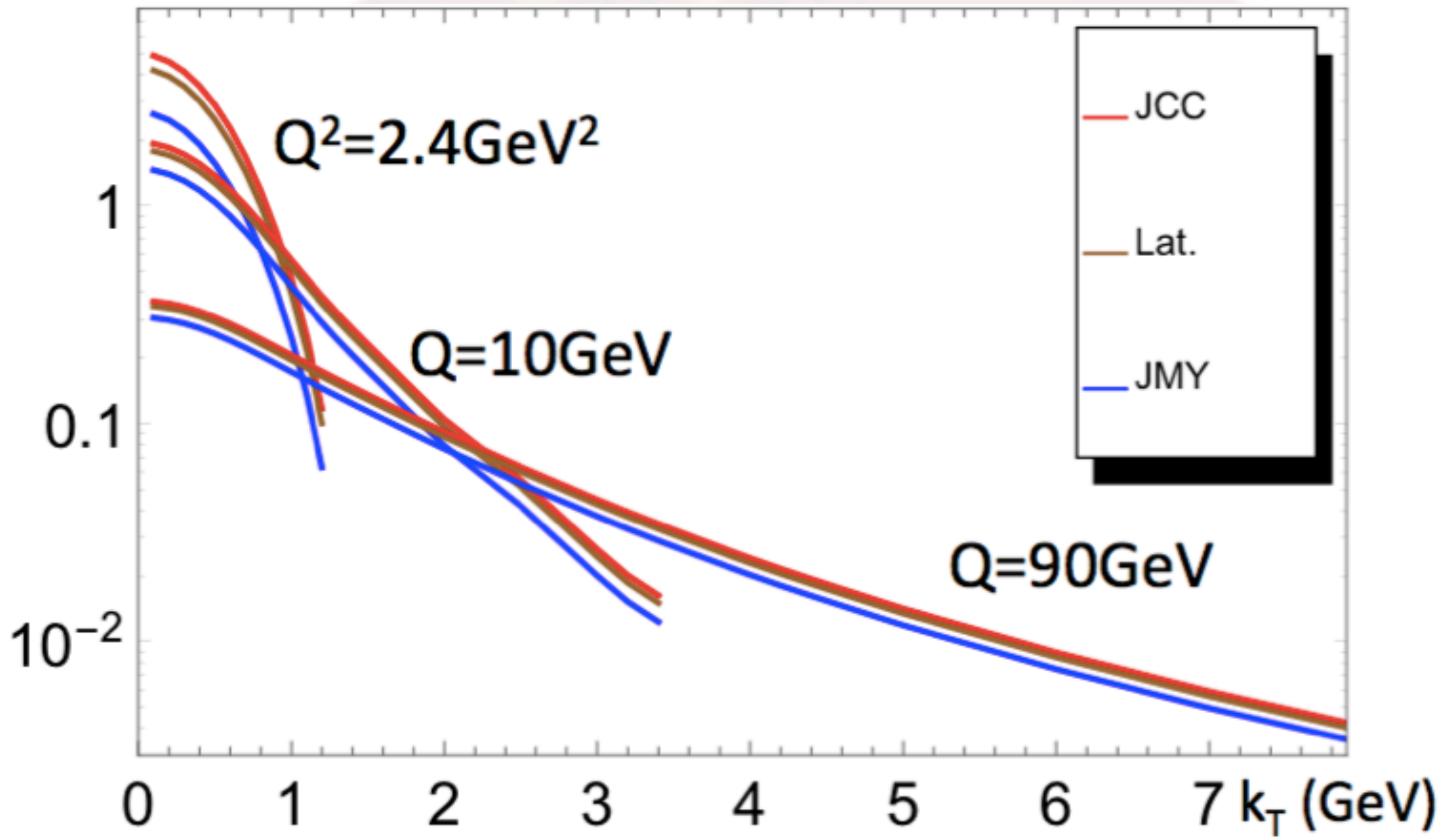
Mueller, Xiao, Yuan, PRL110,082301 (2013);
Phys.Rev. D88 (2013) 114010;
Xiao, Yuan, Zhou, NPB 2017

Gluon tomography at small x (GPDs)



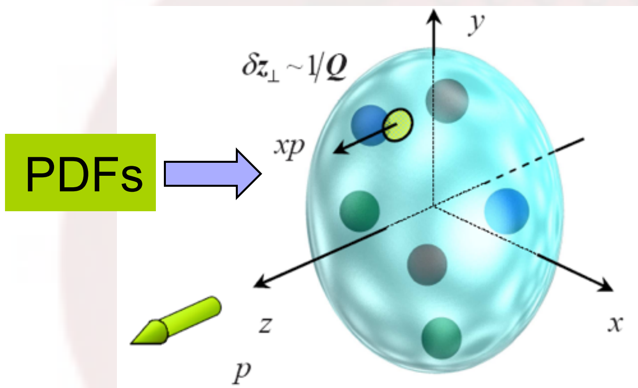
EIC-White paper
arXiv:1212.1701

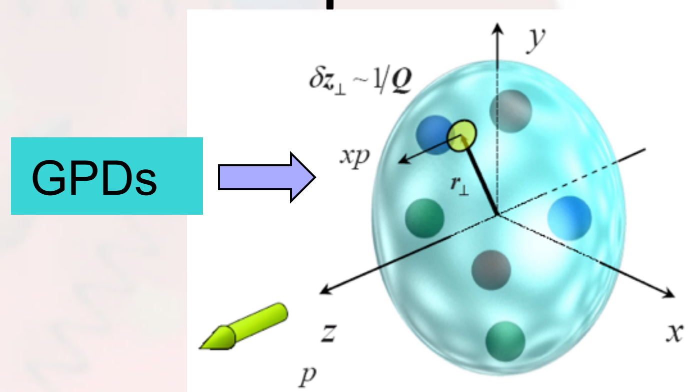
Unpolarized quark distribution



Hadron tomography via GPDs

- GPDs: fully correlated parton distributions in both momentum and coordinate space



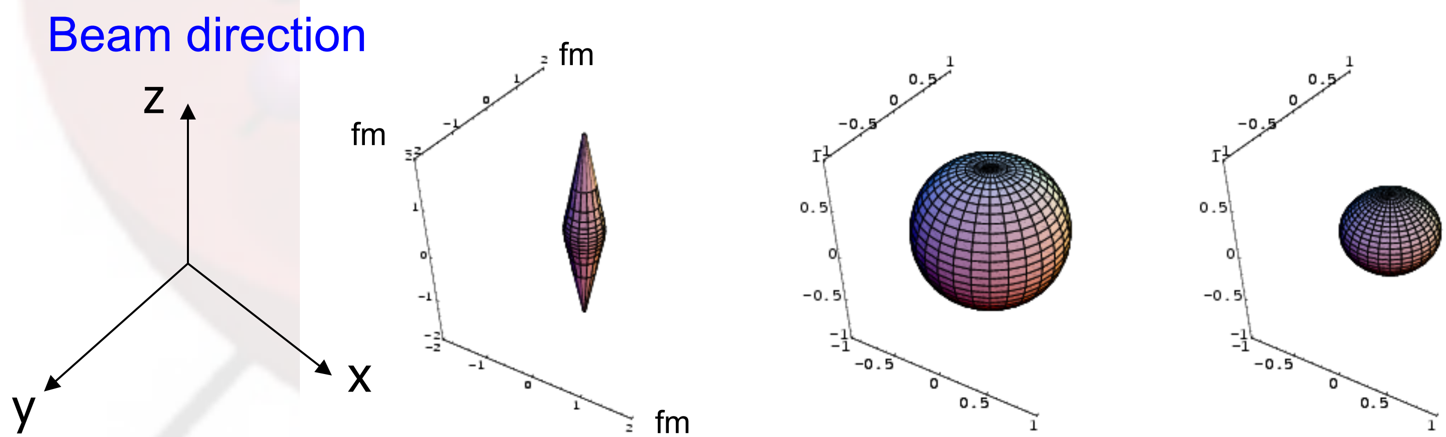


- From the Fourier transform of the momentum transform, we will obtain the partons' 3-d image in nucleon

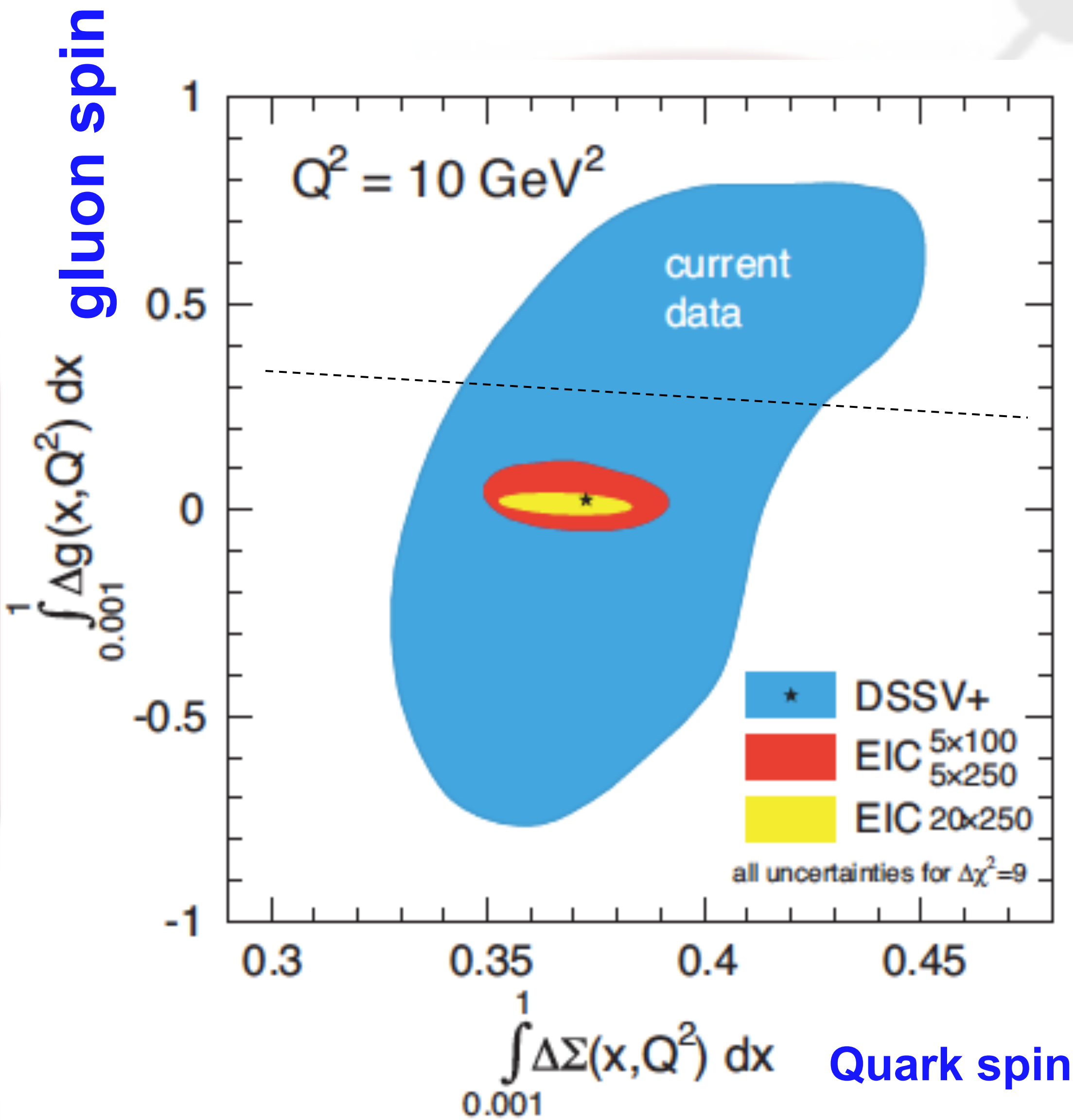
Burkardt 00,02;
Belitsky-Ji-Yuan, PRD04

3D image of quarks at fixed-x

- GPDs can be used to picture quarks in the proton (Belitsky-Ji-Yuan, PRD 04)
  - Fourier transform of the GPDs (respect to the momentum transfer) is a function of position $\vec{r}$ and Feynman momentum x: $f(\vec{r},x)$
  - One can plot this distribution as a 3D function at fixed x

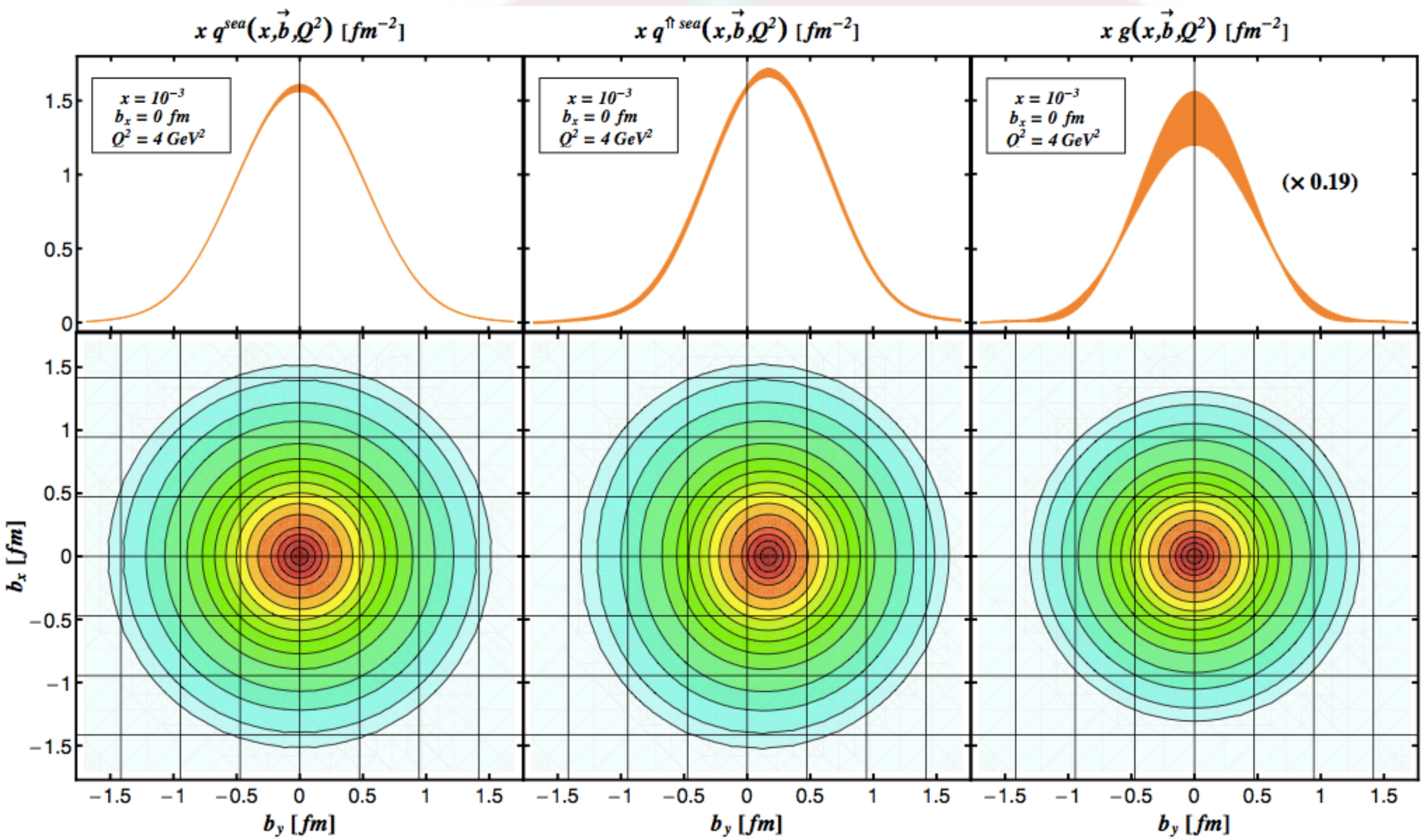


low x moderate x high x



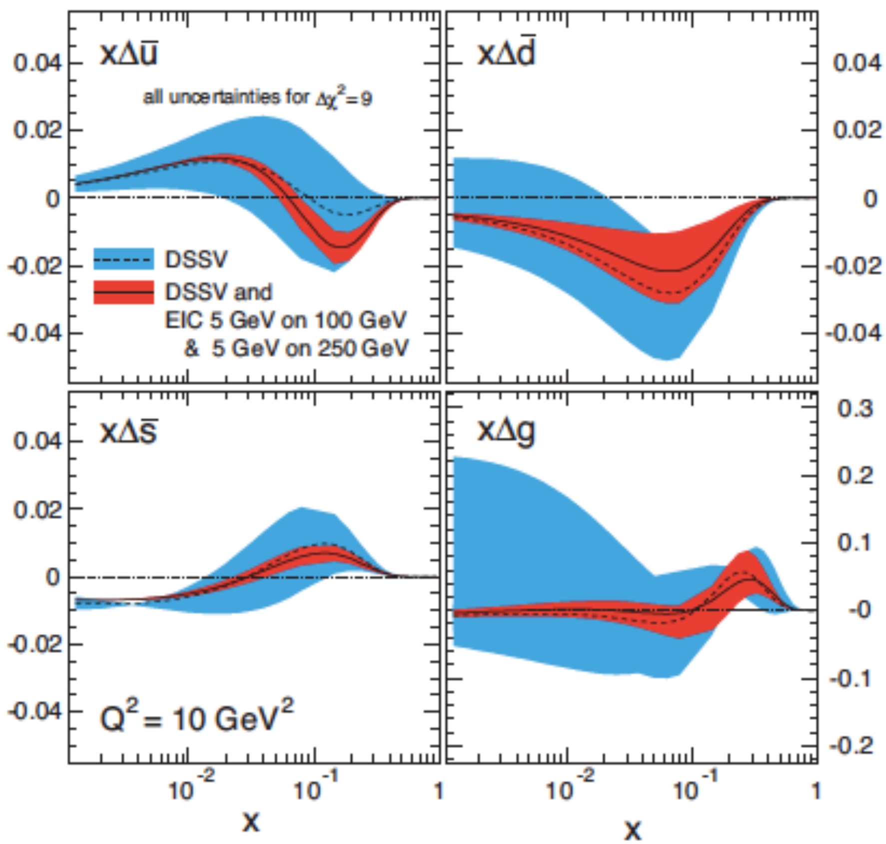
Stratmann, et al.
EIC-White Paper

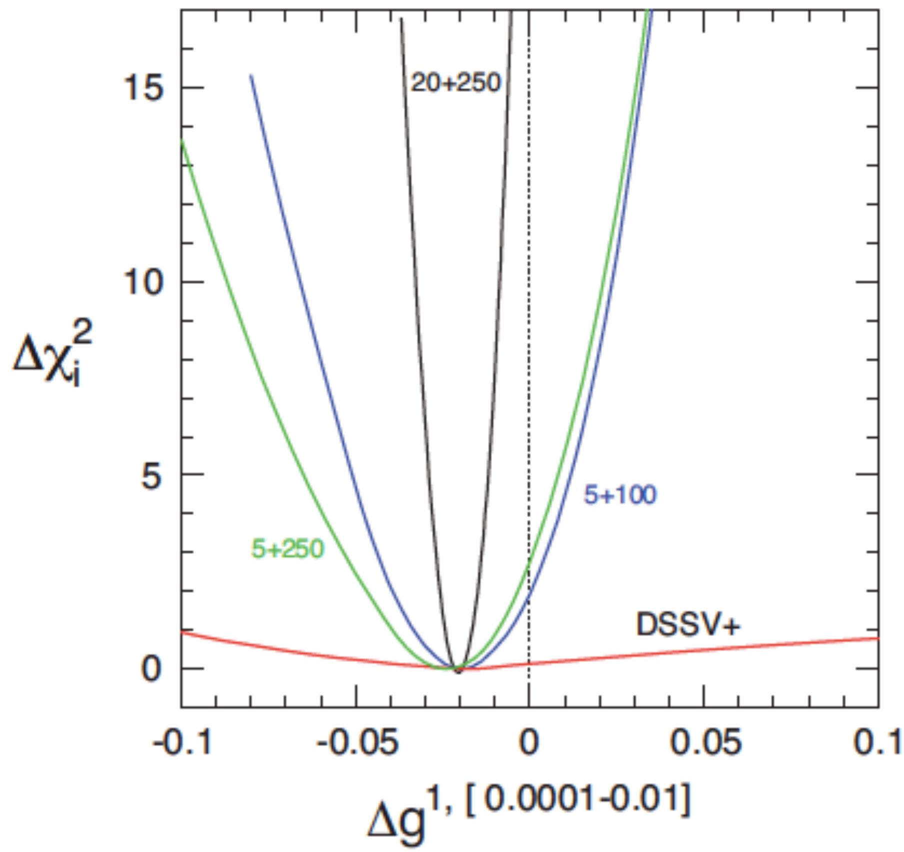
Quark imaging from EIC



Mueller, et al., 1304.0077

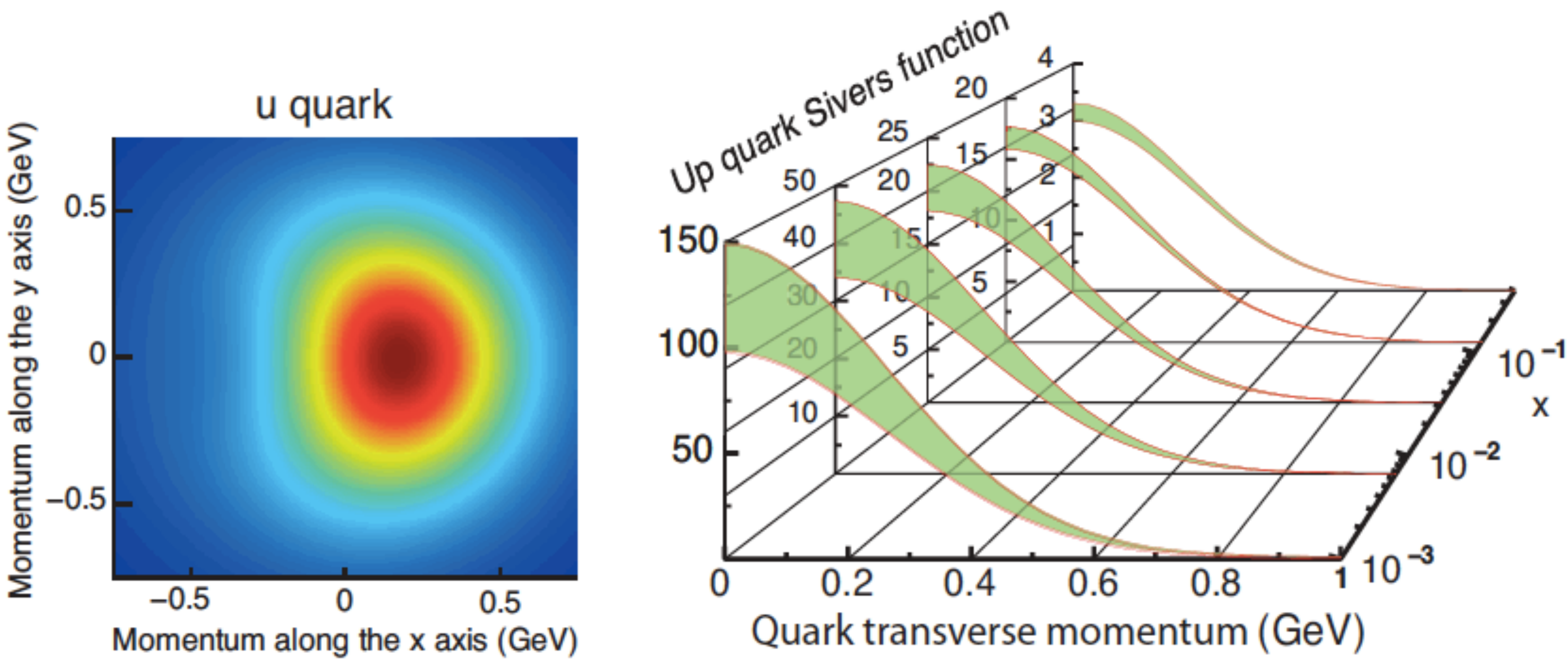
In particular



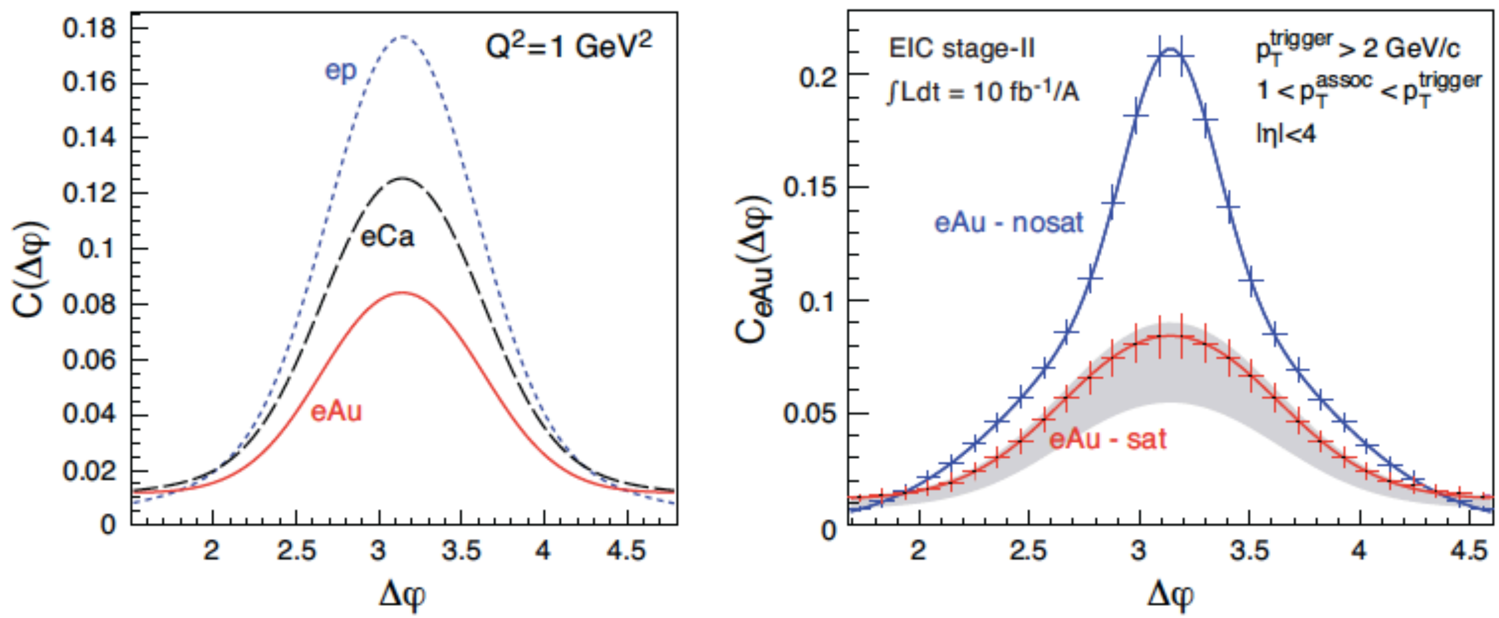


Berkeley Lab

Kt-dependence



Di-hadron correlations



Partonic cross section eq→e'q'

- Cross symmetry with e+e-→qq

$$d\sigma = \frac{d^3k'}{2s|\vec{k}'|}\frac{1}{(q^2)^2}L^{\mu\nu}(k,q)W_{\mu\nu}(p,q)$$

$$L^{\mu\nu} \equiv \frac{e^2}{8\pi^2}tr\left[\not{k}\gamma^\mu \not{k}'\gamma^\nu\right]$$

$$|\overline{\mathcal{M}}|^2 = \frac{1}{(q^2)^2}L_{\mu\nu}W_{\mu\nu} = e_q^2\frac{e^4}{(q^2)^2}2\left[s^2+u^2\right]$$

$$u = (k'-p)^2 = -2k'\cdot p = -s(1-y), \quad y = \frac{q\cdot p}{k\cdot p}$$

$$(s^2+u^2) = s^2(1+(1-y)^2)$$

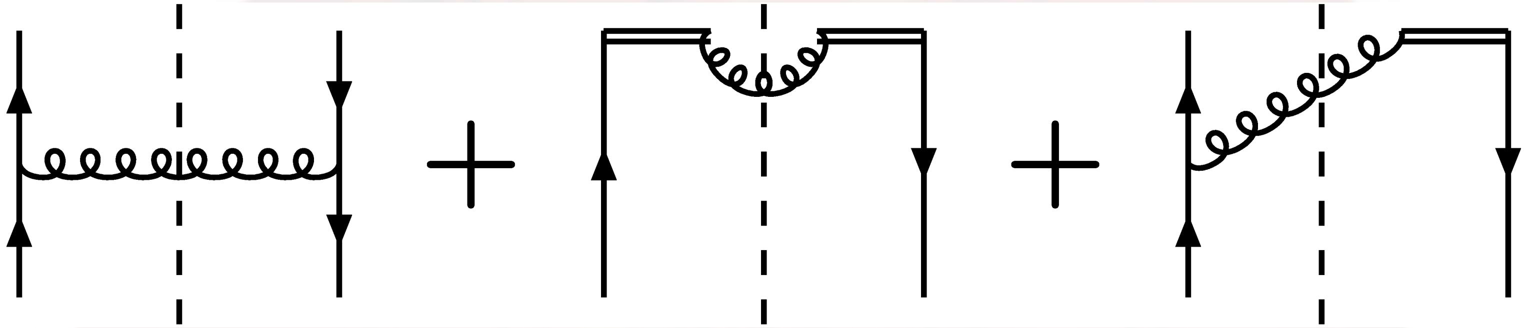
$$d\sigma(ep\to e'+X) = \int dxdy\frac{2\pi\alpha^2}{Q^2}\left[1+(1-y)^2\right]\sum_q e_q^2\phi_{q/P}(x)$$

BACK-UP

SIDIS: at Large P_T

- When $q_T >> \Lambda_{QCD}$, the P_t dependence of the TMD parton distribution and fragmentation functions can be calculated from pQCD, because of hard gluon radiation
- Single Spin Asymmetry at large P_T is not suppressed by $1/Q$, but by $1/P_T$

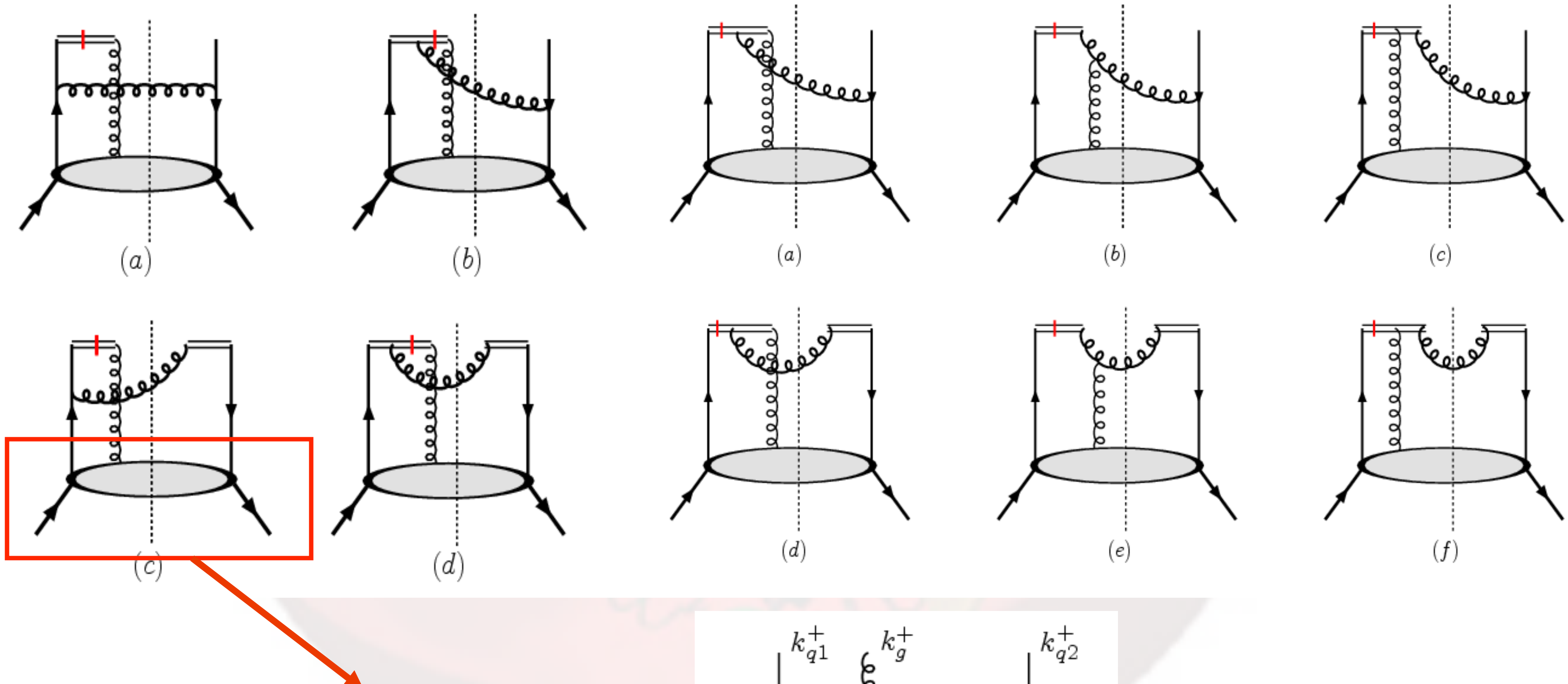
Fragmentation function at $p_T >> \Lambda_{QCD}$



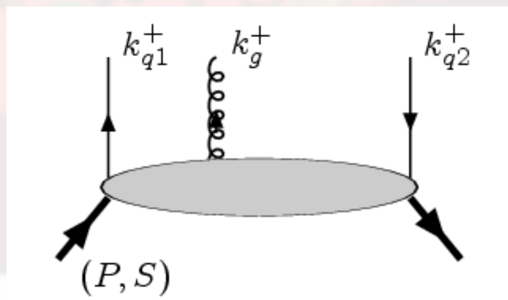
$$\hat{q}(z_h, p_\perp) = \frac{\alpha_s}{2\pi^2}\frac{1}{\vec{p}_\perp^2} C_F \int \frac{dz}{z} \hat{q}(z) \times \left[\frac{1+\hat{\xi}^2}{(1-\hat{\xi})_+} + \delta(\hat{\xi}-1)\left(\ln\frac{\hat{\zeta}^2}{\vec{p}_\perp^2} - 1\right)\right]$$

See, e.g., Ji, Ma, Yuan, 04

Sivers Function at large k_T

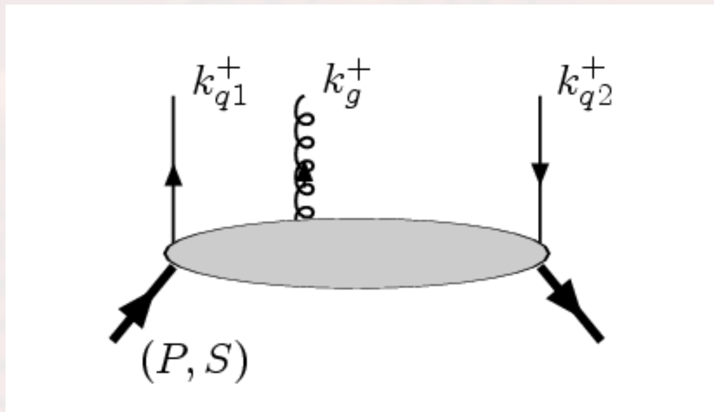


Quark-gluon Correlation



Qiu, Sterman, 91,99

Qiu-Sterman matrix element



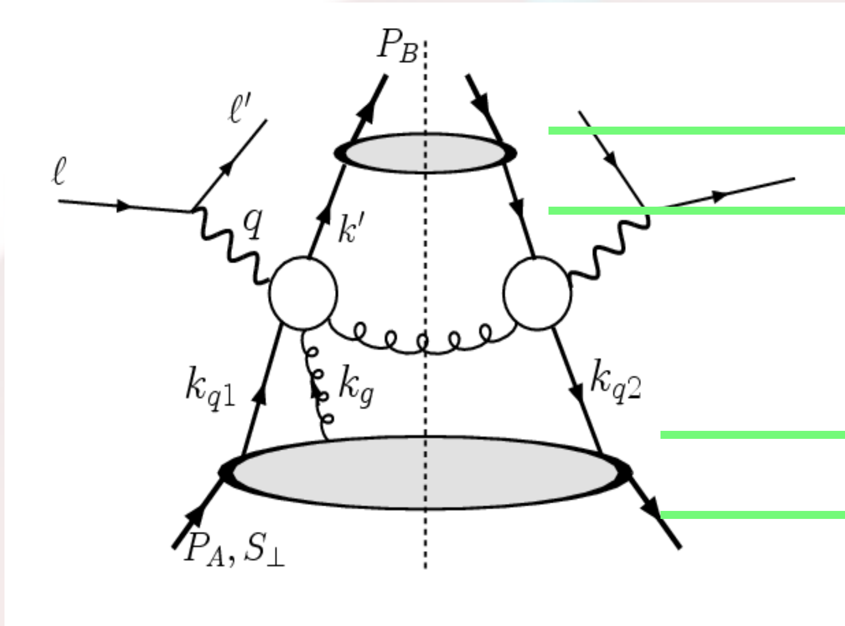
$$T_{a,F}(x_1, x_2) = \int \frac{dy_1^- dy_2^-}{4\pi} e^{ix_1 P^+ y_1^- + i(x_2 - x_1) P^+ y_2^-} \times \langle P, \vec{s}_T | \bar{\psi}_a(0) \gamma^+ \left[\epsilon^{s_T \sigma n \bar{n}} F_\sigma{}^+(y_2^-) \right] \psi_a(y_1^-) | P, \vec{s}_T \rangle$$

Sivers Function at Large k_T

$$q_T(x,k_\perp) = -\frac{\alpha_s}{4\pi^2}\frac{2M_p}{(k_\perp^2)^2}\int\frac{dx}{x}\{A + C_F T_F(x) \times \delta(\xi-1)\left(\ln \zeta^2/\vec{k}_\perp^2 - 1\right)\}$$

- $1/k_T^4$ follows a power counting
- Drell-Yan Sivers function has opposite sign
- Plugging this into the factorization formula, we indeed reproduce the polarized cross section calculated from twist-3 correlation

SSA in the Twist-3 approach



Fragmentation func
\hat q(x')

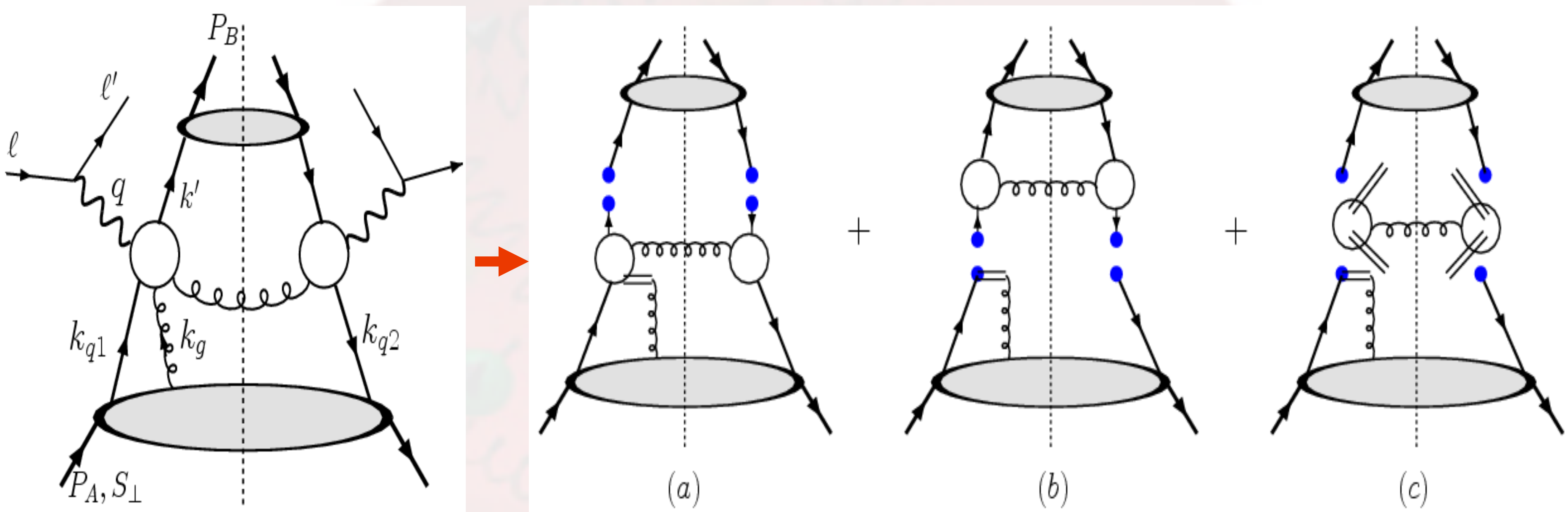
Twist-3 quark-gluo
Correlation: $T_F(x_1,x$

Collinear Factorization:

$$d\sigma \propto \epsilon^{\beta\alpha} S_{\perp\beta} P_{h\perp\alpha} \int \frac{dx}{x}\frac{dz}{z} \hat{q}(z) T_F(x, x - xg) \times \cdots$$

Qiu,Sterman, 91

Factorization guidelines



Reduced diagrams for different regions of the gluon momentum: along P direction, P', and soft Collins-Soper 81

Final Results

- P_T dependence

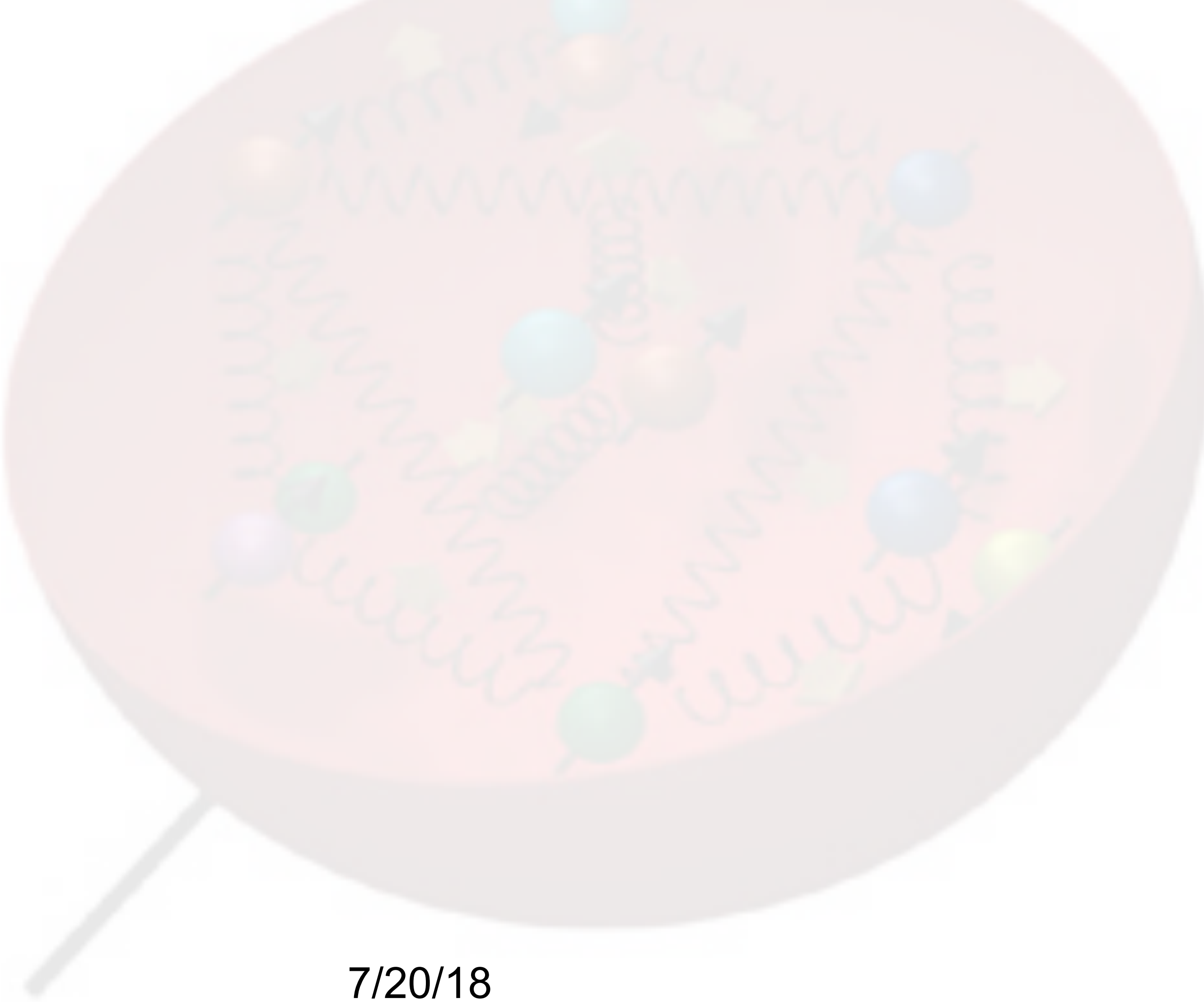
$$\frac{d\triangle\sigma}{d^2q_\perp dy} = \int q_T(z_1,k_\perp)\bar{q}(z_2,k_\perp) + \left(\frac{d\triangle\sigma^{QS}}{d^2q_\perp dy} - \frac{d\triangle\sigma^{QS}}{d^2q_\perp dy}|_{aspt.}\right)$$

Sivers function at low P_T

Qiu-Sterman Twist-three

- Which is valid for all P_T range
  - Resummation can be performed further

Extend to other TMDs



Polarized TMD Quark Distributions

Quark \ Nucleon	Unpol.	Long.	Trans.
Unpol.	$f_1(x,k_\perp)$		$f_{1T}^\perp(x,k_\perp)$
Long.		$g_1(x,k_\perp)$	$g_{1T}(x,k_\perp)$
Trans.	$h_1^\perp(x,k_\perp)$	$h_{1L}(x,k_\perp)$	$h_1(x,k_\perp)$ $h_{1T}^\perp(x,k_\perp)$

Boer, Mulders, Tangerman (96&98)

TMDs and Quark-gluon Correlations (twist-3)

- Kt-odd distribution

$f_{1T}^{\perp}(x, k_\perp)$, $g_{1T}(x, k_\perp)$ ⟺ $G_D(x_1, x_2\, \tilde{G}_D(x_1, x_2)$, $T_F(x_1, x_2)\tilde{T}_F(x_1, x_2)$

$h_1^{\perp}(x, k_\perp)$ ⟺ $T_F^{(\sigma)}(x_1, x_2)$

$h_{1L}(x, k_\perp)$ ⟺ $H_D(x_1, x_2)$

$$T_F(x,x) = -\int \frac{d^2\vec{k}_\perp}{2\pi}\frac{\vec{k}_\perp^2}{M^2} f_{1T}^{\perp}|_{\rm DIS}(x, k_\perp)\,,$$

$$T_F^{(\sigma)}(x,x) = -\int \frac{d^2\vec{k}_\perp}{2\pi}\frac{\vec{k}_\perp^2}{M^2} h_1^{\perp}|_{\rm DIS}(x, k_\perp)$$

$$\tilde{g}(x) = \int d^2\vec{k}_\perp \frac{\vec{k}_\perp^2}{2M^2} g_{1T}(x, k_\perp)$$

$$\tilde{h}(x) = \int d^2\vec{k}_\perp \frac{\vec{k}_\perp^2}{2M^2} h_{1L}(x, k_\perp)$$

Boer-Mulders-Pijlman, 2003

Quark-gluon correlations (twist-three)

- Have long been studied,

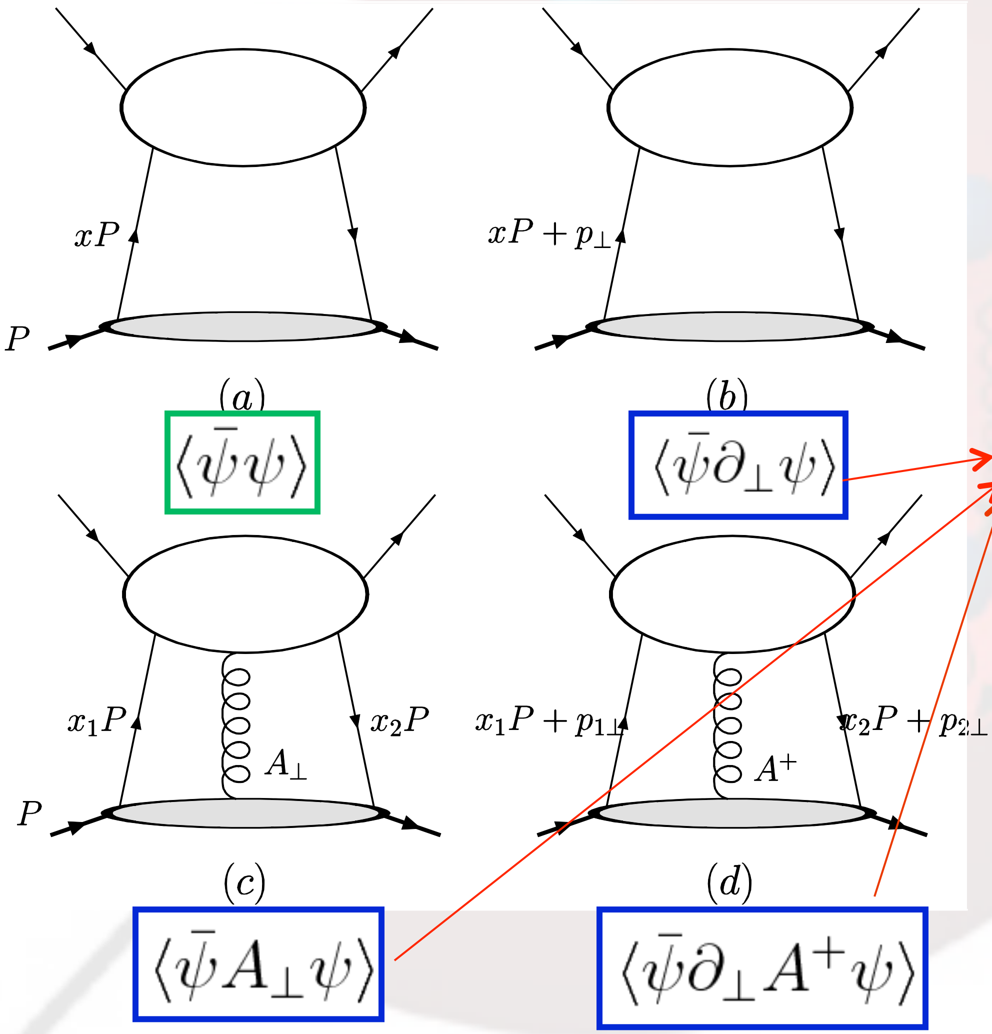
$$D_\Gamma^i(y_1, y_2, s) = \langle P, s|\ \bar{\psi}(0)\ \Gamma\ D^i(y_2)\ \psi(y_1)\ |P, s\rangle$$
$$F_\Gamma^i(y_1, y_2, s) = \langle P, s|\ \bar{\psi}(0)\ \Gamma\ n_\mu F^{i\mu}(y_2)\ \psi(y_1)\ |P, s\rangle$$

- F-type and D-type are related to each other,
Ellis-Furmanski-Petronzio 82, Eguchi-Koike-Tanaka 06

$$G_D(x, x_1) = P\frac{1}{x - x_1}T_F(x, x_1),$$
$$\tilde{G}_D(x, x_1) = P\frac{1}{x - x_1}\tilde{T}_F(x, x_1) + \delta(x - x_1)\tilde{g}(x),$$
$$E_D(x, x_1) = P\frac{1}{x - x_1}T_F^{(\sigma)}(x, x_1),$$
$$H_D(x, x_1) = P\frac{1}{x - x_1}\tilde{T}_F^{(\sigma)}(x, x_1) + \delta(x - x_1)\tilde{h}(x)$$

twist and collinear expansion

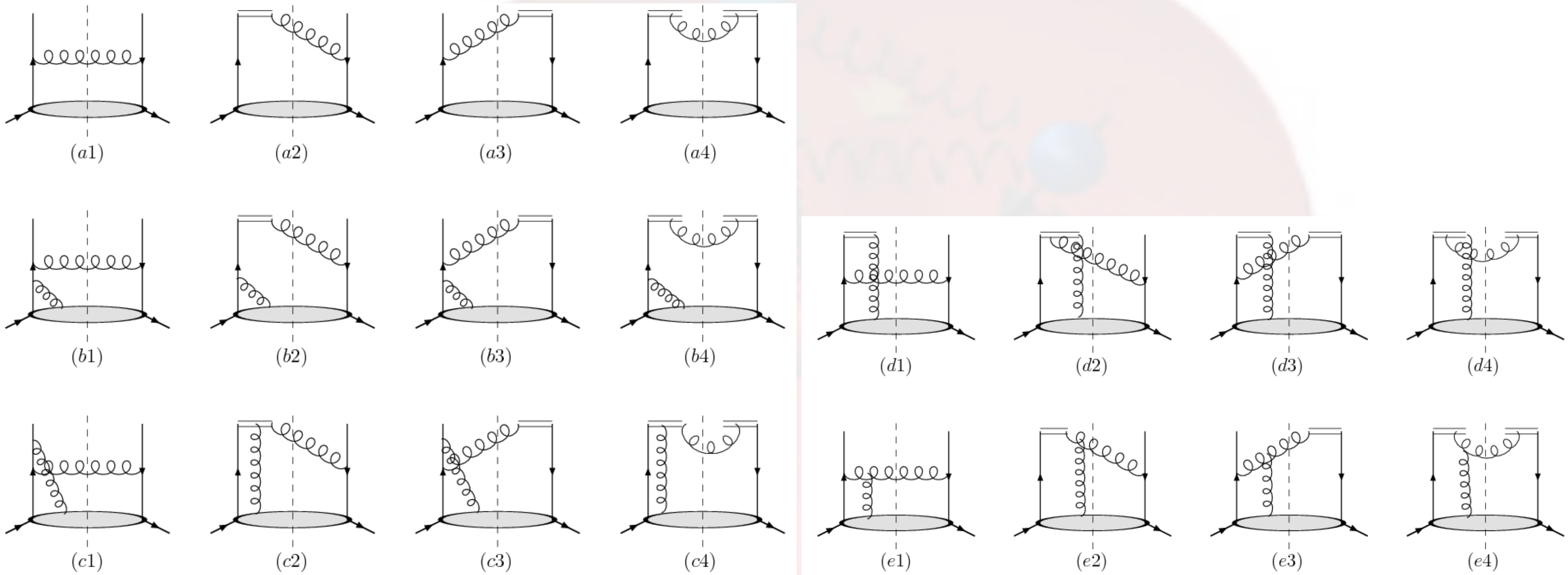
R.K. Ellis et al., 82;
Qiu-Sterman, 90



Twist-three matrix

Gauge invariant twist-3
Quark-gluon correlatio
functions: D- or F-type

Large kt TMDs



- Color factors, C_F: a1-4,b1-4,e2...
  $1/2N_c$: c1,c3, d1...
  $C_A/2$: e1-4

- $\langle\bar{\psi}\partial_\perp\psi\rangle$ a1-4
- $\langle\bar{\psi}A_\perp\psi\rangle$ b1-4,c1,c3,e1-4
- $\langle\bar{\psi}\partial_\perp A^+\psi\rangle$ b1-4,c1-4,d1-4,e1-4

Generic results

- Kt-even TMDs

Zhou,Liang,Yuan,2010

$$f_1(x_B,k_\perp) = \frac{\alpha_s}{2\pi^2}\frac{1}{\vec{k}_\perp^2}C_F\int\frac{dx}{x}f_1(x)\left[\frac{1+\xi^2}{(1-\xi)_+}+\delta(1-\xi)\left(\ln\frac{x_B^2\zeta^2}{\vec{k}_\perp^2}-1\right)\right]$$

$$g_{1L}(x_B,k_\perp) = \frac{\alpha_s}{2\pi^2}\frac{1}{\vec{k}_\perp^2}C_F\int\frac{dx}{x}g_{1L}(x)\left[\frac{1+\xi^2}{(1-\xi)_+}+\delta(1-\xi)\left(\ln\frac{x_B^2\zeta^2}{\vec{k}_\perp^2}-1\right)\right]$$

$$h_1(x_B,k_\perp) = \frac{\alpha_s}{2\pi^2}\frac{1}{\vec{k}_\perp^2}C_F\int\frac{dx}{x}f_1(x)\left[\frac{2\xi}{(1-\xi)_+}+\delta(1-\xi)\left(\ln\frac{x_B^2\zeta^2}{\vec{k}_\perp^2}-1\right)\right]$$

Splitting kernel

Large logs

- Sivers and Boer-Mulders

$$f_{1T}^{\perp}|_{\rm DY}(x_B,k_\perp) = \frac{\alpha_s}{\pi}\frac{M^2}{(\vec{k}_\perp^2)^2}\int\frac{dx}{x}\left[A_{f_{1T}^\perp} + C_F T_F(x,x)\delta(1-\xi)\left(\ln\frac{x_B^2\zeta^2}{\vec{k}_\perp^2}-1\right)\right]$$

$$h_1^{\perp}|_{\rm DY}(x_B,k_\perp) = \frac{\alpha_s}{\pi}\frac{M^2}{(\vec{k}_\perp^2)^2}\int\frac{dx}{x}\left[A_{h_1^\perp} + C_F T_F^{(\sigma)}(x,x)\delta(1-\xi)\left(\ln\frac{x_B^2\zeta^2}{\vec{k}_\perp^2}-1\right)\right]$$

$$A_{f_{1T}^\perp} = -\frac{1}{2N_c}T_F(x,x)\frac{1+\xi^2}{(1-\xi)_+} + \frac{C_A}{2}T_F(x,x_B)\frac{1+\xi}{(1-\xi)_+} + \frac{C_A}{2}\tilde{T}_F(x_B,x)$$

$$A_{h_1^\perp} = -\frac{1}{2N_c}T_F^{(\sigma)}(x,x)\frac{2\xi}{(1-\xi)_+} + \frac{C_A}{2}T_F^{(\sigma)}(x,x_B)\frac{2}{(1-\xi)_+}\ .$$

g_{1T} and h_{1L}

$$g_{1T}(x_B,k_\perp)=\frac{\alpha_s}{\pi^2}\frac{M^2}{(k_\perp^2)^2}\int\frac{dx}{x}\left\{A_{g_{1T}}+C_F\tilde{g}(x)\delta(\xi-1)\left(\ln\frac{x_B^2\zeta^2}{k_\perp^2}-1\right)\right\}$$

$$h_{1L}(x_B,k_\perp)=\frac{\alpha_s}{\pi^2}\frac{M^2}{(k_\perp^2)^2}\int\frac{dx}{x}\left\{A_{h_{1L}}+C_F\tilde{h}(x)\delta(\xi-1)\left(\ln\frac{x_B^2\zeta^2}{k_\perp^2}-1\right)\right\}$$

$$\begin{aligned}A_{g_{1T}} &= \int dx_1\left\{\frac{1}{2N_C}\tilde{g}(x)\frac{1+\xi^2}{(1-\xi)_+}\delta(x_1-x)\right.\\ &+\left[C_F\left(\frac{x_B^2}{x^2}+\frac{x_B}{x_1}-\frac{2x_B^2}{x_1x}-\frac{x_B}{x}-1\right)+\frac{C_A}{2}\frac{(x_B^2+xx_1)(2x_B-x-x_1)}{(x_B-x_1)(x-x_1)x_1}\right]\tilde{G}_D(x,x_1\\ &+\left.\left[C_F\left(\frac{x_B^2}{x^2}+\frac{x_B}{x_1}-\frac{x_B}{x}-1\right)+\frac{C_A}{2}\frac{x_B^2-xx_1}{(x_1-x_B)x_1}\right]G_D(x,x_1)\right\} \qquad (\end{aligned}$$

Asymptotical Freedom and Factorization

- QCD is an asymptotical freedom theory (Gross, Politzer, Wilczek, 1973), where perturbation method becomes relevant at large scale.
- While, because of confinement, a typical hadronic process contains multiple scales, e.g., the nonperturbative scale Λ_{QCD}, meaning that a QCD factorization must be proven in order to successfully separate different scales.

One Large Scale Factorization

- If the physics only involves one large scale, the factorization is the simplest,
  - Inclusive DIS and Drell-Yan
  - Jet production
  - Inclusive particle production at hadron collider
  - Hard exclusive processes, Pi form factor, DVCS, …

$$\sigma(Q)=H(Q/\mu)\ f_1(\mu)\ldots$$

Additional Large Scale Introduces Large Double Logarithms

- For example, a differential cross section depends on Q_1, where $Q^2 \gg Q_1^2 \gg \Lambda^2_{QCD}$

$$\frac{d\sigma}{dQ_1^2} = \frac{1}{Q_1^2} f_1 \otimes f_2 \otimes \sum_i \alpha_s^i \ln^{2i-1} \frac{Q^2}{Q_1^2} + \cdots$$

- We have to resum these large logs to make reliable predictions
  - Q_T: Dokshitzer, Diakonov, Troian, 78; Parisi Petronzio, 79; Collins, Soper, Sterman, 85
  - Threshold: Sterman 87; Catani and Trentadue 89

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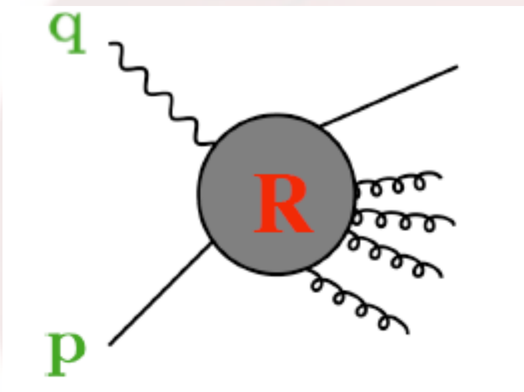
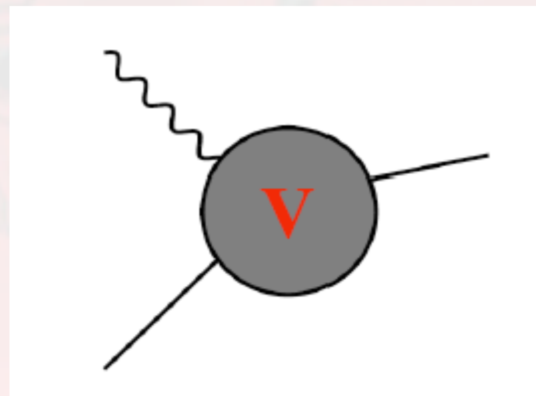
Why Resummation is Relevant

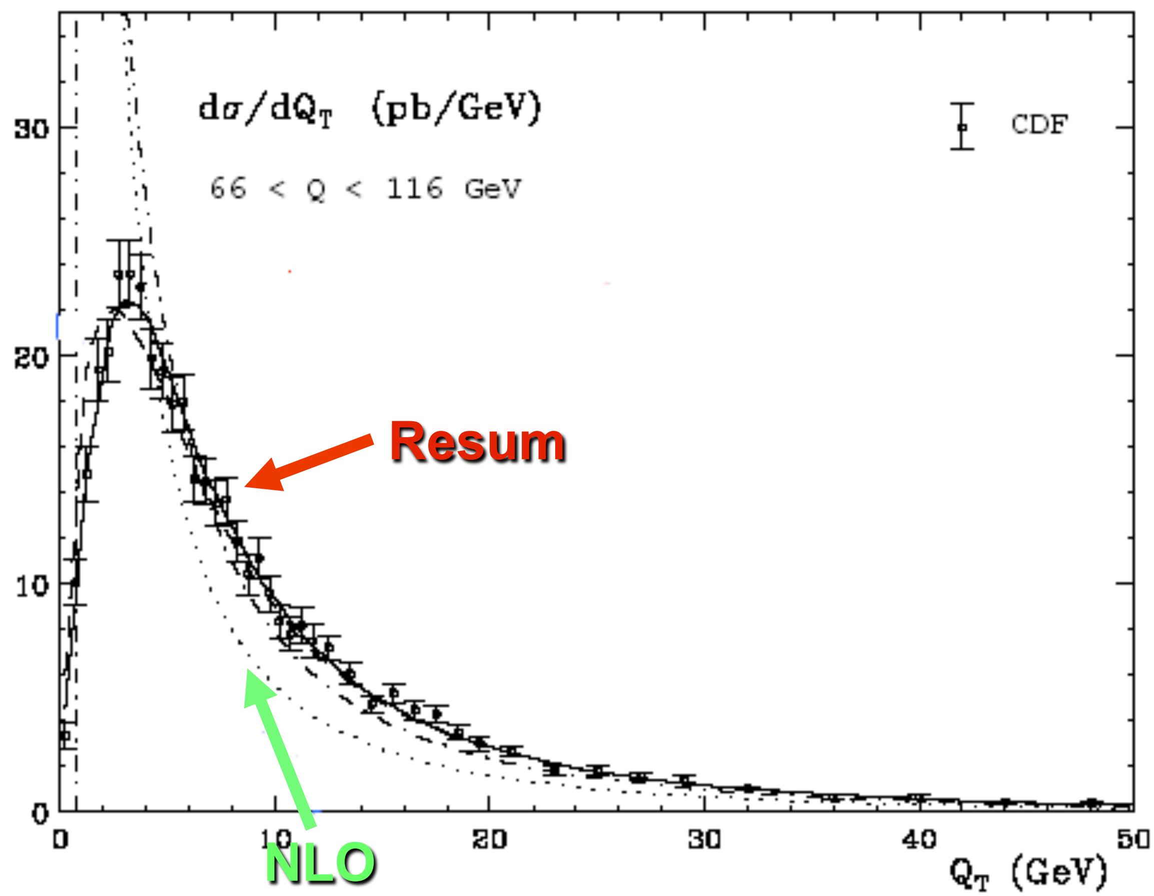
- Soft gluon radiation is very important for this kinematical limit





- Real and Virtual contributions are "imbalanced" IR cancellation leaves large logarithms (implicit)

How Large of the Resummation effects



Kulesza, Sterman, Vogelsang, 02

General Structure of Large Logs

LO	1			
NLO	$\alpha_s L^2$	$\alpha_s L$	α_s	
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^3$	$\alpha_s^2 L^2$	+…
N^{3}LO	$\alpha_s^3 L^6$	$\alpha_s^3 L^5$	$\alpha_s^3 L^4$	+…
N^{k}LO	$\alpha_s^k L^{2k}$	$\alpha_s^k L^{2k-1}$	$\alpha_s^k L^{2k-2}$	+…
	LL	**NLL**	**NNLL**	

Two Large Scales Processes

- Include
  - DIS and Drell-Yan at small P_T (Q_T Resum) ✓
  - DIS and Drell-Yan at large x (Threshold Resum) ✓
  - Higgs production at small P_T or large x ✓
  - Semileptonic B Decays
  - Non-leptonic B Decays
  - Thrust distribution
  - Jet shape function
  - …

Collins-Soper-Sterman Resummation

- Introduce a new concept, the Transverse Momentum Dependent PDF
- Prove the Factorization in terms of the TMDs

$\sigma(P_T,Q)=H(Q)\ f_1(k_{1T},Q)\ f_2(k_{2T},\ Q)\ S(\lambda_T)$

- Large Logs are resummed by solving the energy evolution equation of the TMDs

$$\frac{\partial}{\partial \ln Q} f(k_\perp, Q) = (K(q_\perp,\mu) + G(Q,\mu)) \otimes f(k_\perp, Q)$$

(**Collins-Soper 81, Collins-Soper-St**

CSS Formalism (II)

- K and G obey the renormalization group eq.

$$\frac{\partial}{\partial \ln \mu} K = -\gamma_K = \frac{\partial}{\partial \ln \mu} G$$

- The large logs will be resummed into the exponential form factor

$$W(Q,b) = e^{-\int_{1/b}^{Q} \frac{d\mu}{\mu}\left(\ln \frac{Q}{\mu} A + B\right)} C \otimes f_1 C \otimes f_2$$

  - A,B,C functions are perturbative calculable.

(Collins-Soper-Sterman 85)

SSAs: DY as an example

$$A(P_A, S_\perp) + B(P_B) \to \gamma^*(q) + X \to \ell^+ + \ell^- + X,$$

- P_T dependence

$$\frac{d\triangle\sigma}{d^2q_\perp dy} = \int q_T(z_1, k_\perp)\bar{q}(z_2, k_\perp) + \left(\frac{d\triangle\sigma^{QS}}{d^2q_\perp dy} - \frac{d\triangle\sigma^{QS}}{d^2q_\perp dy}|_{aspt.}\right)$$

Sivers function at low P_T

Qiu-Sterman Twist-three

- Which is valid for all P_T range

CSS Resummation

$$\frac{d\Delta\sigma(S_\perp)}{dydQ^2d^2q_\perp} = \sigma_0\epsilon^{\alpha\beta}S_\perp^\alpha W_{UT}^\beta(Q;q_\perp)$$

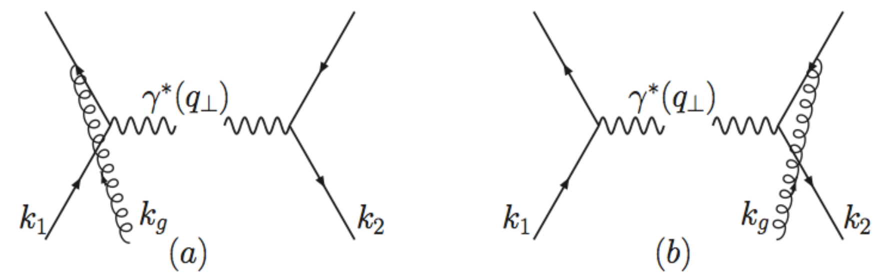
- Separate the singular and regular parts

$$W_{UT}^\alpha(Q;q_\perp) = \int \frac{d^2b}{(2\pi)^2}e^{i\vec{q}_\perp\cdot\vec{b}}\widetilde{W}_{UT}^\alpha(Q;b) + Y_{UT}^\alpha(Q;q_\perp)$$

- TMD factorization in b-space

$$\widetilde{W}_{UT}^\alpha(Q;b) = \tilde{f}_{1T}^{(\perp\alpha)}(z_1,b,\zeta_1)\bar{q}(z_2,b,\zeta_2) \times H_{UT}(Q)\left(S(b,\rho)\right)^{-1} \ ,$$

Leading order

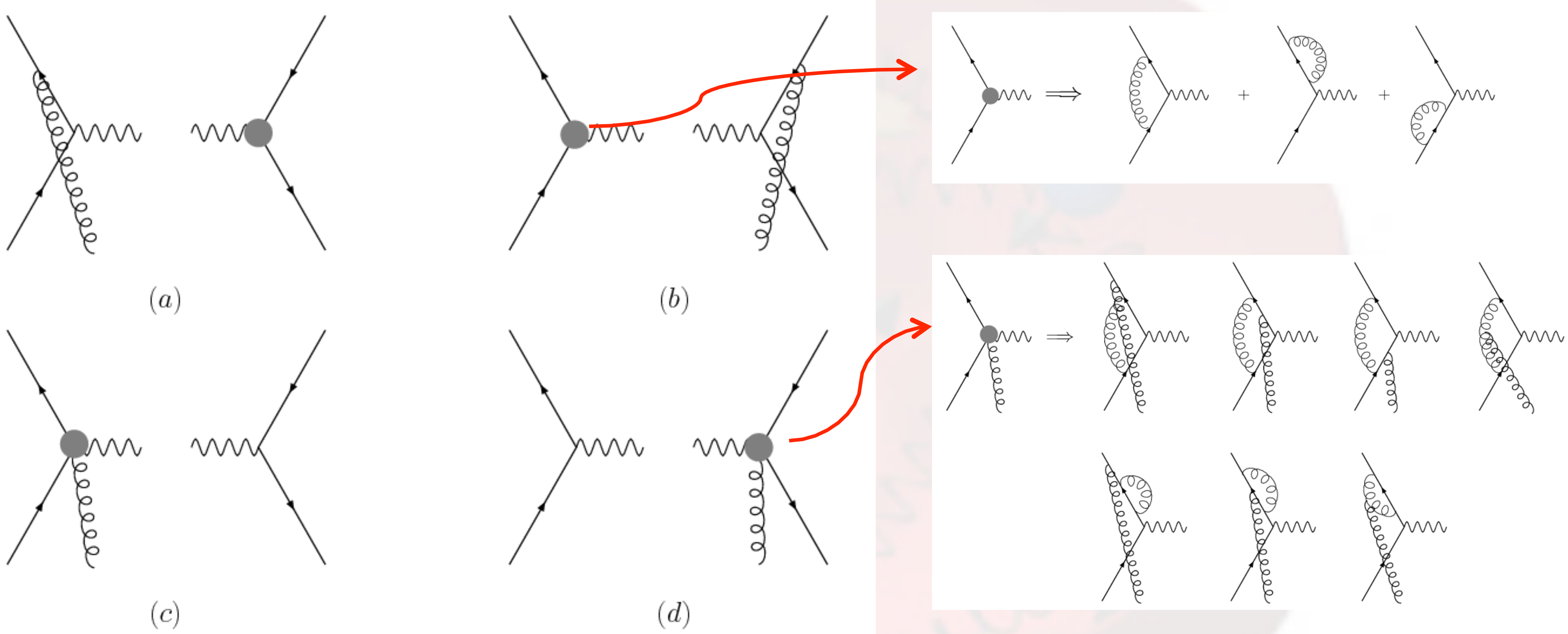


- Small-b expansion, 1/b>>intrinsic kt

$$\text{Fig.2} = \int d^2q_\perp e^{-i\vec{q}_\perp\cdot\vec{b}_\perp}\left(\frac{ig}{-(k_2^+-k_1^+)-i\epsilon}\right)\times\left[\delta(q_\perp-k_{2\perp})-\delta(q_\perp-k_{1\perp})\right]$$
$$=\left(\frac{ig}{-(k_2^+-k_1^+)-i\epsilon}\right)\left[e^{-i\vec{k}_{2\perp}\cdot\vec{b}_\perp}-e^{-i\vec{k}_{1\perp}\cdot\vec{b}_\perp}\right]$$
$$=\frac{ig}{-(k_2^+-k_1^+)-i\epsilon}\left(-ib_\perp^\alpha\right)k_{g\perp}^\alpha\ ,$$

$$\widetilde{W}_{UT}^{\alpha(0)}(Q,b)=\left(\frac{-ib_\perp^\alpha}{2}\right)T_F(z_1,z_1)\bar{q}(z_2)$$

Virtual diagrams

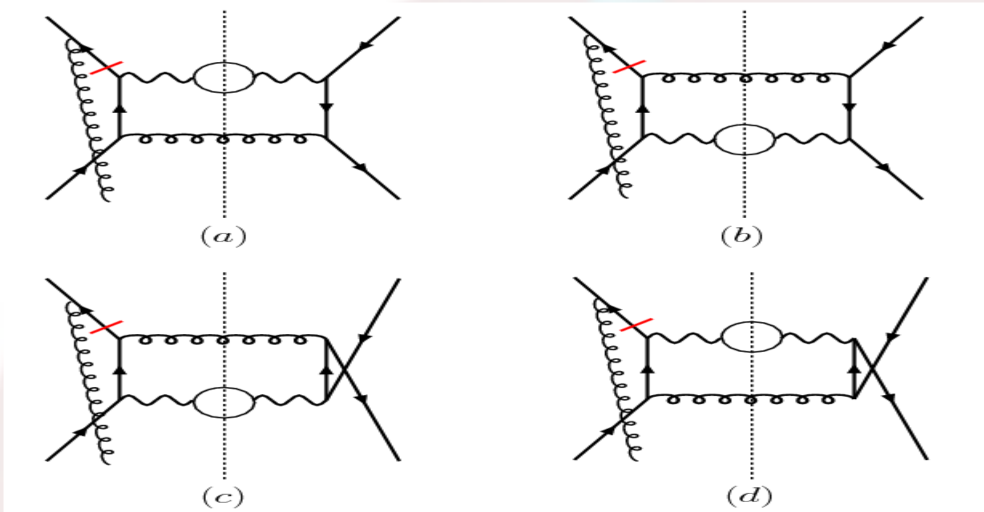


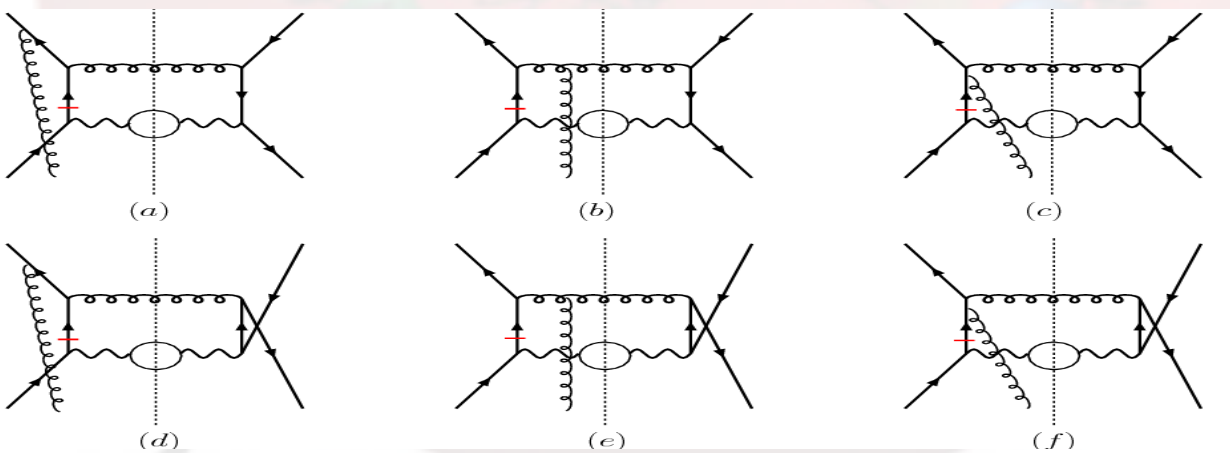
$$\sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int \frac{dx}{x}\frac{dx'}{x'} T_F(x,x)\bar{q}(x')C_F\delta(1-z)\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\pi^2\right]$$

soft divergence

collinear diverge

Soft divergence from real diagrams





Collinear divergence--splitting

$$\frac{\alpha_s}{2\pi}\left(\frac{-ib_\perp^\alpha}{2}\right)\left\{\left[-\frac{1}{\epsilon}+\ln\frac{4}{b^2\mu^2}e^{-2\gamma_E}\right]\left(\mathcal{P}_{q/q}\otimes\bar{q}(z_2')\right.\right.$$
$$\left.+\mathcal{P}^T_{qg\to qg}\otimes T_F(z_1',z_1'')\right)+C_F(1-\xi_2)\delta(1-\xi_1)$$
$$+\left(-\frac{1}{2N_c}\right)(1-\xi_1)\delta(1-\xi_2)+\delta(1-\xi_1)\delta(1-\xi_2)$$
$$\left.\times C_F\left[-\ln^2\left(\frac{Q^2b^2}{4}e^{2\gamma_E-\frac{3}{2}}\right)-\frac{23}{4}+\pi^2\right]\right\}\ ,$$

- Sivers function

$$\tilde{f}_{1T}^\alpha(z_1,b_\perp)=\frac{\alpha_s}{2\pi}\left(\frac{-ib_\perp^\alpha}{2}\right)\left\{\left[-\frac{1}{\epsilon}+\ln\frac{4}{b^2\mu^2}e^{-2\gamma_E}\right]\right.$$
$$\times\mathcal{P}^T_{qg\to qg}\otimes T_F(z_1',,z_1'')+\delta(1-\xi_1)C_F\left[-\frac{3}{2}\ln\frac{4}{b^2\mu^2}e^{-2\gamma_E}\right.$$
$$\left.-\frac{1}{2}\ln^2\left(\frac{z_1^2\zeta_1^2b^2}{4}e^{2\gamma_E-1}\right)-\frac{3+\pi^2}{2}\right]$$
$$\left.+\left(-\frac{1}{2N_c}\right)(1-\xi_1)\right\}\ .$$

Hard factor at one-loop order

- Same as the spin-average case

$$H_{UT}^{(1)\mathrm{DY}} = H_{UU}^{(1)}|_{\mathrm{DY}}$$
$$= \frac{\alpha_s}{2\pi} C_F \left[\ln \frac{Q^2}{\mu^2}(1 + \ln \rho^2) - \ln \rho^2 + \ln^2 \rho + 2\pi^2 - 4 \right],$$

Final resum form

$$\widetilde{W}_{UT}^{\alpha}(Q;b) = e^{-\mathcal{S}_{UT}(Q^2,b)}\widetilde{W}_{UT}^{\alpha}(C_1/b,b)$$
$$= (-ib_\perp^\alpha/2)\, e^{-\mathcal{S}_{UT}(Q^2,b)}\Sigma_{i,j}$$
$$\times \Delta C_{qi}^T \otimes f_{i/A}^{(3)}(z_1) C_{\bar{q}j} \otimes f_{j/B}(z_2)$$

- Sudakov the same

$$\mathcal{S}_{UT}(Q^2,b) = \int_{C_1^2/b^2}^{C_2^2Q^2} \frac{d\mu^2}{\mu^2}\left[\ln\left(\frac{C_2^2Q^2}{\mu^2}\right)A_{UT}(C_1;g(\mu)) + B_{UT}(C_1,C_2;g(\mu))\right] ,$$

Coefficients at one-loop order

$$A_{UT}^{(1)} = C_F,\ B_{UT}^{(1)} = -3/2C_F,\ \Delta C_{qq}^{T(0)} = \delta(1-x)\ ,$$
$$\Delta C_{qq}^{T(1)} = -\frac{1}{4N_c}(1-x) + \frac{C_F}{2}\delta(x-1)\left[\frac{\pi^2}{2} - 4\right]\ ,$$

- It will be important to apply this resummation formalism to study the energy dependence of the SSAs
  - Work in progress…