Analytical estimation of the beam–beam interaction limited dynamic apertures and lifetimes in e\(^+\)e\(^-\) circular colliders

J. Gao*

Laboratoire de L’Accélérateur Linéaire, IN2P3-CNRS et Université de Paris-Sud, B.P. 34, 91898 Orsay cedex, France

Received 30 July 2000; accepted 23 September 2000

Abstract

Physically speaking, the delta function like beam–beam nonlinear forces at interaction points act as a sum of delta function nonlinear multipoles. By applying the general theory established by J. Gao (Nucl. Instr. and Meth. A 451 (2000) 545), in this paper we examine analytically the beam–beam interaction limited dynamic apertures and the corresponding beam lifetimes for both the round and the flat beams. Relations between the beam–beam limited beam lifetimes and the beam–beam tune shifts are established, which show clearly why experimentally one has always a maximum beam–beam tune shift, \(\xi_{y,\text{max}}\), around 0.045 for e\(^+\)e\(^-\) circular colliders, and why one can use round beams to double this value approximately. Comparisons with some machine parameters are given. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 29.20.–c

Keywords: Beam–beam effect; Dynamic aperture; Circular colliders

1. Introduction

Beam–beam interactions in circular colliders have many influences on the performance of the machines, and the most important effect is that beam–beam interactions contribute to the limitations on dynamic apertures and beam lifetimes. Due to the importance of this subject, enormous efforts have been made to calculate incoherent and coherent beam–beam forces, to simulate beam–beam effects, to find the difference between flat and round colliding beams, and to establish analytical formulae to estimate the maximum beam–beam tune shift [1–19]. Physically speaking, the delta function like beam–beam nonlinear forces at interaction points (IPs) act as a sum of delta function nonlinear multipoles. In Ref. [20] we have established a general theory to study analytically in detail the delta function multipoles and their combined effects on the dynamic apertures in circular storage rings, and in this paper we will apply these general analytical formulae to the case of beam–beam interactions and find the corresponding beam dynamic apertures and beam lifetimes. Finally, we will show quantitatively why there exists a maximum beam–beam tune shift,

*Fax: +33-16907-1499.
E-mail address: gao@lal.in2p3.fr (J. Gao).
around 0.045 for flat beams in $e^+e^-$ circular colliders, and why this number can be almost doubled for round colliding beams. Applications to some machine parameters are also given. In this paper we will restrict ourselves to the discussion of $e^+e^-$ circular colliders since the treatment for the hadron colliders will be somewhat different and more difficult.

2. Beam–beam interactions

For two head-on colliding bunches, the incoherent kick felt by each particle can be calculated as [12]

$$\delta y' + i \delta x' = -\frac{N_e r_e}{\gamma_*} f(x, y, \sigma_x, \sigma_y)$$  (1)

where $x'$ and $y'$ are the horizontal and vertical slopes, $N_e$ is the particle population in the bunch, $r_e$ is the electron classical radius ($2.818 \times 10^{-15}$ m), $\sigma_x$ and $\sigma_y$ are the standard deviations of the transverse charge density distribution of the counter-rotating bunch at IP, $\gamma_*$ is the normalized particle’s energy, and $*$ denotes the test particle and the bunch to which the test particle belongs. When the bunch is Gaussian $f(x, y, \sigma_x, \sigma_y)$ can be expressed by Basseti–Erskine formula [9]:

$$f(x, y, \sigma_x, \sigma_y) = \sqrt{\frac{2\pi}{\sigma_x^2 - \sigma_y^2}} \left( \frac{x + iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right) - \exp \left( -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} \right) w \left( \frac{\sigma_x}{\sigma_y} x + i(\sigma_x/\sigma_y)y \right)$$  (2)

where $w$ is the complex error function expressed as

$$w(z) = \exp(-z^2)(1 - \text{erf}(-iz)).$$  (3)

For the round beam (RB) and the flat beam (FB) cases one has the incoherent beam–beam kicks expressed as [3,11,12]

$$\delta y' = \frac{2N_e r_e}{\gamma_* r} \left( 1 - \exp \left( -\frac{r^2}{2\sigma_y^2} \right) \right), \text{ (RB, } \sigma_x = \sigma_y = \sigma)$$  (4)

$$\delta x' = -\frac{2\sqrt{2}N_e r_e}{\gamma_* \sigma_x} \exp \left( -\frac{x^2}{2\sigma_y^2} \right) \int_0^{x/\sqrt{2}\sigma_y} \exp(u^2) du \text{, (FB, } \sigma_x \gg \sigma_y)$$  (5)

$$\delta y' = \frac{\sqrt{2}\pi N_e r_e}{\gamma_* \sigma_x} \exp \left( -\frac{x^2}{2\sigma_y^2} \right) \text{erf} \left( \frac{y}{\sqrt{2}\sigma_y} \right), \text{ (FB, } \sigma_x \gg \sigma_y)$$  (6)

where $r = \sqrt{x^2 + y^2}$. Now we want to calculate the average kick felt by the test particle since the possibility to find the transverse displacement of the test particle is not constant (in fact, the possibility function is the same as the charge distribution of the bunch to which the test particle belongs in lepton machines due to synchrotron radiations). In the following we assume that the transverse sizes for the two colliding bunches at IP are exactly the same. For the round beam case after averaging one gets [3,13]

$$\delta y'' = -\frac{2N_e r_e}{\gamma_* \bar{r}} \left( 1 - \exp \left( -\frac{\bar{r}^2}{4\sigma_y^2} \right) \right) \text{ (RB).}$$  (7)

Even this expression is found to be the same as that of the coherent beam–beam kick for round beams, one cannot misunderstand that we are finding coherent beam–beam kick, and the difference will be obvious when we treat the vertical motion of the flat beams. For the flat beam case, we will treat the horizontal and vertical planes separately. As far as the horizontal kick is concerned, the horizontal kick depends only on
one displacement variable just similar to the round beam case, we will use its coherent form expressed as follows [11,13]:

$$\delta x' = -\frac{2N \sigma x}{\gamma^* x} \exp\left( -\frac{x^2}{4\sigma x^2} \right) \int_0^{x/2\sigma_x} \exp(u^2) \, du \quad \text{(FB)}$$

where $\sigma_x$ in the incoherent formula in Ref. [11] has been replaced by $\Sigma_x = \sqrt{2}\sigma_x$ (for two identical Gaussian colliding beams) according to Hirata theorem demonstrated in Appendix A of Ref. [13]. As for the vertical kick, however, one has to make an average over Eq. (6) with the horizontal possibility distribution function of the test particle, and one gets [12]

$$\delta y' = -\frac{2\pi N \sigma c}{\gamma^* x} \left( \exp\left( -\frac{x^2}{2\sigma_x^2} \right) \right) \text{erf}\left( \frac{y}{\sqrt{2}\sigma_y} \right) \quad \text{(FB)}$$

where $\langle \cdot \rangle_x$ means the average over the horizontal possibility distribution function of the test particle, and for two identical colliding Gaussian beams $\langle \cdot \rangle_x = 1/\sqrt{2}$. It is obvious that Eq. (9) is not the expression for the coherent beam–beam kick. The average over Eqs. (4) and (6) is only a technical operation to simplify (or to make equivalence) a two dimensional problem to a one dimensional one. To study both round and flat beam cases, we expand $\delta y$ at $x = 0$ (for round beam we study only vertical plane since the formalism in the horizontal plane is the same), $\delta x'$ and $\delta y'$ expressed in Eqs. (7), (8) and (9), respectively, into Taylor series:

$$\delta y = \frac{N \sigma c}{\gamma^* x} \left( \frac{1}{2\sigma_x^2} y - \frac{1}{16\sigma_x^4} y^3 + \frac{1}{192\sigma_x^6} y^5 - \frac{1}{3072\sigma_x^8} y^7 + \frac{1}{61440\sigma_x^{10} y^9} - \frac{1}{1474560\sigma_x^{12} y^{11}} + \frac{1}{41287680\sigma_x^{14} y^{13}} - \cdots \right) \quad \text{(RB)}$$

$$\delta x' = -\frac{N \sigma c}{2\gamma^* x} \left( \frac{2}{\sigma_x^2} x - \frac{1}{3\sigma_x^4} x^3 + \frac{1}{30\sigma_x^6} x^5 - \frac{1}{420\sigma_x^8} x^7 + \frac{1}{7560\sigma_x^{10} x^9} - \frac{1}{166320\sigma_x^{12} x^{11}} + \frac{1}{4324320\sigma_x^{14} x^{13}} - \cdots \right) \quad \text{(FB)}$$

$$\delta y' = \frac{N \sigma c}{\sqrt{2}\gamma^* x} \left( \frac{2}{\sigma_x\sigma_y} y - \frac{1}{3\sigma_x^2\sigma_y^2} y^3 + \frac{1}{20\sigma_x^4\sigma_y^4} y^5 - \frac{1}{168\sigma_x^6\sigma_y^6} y^7 + \frac{1}{1728\sigma_x^8\sigma_y^8} y^9 - \frac{1}{21120\sigma_x^{10}\sigma_y^{10} y^{11}} + \frac{1}{299520\sigma_x^{12}\sigma_y^{12} y^{13}} - \cdots \right) \quad \text{(FB)}.$$

The differential equations of the motion of the test particle in the transverse planes can be expressed as

$$\frac{d^2 y}{ds^2} + K_y(s) y = -\frac{N \sigma c}{\gamma^* x} \left( \frac{1}{2\sigma_x^2} y - \frac{1}{16\sigma_x^4} y^3 + \frac{1}{192\sigma_x^6} y^5 - \frac{1}{3072\sigma_x^8} y^7 + \frac{1}{61440\sigma_x^{10} y^9} - \frac{1}{1474560\sigma_x^{12} y^{11}} + \frac{1}{41287680\sigma_x^{14} y^{13}} - \cdots \right)$$

$$\sum_{k=-\infty}^{\infty} \delta(s - kL) \quad \text{(RB)}$$
\[
\frac{d^2 x}{ds^2} + K_x(s) x
= -\frac{N e \epsilon_c}{2 \gamma_*} \left( \frac{2}{\sigma_x^3} x - \frac{1}{3 \sigma_x^3} x^3 + \frac{1}{30 \sigma_x^5} x^5 \right)
- \frac{1}{420 \sigma_x^8} x^7 + \frac{1}{7560 \sigma_x^{10}} x^9 - \frac{1}{166320 \sigma_x^{12}} x^{11}
+ \frac{1}{4324320 \sigma_x^{13}} x^{13} - \cdots 
\]
\[
\sum_{k=-\infty}^{\infty} \delta(s - kL) \quad \text{(FB)}
\]

\[
\frac{d^2 y}{ds^2} + K_y(s) y
= -\frac{N e \epsilon_c}{\sqrt{2} \gamma_*} \left( \frac{2}{\sigma_y} y - \frac{1}{3 \sigma_y^3} y^3 + \frac{1}{20 \sigma_y^5} y^5 - \frac{1}{168 \sigma_y^7} y^7 + \frac{1}{1728 \sigma_y^9} y^9 \right)
- \frac{1}{21120 \sigma_x \sigma_y^3} y^{11} + \frac{1}{299520 \sigma_x \sigma_y^{13}} y^{13} - \cdots 
\sum_{k=-\infty}^{\infty} \delta(s - kL) \quad \text{(FB)}
\]

where \( K_x(s) \) and \( K_y(s) \) describe the linear focusing of the lattice in the horizontal and vertical planes. The corresponding Hamiltonians are expressed as

\[
H = \frac{p_x^2}{2} + \frac{K_x(s)}{2} x^2 + \frac{N e \epsilon_c}{\gamma_*} \left( \frac{1}{4 \sigma_x^3} x^2 - \frac{1}{64 \sigma_x^4} x^4 + \frac{1}{1152 \sigma_x^6} x^6 - \frac{1}{24576 \sigma_x^8} x^8 + \cdots \right)
\times \sum_{k=-\infty}^{\infty} \delta(s - kL) \quad \text{(RB)}
\]

\[
H_x = \frac{p_x^2}{2} + \frac{K_x(s)}{2} x^2 + \frac{N e \epsilon_c}{2 \gamma_*} \left( \frac{1}{\sigma_x^3} x^2 - \frac{1}{120 \sigma_x^4} x^4 + \frac{1}{180 \sigma_x^6} x^6 - \frac{1}{3360 \sigma_x^8} x^8 + \cdots \right)
\times \sum_{k=-\infty}^{\infty} \delta(s - kL) \quad \text{(FB)}
\]

\[
H_y = \frac{p_y^2}{2} + \frac{K_y(s)}{2} y^2 + \frac{N e \epsilon_c}{\sqrt{2} \gamma_*} \left( \frac{1}{\sigma_y} y^2 - \frac{1}{120 \sigma_y^4} y^4 + \frac{1}{120 \sigma_y^6} y^6 - \frac{1}{1344 \sigma_y^8} y^8 + \cdots \right)
\times \sum_{k=-\infty}^{\infty} \delta(s - kL) \quad \text{(FB)}
\]

where \( p_x = \frac{dx}{ds} \) and \( p_y = \frac{dy}{ds} \).

3. Review of the general analytical formulae for dynamic apertures

In Ref. [20] we have studied analytically the one dimensional \((y = 0)\) dynamic aperture of a storage ring described by the following Hamiltonian:
\[
H = \frac{p^2}{2} + \frac{K(s)}{2} x^2 + \frac{1}{3! \beta_x} \frac{\partial^2 B_x}{\partial s^2} x^3 L \sum_{k=0}^{\infty} \delta(s - kL) + \frac{1}{4! \beta_x} \frac{\partial^3 B_x}{\partial s^3} x^4 L \sum_{k=0}^{\infty} \delta(s - kL) + \cdots
\]

(19)

where

\[
B_z = B_0(1 + x b_1 + x^2 b_2 + x^3 b_3 + x^4 b_4 + \cdots + x^{m-1} b_{m-1} + \cdots).
\]

(20)

The dynamic aperture corresponding to each multipole is given as

\[
A_{\text{dyna},2m,x}(s) = \sqrt{2 \beta_x(s)} \left( \frac{1}{m \beta_x^{m+1}(s_{2m})} \right)^{1/2} \left( \frac{\rho}{|b_{m-1}|L} \right)^{1/(m-2)}
\]

(21)

where \(s_{2m}\) is the location of the \(2m\)th multipole, \(\beta_x(s)\) is the beta function in \(x\) plane. Since these results are general, we have tried to avoid to assign the freedom \(x\) a specific name, such as horizontal, or vertical plane.

4. Beam–beam limited dynamic apertures

To make use of the general dynamic aperture formulae shown in Section 3, one needs only to find the equivalence relations by comparing three Hamiltonians expressed in Eqs. (16)–(18), with Eq. (19), and it is found by analogy that

\[
an_{m-1} = \frac{N_e r_e}{C_{m,RB} \gamma_s \sigma_x^{m}} \quad (\text{RB})
\]

(22)

\[
an_{m-1} = \frac{N_e r_e}{C_{m,FB,x} 2 \gamma_s \sigma_x^{m-1}} \quad (\text{FB},x)
\]

(23)

\[
an_{m-1} = \frac{N_e r_e}{C_{m,FB,y} \sqrt{2 \gamma_s \sigma_x \sigma_y^{m-1}}} \quad (\text{FB},y)
\]

(24)

where \(C_{m,RB}, C_{m,FB,x},\) and \(C_{m,FB,y}\) are given in Table 1. Now by inserting Eqs. (22)–(24) into Eq. (21) one can calculate dynamic apertures of different multipoles due to nonlinear beam–beam forces. For example, one can get the dynamic apertures due to the beam–beam octupole nonlinear force:

\[
A_{\text{dyna},8,y}(s) = \sqrt{\frac{\beta_y(s)}{\beta_y(s_{IP})}} \sqrt{\frac{\rho}{|b_3|L}} \left( \frac{16 \gamma_s \sigma_x^4}{N_e r_e} \right)^{1/2}
\]

(25)

(\text{RB})

\[
A_{\text{dyna},8,x}(s) = \sqrt{\frac{\beta_x(s)}{\beta_x(s_{IP})}} \sqrt{\frac{\rho}{|b_3|L}} \left( \frac{6 \gamma_s \sigma_x^4}{N_e r_e} \right)^{1/2}
\]

(26)

(\text{FB})

Table 1

Summary of multipole coefficients

<table>
<thead>
<tr>
<th>(m)</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{m,RB})</td>
<td>16</td>
<td>192</td>
<td>3072</td>
<td>61440</td>
<td>1474560</td>
<td>41287680</td>
</tr>
<tr>
<td>(C_{m,FB,x})</td>
<td>3</td>
<td>30</td>
<td>420</td>
<td>7560</td>
<td>166320</td>
<td>4324320</td>
</tr>
<tr>
<td>(C_{m,FB,y})</td>
<td>3</td>
<td>20</td>
<td>168</td>
<td>1728</td>
<td>21120</td>
<td>299520</td>
</tr>
</tbody>
</table>
\[ A_{\text{dyna}, s, p}(s) = \frac{\sqrt{\beta_y(s)}}{\beta_y(s_{\text{IP}})} \sqrt{\frac{\rho}{|b_2|L}} = \frac{\sqrt{\beta_y(s)}}{\beta_y(s_{\text{IP}})} \left( \frac{3 \sqrt{2 \gamma_s \sigma_x \sigma_y}}{N_{e\epsilon}} \right)^{1/2} \]  

(FB)  

(27)

where \( s_{\text{IP}} \) is the IP position. Given the dynamic aperture of the ring without the beam–beam effect as \( A_{x,y} \), the total dynamic aperture including the beam–beam effect can be estimated usually as

\[ A_{\text{total, } x,y}(s) = \frac{1}{\sqrt{(1/A_{x,y}(s^2) + (1/A_{bb, x,y}(s^2))^2)}}. \]  

(28)

In the following we will consider the case of \( A_{\text{total, } x,y}(s) \approx A_{bb, x,y}(s) \). If we measure the beam–beam interaction limited dynamic apertures by the beam sizes (the normalized dynamic aperture), one gets

\[ \mathcal{R}_{y,k} = \frac{A_{\text{dyna}, x,y}(s)}{\sigma_{*,k}(s)} = \left( \frac{16 \gamma_s \sigma_x^2}{N_{e\epsilon} \beta_y(s_{\text{IP}})} \right)^{1/2} \]  

(RB)  

(29)

\[ \mathcal{R}_{x,k} = \frac{A_{\text{dyna}, x,y}(s)}{\sigma_{*,x}(s)} = \left( \frac{6 \gamma_s \sigma_y^2}{N_{e\epsilon} \beta_y(s_{\text{IP}})} \right)^{1/2} \]  

(FB)  

(30)

\[ \mathcal{R}_{y,k} = \frac{A_{\text{dyna}, x,y}(s)}{\sigma_{*,y}(s)} = \left( \frac{3 \sqrt{2 \gamma_s \sigma_x \sigma_y}}{N_{e\epsilon} \beta_y(s_{\text{IP}})} \right)^{1/2} \]  

(FB).  

(31)

Recalling and using the definitions of the beam–beam tune shifts \( \xi_x \) and \( \xi_y \) in Eqs. (32) and (33):

\[ \xi_x = \frac{N_{e\epsilon} \beta_{x, IP}}{2 \pi \gamma_s \sigma_x (\sigma_x + \sigma_y)} \]  

(32)

\[ \xi_y = \frac{N_{e\epsilon} \beta_{y, IP}}{2 \pi \gamma_s \sigma_y (\sigma_x + \sigma_y)} \]  

(33)

one can simplify the above defined normalized dynamic apertures. As general results one finds

\[ \mathcal{R}_{y,2m} = \frac{A_{\text{dyna}, 2m, y}(s)}{\sigma_{*,y}(s)} = \left( \frac{2^{(m-2)/2} C_{m, \text{RB}}}{4\pi \sqrt{m \xi_y^*}} \right)^{1/m-2} \]  

(RB)  

(34)

\[ \mathcal{R}_{x,2m} = \frac{A_{\text{dyna}, 2m, x}(s)}{\sigma_{*,x}(s)} = \left( \frac{2^{(m-2)/2} C_{m, \text{FB}, x}}{2\sqrt{m \pi \xi_x^*}} \right)^{1/m-2} \]  

(FB)  

(35)

\[ \mathcal{R}_{y,2m} = \frac{A_{\text{dyna}, 2m, y}(s)}{\sigma_{*,y}(s)} = \left( \frac{2^{(m-2)/2} C_{m, \text{FB}, y}}{\sqrt{2m \pi \xi_y^*}} \right)^{1/m-2} \]  

(FB).  

(36)

Obviously, the normalized beam–beam effect limited dynamic apertures are determined only by the beam–beam tune shifts. The impact of this discovery will be more appreciated later. When the higher order multipoles effects \( 2m > 8 \) can be neglected, Eqs. (25)–(27) give very good approximations dynamic apertures limited by one beam–beam IP. If there are \( N_{\text{IP}} \) interaction points in a ring the dynamic apertures described in Eqs. (25) and (27) will be reduced by a factor of \( \sqrt{N_{\text{IP}}} \) (if these \( N_{\text{IP}} \) interaction points can be regarded as independent).
5. Beam lifetime due to beam–beam interactions

We take the beam–beam limited dynamic aperture as the rigid mechanical boundary, i.e., those particles which walk beyond this virtual boundary will be regarded lost instantaneously. Based on this physical point of view we can use the beam quantum lifetime formula [21] to estimate the beam lifetime due to beam–beam effect:

\[
\tau_{bb} = \frac{\tau_y}{2} \left( \frac{\langle y^2 \rangle}{y_{\text{max}}} \right) \exp \left( \frac{y_{\text{max}}^2}{2 \langle y^2 \rangle} \right) = \frac{\tau_y}{2} \left( \frac{\sigma_y(s)}{A_{\text{dynamo},y}(s)} \right)^2 \exp \left( \frac{A_{\text{dynamo},y}(s)^2}{\sigma_y(s)} \right)
\]

(37)

where \( y_{\text{max}} \) is the boundary dimension, \( \langle \cdot \rangle \) denotes the average over the particle distribution, and \( \tau_y \) is the synchrotron radiation damping time in vertical plane. In Eq. (37) we have replaced the \( y_{\text{max}} \) and \( \langle y^2 \rangle \) by \( A_{\text{dynamo},y} \) and \( \sigma_y^2 \) (instead of \( 2\sigma_y^2 \)), respectively, here the meanings of \( y^2_{\text{max}} \) and \( \langle y^2 \rangle \) are the average of the square of the amplitudes over its distribution function and the square of the maximum amplitude, respectively. If the beam–beam octupole nonlinear force dominates the dynamic aperture, by inserting Eqs. (29)–(31) into Eqs. (37), or inserting Eqs. (34)–(36), into Eq. (37) one gets

\[
\tau_{bb,y}^* = \frac{\tau_y^*}{2} \left( \frac{16\gamma_\ast \sigma_y^2}{N_\text{e} e \beta_y(s_{\text{IP}})} \right)^{-1} \exp \left( \frac{16\gamma_\ast^2 \sigma_y^4}{N_\text{e} e \beta_y(s_{\text{IP}})} \right)
\]

(RB)

(38)

\[
\tau_{bb,x}^* = \frac{\tau_x^*}{2} \left( \frac{6\gamma_\ast \sigma_x^2}{N_\text{e} e \beta_x(s_{\text{IP}})} \right)^{-1} \exp \left( \frac{6\gamma_\ast^2 \sigma_x^4}{N_\text{e} e \beta_x(s_{\text{IP}})} \right)
\]

(FB)

(39)

\[
\tau_{bb,y}^* = \frac{\tau_y^*}{2} \left( \frac{3\sqrt{2\gamma_\ast \sigma_x \sigma_y}}{N_\text{e} e \beta_y(s_{\text{IP}})} \right)^{-1} \exp \left( \frac{3\sqrt{2\gamma_\ast \sigma_x \sigma_y}}{N_\text{e} e \beta_y(s_{\text{IP}})} \right)
\]

(FB)

(40)

or

\[
\tau_{bb,y}^* = \frac{\tau_y^*}{2} \left( \frac{\sqrt{2\pi} \gamma_\ast}{\zeta_y} \right)^{-1} \exp \left( \frac{\sqrt{2\pi} \gamma_\ast}{\zeta_y} \right)
\]

(RB)

(41)

\[
\tau_{bb,x}^* = \frac{\tau_x^*}{2} \left( \frac{\sqrt{2\pi} \gamma_\ast}{\zeta_x} \right)^{-1} \exp \left( \frac{\sqrt{2\pi} \gamma_\ast}{\zeta_x} \right)
\]

(FB)

(42)

\[
\tau_{bb,y}^* = \frac{\tau_y^*}{2} \left( \frac{3\sqrt{2\pi} \gamma_\ast}{\sqrt{2\pi} \gamma_\ast} \right)^{-1} \exp \left( \frac{3\sqrt{2\pi} \gamma_\ast}{\sqrt{2\pi} \gamma_\ast} \right)
\]

(FB)

(43)

More generally, one has

\[
\tau_{bb,2m_y}^* = \frac{\tau_y^*}{2} \left( \frac{2(m-2)\gamma_\ast C_{m,\text{RB}}}{4\pi \sqrt{m_{\gamma y}^{*2}}} \right)^{-2/m-2} \exp \left( \frac{2(m-2)\gamma_\ast C_{m,\text{RB}}}{4\pi \sqrt{m_{\gamma y}^{*2}}} \right)^{2/m-2}
\]

(RB)

(44)

\[
\tau_{bb,2m_x}^* = \frac{\tau_x^*}{2} \left( \frac{2(m-2)\gamma_\ast C_{m,\text{FB,x}}}{\pi^2 \sqrt{m_{\gamma x}^{*2}}} \right)^{-2/m-2} \exp \left( \frac{2(m-2)\gamma_\ast C_{m,\text{FB,x}}}{\pi^2 \sqrt{m_{\gamma x}^{*2}}} \right)^{2/m-2}
\]

(FB)

(45)
If we define the lifetime divided by the corresponding damping time as normalized beam lifetime, one finds that the beam–beam effect limited normalized lifetimes depend only on beam–beam tune shifts. Figs. 1 and 2 show the normalized beam lifetime with respect to the beam–beam tune shifts for both flat and round beams.

6. The maximum beam–beam tune shifts for flat and round beams

Now it is high time for us to discuss the maximum beam–beam tune shift problem. In literatures the term “maximum beam–beam tune shift” of a specific machine is not well defined. One of the reasonable definitions would be that the maximum beam–beam tune shift corresponding to a well defined minimum beam–beam limited lifetime. In this paper, we propose to take this well defined minimum beam–beam limited lifetime as 1 h (the idea is to reduce Eq. (28) to $A_{total}(s) \approx A_{bb}(s)$, and to have a machine still working!). Assuming that for both round and flat beam cases one has the same $\tau_{y}$, from Eqs. (41)–(43) one finds the following relations:

$$\tau_{bb,2m,y} = \frac{\tau_{y}}{2} \left( \frac{2(m-2)/2C_{m,FB,y}}{\pi \sqrt{2m \xi_{y}}^*} \right)^{-2/m-2} \exp \left( \frac{2(m-2)/2C_{m,FB,y}}{\pi \sqrt{2m \xi_{y}}^*} \right)^{2/m-2} \text{ (FB).}$$  (46)

If we define the lifetime divided by the corresponding damping time as normalized beam lifetime, one finds that the beam–beam effect limited normalized lifetimes depend only on beam–beam tune shifts. Figs. 1 and 2 show the normalized beam lifetime with respect to the beam–beam tune shifts for both flat and round beams.

For the maximum beam–beam tune shifts, one finds:

$$\xi_{y,\text{max}}^{RB} = \frac{4\sqrt{2}}{3} \xi_{y,\text{max}}^{FB} = 1.89 \xi_{y,\text{max}}^{FB}$$  (47)

and

$$\xi_{x,\text{max}}^{FB} = \sqrt{2} \xi_{y,\text{max}}^{FB}.$$  (48)

It is proved theoretically why round beam scheme can almost double the $\xi_{y,\text{max}}^{FB}$ of flat beam scheme, and why the vertical beam–beam tune shift reaches its limit earlier than the horizontal one. Quantitatively, taking $\tau_{y} = 30$ ms, one finds that $\xi_{y,\text{max},FB}(\tau_{bb} = 1 \text{ h}) = 0.0447$, $\xi_{x,\text{max},FB}(\tau_{bb} = 1 \text{ h}) = 0.0632$, and $\xi_{y,\text{max},RB}(\tau_{bb} = 1 \text{ h}) = 0.0843$.

Now we investigate how the order of nonlinear resonance affects the maximum beam–beam tune shift. By using Eqs. (44)–(46), and assuming that $\tau_{x} = \tau_{y}$, one gets the maximum beam–beam tune shift with
respect to the order of nonlinear resonance, $m$, as shown in Fig. 3, where each maximum beam–beam tune corresponds to each dominating multipole resonance. For flat beams, it is obvious that if the horizontal tune is not well chosen, the $\xi_{x,FB}$ can be 0.032 instead of 0.0447, however, if the vertical resonances have been successfully avoided before $\xi_{x,FB}$ reaches its limit, one could possibly obtain $\xi_{x,max,FB}(\tau_{bb} = 1 \, h) = 0.0632$ even difficulty. What should be stressed is that in choosing the working point in the tune diagram, one has to pay attention to the nonlinear resonances of order as high as 14. To explain qualitatively why the maximum beam–beam tune shifts for both round and flat beams seem to be limited by the lowest order of resonance, i.e., the $\frac{1}{4}$ resonance, we have plotted in Fig. 4 the sum of the multipole strengths from $m = 4$ to 14 assuming that they have the same strength, as expressed below:

$$A(Q_y) = \sum_{m=4}^{14} (-1)^{m/2} \sin(2\pi m Q_y).$$  \hspace{1cm} (49)

In the same figure we have plotted also the first term (octupole) in this summation with two opposite phases as compare references, and it is obvious that except two regions of $Q_y$, (0.2–0.3) and (0.7–0.8), one has always that the amplitude of the sum is almost the same as that of the octupole term, and in this case the dangerous $Q_y$ values are 0.225, 0.275, 0.725 and 0.775. Another reason for the lowest resonance dominating is that the lower the resonance order the more stable the resonance facing to the phase perturbations.

In this paper, under the assumption that the two colliding beams always have the same transverse dimensions, we have arrived at the beam–beam effect determined lifetimes expressed in Eqs. (44)–(46). For a given minimum normalized (with respect to the damping time) beam lifetime one gets universal maximum beam–beam tune shift values corresponding to different cases. In a real machine the situation can be more complicated, such as the flip-flop phenomenon which breaks the symmetry assumed above, and in this case one can continue the discussion starting from Eqs. (25)–(27) by differentiating $\sigma_y$ from $\sigma_{*y}$, and by replacing $\sigma, \sigma_x$ by $\Sigma/\sqrt{2}, \Sigma_x/\sqrt{2}$, respectively, where $\Sigma = \sqrt{\sigma_{*y}^2 + \sigma_x}$ and $\Sigma_x = \sqrt{\sigma_{*x}^2 + \sigma_x^2}$. We will not, however, continue our discussions in this direction in an exhaustive way.
7. Applications to some machines

Let us look at three machines, PEP-II B-Factory [22] and DAΦNE [23], and BTCF [24], and the first two have been put to operation. The relevant machine parameters are shown in Table 2. Figs. 5 and 6 give the theoretical estimations for the beam–beam limited beam lifetimes in both PEP-II Low and High Energy Rings (LER and HER). Figs. 7 and 8 show the beam lifetimes versus the beam–beam tune shifts in both LER and HER. It is obvious that the nominal charge in the bunch of HER is close to the limit which sets the beam lifetime in low energy ring, however, the beam lifetime in HER is much longer than that in LER. The theoretical results consist with the experimental observation [22]. Fig. 9 shows the beam lifetime prediction for the DAΦNE $e^+e^-$ collider with single IP. Finally, we study the beam–beam limited beam lifetime in BTCF (standard scheme) and the theoretical result is given in Fig. 10 where the dot indicates the designed beam lifetime.

8. Conclusion

In this paper we have established analytical formulae for the beam–beam interaction limited dynamic apertures and beam lifetimes in $e^+e^-$ circular colliders for both round and flat beam cases. It is shown

Table 2
Machine parameters

<table>
<thead>
<tr>
<th>Machine</th>
<th>$N_e$</th>
<th>$\beta_{x,IP}$ (cm)</th>
<th>$\sigma_{x,IP}$ (µm)</th>
<th>$\gamma$</th>
<th>$\tau_y$ (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PEP-II LER</td>
<td>$6 \times 10^{10}$</td>
<td>1.5</td>
<td>157</td>
<td>4.78</td>
<td>6067</td>
</tr>
<tr>
<td>PEP-II HER</td>
<td>$2.8 \times 10^{10}$</td>
<td>1.5</td>
<td>157</td>
<td>4.78</td>
<td>17613</td>
</tr>
<tr>
<td>DAΦNE</td>
<td>$8.9 \times 10^{10}$</td>
<td>4.5</td>
<td>2100</td>
<td>21</td>
<td>998</td>
</tr>
<tr>
<td>BTCF</td>
<td>$1.4 \times 10^{11}$</td>
<td>1</td>
<td>450</td>
<td>9</td>
<td>3914</td>
</tr>
</tbody>
</table>

Fig. 5. The beam lifetime due to the beam–beam interaction versus the particle population in the bunch in the low energy ring of PEP-II.

Fig. 6. The beam lifetime due to the beam–beam interaction versus the particle population in the bunch in the high energy ring of PEP-II.
analytically why for flat colliding beams one has always $\xi_y$ around 0.045 and why this value can be almost doubled by using round beams. Applications to the machines, such as PEP-II, DAΦNE, and BTCF have been made.

Acknowledgements

The author thanks J. Maissinski, A. Tkachenko, and J. LeDuff for useful discussions.
References