Analytical estimation of the effects of crossing angle on the luminosity of an e\textsuperscript{+}e\textsuperscript{-} circular collider

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Abstract

Based on the theory established in Nucl. Instr. and Meth. A 463 (2001) 50, in this note we investigate the effects of the crossing angle on the luminosity of an e\textsuperscript{+}e\textsuperscript{-} circular collider. Analytical expression for the luminosity reduction factor is established and compared with simulation result. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

To increase the luminosity of a circular collider one could run in the multibunch operation mode with a finite collision crossing angle. Different from the head-on collision discussed in Ref. [1], the transverse kick received by a test particle due to the space charge field of the counter-rotating bunch will depend on its longitudinal position with respect to the center of the bunch which the test particle belongs to. In this note, we consider first a flat beam colliding with another flat beam with a half crossing angle of $\phi$ in the horizontal plane. Due to the crossing angle, the two curvilinear coordinates of the two colliding beams at the interaction point will no longer coincide. The detailed discussion about the coordinate transformation can be found in Ref. [2]. When the crossing angle is not too large one has

$$x^* = x + z\phi$$  (1)

where $x^*$ is the horizontal displacement of the test particle to the center of the colliding bunch, $z$ and $x$ are the longitudinal and horizontal displacements of the test particle from the center of the bunch to which it belongs. Now we recall Eq. (17) in Ref. [1] which describes the Hamiltonian of the horizontal motion of a test particle in the head-on collision mode

$$H_x = \frac{p_x^2}{2} + \frac{K_x(s)}{2}x^2 + \frac{N_\text{ref}}{2g^*} \left( \frac{1}{\sigma_x^2}x^2 - \frac{1}{12\sigma_x^4}x^4 \right)$$

$$+ \frac{1}{180\sigma_x^6}x^6 - \frac{1}{3360\sigma_x^8}x^8 + \cdots$$

$$\times \sum_{k=-\infty}^{\infty} \delta(s - kL) \quad (FB).$$  (2)
By inserting Eq. (1) into Eq. (2) we get
\[
H_x = \frac{p_x^2}{2} + \frac{K_x(s)}{2}v_x^2 + \frac{N_0r_0}{2\gamma^*_s} \left( \frac{1}{\sigma^2_x}(x + z\phi)^2 \right. \\
- \frac{1}{12\sigma^4_x}(x + z\phi)^4 + \frac{1}{180\sigma^6_x}(x + z\phi)^6 \\
- \frac{1}{3360\sigma^8_x}(x + z\phi)^8 + \cdots \\
\times \sum_{k=-\infty}^{\infty} \delta(s - kL) \quad (FB).
\]
(3)

Since the test particle can occupy a definite \( z \) within the bunch according to a certain probability distribution, say Gaussian, it is reasonable to replace \( z \) in Eq. (3) by \( \sigma_z \), and in this way we reduce a two dimensional Hamiltonian expressed in Eq. (3) into a one dimensional one. What should be noted is that Eq. (3) takes only the test particle’s longitudinal position into consideration which is regarded as a small perturbation to the head-on collision case, and the geometrical effect will be included later. To simplify our analysis we consider only the lowest synchrobetatron non-linear resonance, i.e., \( 3Q_x + Q_y = p \) (where \( Q_y \) is the synchrotron oscillation tune, and \( p \) is an integer) which turns out to be the most dangerous one [3,4]. Following the same procedure as in Section 4 of Ref. [1], one finds the dynamic aperture due to the lowest synchrobetatron non-linear resonance as follows:

\[
A_{syn-beta,x}(s) = \left( \frac{2\beta_x(s)}{3\beta_x(s)l_p} \right)^{1/2}\frac{2\gamma^*_s\sigma^4_x}{N_0r_0\sigma_z\phi}
\]
(4)

and

\[
R_{syn-beta,x} = \frac{A_{syn-beta,x}(s)^2}{\sigma_x(s)^2} = \frac{2}{3\pi^2} \left( \frac{1}{\xi_x\phi} \right)^2
\]
(5)
where \( \Phi = (\sigma_z/\sigma_x)\phi \) is Piwinski angle. Now we are facing a problem of how to combine the two effects: the principal vertical beam–beam effect and the horizontal crossing angle induced perturbation. To solve this problem we assume that the total beam lifetime due to the vertical and the horizontal crossing angle beam–beam effects can be expressed as

\[
\tau_{bb, total}^* = \frac{\tau_y^* + \tau_y^*}{4} \left( \frac{1}{R_{y,8,FB}} + \frac{1}{R_{syn-beta,x}} \right)^{-1} \\
\times \exp \left( \frac{1}{1/R_{y,8,FB} + 1/R_{syn-beta,x}} \right) \quad (FB).
\]
(6)

where \( R_{y,8,FB} \) corresponds to Eq. (31) of Ref. [1]. After the necessary preparations, we can try to answer two frequently asked questions. Firstly, for a machine working at the head-on collision beam–beam limit, how the beam lifetime depends on the crossing angle? Secondly, for a finite crossing angle, to keep the beam lifetime the same as that of the head-on collision at the beam–beam limit, how much one has to operate the machine below the designed head-on peak luminosity? To answer the first question we define a lifetime reduction factor

\[
R(\Phi) = \frac{\tau_{bb, total}^*}{\tau_{bb,y}^*} \quad (FB)
\]
(7)

where \( \tau_{bb,y}^* \) is given in Eq. (43) of Ref. [1], and \( R(\Phi) \) will tell us to what extent one can increase \( \Phi \). Concerning the second question, one can imagine to reduce the luminosity at beam–beam limit by a factor of \( f(\Phi) \) in order to keep the lifetime the same as that without the crossing angle. Physically, from Eq. (6) one requires

\[
\left( \frac{A_{syn-beta,x}(s)^2}{\sigma_x(s)^2} \right)^{-1} + \left( \frac{A_{dy,8,FB}(s)^2}{\sigma_y(s)^2} \right)^{-1} = \left( \frac{A_{dy,head-on,8,FB}(s)^2}{\sigma_y(s)^2} \right)^{-1} \quad (FB).
\]
(8)

Mathematically, one has to solve the following equation to find the peak luminosity reduction factor \( f(\Phi) \):

\[
\frac{3\pi^2\xi^2_{x,design,FB}f(\Phi)^2\Phi^2}{2} + \frac{\sqrt{2\pi\xi_{y,max,FB}f(\Phi)}}{3} = \frac{\sqrt{2\pi\xi_{y,max,FB}}}{3} \quad (FB)
\]
(9)
depend on Piwinski angle where $a = 3\pi^2 \xi_x,\text{design},FB \Phi^2 / 2$, $b_0 = c_0 = \sqrt{2\pi} \xi_{y,\text{max},FB} / 3$, and $\xi_{y,\text{max},FB} \approx 0.0447$. In fact, $f(\Phi)$ corresponds to the luminosity reduction due to the synchrobetatron resonance, and to find out the total luminosity reduction factor, one has to include the geometrical effects [5,6]. The total luminosity reduction factor can be expressed as follows:

$$f(\Phi) = f(\Phi)(1 + \Phi^2)^{-1/2} \quad \text{(FB)}$$

where hourglass effect is not taken into account (i.e., $\beta_{y,\text{IP}} > \sigma_z$). Taking KEKB factory as an interesting example [7], one has $\sigma_x = 90 \, \mu m$, $\sigma_z = 0.4 \, cm$, $\Phi = 11 \, mrad$, $\Phi = 0.49$, $\xi_x,\text{design},FB = 0.039$, and by putting $\Phi = 0.49$ into Eq. (10) one finds $F(0.49) = 83.5\%$ which is very close to a three dimensional numerical simulation result, i.e., 85% of designed head-on luminosity, given in Ref. [8].

In Figs. 1 and 2 we show how $R(\Phi)$ and $F(\Phi)$ depend on Piwinski angle where $\xi_x,\text{design},FB = 0.039$ has been used in Fig. 2. Finally, when the crossing angle is in the vertical plane or the beam is round, one gets

$$F(\Phi) = \frac{1}{3\pi^2} \left( \frac{r}{\xi_y \Phi} \right)^2 \quad \text{(RB)}$$

where $r = \sigma_y / \sigma_x$ and $\Phi = (\sigma_z / \sigma_x) \Phi$ as defined before. Replacing $\mathcal{R}_{\text{syn-beta,x}}$ in Eq. (6) by Eq. (12) or Eq. (13) and following the same procedure shown above one can easily make the corresponding discussion about the luminosity reduction effects. What should be remembered is that the geometrical luminosity reduction factors for the vertical crossing angle and the round beam cases are $(1 + (\Phi/r)^2)^{-1/2}$ and $(1 + \Phi^2)$, respectively.

Finally, we discuss briefly the choice of tunes (working point). Limited to one IP and the flat beam case, based on the original work of Bassetti (LNF-135, Frascati, Italy), Richter has shown [9] that the tune should be chosen just above an integer or half integer to make a best use of dynamic beta effect, and this conclusion has been experimentally observed in CESR [10]. Combining this information with that suggested by Eq. (49) of Ref. [1], one concludes that the tune $Q$ should be chosen in the regions $(0, 0.2)$ or $(0.5, 0.7)$ to obtain a maximum luminosity. If the collisions take place with a finite crossing angle some important synchrobetatron nonlinear resonances, such as $3Q_x \pm Q_y = p$, should also be avoided.

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**Fig. 1.** The lifetime reduction factor $R(\Phi)$ vs. Piwinski angle $\Phi$.

**Fig. 2.** The luminosity reduction factor $F(\Phi)$ vs. Piwinski angle $\Phi$. The curve is obtained by taking $\xi_x,\text{design},FB = 0.039$ (KEKB), and the dot is the numerical simulation result given in Ref. [8].
2. Conclusion

A finite crossing angle is shown to reduce further beam–beam limited dynamic aperture and beam–beam limited lifetime, and in consequence, the resultant luminosity is smaller than that of head-on collision case. An analytical expression for the luminosity reduction factor is established.

References