Contractions of 1-loop 5-point tensor Feynman integrals

Tord Riemann   DESY, Zeuthen, Germany

in cooperation with **J. Fleischer** (1937-2013)**, J. Gluza, A. Almasy**

Talk held at

ACAT2013 – 15th Int. WS on Advanced Computing and Analysis Techniques in Physics Research

May 16 – 21, 2013, IHEP, Beijing, China

http://indico.ihep.ac.cn/conferenceOtherViews.py?view=standard&confId=2813

Introduction

Recent more detailed overviews of our approach to tensor reduction:

J. Fleischer, at:
"Frontiers in Perturbative Quantum Field Theory"
10-12 September 2012, Bielefeld University, Germany
http://www2.physik.uni-bielefeld.de/fileadmin/user_upload/workshops/fleischer.pdf

T. Riemann, at:
4th Workshop "HP2 – High Precision for Hard Processes"
4-7 September 2012, Max Planck Institute for Physics, Munich, Germany
https://indico.mpp.mpg.de/contributionDisplay.py?contribId=27&confId=1369

T. Riemann, at:
5th Helmholtz International Summer School - Workshop
Dubna International Advanced School of Theoretical Physics - DIAS TH
Calculations for Modern and Future Colliders
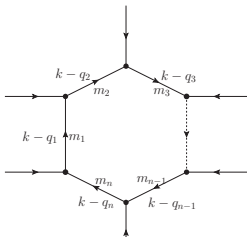July 23 - August 2, 2012, Dubna, Russia
http://theor.jinr.ru/ calc2012/

Definitions

n-point tensor integrals of rank R: (n,R)-integrals

$$I_n^{\mu_1 \cdots \mu_R} = \int \frac{d^d k}{i \pi^{d/2}} \; \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j^{\nu_j}},$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon$$

tensor integrals due to, e.g.:

- fermion propagators
- three-gauge boson couplings

A simple example

1-loop self-energy:

$$I_2^\mu = \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2][(k+p)^2 - M_2^2]}$$

Ansatz : $\quad I_2^\mu = p_\mu \cdot B_1(p, M_1, M_2)$

Solve:

$$
\begin{aligned}
p_\mu \cdot I_2^\mu &= p^2 \cdot B_1(p, M_1, M_2) \\
&= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2][(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1 \, D_2} \\
&= \int \frac{d^d k}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1 \, D_2} \right], \\
B_1(p, M_1, M_2) &= \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]
\end{aligned}
$$

A tensor Feynman integral may be expressed in terms of scalar Feynman integrals.

Passarino-Veltman algorithm

1 Contract n-point and R-rank Feynman integral with *external momenta p_i^μ* and with $g^{\mu\nu}$, and cancel propagators

2 Invert the resulting system of linear equations

3 The result consists of $(n-1)$-point and $(R-1)$-rank functions

Reducing tensor rank introduces inverse Gram determinant:

$$I_5^{\mu_1\cdots\mu_{R-1}\mu_R} = \sum_{i=1}^{5} \frac{q_i^{\mu_R}}{\det(G_5)} \left[A_{0i}\, I_5^{\mu_1\cdots\mu_{R-1}} - \sum_{s=1}^{5} A_{si}\, I_4^{\mu_1\cdots\mu_{R-1},s} \right]$$

Gram determinant G_n:

$$G_n = |2q_i q_j|, \; i,j = 1,\ldots n-1 \tag{1}$$

and A_{0i}, A_{si} are kinematic coefficients. The q_i are **internal** momenta.

Systematic approach to tensor reductions:

1,2,3,4-point functions:
- Passarino, Veltman 1978 [1]

~~Open source~~ Source-open programs for 5,6-point reductions:

- LoopTools/FF ($n \leq 5$, $rank \leq 4$), T. Hahn [2, 3] 1998,1990.

- Golem95 T. Binoth et al. [4] 2008

- PJFry ($n \leq 5$, $rank \leq 5$), V. Yundin et al. [5, 6] July 2011

Need in addition a library of scalar functions:

- 't Hooft, Veltman 1979 [7]

- LoopTools/FF T. Hahn [2, 3] 1998,1990

- QCDloop/FF K. Ellis and G. Zanderighi [8, 3] 2007,1990

- OneLOop (complex masses) van Hameren [9] 2010

This talk: Efficient reduction formulae in the algebraic Davydychev-Tarasov-Fleischer-Jegerlehner-TR approach

- Get $n > 4$ tensor reduction with $\cdots$:
- $\cdots$ arbitrary masses
- $\cdots$ inverse pentagon Gram determinants killed
- $\cdots$ full kinematics treated, also with small inverse sub-diagram Gram determinants

- **new:** $\cdots$ multiple sums over tensor coefficients made efficient by **contracting with external momenta**

  Fleischer, TR [10] PLB 701(2011)646 + further simplifications

  Programs:
  OLEC (C++), J. Fleischer, J. Gluza, M. Gluza, TR [11]
  CONTRACTIONS (F95), Andrea Almasy, J. Fleischer, TR [12]

- $\cdots$ higher n point functions, $n \geq 7$ Fleischer, TR [13] PLB 707(2012)375
  Programs: to be done

History of the Approach - not a complete list of references

[14] Melrose 1965: Reduction of Feynman diagrams and Cayley determinants

[15] Davydychev 1991: Integrals in different space-time dimension.

[16] See also Bern et al. 1993

[17] Tarasov 1996: Dimensional recurrence relations

[18] Fleischer,Jegerlehner,Tarasov 2000: 1-loop reductons and signed minors.

[4] Binoth,Guillet,Heinrich,Pilon,Schubert, 2005: Algebraic/numerical formalism for one-loop multi-leg amplitudes

[19] Fleischer and T.Riemann (since 2007) 2011: Complete reduction of 1-loop tensors.

See also Diakonidis et al. [20] 2009 and [21] 2009

[22] Yundin's package PJFry 2011; https://github.com/Vayu/PJFry.

See also Fleischer, TR, Yundin [5, 6]

[10] Fleischer and T.Riemann 2011: Contracted tensor Feynman integrals.

[23] Fleischer and T.Riemann 2012: A solution for tensor reduction of one-loop n-point functions with $n \geq 6$

Tensor integrals expressed in terms of scalar integrals in higher dimensions

$D = d + 2l = 4 - 2\epsilon,\ 6 - 2\epsilon, \cdots$ [Davydychev:1991] , also [Fleischer et al.:2000]

$n_{ij} = \nu_{ij} = 1 + \delta_{ij},\ n_{ijk} = \nu_{ij}\nu_{ijk},\ \nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$

$$I_n^{\mu} = \int^d k^{\mu} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i=1}^{n} q_i^{\mu}\, I_{n,i}^{[d+]}$$

$$I_n^{\mu\,\nu} = \int^d k^{\mu} k^{\nu} \prod_{r=1}^{n} c_r^{-1} = \sum_{i,j=1}^{n} q_i^{\mu} q_j^{\nu}\, n_{ij}\, I_{n,ij}^{[d+]^2} - \frac{1}{2}\, g^{\mu\nu}\, I_n^{[d+]}$$

$$I_n^{\mu\,\nu\,\lambda} = \int^d k^{\mu} k^{\nu} k^{\lambda} \prod_{r=1}^{n} c_r^{-1} = -\sum_{i,j,k=1}^{n} q_i^{\mu} q_j^{\nu} q_k^{\lambda}\, n_{ijk}\, I_{n,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^{n} g^{[\mu\nu} q_i^{\lambda]}\, I_{n,i}^{[d+]^2}$$

$$
\begin{aligned}
I_n^{\mu\,\nu\,\lambda\,\rho} &= \int \frac{d^d k}{i\pi^{d/2}}\, \frac{k^{\mu} k^{\nu} k^{\lambda} k^{\rho}}{\prod_{r=1}^{n} c_r} = \sum_{i,j,k,l=1}^{n} q_i^{\mu} q_j^{\nu} q_k^{\lambda} q_l^{\rho}\, n_{ijkl}\, I_{n,ijkl}^{[d+]^4} \\
&\quad - \frac{1}{2} \sum_{i,j=1}^{n} g^{[\mu\nu} q_i^{\lambda} q_j^{\rho]}\, n_{ij} I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{[\mu\nu} g^{\lambda\rho]} I_n^{[d+]^2} \qquad (2)
\end{aligned}
$$

$$I_n^{\mu\nu\lambda\rho\sigma} = \int \frac{d^dk}{i\pi^{d/2}} \frac{k^\mu k^\nu k^\lambda k^\rho k^\sigma}{\prod_{j=1}^n c_j}$$

$$= -\sum_{i,j,k,l,m=1}^n q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma n_{ijklm} I_{n,ijklm}^{[d+]^5}$$

$$+ \frac{1}{2}\sum_{i,j,k=1}^n g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} n_{ijk} I_{n,ijk}^{[d+]^4} - \frac{1}{4}\sum_{i=1}^n g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} I_{n,i}^{[d+]^3}.$$

The integrals are defined in $[d+]^l = 4 - 2\epsilon + 2l$ dimensions.

$$I_{n-1,ab}^{\{\mu_1,\cdots\},s}, \ a,b \neq s \text{ is obtained from } I_n^{\{\mu_1,\cdots\}}$$

by

- shrinking line s
- raising the powers of inverse propagators a, b.

Notations: Gram and modified Cayley determinant, signed minors

[Melrose:1965]

Gram determinant G_n:

$$G_n \quad = \quad |2q_i q_j|, i, j = 1, \ldots n-1 \tag{3}$$

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & Y_{11} & Y_{12} & \ldots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{vmatrix} \tag{4}$$

with the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \ldots N) \tag{5}$$

The propagators are: $c_i = (k - q_i)^2 - m_i^2$

For the choice $q_n = 0$, both determinants are related:

$$()_N = -G_N$$

$\Rightarrow$ The modfied Cayley determinant $()_N$ does not depend on masses.

Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\left(\begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array} \right)_N \equiv$$

$$\equiv (-1)^{\sum_l (j_l + k_l)} \text{sgn}_{\{j\}} \, \text{sgn}_{\{k\}} \left| \begin{array}{l} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \qquad (6)$$

where $\text{sgn}_{\{j\}}$ and $\text{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\left(\begin{array}{c} 0 \\ 0 \end{array} \right)_N \equiv \left| \begin{array}{cccc} Y_{11} & Y_{12} & \ldots & Y_{1N} \\ Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{array} \right|, \qquad (7)$$

Tarasov's dimensional recurrences for scalars

Following [Tarasov:1996 [17], Fleischer:1999 [18]]
apply recurrence relations, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

shift dimension + index:

$$\nu_j(\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{()_5} \left[-\binom{j}{0}_5 + \sum_{k=1}^{5} \binom{j}{k}_5 \mathbf{k}^- \right] I_5 \tag{8}$$

shift dimension:

$$(d - \sum_{i=1}^{5} \nu_i + 1) I_5^{[d+]} = \frac{1}{()_5} \left[\binom{0}{0}_5 - \sum_{k=1}^{5} \binom{0}{k}_5 \mathbf{k}^- \right] I_5, \tag{9}$$

shift index:

$$\nu_j \mathbf{j}^+ I_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^{5} \binom{0j}{0k}_5 \left[d - \sum_{i=1}^{5} \nu_i(\mathbf{k}^- \mathbf{i}^+ + 1) \right] I_5 \tag{10}$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

Recursions for tensors – Alternative to dimensional recurrences of scalars

Express any $(5, R)$ pentagon by a $(5, R - 1)$ pentagon plus $(4, R - 1)$ boxes

[Diakonidis,Fleischer,T.Riemann,Tausk: Phys.Lett. **B683** (2010) [21]]

5-point tensor recursion:

$$I_5^{\mu_1 \cdots \mu_{R-1} \mu} = I_5^{\mu_1 \cdots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1 \cdots \mu_{R-1}, s} Q_s^\mu,$$

For $n = 6, 7, 8, \cdots$ things are close but differ a bit; see later.

auxiliary vectors with inverse Gram determinants

$$Q_s^\mu = \sum_{i=1}^{5} q_i^\mu \frac{\binom{s}{i}_5}{()_5}, \quad s = 0, \ldots, 5$$

For e.g. $R = 3$, again $[1/()_5]^3$ will occur.

Contractions

One may combine now *Tarasov's dimensional recurrence relations* and the *tensor rank recursions* in order to derive especially useful representations.

After that, we will perform contractions with external momenta.
So, the following equations are symbolic:

$$q_{i_1\mu_1}\cdots q_{i_R\mu_R}\ I_5^{\mu_1\cdots\mu_R} = \int \frac{d^d k}{i\pi^{d/2}}\ \frac{\prod_{r=1}^R (q_{i_r}\cdot k)}{\prod_{j=1}^5 c_j},$$

$$g_{\mu_1,\mu_2}q_{i_1\mu_3}\cdots q_{i_R\mu_R}\ I_5^{\mu_1\cdots\mu_R} \neq \int \frac{k^2 d^d k}{i\pi^{d/2}}\ \frac{\prod_{r=3}^R (q_{i_r}\cdot k)}{\prod_{j=1}^5 c_j}$$

One may arrange a one-loop calculation such that all the one-loop integrals appear only in such contractions.
Important:
The contraction with g_{μ_1,μ_2} is shown here in a symbolic form; in practice we work strictly 4-dimensional with g_{μ_1,μ_2}.

One option was to avoid the appearance of inverse Gram determinants $1/()_5$.

For rank $R = 5$, e.g.:

$$I_5^{\mu\,\nu\,\lambda\rho\sigma} = \sum_{s=1}^{5}\left[\sum_{i,j,k,l,m=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma E_{ijklm}^s + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} E_{00ijk}^s\right.$$
$$\left. + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^s\right] \tag{11}$$

The tensor coefficients are expressed in terms of integrals $I_{4,i\cdots}^{[d+]^l,s}$, e.g.:

$$
\begin{aligned}
E_{ijklm}^s \;=\; & -\frac{1}{\binom{0}{0}_5} \left\{ \left[\begin{pmatrix} 0l \\ sm \end{pmatrix}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l) \right] \right. \\
& \left. + \begin{pmatrix} 0s \\ 0m \end{pmatrix}_5 n_{ijkl} I_{4,ijkl}^{[d+]^4,s} \right\}.
\end{aligned}
$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $()_4$.

Further, the complete dependence on the indices i of the tensor coefficients is contained now in the pre-factors with signed minors.

One can say that **the indices *decouple* from the integrals**.

As an example, we reproduce the 4-point part of $I_{4,ijkl}^{[d+]^4}$:

$$n_{ijkl} I_{4,ijkl}^{[d+]^4} = \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} \frac{\binom{0}{k}}{\binom{0}{0}} \frac{\binom{0}{l}}{\binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]^4}$$

$$+ \frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0i}{0k}\binom{0}{j}\binom{0}{l} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0j}{0l}\binom{0}{i}\binom{0}{k} + \binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3}$$

$$\times\ d(d+1) I_4^{[d+]^3}$$

$$+ \frac{\binom{0i}{0l}\binom{0j}{0k} + \binom{0j}{0l}\binom{0i}{0k} + \binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]^2} + \cdots \qquad (12)$$

In (12), one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are $\binom{0}{k} \to \binom{0s}{ks}_5$ etc.

✓ no scalar 5-point integrals in higher dimensions
✓ no inverse Gram det. $()_5$
✓ 4-point integrals without indices
† scalar 4-point integrals in higher dimensions: $I_4^{[d+]^2,s}$ etc.
† inverse Gram det. $\binom{0}{0}_5 \equiv ()_4$

Contractions for the 5-point functions with rank $R = 1$

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + \cdots + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ is ([19], eq. (4.6)):

$$I_5^\mu = -\sum_{i=1}^5 q_i^\mu I_{5,i}^{[d+]} \tag{13}$$

$$= -\sum_{i=1}^4 q_i^\mu \sum_{s=1}^5 \frac{\binom{0i}{0s}_5}{\binom{0}{0}_5} I_4^s \tag{14}$$

This yields, when contracted with a chord,

$$q_{a\mu} I_5^\mu = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left[\sum_{i=1}^4 (q_a \cdot q_i) \binom{0i}{0s}_5 \right] I_4^s. \tag{15}$$

In fact, the sum over i may be performed explicitly:

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0s \\ 0i \end{pmatrix}_5 = +\frac{1}{2} \left\{ \begin{pmatrix} s \\ 0 \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\},$$

We get immediately

$$q_{a\mu} I_5^{\mu} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \Sigma_a^{1,s} \; I_4^s. \tag{16}$$

Contractions for the 5-point functions with rank $R = 2$

$$I_5^{\mu\nu} = \sum_{i,j=1}^{4} q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \tag{17}$$

has the following tensor coefficients free of $1/()_5$:

$$E_{00} = -\sum_{s=1}^{5} \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \tag{18}$$

$$E_{ij} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \tag{19}$$

Equation (17) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \tag{20}$$

and finally (20) simply reads

$$
\begin{aligned}
q_{a\mu} q_{b\nu} I_5^{\mu\,\nu} &= \frac{1}{4} \sum_{s=1}^{5} \Bigg\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab}\delta_{as} + \delta_{5s}) + \frac{\binom{s}{s}_5}{\binom{0s}{0s}_5} \Big[(\delta_{as} - \delta_{5s})(Y_{b5} - Y_{55}) \\
&\quad + (\delta_{bs} - \delta_{5s})(Y_{a5} - Y_{55}) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} (Y_{a5} - Y_{55})(Y_{b5} - Y_{55}) \Big] \Bigg\} I_4^{[d+],s} \\
&\quad + \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\Sigma_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \Sigma_a^{2,st} I_3^{st},
\end{aligned}
$$

with

$$\Sigma_a^{2,st} \equiv \sum_{i=1}^{4} (q_a \cdot q_i) \begin{pmatrix} 0st \\ 0si \end{pmatrix}_5$$

$$= \frac{1}{2}(1 - \delta_{st}) \left\{ \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 (\delta_{at} - \delta_{5t}) - \begin{pmatrix} 0s \\ 0t \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right.$$

This has been extended also to higher ranks.
We need at most double sums, e.g.:

$$\Sigma_{ab}^{2,s} \equiv \sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_5$$

$$= \frac{1}{2}(q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_5 - \frac{1}{4}()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \qquad (21)$$

The sums over signed minors, weighted with scalar products of chords are given in J. Fleischer, T.R., PLB 2011 [10].

Contractions for the 5-point functions with rank $R = 4$

$$I_5^{\mu\nu\lambda\rho} = I_5^{\mu\nu\lambda} \cdot Q_0^\rho - \sum_{s=1}^{5} I_4^{\mu\nu\lambda,s} \cdot Q_s^\rho. \tag{22}$$

Contracted with chords (differences of external momenta):

$$q_a^\mu q_b^\nu q_c^\lambda q_d^\rho \, I_5^{\mu\nu\lambda\rho} = q_a^\mu q_b^\nu q_c^\lambda q_d^\rho \, I_5^{\mu\nu\lambda} \, Q_0^\rho - C_{5,abcd} \tag{23}$$

Here:

$$Q_\sigma^\mu \;\; = \;\; \sum_{i=1}^{5} q_i^\mu \, \frac{\binom{\sigma}{i}_5}{()_5}, \;\; \sigma = 0 \cdots 5 \tag{24}$$

The first term $q_a^\mu q_b^\nu q_c^\lambda \, I_5^{\mu\nu\lambda}$ is known, the second term has to be determined:

$$C_{5,abcd} = -\sum_{s=1}^{5} q_{a\mu} q_{b\nu} q_{c\lambda} I_4^{\mu\nu\lambda,s} \frac{1}{2} \left(\delta_{ds} - \delta_{5s}\right), \tag{25}$$

becomes:

$$
\begin{aligned}
C_{5,abcd} &= \frac{1}{16} \left\{ G^5 + \delta_{ab}\delta_{ac}\delta_{ad}G^d - I_1^{5abc} - I_1^{5abd} - I_1^{5acd} - I_1^{5bcd} + I_1^{abcd} - J_3^{a5} - J_3^{b5} - J_3^{c5} - J_3^{d5} \right. \\
&+ R^{5ab} + R^{5ac} + R^{5bc} + R^{5da} + R^{5db} + R^{5dc} + \delta_{bc}\delta_{bd}\left(J_3^{ad} - J_3^{5d}\right) + \delta_{ac}\delta_{ad}\left(J_3^{bd} - J_3^{5d}\right) \\
&+ \delta_{ab}\delta_{ad}\left(J_3^{cd} - J_3^{5d}\right) + \delta_{ab}\delta_{ac}\left(J_3^{dc} - J_3^{5c}\right) + \delta_{ab}\delta_{cd}\tilde{J}_3^{db} + \delta_{ad}\delta_{bc}\tilde{J}_3^{dc} + \delta_{ac}\delta_{bd}\tilde{J}_3^{dc} \\
&+ \delta_{ab}\left(\tilde{J}_3^{5b} - R^{b5c} - R^{bd5} + R^{bdc}\right) + \delta_{ac}\left(\tilde{J}_3^{5c} - R^{c5b} - R^{cd5} + R^{cdb}\right) \\
&+ \delta_{ad}\left(\tilde{J}_3^{d5} - R^{d5b} - R^{d5c} + R^{dbc}\right) + \delta_{bc}\left(\tilde{J}_3^{5c} - R^{c5a} - R^{cd5} + R^{cda}\right) \\
&+ \delta_{bd}\left(\tilde{J}_3^{d5} - R^{d5a} - R^{d5c} + R^{dac}\right) + \delta_{cd}\left(\tilde{J}_3^{d5} - R^{d5a} - R^{d5b} + R^{dab}\right) \\
&+ \delta_{ab}\delta_{ad}J_4^d Y_c + \delta_{ac}\delta_{ad}J_4^d Y_b + \delta_{bc}\delta_{bd}J_4^d Y_a + \delta_{ab}\left(R^{bd} - R^{b5}\right)Y_c + \delta_{ac}\left(R^{cd} - R^{c5}\right)Y_b \\
&+ \delta_{ad}\left(R^{dc} - R^{d5}\right)Y_b + \delta_{ad}\left(R^{db} - R^{d5}\right)Y_c + \delta_{bc}\left(R^{cd} - R^{c5}\right)Y_a + \delta_{bd}\left(R^{dc} - R^{d5}\right)Y_a \\
&+ \delta_{bd}\left(R^{da} - R^{d5}\right)Y_c + \delta_{cd}\left(R^{da} - R^{d5}\right)Y_b + \delta_{cd}\left(R^{db} - R^{d5}\right)Y_a + \left(I_4^d - I_4^5\right)Y_a Y_b Y_c \\
&+ \left(I_3^{cd} - I_3^{5c} - I_3^{5d} + R^5 + \delta_{cd}R^d\right)Y_a Y_b + \left(I_3^{bd} - I_3^{5b} - I_3^{5d} + R^5 + \delta_{bd}R^d\right)Y_a Y_c \\
&+ \left(I_3^{ad} - I_3^{5a} - I_3^{5d} + R^5 + \delta_{ad}R^d\right)Y_b Y_c + \left(I_2^{bcd} - I_2^{5bc} - I_2^{5bd} - I_2^{5cd} - J_4^5 + R^{5b} + R^{5c} + R^{5d}\right)Y_a \\
&+ \left(I_2^{acd} - I_2^{5ac} - I_2^{5ad} - I_2^{5cd} - J_4^5 + R^{5a} + R^{5c} + R^{5d}\right)Y_b \\
&\left. + \left(I_2^{abd} - I_2^{5ab} - I_2^{5ad} - I_2^{5bd} - J_4^5 + R^{5a} + R^{5b} + R^{5d}\right)Y_c \right\}
\end{aligned}
\tag{26}
$$

where we have introduced:

$$J_3^{st} \equiv \frac{1}{\binom{st}{st}_5} \left\{ -\binom{s}{s}_5 I_3^{[d+],st} + \binom{ts}{0s}_5 R^{ts} - \sum_{u=1}^{5} \binom{ts}{us}_5 R^{tsu} \right\}, \tag{27}$$

$$\tilde{J}_3^{st} \equiv \frac{1}{\binom{st}{st}_5} \left\{ \binom{s}{t}_5 I_3^{[d+],st} + \binom{st}{0t}_5 R^{ts} - \sum_{u=1}^{5} \binom{st}{ut}_5 R^{tsu} \right\}, \tag{28}$$

$$G^s \equiv \frac{1}{\binom{s}{s}_5} \left\{ -2\binom{}{}_5 R^{[d+],s} + \binom{s}{0}_5 J_4^s - \sum_{t=1}^{5} \binom{s}{t}_5 J_3^{ts} \right\}. \tag{29}$$

J_4^s and $R^{[d+],s}$ are given in Eqs. (2.24) and (2.44) of of [19] respectively.

We assume throughout that $q_5 = 0$, where $q_1, \ldots, q_5$ are chords – differences of external momenta.

Introduction
ooooooo

Approaches
ooooooo

Recursions

Contractions
ooooooooooooo●o

Summary
oo

Further abbreviations (see (2.24, 2.49, 2.9, 2.17, 2.34, 2.41) of [19]) :

$$J_4^s \equiv \frac{1}{\binom{s}{s}_5} \left\{ -()_5 I_4^{[d+],s} + \binom{s}{0}_5 R^s - \sum_{t=1}^{5} \binom{s}{t}_5 R^{st} \right\} \tag{30}$$

$$= \frac{-1}{\binom{0s}{0s}_5} \left\{ ()_5 (d-2)(d-1) I_4^{[d+]^2,s} - \binom{0}{0}_5 I_4^{[d+],s} + \sum_{t=1}^{5} \binom{t}{0}_5 (d-2) I_3^{[d+],st} + \sum_{t=1}^{5} \binom{0s}{0t}_5 R^{st} \right\}$$

$$R^{[d+],s} = \frac{1}{\binom{s}{s}_5} \left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^{5} \binom{s}{t}_5 I_3^{[d+],st} \right] = \frac{1}{\binom{0s}{0s}_5} \left[\binom{s}{0}_5 (d-1) I_4^{[d+]^2,s} - \sum_{t=1}^{5} \binom{0s}{0t}_5 I_3^{[d+],st} \right] \tag{31}$$

$$R^s \equiv \frac{1}{\binom{s}{s}_5} \left[\binom{s}{0}_5 I_4^s - \sum_{t=1}^{5} \binom{s}{t}_5 I_3^{st} \right] = \frac{1}{\binom{0s}{0s}_5} \left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^{5} \binom{0s}{0t}_5 I_3^{st} \right] . \tag{32}$$

$$R^{st} \equiv \frac{1}{\binom{st}{st}_5} \left[\binom{st}{0t}_5 I_3^{st} - \sum_{u=1}^{5} \binom{st}{ut}_5 I_2^{stu} \right] = \frac{1}{\binom{0st}{0st}_5} \left[\binom{st}{0t}_5 (d-2) I_3^{[d+],st} - \sum_{u=1}^{5} \binom{0st}{0ut}_5 I_2^{stu} \right] . \tag{33}$$

$$R^{tsu} = \frac{1}{\binom{stu}{stu}_5} \left[\binom{tsu}{0su}_5 I_2^{stu} - \sum_{v=1}^{5} \binom{tsu}{vsu}_5 I_1^{stuv} \right]$$

$$= \frac{1}{\binom{0stu}{0stu}_5} \left[\binom{tsu}{0su}_5 (d-1) I_2^{[d+],stu} - \sum_{v=1}^{5} \binom{0tsu}{0vsu}_5 I_1^{stuv} \right] . \tag{34}$$

$$Y_a = Y_{a5} - Y_{55}, \quad Y_{ab} = -(q_a - q_b)^2 + m_a^2 + m_b^2 \tag{35}$$

Some numerics for 5-point function with rank $R = 4$

We compare with LoopTools/FF/OneLOop

```
  the kinematics:
p1s =  1.1163688400000000E-002  p2s =  2.6109999999999998E-007  p3s =  0.0000000000000000
p4s =  2.6109999999999998E-007  p5s =  1.1163688400000000E-002
s12 = -0.70858278190000001       s23 = -1.5343299000000002E-003  s34 = -0.12851860429999998
s45 = -0.61023937949999996       s15 =  0.92668942420000000
m1s =  1.1163688361676107E-002  m2s =  0.0000000000000000      m3s =  2.6112003932088364E-00
m4s =  2.6112003932088364E-007  m5s =  0.0000000000000000

OneLOop-3.3.1
 for the evaluation of 1-loop scalar 1-, 2-, 3- and 4-point functions
van Hameren arXiv:1007.4716 and van Hameren, Papadopoulos, Pittau arXiv:0903.4665

 the R=4 contractions, a,b,c,d=3,3,3,3  ( -48094.1074 54542318     , -47802.08746 5035322 )
      LoopTools                        ( -48094.1074 65            , -47802.08746 05       )

 the R=4 contractions, a,b,c,d=3,3,3,4  ( -18463.1204 24842149     , -23446.4704 12257226 )
            LoopTools                  ( -18463.1204 31            , -23446.4704 09        )

 the R=4 contractions, a,b,c,d=3,3,3,5  (  0.0000000000000000      ,  0.0000000000000000 )
            LoopTools                  (  0.0000000000000000      ,  0.0000000000000000 )

compared to: LoopTools/FF, Hahn, arXiv:hep-ph/9807565 and van Oldenborgh, CPC66(1991)
```

Summary

- Explicit Analytical, recursive treatment of **heptagon, hexagon and pentagon tensor integrals** of rank R in terms of pentagons and boxes of rank $R - 1$

- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R = 5$

- Numerical tensor reduction package PJFry for C, C++, Mathematica, Fortran

- Numerical packages for contracted tensor integrals OLEC and CONTRACTIONS for C++ and Fortran under development

References I

G. Passarino, M. Veltman, One loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model, Nucl. Phys. B160 (1979) 151.
doi:10.1016/0550-3213(79)90234-7.

T. Hahn, M. Perez-Victoria, Automatized one loop calculations in four-dimensions and D-dimensions, Comput. Phys. Commun. 118 (1999) 153–165.
arXiv:hep-ph/9807565, doi:10.1016/S0010-4655(98)00173-8.

G. J. van Oldenborgh, FF: A Package to evaluate one loop Feynman diagrams, Comput. Phys. Commun. 66 (1991) 1–15, doi:10.1016/0010-4655(91)90002-3, scanned version at http://ccdb3fs.kek.jp/cgi-bin/img_index?9004168.
doi:10.1016/0010-4655(91)90002-3.

T. Binoth, J. P. Guillet, G. Heinrich, E. Pilon, T. Reiter, Golem95: a numerical program to calculate one-loop tensor integrals with up to six external legs, Comput. Phys. Commun. 180 (2009) 2317–2330.
arXiv:0810.0992, doi:10.1016/j.cpc.2009.06.024.

V. Yundin, Massive loop corrections for collider physics, PhD thesis, Humboldt-Universität zu Berlin, 2012, see http://edoc.hu-berlin.de/docviews/abstract.php?id=39163 and urn:nbn:de:kobv:11-100199626.

J. Fleischer, T. Riemann, V. Yundin, PJFry: A C++ package for tensor reduction of one-loop Feynman integrals. In: [22], preprint DESY 11-25 (2011), http://www-library.desy.de/cgi-bin/showprep.pl?desy11-252.

G. 't Hooft, M. Veltman, Scalar One Loop Integrals, Nucl. Phys. B153 (1979) 365–401.
doi:10.1016/0550-3213(79)90605-9.

References II

R. K. Ellis, G. Zanderighi, Scalar one-loop integrals for QCD, JHEP 02 (2008) 002.
`arXiv:arXiv:0712.1851`, `doi:10.1088/1126-6708/2008/02/002`.

A. van Hameren, Oneloop: For the evaluation of one-loop scalar functions, Comput. Phys. Commun. 182 (2011) 2427–2438.
`arXiv:arXiv:1007.4716`, `doi:10.1016/j.cpc.2011.06.011`.

J. Fleischer, T. Riemann, Calculating contracted tensor Feynman integrals, Phys. Lett. B701 (2011) 646–653.

`arXiv:arXiv:1104.4067`, `doi:10.1016/j.physletb.2011.06.033`.

J. Fleischer, J. Gluza, M. Gluza, and T. Riemann, OLEC project; C++ and Mathematica packages OLEC v.0.9 available at Silesian Univ., Katowice, webpage http://prac.us.edu.pl/ gluza/olec/.

A. Almasy, Mathematica and Fortran packages for contracted 1-loop tensor reduction, to be published.

J. Fleischer, T. Riemann, A solution for tensor reduction of one-loop n-point functions with $n \geq 6$, Physics Letters B 707 (2012) 375 – 380.
`arXiv:arXiv:1111.5821`, `doi:10.1016/j.physletb.2011.12.060`.
URL `http://www.sciencedirect.com/science/article/pii/S0370269311015280`

D. B. Melrose, Reduction of Feynman diagrams, Nuovo Cim. 40 (1965) 181–213.
`doi:10.1007/BF028329`.

A. I. Davydychev, A simple formula for reducing Feynman diagrams to scalar integrals, Phys. Lett. B263 (1991) 107–111, doi:10.1016/0370-2693(91)91715-8.
`doi:10.1016/0370-2693(91)91715-8`.

References III

Z. Bern, L. J. Dixon, D. A. Kosower, Dimensionally Regulated One-Loop Integrals, Phys. Lett. B302 (1993) 299–308 [Erratum–ibid. B **318** (1993) 649].
`arXiv:hep-ph/9212308`, `doi:10.1016/0370-2693(93)90400-C`.

O. Tarasov, Connection between Feynman integrals having different values of the space-time dimension, Phys.Rev. D54 (1996) 6479–6490.
`arXiv:hep-th/9606018`, `doi:10.1103/PhysRevD.54.6479`.

J. Fleischer, F. Jegerlehner, O. Tarasov, Algebraic reduction of one-loop Feynman graph amplitudes, Nucl. Phys. B566 (2000) 423–440.
`arXiv:hep-ph/9907327`, `doi:10.1016/S0550-3213(99)00678-1`.

J. Fleischer, T. Riemann, Complete algebraic reduction of one-loop tensor Feynman integrals, Phys. Rev. D83 (2011) 073004.
`arXiv:arXiv:1009.4436`, `doi:10.1103/PhysRevD.83.073004`.

T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, J. Tausk, A complete reduction of one-loop tensor 5- and 6-point integrals, Phys. Rev. D80 (2009) 036003.
`arXiv:arXiv:0812.2134`, `doi:10.1103/PhysRevD.80.036003`.

T. Diakonidis, J. Fleischer, T. Riemann, J. B. Tausk, A recursive reduction of tensor Feynman integrals, Phys. Lett. B683 (2010) 69–74.
`arXiv:arXiv:0907.2115`, `doi:10.1016/j.physletb.2009.11.049`.

Introduction
ooooooo

Approaches
ooooooo

Recursions

Contractions
oooooooooooooo

Summary
o●

References IV

📄 J. Alcaraz Maestre, et al., The SM and NLO Multileg and SM MC Working Groups: Summary Report, report
of the SM and NLO Multileg and SM MC Working Groups for the Workshop 'Physics at TeV Colliders', Les
Houches, France, 31 May – 8 June, 2011. See [arXiv:1203.1488] and [arXiv:1203.6803] (2012).
arXiv:1203.6803.

📄 J. Fleischer, T. Riemann, A solution for tensor reduction of one-loop N-point functions with $N \geq 6$, Phys. Lett.
B707 (2012) 375–380.
arXiv:1111.5821, doi:10.1016/j.physletb.2011.12.060.