Nucleon-nucleon interaction in covariant chiral effective field theory

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Peter Ring, and Jie Meng
OUTLINE

- Introduction
- Theoretical framework
- Results and discussion
- Summary and perspectives
Basic for all nuclear physics

- Precise understanding of the nuclear force

- Complexity of the nuclear force (vs. electromagnetic force)
  - Finite range
  - Intermediate-range attraction
  - Short-range repulsion-“hard core”
  - Spin-dependent non-central force
    - Tensor interaction
    - Spin-orbit interaction
  - Charge independent (approximate)

M. Taketani, Suppl.PTP3(1956)1
Nuclear force (NF) from QCD

- **Residual** quark-gluon strong interaction

- **Understood from** QCD

At low-energy region
- Running coupling constant $\alpha_s \geq 1$
- Nonperturbative QCD -- unsolvable

Phenomenological models
- Lattice QCD simulation
- Chiral effective field theory

---

S. Bethke, PPNP(2013)
NF from Chiral EFT

- Chiral effective field theory
  - Effective field theory (EFT) of low-energy QCD
  - Model independent to study the nuclear force

Main advantages of chiral nuclear force

- Self-consistently include many-body forces
  \[ V = V_{2N} + V_{3N} + \cdots + V_{iN} + \cdots \]

- Systematically improve NF order by order
  \[ V_{iN} = V_{iN}^{\text{LO}} + V_{iN}^{\text{NLO}} + V_{iN}^{\text{NNLO}} + \cdots \]

- Systematically estimate theoretical uncertainties
  \[ |V_{iN}^{\text{LO}}| > |V_{iN}^{\text{NLO}}| > |V_{iN}^{\text{NNLO}}| > \cdots \]

S. Weinberg, Phys. A1979
S. Weinberg, PLB1990
Current status of chiral NF

- Nonrelativistic (NR) chiral NF
  - NN interaction
    - up to NLO  U. van Kolck et al., PRL, PRC1992-94; N. Kaiser, NPA1997
    - up to NNLO  U. van Kolck et al., PRC1994; E. Epelbaum, et al., NPA2000
    - up to $N^3$LO  R. Machleidt et al., PRC2003; E. Epelbaum et al., NPA2005
    - up to $N^4$LO  E. Epelbaum et al., PRL2015, D.R. Entem, et al., PRC2015
    - up to $N^5$LO  (dominant terms)  D.R. Entem, et al., PRC2015
  - 3N interaction
    - up to NNLO  U. van Kolck, PRC1994
    - up to $N^4$LO  H. Krebs, et al., PRC2012-13
  - 4N interaction
    - up to $N^3$LO  E. Epelbaum, PLB 2006, EPJA 2007

R. Machleidt, D. R. Entem, Phys. Rept. 503 (2011) 1
Chiral NN potential is of high precision

<table>
<thead>
<tr>
<th>Phenomenological forces</th>
<th>NR Chiral nuclear force</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LO</td>
</tr>
<tr>
<td>Reid93</td>
<td>2+2</td>
</tr>
<tr>
<td>AV18</td>
<td>94</td>
</tr>
<tr>
<td>CD-Bonn</td>
<td>1.03</td>
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</tbody>
</table>

P.Reinert's talk, Bochum-Juelich (2017)
D.Entem, et al., PRC96(2017)024004

Chiral force has been extensively applied in the studies of nuclear structure and reactions within the non-relativistic few-/many-body theories.

Motivation for the relativistic chiral force

- The success of covariant density functional theory (CDFT) in the nuclear structure studies.
  

- Relativistic Brueckner-Hartree-Fock theory in nuclear matter and finite nuclei **(input: relativistic Bonn)**

Relativistic nuclear force based on ChEFT is needed

\[R. \text{Brockmann \& R. Machleidt, PRC (1990)}\]  
\[S.H. \text{Shen, et al., CPL (2016), PRC (2017)}\]
In this work

We extend covariant ChEFT to the nucleon-nucleon sector and construct a relativistic nuclear force up to leading order

- Construct the interaction kernel in **covariant power counting**
  - Employ the Lorentz invariant chiral Lagrangains
  - Retain the complete form of Dirac spinor

\[
    u(\vec{p}, s) = N_p \left( \frac{1}{\epsilon_p} \frac{\vec{\sigma} \cdot \vec{p}}{\epsilon_p} \right) \chi_s, \quad N_p = \sqrt{\frac{\epsilon_p}{2M_N}}.
\]

- Use naïve dimensional analysis to determine the chiral dimension

- Employ the 3D-reduced Bethe-Salpeter equation, such as Kadyshevsky equation, to re-sum the potential.
OUTLINE

- Introduction
- Theoretical framework
  - NN potential concept
  - Relativistic chiral force up to LO
- Results and discussion
- Summary and perspectives
NN potential concept

- Often-thought as a nonrelativistic quantity
  - Appear in the Schrödinger equation
    \[ -\frac{\hbar^2}{2m} \nabla^2 \Psi(t, \mathbf{r}) + V(\mathbf{r}) \Psi(t, \mathbf{r}) = i\hbar \frac{\partial}{\partial t} \Psi(t, \mathbf{r}). \]
  - (or) Appear in the Lippmann-Schwinger equation
    \[ T(\mathbf{p}', \mathbf{p}) = V(\mathbf{p}', \mathbf{p}) + \int \frac{d\mathbf{k}}{(2\pi)^3} V(\mathbf{p}', \mathbf{k}) \frac{m_N}{\mathbf{p}^2 - \mathbf{k}^2 + i\epsilon} T(\mathbf{k}, \mathbf{p}). \]

- Generalize the definition of NN potential
  - An interaction quantity appearing in a three-dimensional scattering equation can be referred as a NN potential.

⇒ Relativistic potential

Bethe-Salpeter equation

- For the relativistic nucleon-nucleon scattering

\[ p \ T \ p' = p \ A \ p' + p \ T \ G \ k \ A \ p' \]

Bethe-Salpeter equation with an operator form:

\[ \mathcal{T}(p', p|W) = \mathcal{A}(p', p|W) + \int \frac{d^4k}{(2\pi)^4} \mathcal{A}(p', p|W)G(k|W)T(k, p|W), \]

- \( \mathcal{T} \): Invariant scattering amplitude
- \( \mathcal{A} \): Interaction kernel (sum all the irreducible diagrams)
- \( G \): Two-nucleon's Green function

\[ G(k|W) = i \frac{1}{[\gamma^\mu(W + k)_\mu - m_N + i\epsilon]^{(1)} [\gamma^\mu(W - k)_\mu - m_N + i\epsilon]^{(2)}}. \]
Bethe-Salpeter equation

For the relativistic nucleon-nucleon scattering

\[
p \mathcal{T} p' = p \mathcal{A} p' + p \mathcal{T} G k \mathcal{A} p'
\]

Bethe-Salpeter equation with an operator form:

\[
\mathcal{T}(p', p|W) = \mathcal{A}(p', p|W) + \int \frac{d^4 k}{(2\pi)^4} \mathcal{A}(p', p|W) G(k|W) T(k, p|W),
\]

- \( \mathcal{T} \): Invariant scattering amplitude
- \( \mathcal{A} \): Interaction kernel (sum all the irreducible diagrams)
- \( G \): Two-nucleon’s Green function

It is hard to solve the BS equation, one always perform the 3-dimensional reduction.
Reduction of BS equation

- Introduce a three dimensional Green function $g$
  - Maintain the same elastic unitarity of $G$ at physical region
  - We choose the Kadyshevsky propagator $V. Kadyshevsky, NPB (1968)$.

\[
g = 2\pi \frac{m_N^2 \Lambda_+^{(1)}(k) \Lambda_+^{(2)}(-k)}{E_k^2 \sqrt{s - 2E_k + i\epsilon}} \delta[k_0 - (E_k - \frac{\sqrt{s}}{2})].
\]

- To replace $G$ with $g$, one can introduce the effective interaction kernel $\mathcal{V}$

\[
\mathcal{T} = \mathcal{A} + \mathcal{A}G\mathcal{T}.
\]

\[
\mathcal{T} = \mathcal{V} + \mathcal{V} g \mathcal{T}.
\]

\[
\mathcal{V} = \mathcal{A} + \mathcal{A} (G - g) \mathcal{V}.
\]
Reduction of BS equation

- BS equation reduces to the Kadyshevsky equation:

\[ \mathcal{T} = \mathcal{V} + \mathcal{V}_g \mathcal{T} \]

\[ = \mathcal{V} + \int \frac{dk}{(2\pi)^3} \int \frac{dk_0}{2\pi} \mathcal{V} \times 2\pi \frac{m_N^2 \Lambda_+^{(1)}(\mathbf{k}) \Lambda_+^{(2)}(-\mathbf{k})}{E_k^2 \sqrt{s - 2E_k + i\epsilon}} \delta[k_0 - (E_k - \frac{\sqrt{s}}{2})] \times \mathcal{T} \]

\[ = \mathcal{V} + \int \frac{dk}{(2\pi)^3} \mathcal{V} \frac{m_N^2 \Lambda_+^{(1)}(\mathbf{k}) \Lambda_+^{(2)}(-\mathbf{k})}{E_k^2 \sqrt{s - 2E_k + i\epsilon}} \mathcal{T}, \quad \text{with } k_0 = E_k - \frac{\sqrt{s}}{2}. \]

- Sandwiched by Dirac spinors:

\[ T(p', p) = V(p', p) + \int \frac{d^3k}{(2\pi)^3} V(p', k) \frac{m_N^2}{2E_k^2} \frac{1}{E_p - E_k + i\epsilon} T(k, p), \]

- Relativistic potential definition:

\[ V(p', p) = \bar{u}(p', s_1)\bar{u}(-p', s_2) \times \mathcal{V}(p'_0 = E_{p'} - \sqrt{s}/2, p'; p_0 = E_p - \sqrt{s}/2, p|W) \times u(p, s_1)u(p', s_2). \]

V. Kadyshevsky, NPB (1968).
Calculate potential in ChEFT

- To obtain the potential
  \[ V(p', p) = \bar{u}_1 \bar{u}_2 \, V(p, p') \, u_1 u_2. \]
- Solve the iterated equation perturbatively
  \[ V = A + A(G - g)V. \]

\[ V^{(2)} = A^{(2)}, \]
\[ V^{(4)} = A^{(4)} + A^{(2)}(G - g)A^{(2)}, \]

- Interaction kernel, \( A \), can be calculated by using covariant chiral perturbation theory order by order.
Relativistic chiral NF up to LO

\[ V^{\text{LO}}_{2N} = \bar{u}_1 \bar{u}_2 \mathcal{A}^{\text{LO}} u_1 u_2 \]

\[ = \bar{u}_1 \bar{u}_2 \left( \mathcal{A}_{\text{CTP}} + \mathcal{A}_{\text{OPEP}} \right) u_1 u_2. \]
Covariant chiral Lagrangians

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{NN}^{(0)}. \]

- **Pion-pion interaction:**
  \[ \mathcal{L}_{\pi\pi}^{(2)} = \frac{f_{\pi}^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger + (U + U^\dagger) m_{\pi}^2 \rangle. \]

- **Pion-nucleon interaction:**
  \[ \mathcal{L}_{\pi N}^{(1)} = \bar{\Psi} (i \partial - m_N) \Psi + \frac{g_A}{2} \bar{\Psi} \gamma^\mu \gamma^5 u_\mu \Psi. \]

- **Nucleon-nucleon interaction:**
  \[ \mathcal{L}_{NN}^{(0)} = -\frac{1}{2} \left[ C_S (\bar{\Psi} \Psi)(\bar{\Psi} \Psi) + C_A (\bar{\Psi} \gamma_5 \Psi)(\bar{\Psi} \gamma_5 \Psi) + C_V (\bar{\Psi} \gamma_\mu \Psi)(\bar{\Psi} \gamma^\mu \Psi) + C_{AV} (\bar{\Psi} \gamma_5 \gamma_\mu \Psi)(\bar{\Psi} \gamma_5 \gamma^\mu \Psi) + C_T (\bar{\Psi} \sigma_{\mu\nu} \Psi)(\bar{\Psi} \sigma^{\mu\nu} \Psi) \right]. \]

\[ U = 1 + i \frac{\Phi}{f_{\pi}} - \ldots \]
\[ \Phi = \tau_\sigma \pi^\sigma \]
\[ f_{\pi} = 92.4 \text{ MeV} \]
\[ u_\mu = -\frac{1}{f_{\pi}} \partial_\mu \Phi + \ldots \]
\[ \Psi = (p, n)^\dagger \]
\[ g_A = 1.26 \]

5 unknown low-energy constants (LECs)
Relativistic chiral potential at LO

- Contact potential (momentum space):

\[ V_{\text{CTP}} = C_S(\bar{u}_2 u_2)(\bar{u}_1 u_1) + C_A(\bar{u}_2 \gamma_5 u_2)(\bar{u}_1 \gamma_5 u_1) + C_V(\bar{u}_2 \gamma_\mu u_2)(\bar{u}_1 \gamma^\mu u_1) + C_{AV}(\bar{u}_2 \gamma_\mu \gamma_5 u_2)(\bar{u}_1 \gamma^\mu \gamma_5 u_1) + C_T(\bar{u}_2 \sigma_{\mu\nu} u_2)(\bar{u}_1 \sigma_{\mu\nu} u_1). \]

- One-pion-exchange potential (momentum space):

\[
V_{\text{OPEP}} = -\frac{g_A^2}{4 f_\pi^2} \tau_1 \cdot \tau_2 \frac{(\bar{u}_1 \gamma^\mu \gamma_5 q_\mu u_1)(\bar{u}_2 \gamma^\nu \gamma_5 q_\nu u_2)}{(E_{p'} - E_p)^2 - q^2 - m_{\pi}^2}.
\]

Retardation effect is included

- In the static limit \((m_N \rightarrow \text{infinity})\), the NR results can be recovered

\[
V_{\text{NonRel.}} = (C_S + C_V) - (C_{AV} - 2 C_T) \sigma_1 \cdot \sigma_2 - \frac{g_A^2}{4 f_\pi^2} \tau_1 \cdot \tau_2 \frac{\sigma_1 \cdot q \sigma_2 \cdot q}{q + m_{\pi}^2 + i\epsilon} + \mathcal{O}\left(\frac{1}{M_N}\right).
\]

\[ S. \ Weinberg, \ PLB1990 \]
Scattering equation and Phase shifts

- Perform the partial wave projection, one can obtain the Kadyshevesky equation in $|LSJ\rangle$ basis

\[
T_{L',L}^{SJ}(p',p) = V_{L',L}^{SJ}(p',p) + \sum_{L''} \int_{0}^{+\infty} \frac{k^2 dk}{(2\pi)^3} V_{L'',L}^{SJ}(p',k) \frac{M_N^2}{2(k^2 + M_N^2)\sqrt{p^2 + M_N^2} - \sqrt{k^2 + M_N^2} + i\epsilon} T_{L'',L}^{SJ}(k,p).
\]

- Cutoff renormalization for scattering equation
  - Potential regularized by an \textbf{exponential regulator function}

\[
V(p',p) \rightarrow V(p',p) \exp[-(|p'|/\Lambda)^{2n} - (|p|/\Lambda)^{2n}]. \quad n = 2
\]

- On-shell $S$ matrix and phase shift $\delta$

\[
S_{L'\rightarrow L}^{SJ} = \delta_{L'\rightarrow L} - \frac{i}{8\pi^2} \frac{M_N^2 |p|}{E_p} T_{L'\rightarrow L}^{SJ}.
\]

\[
S = \exp(2i\delta)
\]

Coupel channel: Stapp parameterization
Numerical details

- 5 LECs $C_{S,A,V,AV,T}$ are determined by fitting
  - **NPWA**: $p-n$ scattering phase shifts of Nijmegen 93
  - 7 partial waves: $J=0, 1$ \[1S_0, 3P_0, 1P_1, 3P_1, 3D_1, 3S_1, \epsilon_1\]
  - 42 data points: 6 data points for each partial wave
    \[(E_{lab} = 1, 5, 10, 25, 50, 100 \text{ MeV})\]

\[
\tilde{\chi}^2 = \sum_i \left( \delta_i^{\text{Theory}} - \delta_i^{\text{Nij93}} \right)^2
\]

<table>
<thead>
<tr>
<th>LECs</th>
<th>Values $[10^4 \text{ GeV}^{-2}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_S$</td>
<td>-0.125</td>
</tr>
<tr>
<td>$C_A$</td>
<td>0.040</td>
</tr>
<tr>
<td>$C_V$</td>
<td>0.248</td>
</tr>
<tr>
<td>$C_{AV}$</td>
<td>0.221</td>
</tr>
<tr>
<td>$C_T$</td>
<td>0.059</td>
</tr>
</tbody>
</table>
Description of $J=0$, 1 partial waves

- Red variation bands: cutoff $500 \sim 1000$ MeV
- Improve description of $^1S_0$, $^3P_0$ phase shifts
- Quantitatively similar to the nonrelativistic case for $J=1$ partial waves

arXiv:1611.08475
Description of J=0, 1 partial waves

- Red variation bands: cutoff 500~1000 MeV
- Improve description of $^1S_0, ^3P_0$ phase shifts

<table>
<thead>
<tr>
<th></th>
<th>Relativistic Chiral NF</th>
<th>Non-relativistic Chiral NF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chiral order</td>
<td>LO</td>
<td>LO*</td>
</tr>
<tr>
<td>No. of LECs</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>$\chi^2$/d.o.f.</td>
<td>2.0~11.8</td>
<td>147.9</td>
</tr>
</tbody>
</table>

*E. Epelbaum, et. al., NPA(2000)
Higher partial waves

- The relativistic results are almost the same as the non-relativistic case.
- Relativistic correction of OPEP is small!
1S0 wave phenomena

- Interesting phenomena of 1S0 wave
  - Large variance of phase shift from 60 to -10 (zero point: \( k_0 = 340.5 \text{ MeV} \))
  - Virtual bound state at very low-energy region (pole: \(-10 \text{ MeV}\))
  - Significantly large scattering length (\( a = -23.7 \text{ fm} \))

Energy scales smaller than chiral symmetry breaking scale

The 1S0 phenomena should be roughly reproduced simultaneously at the lowest order of chiral nuclear force

\[ \text{Bira van Kolck, et al., 1704.08524.} \]

However, the NR chiral force at LO cannot achieve such description.
1S0 in relativistic chiral force (LO)

- Rather good description of phase shift:

- Predicted results: *(reproduced simultaneously)*

<table>
<thead>
<tr>
<th></th>
<th>Nijmegen PWA [60, 61]</th>
<th>Global-Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$ [MeV]</td>
<td>-</td>
<td>750</td>
</tr>
<tr>
<td>scattering length $a$ [fm]</td>
<td>-23.7</td>
<td>-20.3</td>
</tr>
<tr>
<td>effective range $r$ [fm]</td>
<td>2.70</td>
<td>2.45</td>
</tr>
<tr>
<td>virtual pole position $i\gamma$ [MeV]</td>
<td>$-i10$</td>
<td>$-i9.2$</td>
</tr>
<tr>
<td>zero phase shift position $k_0$ [MeV]</td>
<td>340.5</td>
<td>326.5</td>
</tr>
</tbody>
</table>
Summary and perspectives

- We performed an exploratory study to construct the relativistic nuclear force up to leading order in covariant ChEFT
  - Relativistic chiral force can improve the description of $^1S_0$ and $^3P_0$ phase shifts at LO
  - For the phase shifts of partial waves with high angular momenta ($J\geq1$), the relativistic results are quantitatively similar to the nonrelativistic counter parts.

- We are now working on the NLO studies
  - Calculate the two-pion exchange potentials (almost finished)
  - Construct the contact Lagrangians with two derivatives
  - Expect to achieve a better description of phase shifts  
    ☝ Stay tuned
Thank you very much for your attention!
Back up slides
Hint at a more efficient formulation

- $V_{1S0}$: $1/m_N$ expansion

$$V_{1S0} = 4\pi \left[ C_{1S0} + \left( C_{1S0} + \hat{C}_{1S0} \right) \left( \frac{p^2 + p'^2}{4M_N^2} + \cdots \right) \right]$$

$$+ \frac{\pi g_A^2}{2f_\pi^2} \int_{-1}^{1} \frac{dz}{\bar{q}^2 + m_\pi^2} \left[ \bar{q}^2 - \left( \frac{(\bar{p}^2 - p'^2)^2}{4M_N^2} + \cdots \right) \right].$$

- Relativistic corrections are suppressed
- One has to be careful with the new contact term, the momentum dependent term, which is desired to achieve a reasonable description of the phase shifts of $1S0$ channel.

*J. Soto et al., PRC(2008), B. Long, PRC (2013)*
Relativistic effects in nuclear physics

- **Kinematical effect**: safely neglected or perturbatively treated

\[ \sqrt{p^2 + m_N^2} = m_N \sqrt{1 + 0.102} \]

- **Dynamical effect**: nucleon spin, spin-orbit splitting, anti-nucleon …

Relativistic (dynamical) effects are important

- Nuclear system:
  - **Covariant density functional theory (CDFT)**
  - One-nucleon system:
    - **Covariant ChEFT with extended-on-mass-shell (EOMS) scheme**
## Errors and correlation matrix

<table>
<thead>
<tr>
<th></th>
<th>$C_S$</th>
<th>$C_A$</th>
<th>$C_V$</th>
<th>$C_{AV}$</th>
<th>$C_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_S$</td>
<td>1.00</td>
<td>0.21</td>
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<td>-0.58</td>
<td>-0.39</td>
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<td>$C_A$</td>
<td>0.23</td>
<td>1.00</td>
<td>-0.15</td>
<td>0.45</td>
<td>0.21</td>
</tr>
<tr>
<td>$C_V$</td>
<td>-0.93</td>
<td>-0.15</td>
<td>1.00</td>
<td>0.77</td>
<td>0.69</td>
</tr>
<tr>
<td>$C_{AV}$</td>
<td>-0.57</td>
<td>0.45</td>
<td>0.77</td>
<td>1.00</td>
<td>0.89</td>
</tr>
<tr>
<td>$C_T$</td>
<td>-0.39</td>
<td>0.21</td>
<td>0.69</td>
<td>0.89</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Only two LECs fit:

\[ V_{\text{CTP}}^{\text{NonRel.}} = (C_S + C_V) - (C_{AV} - 2C_T)\sigma_1 \cdot \sigma_2 + \mathcal{O}\left(\frac{1}{M_N}\right). \]

- Take CS and CAV as free parameters
- Best fit result:
  - \( \chi^2/\text{d.o.f.} = 84.5 \)

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<td>2.0-11.0</td>
<td>147.9</td>
</tr>
<tr>
<td>T_{lab} [MeV]</td>
<td>1</td>
<td>50</td>
</tr>
<tr>
<td>--------------</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>P_{cm} [MeV]</td>
<td>21.67</td>
<td>153.22</td>
</tr>
<tr>
<td>V_{cm}</td>
<td>0.023 \text{c}</td>
<td>0.16 \text{c}</td>
</tr>
<tr>
<td>E_{corr}(2n) [MeV]</td>
<td>0.25</td>
<td>12.5</td>
</tr>
</tbody>
</table>

\[ p_{cm} = \sqrt{\frac{m_N T_{lab}}{2}} \quad V_{cm} = \frac{p_{cm}}{m_N} c \]

\[ E_{corr}^{T} = \frac{p_{cm}^2}{2m_N} \]