A crisis of NRQCD in describing charmonium?

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- Brief review of NRQCD factorization to quarkonium production/decay
- NNLO QCD correction to $\gamma\gamma^* \rightarrow \eta_c$ form factor and confront BaBar data
- NNLO QCD correction to $\eta_c \rightarrow \gamma\gamma$ (including “light-by-light”)  
- NNLO QCD correction to $\eta_c \rightarrow$ light hadrons and $\text{Br}[\eta_c \rightarrow \gamma\gamma]$, then confront the PDG data
- Summary
Nonrelativistic QCD (NRQCD): Paradigm of EFT, tailored for describing heavy quarkonium dynamics: exploiting NR nature of quarkonium
caswell, Lepage (1986); Bodwin, Braaten, Lepage (1995)

This scale separation is usually referred to as NRQCD factorization.
The NRQCD short-dist. coefficients can be computed in perturbation theory, order by order
NRQCD Lagrangian (characterized by velocity expansion)

\[
\mathcal{L}_{\text{NRQCD}} = \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{heavy}} + \delta \mathcal{L}.
\]

\[
\mathcal{L}_{\text{light}} = -\frac{1}{2} \text{tr} G_{\mu \nu} G^{\mu \nu} + \sum \bar{q} i \not{D} q,
\]

\[
\mathcal{L}_{\text{heavy}} = \psi^\dagger \left( iD_t + \frac{D^2}{2M} \right) \psi + \chi^\dagger \left( iD_t - \frac{D^2}{2M} \right) \chi,
\]

\[
\delta \mathcal{L}_{\text{bilinear}} = \frac{c_1}{8M^3} \left( \psi^\dagger (D^2)^2 \psi - \chi^\dagger (D^2)^2 \chi \right)
\]

\[
+ \frac{c_2}{8M^2} \left( \psi^\dagger (D \cdot gE - gE \cdot D) \psi + \chi^\dagger (D \cdot gE - gE \cdot D) \chi \right)
\]

\[
+ \frac{c_3}{8M^2} \left( \psi^\dagger (iD \times gE - gE \times iD) \cdot \sigma \psi + \chi^\dagger (iD \times gE - gE \times iD) \cdot \sigma \chi \right)
\]

\[
+ \frac{c_4}{2M} \left( \psi^\dagger (gB \cdot \sigma) \psi - \chi^\dagger (gB \cdot \sigma) \chi \right),
\]

Identical to HQET, but with different power counting
NRQCD is the mainstream tool in studying quarkonium \((\text{see Brambilla et al. EPJC 2011 for a review})\)

Nowadays, NRQCD becomes standard approach to tackle various quarkonium production and decay processes:

- charmonium: \(v^2/c^2 \sim 0.3\) not truly non-relativistic to some extent
- bottomonium: \(v^2/c^2 \sim 0.1\) a better “non-relativistic” system

Exemplified by

\[ e^+ e^- \rightarrow J/\psi + \eta_c \] at B factories  \((\text{exclusive charmonium production})\)

Unpolarized/polarized \(J/\psi\) production at hadron colliders \((\text{inclusive})\)

Very active field in recent years \((\text{Chao’s group, Kniehl’s group, Wang’s group, Bodwin’s group, Qiu’s group …})\) marked by a plenty of PRLs
The ubiquitous symptom of NRQCD factorization: often plagued with huge QCD radiative correction

Most of the NRQCD successes based on the NLO QCD predictions.

However, the NLO QCD corrections are often large:

\[ e^+ e^- \rightarrow J/\psi + \eta_c \quad \text{K factor: } 1.8 \sim 2.1 \quad \text{Zhang et.al.} \]
\[ e^+ e^- \rightarrow J/\psi + J/\psi \quad \text{K factor: } -0.31 \sim 0.25 \quad \text{Gong et.al.} \]
\[ p + p \rightarrow J/\psi + X \quad \text{K factor: } \sim 2 \quad \text{Campbell et.al.} \]
\[ J/\psi \rightarrow \gamma\gamma\gamma \quad \text{K factor: } \leq 0 \quad \text{Mackenzie et.al.} \]

... ...
The existing NNLO corrections are rather few: all related to S-wave quarkonium decay.

1. $\Upsilon(J/\Psi) \to e^+ e^-$

NNLO corrections were first computed by two groups in 1997: Czarnecki and Melkinov; Beneke, Smirnov, and Signer; NNNLO correction available very recently: Steinhauser et al. (2013)

2. $\eta_c \to \gamma\gamma$

NNLO correction was computed by Czarnecki and Melkinov (2001): (neglecting light-by-light)

3. $B_c \to l \nu$:

NNLO correction computed by Onishchenko, Veretin (2003); Chen and Qiao, (2015)
Perturbative convergence of these decay processes appears to be rather poor.

\[
\Gamma(J/\psi \rightarrow \ell\ell) = \Gamma^{(0)} \left[ 1 - \frac{8}{3} \frac{\alpha_s}{\pi} - (44.55 - 0.41 n_f) \left( \frac{\alpha_s}{\pi} \right)^2 \right]^2 \\
+ (-2091 + 120.66 n_f - 0.82 n_f^2) \left( \frac{\alpha_s}{\pi} \right)^3
\]

\[
\Gamma(B_c \rightarrow \ell\nu) = \Gamma^{(0)} \left[ 1 - 1.39 \frac{\alpha_s}{\pi} - 23.7 \left( \frac{\alpha_s}{\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right]^2
\]

\[
\Gamma(\eta_c \rightarrow \gamma\gamma) = \Gamma^{(0)} \left[ 1 - 1.69 \frac{\alpha_s}{\pi} - 56.52 \left( \frac{\alpha_s}{\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right]^2
\]

So calculating the higher order QCD correction is imperative to test the usefulness of NRQCD factorization!
Investigation on $\gamma \gamma^* \to \eta_c$ form factor Experiment


Babar measures the $\gamma \gamma^* \to \eta_c$ transition form factor in the momentum transfer range from 2 to 50 GeV$^2$. 

$q_2^2 \approx 0$

$q_1^2 = -Q^2 = (p' - p)^2$
Digression: recall the surprise brought by BaBar two-photon experiment on $\gamma\gamma^* \rightarrow \pi^0$

The $\pi^0$ Transition Form Factor

Comparison of the result of experiment to the QCD limit

- Experiment:
  In $Q^2$ range 4-9 GeV$^2$ CLEO results are consistent with more precise BaBar data

- QCD prediction (Brodsky-Lepage ’79):
  at high $Q^2$ data should reach asymptotic limit (either from below or from above)

  $Q^2 F(Q^2) = \sqrt{2}f_\pi = 0.185$ GeV
  assuming the asymptotic DA
Belle did not confirm BaBar measurement on $\gamma\gamma^* \rightarrow \pi^0$! Situation needs clarification

Comparison with BELLE, arXiv:1205.3249

- Difference BABAR – BELLE $\sim 2\sigma_{\text{syst}}$
- BELLE has lower detection efficiency ($\sim$ factor 2)
- BELLE has higher systematic uncertainties
Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor: There also exists BaBar measurements!

\[ F(Q^2) : \quad \gamma^*\gamma \rightarrow \eta_c \quad \text{form factor} \]

\[ F(0) : \quad \eta_c \rightarrow \gamma\gamma \quad \text{form factor} \]

| $Q^2$ interval | $\overline{Q}^2$ (GeV$^2$) | $d\sigma/dQ^2(\overline{Q}^2)$ (fb/GeV$^2$) | $|F(\overline{Q}^2)/F(0)|$ |
|----------------|--------------------------|-----------------|-----------------|
| $2-3$          | $2.49$                   | $18.7\pm 4.2\pm 0.8$ | $0.740\pm 0.085$ |
| $3-4$          | $3.49$                   | $10.6\pm 2.1\pm 0.8$ | $0.680\pm 0.073$ |
| $4-5$          | $4.49$                   | $6.62\pm 1.18\pm 0.19$ | $0.629\pm 0.057$ |
| $5-6$          | $5.49$                   | $4.00\pm 0.80\pm 0.10$ | $0.555\pm 0.056$ |
| $6-8$          | $6.96$                   | $3.00\pm 0.43\pm 0.17$ | $0.563\pm 0.043$ |
| $8-10$         | $8.97$                   | $1.58\pm 0.30\pm 0.08$ | $0.490\pm 0.049$ |
| $10-12$        | $10.97$                  | $0.72\pm 0.17\pm 0.05$ | $0.385\pm 0.048$ |
| $12-15$        | $13.44$                  | $0.55\pm 0.13\pm 0.03$ | $0.395\pm 0.047$ |
| $15-20$        | $17.35$                  | $0.34\pm 0.07\pm 0.01$ | $0.385\pm 0.038$ |
| $20-30$        | $24.53$                  | $0.084\pm 0.026\pm 0.004$ | $0.261\pm 0.041$ |
| $30-50$        | $38.68$                  | $0.019\pm 0.009\pm 0.001$ | $0.204\pm 0.049$ |
Investigation on $\gamma\gamma^* \to \eta_c$ form factor

The solid curve is from a simple monopole fit:

$$|F(Q^2)/F(0)| = \frac{1}{1 + Q^2/\Lambda}$$

with $\Lambda = 8.5 \pm 0.6 \pm 0.7$ GeV$^2$

The dotted curve is from pQCD prediction

Investigation on $\gamma \gamma^{*} \rightarrow \eta_{c}$ form factor

Previous investigation

- $k_{\perp}$ factorization: Feldmann et al., Cao and Huang
- Lattice QCD: Dudek et al.,
- $J/\psi$ -pole-dominance: Lees et al.,
- QCD sum rules: Lucha et al.,
- light-front quark model: Geng et al.,
- Dyson-Schwinger approach: Chang, Chen, Ding, Liu, Roberts, 2016

All yield predictions compatible with the data, at least in the small $Q^{2}$ range.

So far, so good. Unlike $\gamma \gamma^{*} \rightarrow \pi^{0}$, there is no open puzzle here.
**Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor**

**Motivation**

- **Model-independent method is always welcome.**
  (NRQCD is the first principle approach from QCD)

- In the normalized form factor, nonperturbative NRQCD matrix element cancels out. Therefore, our predictions are free from any freely adjustable parameters!

- Is LO/NLO NRQCD prediction sufficient?

- The momentum transfer is not large enough, we are not bothered by resumming the large collinear logarithms.
The first \textbf{NNLO} calculation for (exclusive) quarkonium production process

Can Nonrelativistic QCD Explain the $\gamma\gamma^* \to \eta_c$ Transition Form Factor Data?

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Unlike the bewildering situation in the $\gamma\gamma^* \to \pi$ form factor, a widespread view is that perturbative QCD can decently account for the recent BABAR measurement of the $\gamma\gamma^* \to \eta_c$ transition form factor. The next-to-next-to-leading-order perturbative correction to the $\gamma\gamma^* \to \eta_{c,b}$ form factor, is investigated in the non-relativistic QCD (NRQCD) factorization framework for the first time. As a byproduct, we obtain, by far, the most precise order-$\alpha_s^2$ NRQCD matching coefficient for the $\eta_{c,b} \to \gamma\gamma$ process. After including the substantial negative order-$\alpha_s^2$ correction, the good agreement between NRQCD prediction and the measured $\gamma\gamma^* \to \eta_c$ form factor is completely ruined over a wide range of momentum transfer squared. This eminent discrepancy casts some doubts on the applicability of the NRQCD approach to hard exclusive reactions involving charmonium.

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Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor

Definition for form factor:

$$\langle \eta_c(p) | J^\mu | \gamma(k, \varepsilon) \rangle = i e^2 \varepsilon^{\mu\nu\rho\sigma} \varepsilon_\nu q_\rho k_\sigma F(Q^2)$$

NRQCD factorization demands:

$$F(Q^2) = C(Q, m, \mu_R, \mu_\Lambda) \frac{\langle \eta_c | \psi^\dagger \chi(\mu_\Lambda) | 0 \rangle}{\sqrt{m}} + O(\nu^2)$$

Short-distance coefficient (SDC)
We are going to compute it to NNLO

$$\overline{R}_{\eta_c}(\Lambda) \equiv \sqrt{\frac{2\pi}{N_c}} \langle 0 | \chi^\dagger \psi(\Lambda) | \eta_c \rangle,$$

$$\overline{R}_{\psi}(\Lambda) \epsilon \equiv \sqrt{\frac{2\pi}{N_c}} \langle 0 | \chi^\dagger \sigma \psi(\Lambda) | \psi(\epsilon) \rangle,$$
Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor

Perturbative series for NRQCD SDCs

Upon general consideration, the SDC can be written as

$$C(Q, m, \mu_R, \mu_\Lambda) = C^{(0)}(Q, m) \left\{ 1 + C_F \frac{\alpha_s(\mu_R)}{\pi} f^{(1)}(\tau) ight. + \frac{\alpha_s^2}{\pi^2} \left[ \frac{\beta_0}{4} \ln \frac{\mu_R^2}{Q^2 + m^2} C_F f^{(1)}(\tau) - \pi^2 C_F \left( C_F + \frac{C_A}{2} \right) \right] \times \ln \frac{\mu_\Lambda}{m} + f^{(2)}(\tau) \right\} + O(\alpha_s^3) \right\},$$

IR pole matches anomalous dimension of NRQCD pseudo-scalar density

RG invariance
Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor

Theoretical calculation

$$C^{(0)}(Q, m) = \frac{4e_c^2}{Q^2 + 4m^2}$$

Tree-level SDC

$$f^{(1)}(\tau) = \frac{\pi^2(3 - \tau)}{6(4 + \tau)} - \frac{20 + 9\tau}{4(2 + \tau)} - \frac{\tau(8 + 3\tau)}{4(2 + \tau)^2} \ln \frac{4 + \tau}{2} + 3\sqrt{\frac{\tau}{4 + \tau}} \tanh^{-1}\sqrt{\frac{\tau}{4 + \tau}}$$

$$+ \frac{2 - \tau}{4 + \tau} \left( \tanh^{-1}\sqrt{\frac{\tau}{4 + \tau}} \right)^2 - \frac{\tau}{2(4 + \tau)} \text{Li}_2 \left( -\frac{2 + \tau}{2} \right),$$

$$\tau \equiv \frac{Q^2}{m^2}$$

NLO QCD correction
Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor

Feynman diagrams

Numer of diagrams

LO: 2
NLO: 8
NNLO: 108
"light by light": 12
Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor

NNLO corrections

\[ f^{(2)}(\tau) = f^{(2)}_{\text{reg}}(\tau) + f^{(2)}_{\text{lbl}}(\tau). \]

At $\tau \gg 0$, the value of $f^{(2)}_{\text{reg}}(\tau)$ is compatible with asymptotic behavior $\ln^2 \tau$ solving ERBL equation by Yang, NPB 2009

Reproduce known NNLO corr. to $\eta_c \rightarrow \gamma\gamma$

Czarnecki et al. 2001
Investigation on $\gamma\gamma^* \to \eta_c$ form factor

NNLO corrections

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>25</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\text{reg}}^{(2)}$</td>
<td>-59.420(6)</td>
<td>-61.242(6)</td>
<td>-61.721(7)</td>
<td>-61.843(8)</td>
<td>-61.553(8)</td>
</tr>
<tr>
<td>$f_{\text{lbl}}^{(2)}$</td>
<td>0.49(1)</td>
<td>-0.48(1)</td>
<td>-1.10(1)</td>
<td>-2.13(1)</td>
<td>-3.07(1)</td>
</tr>
<tr>
<td></td>
<td>-0.65(1)i</td>
<td>-0.72(1)i</td>
<td>-0.71(1)i</td>
<td>-0.69(1)i</td>
<td>-0.68(1)i</td>
</tr>
<tr>
<td>$f_{\text{reg}}^{(2)}$</td>
<td>-59.636(6)</td>
<td>-61.278(6)</td>
<td>-61.716(7)</td>
<td>-61.864(8)</td>
<td>-61.668(8)</td>
</tr>
<tr>
<td>$f_{\text{lbl}}^{(2)}$</td>
<td>0.79(1)</td>
<td>-5.61(1)</td>
<td>-9.45(1)</td>
<td>-15.32(1)</td>
<td>-20.26(1)</td>
</tr>
<tr>
<td></td>
<td>-12.45(1)i</td>
<td>-13.55(1)i</td>
<td>-13.83(1)i</td>
<td>-14.03(1)i</td>
<td>-14.10(1)i</td>
</tr>
</tbody>
</table>

Table 1: $f_{\text{reg}}^{(2)}(\tau)$ and $f_{\text{lbl}}^{(2)}(\tau)$ at some typical values of $\tau$. The first two rows for $\eta_c$ and the last two for $\eta_b$.

Contribution from light-by-light is not always negligible!
Investigation on $\gamma\gamma^* \rightarrow \eta_c$ form factor

Theory vs Experiment

Our Prediction is free of any nonperturbative parameters!

$\gamma\gamma^* \rightarrow \eta_c :$ NNLO predictions seriously fails to describe data!
Prediction to $\gamma\gamma^* \rightarrow \eta_b$ form factor

Convergence of perturbation series is reasonably well. Await CEPC/ILC to test our predictions?
As a by-product, we also have a complete NNLO prediction for $\eta_c \rightarrow 2\gamma$ (including “light-by-light” diagrams).

We can focus on form factor at $Q^2 = 0$:

- **LO**
- **NLO**
- **NNLO (“light by light”)**
- **NNLO (regular)**
Updated NNLO predictions to $\eta_c \rightarrow 2\gamma$

NNLO correction was previously computed by Czarnecki and Melkinov (2001) (neglecting light-by-light);

Here we present a complete/highly precise NNLO predictions

Form factor at $Q^2 = 0$:

$$F(0) = \frac{e^2}{m^{5/2}} \langle \eta_c | \psi \bar{\chi}(\mu_\Lambda) | 0 \rangle \left\{ 1 + C_F \frac{\alpha_s(\mu_R)}{\pi} \left( \frac{\pi^2}{8} - \frac{5}{2} \right) + \frac{\alpha_s^2}{\pi^2} \left[ C_F \left( \frac{\pi^2}{8} - \frac{5}{2} \right) \ln \frac{\mu_R^2}{m^2} - \pi^2 C_F \left( C_F + \frac{C_A}{2} \right) \ln \frac{\mu_\Lambda}{m} \right] + f^{(2)}_{\text{reg}}(0) + f^{(2)}_{\text{lbl}}(0) + \mathcal{O}(\alpha_s^3) \right\}.$$  

NRQCD factorization scale dependence

$$\Gamma(\eta_c \rightarrow 2\gamma) = \frac{(\pi \alpha^2/4)|F(0)|^2 M^3_{\eta_c}}{}.$$
Complete NNLO correction to $\eta_c \rightarrow$ light hadrons
(first NNLO calculation for inclusive process involving quarkonium)

NRQCD factorization for $\eta_c \rightarrow$ light hadrons — up to relative order-$v^4$ corrections

Bodwin, Petrelli PRD (2002)

\[
\Gamma(1S_0 \rightarrow LH) = \frac{F_1(1S_0)}{m^2} \langle 1S_0 | \mathcal{O}_1(1S_0) | 1S_0 \rangle \\
+ \frac{G_1(1S_0)}{m^4} \langle 1S_0 | \mathcal{P}_1(1S_0) | 1S_0 \rangle \\
+ \frac{F_8(3S_1)}{m^2} \langle 1S_0 | \mathcal{O}_8(3S_1) | 1S_0 \rangle \\
+ \frac{F_8(1P_1)}{m^4} \langle 1S_0 | \mathcal{O}_8(1P_1) | 1S_0 \rangle \\
+ \frac{H_1^1(1S_0)}{m^6} \langle 1S_0 | \mathcal{Q}_1^1(1S_0) | 1S_0 \rangle \\
+ \frac{H_2^2(1S_0)}{m^6} \langle 1S_0 | \mathcal{Q}_1^2(1S_0) | 1S_0 \rangle.
\]

(2.2a)

\[
\mathcal{O}_1(1S_0) = \psi^\dagger \chi \chi^\dagger \chi \psi.
\]

(2.2b)

\[
\mathcal{P}_1(1S_0) = \frac{1}{2} \left[ \psi^\dagger \chi \chi^\dagger \left( -\frac{i}{2} \vec{D} \right)^2 \chi \psi + \psi^\dagger \left( -\frac{i}{2} \vec{D} \right)^2 \chi \chi^\dagger \psi \right].
\]

(2.2c)

\[
\mathcal{O}_8(3S_1) = \psi^\dagger \alpha \tau_3 \chi \chi^\dagger \chi \alpha \tau_3 \psi.
\]

(2.2d)

\[
\mathcal{O}_8(1S_0) = \psi^\dagger \tau_3 \chi \chi^\dagger \tau_3 \psi.
\]

(2.2e)

\[
\mathcal{O}_8(1P_1) = \psi^\dagger \left( -\frac{i}{2} \vec{D} \right) \chi \chi^\dagger \chi \alpha \tau_3 \psi.
\]

(2.2f)

\[
\mathcal{O}_1^1(1S_0) = \psi^\dagger \left( -\frac{i}{2} \vec{D} \right)^2 \chi \chi^\dagger \chi \psi.
\]

(2.2g)

\[
\mathcal{O}_1^2(1S_0) = \frac{1}{2} \left[ \psi^\dagger \chi \chi^\dagger \left( -\frac{i}{2} \vec{D} \right)^4 \psi + \psi^\dagger \left( -\frac{i}{2} \vec{D} \right)^4 \chi \chi^\dagger \psi \right].
\]

(2.2h)
NRQCD factorization for $\eta_c \rightarrow$ light hadrons — up to relative order-$v^4$ corrections

Brambilla, Mereghetti, Vairo, 0810.2259

\[
\Gamma^{(1S_0 \rightarrow \text{l.h.})} = \frac{2 \text{Im} f_1(1S_0)}{M^2} \langle H(1S_0)|O_1(1S_0)|H(1S_0)\rangle \\
+ \frac{2 \text{Im} g_1(1S_0)}{M^4} \langle H(1S_0)|P_1(1S_0)|H(1S_0)\rangle + \frac{2 \text{Im} f_8(3S_1)}{M^2} \langle H(1S_0)|O_8(3S_1)|H(1S_0)\rangle \\
+ \frac{2 \text{Im} f_8(1S_0)}{M^2} \langle H(1S_0)|O_8(1S_0)|H(1S_0)\rangle + \frac{2 \text{Im} f_8(1P_1)}{M^4} \langle H(1S_0)|O_8(1P_1)|H(1S_0)\rangle \\
+ \frac{2 \text{Im} s_{1-8}(1S_0,3S_1)}{M^4} \langle H(1S_0)|S_{1-8}(1S_0,3S_1)|H(1S_0)\rangle + \frac{2 \text{Im} f_{8\text{cm}}(1S_0)}{M^4} \langle H(1S_0)|O_{8\text{cm}}|H(1S_0)\rangle \\
+ \frac{2 \text{Im} g_{8\text{cm}}(1S_0)}{M^4} \langle H(1S_0)|P_{8\text{cm}}|H(1S_0)\rangle + \frac{2 \text{Im} f_{1\text{cm}}(1S_0)}{M^4} \langle H(1S_0)|O_{1\text{cm}}|H(1S_0)\rangle \\
+ \frac{2 \text{Im} h'_1(1S_0)}{M^6} \langle H(1S_0)|Q'_1(1S_0)|H(1S_0)\rangle + \frac{2 \text{Im} h''_1(1S_0)}{M^6} \langle H(1S_0)|Q''_1(1S_0)|H(1S_0)\rangle \\
+ \frac{2 \text{Im} g_8(3S_1)}{M^4} \langle H(1S_0)|P_8(3S_1)|H(1S_0)\rangle + \frac{2 \text{Im} g_8(1S_0)}{M^4} \langle H(1S_0)|P_8(1S_0)|H(1S_0)\rangle \\
+ \frac{2 \text{Im} g_8(1P_1)}{M^4} \langle H(1S_0)|P_8(1P_1)|H(1S_0)\rangle + \frac{2 \text{Im} h'_8(1S_0)}{M^6} \langle H(1S_0)|Q'_8(1S_0)|H(1S_0)\rangle \\
+ \frac{2 \text{Im} h''_8(1S_0)}{M^6} \langle H(1S_0)|Q''_8(1S_0)|H(1S_0)\rangle + \frac{2 \text{Im} h''_8(1S_0)}{M^6} \langle H(1S_0)|Q''_8(1S_0)|H(1S_0)\rangle \\
+ \frac{2 \text{Im} d_8(1S_0,1P_1)}{M^5} \langle H(1S_0)|D_8(1S_0,1P_1)|H(1S_0)\rangle,
\]

Notice the explosion of number of higher-dimensional operators!
NRQCD factorization for $\eta_c \rightarrow$ light hadrons

Current status of radiative corrections

$$\Gamma(\eta_c \rightarrow LH) = \frac{F_1(1^1S_0)}{m^2} \langle \eta_c | \mathcal{O}_1(1^1S_0) | \eta_c \rangle$$
$$+ \frac{G_1(1^1S_0)}{m^4} \langle \eta_c | \mathcal{P}_1(1^1S_0) | \eta_c \rangle + \mathcal{O}(v^3 \Gamma),$$

To warrant predictive power, we only retain terms through relative order-$v^2$

$$F_1(1^1S_0) = \frac{\pi \alpha_s^2 C_F}{N_c} \left\{ 1 + \frac{\alpha_s}{\pi} f_1 + \frac{\alpha_s^2}{\pi^2} f_2 + \cdots \right\}$$

$$G_1(1^1S_0) = -\frac{4\pi \alpha_s^2 C_F}{3N_c} \left\{ 1 + \frac{\alpha_s}{\pi} g_1 + \cdots \right\}.$$
Our calculation of short-distance coefficient utilizes Method of Region (Beneke and Smirnov 1998) to directly extract the hard region contribution from multi-loop diagrams.

![Diagram](image)

FIG. 1: Representative cut Feynman diagrams responsible for the quark reaction $c\bar{c}(1S_0^{(1)}) \rightarrow c\bar{c}(1S_0^{(1)})$ through NNLO in $\alpha_s$. The vertical dashed line denotes the Cutkosky cut.

Roughly 1700 3-loop forward-scattering diagrams, divided into 4 distinct cut topologies; Cutkosky rule is imposed.
Employ a well-known trick to deal with phase-space type integrals

Key technique: using IBP to deal with phase-space integral

\[ \int \frac{d^D p_i}{(2\pi)^D} 2\pi i \delta^+ (p_i^2) = \int \frac{d^D p_i}{(2\pi)^D} \left( \frac{1}{p_i^2 + i\varepsilon} - \frac{1}{p_i^2 - i\varepsilon} \right). \]

duction. Finally, we end up with 93 MIs for the “Double Virtual” type of diagrams, 89 MIs for the “Virtual-Real” type of diagrams, and 32 MIs for “Double Real” type of diagrams, respectively. To the best of our knowledge, this work represents the first application of the trick (4) in higher-order calculation involving quarkonium.
The nontrivial aspects of the calculation

Encounter some rather time-consuming MIs using sector decomposition method (Fiesta)

Roughly speaking, $10^5$ CPU core hour is expensed; Run numerical integration at the GuangZhou Tianhe Supercomputer Center/China Grid.

Explicitly verify the cancellation of IR poles among the 4 types of cut diagrams. Starting from the $\frac{1}{\epsilon^4}$ poles, observe the exquisite cancelation until $\frac{1}{\epsilon}$
Our key results

\[ f_2 = \hat{f}_2 + \frac{3\beta_0^2}{16} \ln^2 \frac{\mu_R^2}{4m^2} + \left( \frac{\beta_1}{8} + \frac{3}{4} \beta_0 \hat{f}_1 \right) \ln \frac{\mu_R^2}{4m^2} - \pi^2 \left( C_F^2 + \frac{C_A C_F}{2} \right) \ln \frac{\mu_A^2}{m^2}, \]  

(5)

Same IR divergence as \( \eta_c \rightarrow 2\gamma \)!

\[ \hat{f}_2 = -0.799(13)N_c^2 - 7.4412(5)n_L N_c - 3.6482(2)N_c + 0.37581(3)n_L^2 + 0.56165(5)n_L + 32.131(5) - 0.8248(3) \frac{n_L}{N_c} - \frac{0.67105(3)}{N_c} - \frac{9.9475(2)}{N_c^2}. \]  

(6)

garms. Concretely, \( \hat{f}_2 = -50.1(1) \) for \( \eta_c \) hadronic decay, and \(-69.5(1) \) for \( \eta_b \) decay. For completeness, here we also enumerate the numerical values of the non-logarithmic parts of \( f_1 \) and \( g_1 \) in (3): \( \hat{f}_1 = 10.62, \hat{g}_1 = 16.20 \) for \( \eta_c \) hadronic decay; \( \hat{f}_1 = 9.73, \hat{g}_1 = 15.06 \) for \( \eta_b \) decay.

Validate the NRQCD factorization for S-wave onium inclusive decay at NNLO!

We also obtain the following RGE for the leading 4-fermion NRQCD operator:

\[ \frac{d\langle O_1(1S_0) \rangle_{\eta_c}}{d \ln \mu_A^2} = \alpha_s^2 \left( C_F^2 + \frac{C_A C_F}{2} \right) \langle O_1(1S_0) \rangle_{\eta_c} - 4 \frac{\alpha_s}{3\pi} C_F \frac{\langle P_1(1S_0) \rangle_{\eta_c}}{m^2} + \cdots, \]  

(7)
Phenomenological study: hadronic width

Input parameters:

\[ \langle \mathcal{O}_1(1S_0) \rangle_{\eta_c} = 0.470 \text{ GeV}^3, \quad \langle v^2 \rangle_{\eta_c} = \frac{0.430 \text{ GeV}^2}{m_c^2}, \]
\[ \langle \mathcal{O}_1(1S_0) \rangle_{\eta_b} = 3.069 \text{ GeV}^3, \quad \langle v^2 \rangle_{\eta_b} = -0.009. \quad (9) \]

PDG values:

\[ \Gamma_{\text{had}}(\eta_c) = 31.8 \pm 0.8 \text{ MeV}, \]
\[ \Gamma_{\text{had}}(\eta_b) = 10^{+5}_{-4} \text{ MeV} \]

FIG. 2: The predicted hadronic widths of \( \eta_c \) (top) and \( \eta_b \) (bottom) as functions of \( \mu_R \), at various level of accuracy in \( \alpha_s \) and \( v \) expansion. The horizontal blue bands correspond to the measured hadronic widths taken from PDG 2016 [4], with \( \Gamma_{\text{had}}(\eta_c) = 31.8 \pm 0.8 \text{ MeV} \) and \( \Gamma_{\text{had}}(\eta_b) = 10^{+5}_{-4} \text{ MeV} \). The label “LO” represents the NRQCD prediction at the lowest-order \( \alpha_s \) and \( v \), and the label “NLO” denotes the “LO” prediction plus the \( \mathcal{O}(\alpha_s^2) \) perturbative correction, while the label “NNLO” signifies the “NLO” prediction plus the \( \mathcal{O}(\alpha_s^4) \) perturbative correction. The label “vLO” represents the “LO” prediction together with the tree-level order-\( v^2 \) correction, and the label “vNLO” designates the “vLO” prediction supplemented with the relative order-\( \alpha_s \) and order-\( \alpha_s v^2 \) correction, while the label “vNNLO” refers to the “vNLO” prediction further supplemented with the order-\( \alpha_s^2 \) correction. The green bands are obtained by varying \( \mu_A \) from 1 GeV to twice heavy quark mass, and the central curve inside the bands are obtained by setting \( \mu_A \) equal to heavy quark mass.
Phenomenological study of $\text{Br}(\eta_{c,b} \to \gamma\gamma)$, Non-Perturbative matrix elements cancel out

For $\eta_c$ more than $10\sigma$ discrepancy!

$$\text{Br}(\eta_c \to \gamma\gamma) = \frac{8\alpha_s^2}{9\alpha_s^2} \left\{ 1 - \frac{\alpha_s}{\pi} \left[ 4.17 \ln\frac{\mu_R^2}{4m_c^2} + 14.00 \right] \right.$$  
$$+ \frac{\alpha_s^2}{\pi^2} \left[ 4.34 \ln^2\frac{\mu_R^2}{4m_c^2} + 22.75 \ln\frac{\mu_R^2}{4m_c^2} + 78.8 \right] \right.$$  
$$+ 2.24 \langle v^2 \rangle_{\eta_c} \left( \frac{\alpha_s}{\pi} \right) \right\}, \quad (10a)$$

$$\text{Br}(\eta_b \to \gamma\gamma) = \frac{\alpha_s^2}{18\alpha_s^2} \left\{ 1 - \frac{\alpha_s}{\pi} \left[ 3.83 \ln\frac{\mu_R^2}{4m_b^2} + 13.11 \right] \right.$$  
$$+ \frac{\alpha_s^2}{\pi^2} \left[ 3.67 \ln^2\frac{\mu_R^2}{4m_b^2} + 20.30 \ln\frac{\mu_R^2}{4m_b^2} + 85.5 \right] \right.$$  
$$+ 1.91 \langle v^2 \rangle_{\eta_b} \left( \frac{\alpha_s}{\pi} \right) \right\}. \quad (10b)$$

To date most refined predictor for $\eta_b \to \gamma\gamma$

$$\text{Br}(\eta_b \to \gamma\gamma) = (4.8 \pm 0.7) \times 10^{-5},$$
Summary

- Investigated NNLO QCD corrections to $\gamma\gamma^* \rightarrow \eta_c$, $(\chi_{c0,2} \rightarrow 2\gamma)$, $\eta_c \rightarrow LH$. Observe significant NNLO corrections. Alarming discrepancy with the existing measurements.

- Perturbative expansion seems to have poor convergence behavior for charmonium

- Perturbative expansion bears much better behavior for bottomonium
Personal biased perspectives

Maybe Nature is just not so mercy to us:

The charm quark is simply not heavy enough to warrant the reliable application of NRQCD to charmonium, just like one cannot fully trust HQET to cope with charmed hadron

Symptom: mc is not much greater than $\Lambda_{QCD}$

  Bigger value of $\alpha_s$ at charm mass scale

But we should still trust NRQCD to be capable of rendering qualitatively correct phenomenology for charmonium

We may need be less ambitious for soliciting precision predictions
Thanks for your attention!

Thanks PKU for warm hospitality!