Scattering Amplitudes
Effective Field Theories & Soft Theorems

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Based on works with Freddy Cachazo & Ellis Yuan **PRD D92 (2015)**
also **PRL 113 (2014), JHEP 1407 033, PRD90 (2014), JHEP 1507 149**...

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Introduction

Revival of S-matrix program: new tools, results & structures

BCFW & unitarity method, NLO revolution (& NNLO) in QCD, all-loops in $\mathcal{N} = 4$ *super Yang-Mills, multi-loops in (super-)gravity, double-copy & black holes, new maths (geometries, polylogs...) ...*

New formulations of QFT: *twistor-string & scattering equations...*

Long history of effective field theories (EFT) (Goldstone bosons *etc.*): renewed interests from S-matrix and IR physics

Long history of soft theorems: renewed interests $\rightarrow$ new soft theorems for EFT's & extended theories from soft limits

- New formulation of S-matrix
- "Discover" EFTs: NLSM, DBI & more
- New soft theorems of EFTs

I

A New Formulation for

S-matrix in (massless) QFTs

Motivations

Witten's twistor string theory for $\mathcal{N} = 4$ SYM ('03): led to revolutions for S-matrix! Still very special $\Rightarrow$ natural questions

- general dimensions? no supersymmetry?
- general theories: Yang-Mills, gravity, standard model...?
- generalize to loop level?

One answer: CHY formulation [Cachazo, He, Yuan '13-]

- compact formulas for gravitons, gluons, scalars,...
- gauge invariance, soft theorems, double-copy etc. manifest
- loops: new rep of loop integrand [Geyer et al '14]

Deep connection to strings & twistor strings [Mason, Skinner '13; ...]

Scattering equations

Universal, independent of dim or theory: scattering equations

$$E_a := \sum_{\substack{b=1 \\ b\neq a}}^{n} \frac{s_{a\,b}}{\sigma_a - \sigma_b} = 0, \quad \text{for } a = 1, 2, \ldots, n\,.$$

Determine locations of n punctures in terms of n-pt kinematics

$n-3$ eqs for $n-3$ variables; non-trivially $(n-3)!$ solutions

key idea: auxiliary space that "knows" locality & unitarity

- kinematic space of n massless particles $\mathcal{K}_n$
- moduli space of n-punctured Riemann spheres $\mathcal{M}_{0,n}$
- the equations map singularities in $\mathcal{K}_n$ to those of $\mathcal{M}_{0,n}$

$\Rightarrow$ massless tree amps from solutions of the equations on $\mathcal{M}_{0,n}$

CHY formulation

Tree amps = contour integral in $\mathcal{M}_{n,0}$ = sum over solutions

$$\mathcal{M}_n = \underbrace{\oint \frac{d^n\sigma}{\text{vol SL}(2,\mathbb{C})} \prod_a{}' \delta(E_a)}_{d\mu_n} \mathcal{I}(\{k,\epsilon,\sigma\}) = \sum_{\{\sigma\}\in\text{solns.}} \frac{\mathcal{I}(\{k,\epsilon,\sigma\})}{J(\{\sigma\})}$$

- $d\mu_n$ contains $n-3$ integrals and $n-3$ delta functions
- J is the Jacobian from solving the equations: $J \sim |\frac{\partial E_a}{\partial \sigma_b}|$

"CHY integrand" $\mathcal{I}$ depends on the theory, determined by

- basic consistency: mass dimension, statistics,...
- importantly, gauge invariance and symmetries
- proof: factorization, soft limits (no spurious pole)

Simplest formula: bi-adjoint ϕ^3 theory

Inspired by Witten: "Parke-Taylor" factor (color part)

$$\mathrm{PT}[1,2,\cdots,n] := \frac{1}{\sigma_{1,2}\cdots\sigma_{n-1,n}\sigma_{n,1}}$$

$$\mathcal{C} := \sum_{\pi\in S_n/Z_n} \mathrm{Tr}(T^{I_{\pi(1)}}\cdots T^{I_{\pi(n)}})\mathrm{PT}[\pi]\,.$$

Simplest amplitudes: ϕ^3 theory with flavors in bi-adjoint of U(N)$\times$ U(N'): $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi^{II'})^2 + \frac{\lambda}{3!}f^{IJK}f^{I'J'K'}\phi_{II'}\phi_{JJ'}\phi_{KK'}$.

CHY formula for bi-adjoint ϕ^3 amplitudes: a sum of cubic scalar graphs with two copies of flavor factors

$$\mathcal{M}_n^{\phi^3} = \oint d\mu_n\,\mathcal{C}\mathcal{C}' = \sum_{\pi,\rho}\mathrm{Tr}(T^{I_{\pi_1}}\cdots T^{I_{\pi_n}})\,\mathrm{Tr}(T^{I'_{\rho_1}}\cdots T^{I'_{\rho_n}})m[\pi|\rho]$$

$$\text{double partial amplitudes :}\quad m[\pi|\rho] = \oint d\mu_n \mathrm{PT}[\pi]\,\mathrm{PT}[\rho]\,.$$

Yang-Mills: a new object

Need a building block to encode gluon polarizations:

- mass dim, permutation invariant, multi-linear in $\{\epsilon_a\}$
- most important: gauge invariance!

Introduce $2n \times 2n$ skew matrix Ψ, with four $n \times n$ blocks

$$\Psi := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

$$A_{a,b} := \begin{cases} \frac{k_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} := \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{a,b}} & a \neq b \\ 0 & a = b \end{cases},$$

$$C_{a,b} := \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{a,b}} & a \neq b \\ -\sum_{c \neq a} C_{a,c} & a = b \end{cases}$$

CHY formula for Yang–Mills

The building block should be pfaffian of Ψ (multilinear in ϵ's)

a subtlety: Ψ is degenerate $\Rightarrow$ reduced pfaffian: $\mathrm{Pf}'\Psi := \frac{\mathrm{Pf}|\Psi|^{i,j}_{i,j}}{\sigma_{i,j}}$

The other copy is the Parke-Taylor factor, or $\mathcal{C}$ for colors:

$$\mathcal{M}_n^{\mathrm{YM}}[\pi] = \oint d\mu_n \, \mathrm{PT}[\pi] \, \mathrm{Pf}' \, \Psi \Rightarrow \mathcal{M}_n^{\mathrm{YM}} = \oint d\mu_n \, \mathcal{C} \, \mathrm{Pf}'\Psi$$

Complete S-matrix for any number of gluons in any dimension

The origin of $\mathrm{Pf}'\Psi$: by scattering equations, it is exactly given by open-string correlators in the field-theory limit

$$\mathrm{Pf}'\Psi \sim \langle V^{(0)}(\sigma_1) \ldots V^{(-1)}(\sigma_i) \ldots V^{(-1)}(\sigma_j) \ldots V^{(0)}(\sigma_n) \rangle$$

Gauge invariance

Gauge invariance of gluons: $\epsilon_a^\mu \sim \epsilon_a^\mu + \alpha k_a^\mu$

$$\left(\begin{array}{cc|cc}
0 & \cdots & \sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots \\
\frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_2\cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{k_n\cdot k_1}{\sigma_{2,1}} & \cdots \\
-\sum_{b=2}^{n} \frac{k_1\cdot k_b}{\sigma_{1,b}} & \cdots & 0 & \cdots \\
\frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_2\cdot k_1}{\sigma_{2,1}} & \cdots \\
\vdots & & \vdots & \\
\frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots & \frac{\epsilon_n\cdot k_1}{\sigma_{2,1}} & \cdots
\end{array}\right)$$

Substituting $\epsilon_1 \to k_1$ $\mathrm{Pf}'\Psi = 0$ for each solution of scattering equations $\Longrightarrow$ gauge invariance manifest from CHY formula!

CHY formula for gravity

$\mathcal{C} \times \mathcal{C}' \Rightarrow$ bi-adjoint scalars, $\mathcal{C} \times \mathrm{Pf}'\Psi \Rightarrow$ Yang-Mills

How about gravity? no color, polarization tensor $h^{\mu\nu} = \epsilon^{\mu}\epsilon^{\nu}$

In general $\epsilon^{\mu}\epsilon'^{\nu}$ gives $h^{\mu\nu} + B^{\mu\nu} + \phi$; CHY formula for gravity

$$\mathcal{M}_n^{h+B+\phi} = \oint d\mu_n\, \mathrm{Pf}'\Psi(\epsilon)\, \mathrm{Pf}'\Psi(\epsilon') \longrightarrow \mathcal{M}_n^{\mathrm{GR}} = \oint d\mu_n\, \det{}'\, \Psi(\epsilon)$$

(linearized) diffeomorphism invariance manifest in CHY

Compete S-matrix of gravitons $\Rightarrow$ hidden simplicity of GR

Most natural way for double copy "GR = YM$^2/\phi^3$" ($\Longrightarrow$ [KLT '86, BCJ' 08]); "Direct Product" of theories: GR=YM $\otimes$ YM.

II

“Discover” new EFTs in CHY

More theories in CHY

We can generate new formulas from old ones, e.g. compactify $R^{d+m} \to R^d$ with $K = (k^{(d)} \mid 0)$, $\mathcal{E} = (\epsilon^{(d)} \mid 0)$ or $(0 \mid e^{(m)})$:

YM $\to$ Yang-Mills-Scalar or GR $\to$ Einstein-Maxwell,

$$\mathrm{Pf}'\Psi(K,\mathcal{E}) \to \mathrm{Pf}'[\Psi](k,\epsilon_g)\,\mathrm{Pf}[X]_s\,, \quad X_{a\,b} := \frac{\delta^{I_a\,I_b}}{\sigma_a - \sigma_b}(1-\delta_{a\,b})\,,$$

Pure-photon (scalar) amps in EM (YMS) particularly simple:

$$\mathcal{M}^{\mathrm{EM}}_{\gamma^n} = \oint d\mu_n\,\mathrm{Pf}'A\,\mathrm{Pf}X\,\mathrm{Pf}'\Psi(\tilde{\epsilon})\,, \quad \mathcal{M}^{\mathrm{YMS}}_{s^n} = \oint d\mu_n\,\mathrm{Pf}'A\,\mathrm{Pf}X\,\mathcal{C}\,.$$

New operations: amps in Einstein-Yang-Mills theory *etc.*
"Direct Sum" of theories: *e.g.* EYM=EM $\oplus$ YM

EFTs in CHY

New formulas based on $\mathrm{Pf}'A \to$ EFT's with NG scalars

$\oint d\mu_n\, (\mathrm{Pf}'A)^2\, \mathcal{C}$? U(N)-flavored scalars with two derivatives?

Non-linear sigma model (NLSM), $\mathcal{L} = \mathrm{Tr}(\partial_\mu U^+ \partial^\mu U)$!

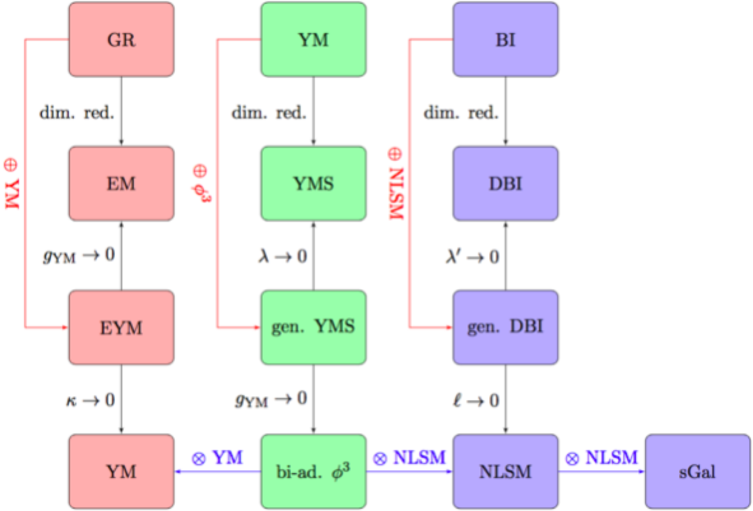
$\int d\mu_n\, (\mathrm{Pf}'A)^2\, \mathrm{Pf}'\Psi(\tilde{\epsilon})$? photons with higher derivatives

Born-Infeld theory, $\mathcal{L} = \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})}$!

Compactify $\to$ DBI: $\mathcal{M}_n^{\mathrm{scalar-DBI}} = \int d\mu_n\, (\mathrm{Pf}'A)^3\, \mathrm{Pf}X$.

The strangest is a special Galileon theory (a scalar theory with many derivatives) [Cheung et al '14,...], $\mathcal{M}_n^{\mathrm{sGal}} = \int d\mu_n (\mathrm{Pf}'A)^4$.

An (incomplete) landscape



EFTs in CHY

New double-copy: DBI=NLSM $\otimes$ YMS sGal=NLSM $\otimes$ NLSM

What is special about these EFTs: Goldstone bosons with enhanced "Adler's zero"! [Cheung et al '14; CHY '14]

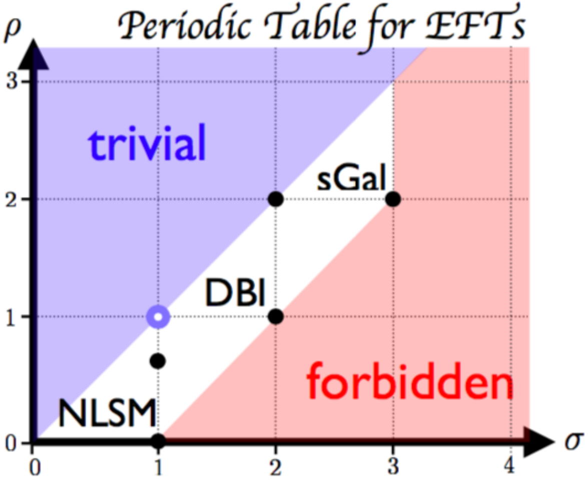
For NLSM, scalar DBI, sGal $\mathcal{M}_n \sim (\mathrm{Pf}'A)^2, (\mathrm{Pf}'A)^3, (\mathrm{Pf}'A)^4$, with soft emission $p^\mu \sim \tau$, $\mathcal{M}_n \sim \tau^\sigma$ for $\sigma = 1, 2, 3$!

Exceptional EFTs [Cheung et al]: softest behavior given the power counting order in derivatives. In fact, (enhanced) Adler's zero completely fix these theories! *e.g.* only possible $\sigma = 2, 3$ theory with a single scalar field is DBI & sGal.

Classify all scalar EFTs by σ and a power counting parameter.

Periodic Table of EFTs

Scalar EFT: $\mathcal{L} = (\partial\phi)^2 \sum_{m,n=0}^{\infty} \lambda_{m,n} \partial^m \phi^n$, $\rho = m/n$ characterizes power counting in derivatives (*e.g.* $\rho = 0, 1$ for free and NG fields). All theories can be classified by ρ and σ [Cheung et al] .



III

New Soft Theorems of EFT's

Soft theorems: gravitons

Soft theorems for gravitons & gauge bosons [Weinberg; Low, ...; Strominger *et al*]. Any amplitudes with an additional graviton:

$$\mathcal{M}_{n+1} = (S^{(0)}_{\text{gravity}} + S^{(1)}_{\text{gravity}} + S^{(2)}_{\text{gravity}})\mathcal{M}_n + \mathcal{O}(\tau^2)$$

in the soft limit $p^\mu = \tau p^\mu$; Weinberg's universal soft factor

$$S^{(0)}_{\text{gravity}} = \sum_{a=1}^{n} \frac{\epsilon_{\mu\nu} k_a^\mu \, k_a^\nu}{p \cdot k_a} \quad \Longrightarrow \quad \text{Equivalence Principle}$$

New: sub-leading soft theorems (proven using CHY *etc.*)

$$S^{(1)}_{\text{gravity}} = \sum_{a=1}^{n} \frac{\epsilon_{\mu\nu} k_a^\mu (p_\rho J_a^{\rho\nu})}{p \cdot k_a}, \quad S^{(2)}_{\text{gravity}} = \frac{1}{2} \sum_{a=1}^{n} \frac{\epsilon_{\mu\nu} (p_\rho J_a^{\rho\mu})(p_\sigma J_a^{\sigma\nu})}{p \cdot k_a}.$$

with $J_a^{\mu\nu}$ a-th particle's total angular momentum, *e.g.* for scalar

$$J^{\mu\nu}_{a,\text{scalar}} \equiv k_a^\mu \frac{\partial}{\partial k_{a,\nu}} - k_a^\nu \frac{\partial}{\partial k_{a,\mu}}.$$

Soft theorems: gluons

Similar to soft photon in QED, in Yang-Mills theory the emission of a soft gluon for color-ordered partial amplitude

$$\mathcal{M}(1,2,\ldots,n,n+1) = (S^{(0)}_{\rm YM} + S^{(1)}_{\rm YM})\mathcal{M}(1,2,\ldots,n) + \mathcal{O}(\tau),$$

where the universal leading and sub-leading soft factors

$$S^{(0)}_{\rm YM} = \frac{\epsilon\cdot k_n}{p\cdot k_n} - \frac{\epsilon\cdot k_1}{p\cdot k_1}, \qquad S^{(1)}_{\rm YM} = \frac{\epsilon_\mu(p_\rho J_n^{\mu\rho})}{p\cdot k_n} - \frac{\epsilon_\mu(p_\rho J_1^{\mu\rho})}{p\cdot k_1}.$$

Q: EFTs? single soft vanishes, double: $p^\mu = \tau p^\mu$, $q^\mu = \tau q^\mu$

In [CHY 15], double soft theorems discovered for many EFTs:
(I): EMS, DBI, & sGal; a single scalar field
(II): YMS, NLSM: scalars under $U(N)$ color/flavor structure

similar to graviton & gluons : $\quad \epsilon^\mu_{\rm soft} \mapsto p^\mu - q^\mu, \quad p^\mu_{\rm soft} \mapsto p^\mu + q^\mu$

Double soft theorems I

Class (I) similar to soft graviton: with $p^\mu = \tau p^\mu$, $q^\mu = \tau q^\mu$, a $(n+2)$-particle amplitude in these theories behaves as

$$\mathcal{M}_{n+2} = (p \cdot q)^m (S^{(0)} + S^{(1)} + S^{(2)}) \mathcal{M}_n + \mathcal{O}(\tau^{2m+4})$$

where $m = 1, 0, -1$ for sGal, DBI and EMS respectively, and

$$S^{(0)} = \frac{1}{4} \sum_{a=1}^{n} \left(\frac{(k_a \cdot (p-q))^2}{k_a \cdot (p+q) + p \cdot q} + k_a \cdot (p+q) + p \cdot q \right),$$

$$S^{(1)} = \frac{1}{2} \sum_{a=1}^{n} \frac{k_a \cdot (p-q)}{k_a \cdot (p+q) + p \cdot q} (p_{mu} q_\nu J_a^{\mu\nu}),$$

$$S^{(2)} = \frac{1}{2} \sum_{a=1}^{n} \frac{1}{k_a \cdot (p+q) + p \cdot q} \left((p_\mu q_\nu J_a^{\mu\nu})^2 + (\frac{3}{2} - 2m)(p \cdot q)^2 \right),$$

$\sim \mathcal{O}(\tau^{1,2,3})$; only multiplicative part of $S^{(2)}$ depends on theory.

Double soft theorems II

$$\mathcal{M}(1,2,\ldots,n,p,q) = (p\cdot q)^m (S^{(0)}+S^{(1)})\mathcal{M}(1,2,\ldots,n) + \mathcal{O}(\tau^{2m+2})$$

with $m = 0, -1$ for NLSM and YMS respectively, and

$$S^{(0)} = \frac{1}{2}\left(\frac{k_n\cdot(p-q)+p\cdot q}{k_n\cdot(p+q)+p\cdot q} + \frac{k_1\cdot(q-p)+q\cdot p}{k_1\cdot(q+p)+q\cdot p}\right),$$

$$S^{(1)} = \frac{p_\mu q_\nu}{k_n\cdot(p+q)+p\cdot q} J_n^{\mu\nu} + \frac{q_\mu p_\nu}{k_1\cdot(q+p)+q\cdot p} J_1^{\mu\nu},$$

$\sim \mathcal{O}(\tau^{0,1})$. In both cases $S^{(0)}$ itself takes very suggestive form:

$$S_I^{(0)} = \frac{1}{4}\sum_{a=1}^{n} \frac{(k_a\cdot(p-q))^2}{k_a\cdot(p+q)} + \mathcal{O}(\tau^2),$$

$$S_{II}^{(0)} = \frac{1}{2}\left(\frac{k_n\cdot(p-q)}{k_n\cdot(p+q)} + \frac{k_1\cdot(q-p)}{k_1\cdot(q+p)}\right) + \mathcal{O}(\tau).$$

Also new double-soft photon theorem: only leading soft factor

$$\mathcal{M}_{n+2} = (p\cdot q)^m \frac{1}{4}\sum_{a=1}^{n} \frac{(k_a\cdot(p-q))^2}{k_a\cdot(p+q)}\,(\epsilon_p\cdot\epsilon_q+\cdots)\,\mathcal{M}_n+\mathcal{O}(\tau^{2m+2})\,,$$

where $m = 0, -1$ for DBI and EMS respectively. In 4d, generalize to susy soft theorems for super-YM and super-DBI.

All double soft theorems reveal non-linearly realized symmetries of the theory, as already hinted by Adler's zero.

e.g. double soft in EMS can be extended to that of $\mathcal{N} = 8$ supergravity, which reveals $E_{7(7)}$ symmetry [Arkani-Hamed et al].

For NLSM, fully understood from Ward id. of shift symmetry [I. Low et al]. What about DBI, sGal *etc.*?

Extended theories from soft limits

Again using CHY [Cachazo et al], mysterious theories discovered underlying sub-leading order of Adler's zero! The idea is:

$$\mathcal{M}_n^{\text{theory}_1} \xrightarrow{\text{soft limit}} \tau^p \, \mathcal{M}_{n-1}^{\text{theory}_1 \oplus \text{theory}_2} + \mathcal{O}(\tau^{p+1}), \quad p > 0.$$

Basic example: single soft limit of NLSM given by (mixed) amplitudes of an extended theory, *i.e.* NLSM $\oplus\ \phi^3$

$$\mathcal{M}_{n+1}^{\text{NLSM}} = \tau \sum_{a=2}^{n-1} s_{p,a} \mathcal{M}_n^{\text{NLSM}\oplus\phi^3}(n,a,1) + \mathcal{O}(\tau^2),$$

which has a nice CHY formula (factorize correctly!)

$$\mathcal{M}_n^{\text{NLSM}\oplus\phi^3}(\alpha|\beta) = \oint d\mu_n \left(\text{PT}_n(\alpha)\right) \left(\text{PT}(\beta)\,(\text{Pf}A_{\bar{\beta}})^2\right).$$

Single soft limit of BI given by amplitudes in BI $\oplus$ YM theory:

$$\mathcal{M}_{n+1}^{\mathrm{BI}} = \tau \sum_a \sum_b s_{p,a} s_{p,b} \left(\frac{\epsilon_p \cdot k_n}{p \cdot k_n} - \frac{\epsilon_p \cdot k_a}{p \cdot k_a} \right) \mathcal{M}_n^{\mathrm{BI}\oplus\mathrm{YM}}(a, b, 1) + \mathcal{O}(\tau^2).$$

which is given by a CHY formula (photons + gluons)

$$\mathcal{M}_n^{\mathrm{BI}\oplus\mathrm{YM}}(\alpha) = \oint d\mu_n \left(\mathrm{PT}(\alpha) \left(\mathrm{Pf}\, A_{\bar{\alpha}} \right)^2 \right) \left(\mathrm{Pf}' \Psi_n \right).$$

What is the Lagrangian? Symmetries of the theory? More mysterious when uplifted to $\mathcal{N} = 4$ super-DBI $\oplus$ SYM.

Even more exotic: the single soft limit of sGal is given by amplitudes in sGal $\oplus$ NLSM^2 $\oplus$ bi-adjoint ϕ^3!

Outlook

New picture: massless particles scattering via punctures on a sphere. Suggest a weak-weak duality for S-matrix in massless QFTs?

Web of theories connected by e.g. $\oplus$ (interaction) & $\otimes$ (double-copy)

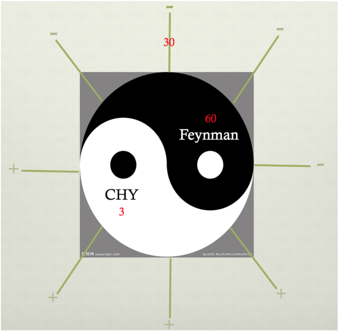
Naturally lead to exceptional EFTs, new single & double soft theorems; symmetry foundation of “soft bootstrap”?

QCD, Higgs, form factor? Scope of QFTs natural in CHY?

Loops: integrands $\to$ CHY for integrated amplitudes?

Origin: strings vs. twistor strings, path-integral derivation???

Thank you!



taken from C.S. Lam's talk