# Linearized Gravity

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# Gravitons

- We don’t know how to unify gravity with the other forces (Strong and Electroweak forces) in a consistent way.
- In fact we do not have a consistent formalism that can unify gravity and quantum field theory itself.
- However, we do have a low energy effective approach to quantize gravity – Great practical use , e.g. Gravitational wave calculations.
- Gravitons (the force carriers of gravity) are massless spin-2 particles – Why?
- So how do we quantize gravity – Simply promote the metric to a field.

# Setting the stage.

- Our starting point is the Einstein Hilbert action $S = \int d^4x \, \mathcal{L}_{4D}$

$$\mathcal{L}_{4D} = \sqrt{-\det \overline{g}_{4D}} \left\{ \frac{2}{\kappa_{4D}^2}(R_{4D} - 2\Lambda_{4D}) + \mathcal{L}_{matter} \right\}$$

$$g^{\mu\nu}\tilde{g}_{\nu\rho} = \delta^{\mu}_{\rho}$$

Christoffel Symbol:

$$\Gamma^{\mu}_{\nu\sigma} \equiv \frac{1}{2}\tilde{g}^{\mu\alpha}\left[\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\alpha\nu} - \partial_\alpha g_{\nu\sigma}\right]$$

Riemann Curvature:

$$R^{\rho}{}_{\sigma\mu\nu} \equiv -\partial_\mu \Gamma^{\rho}{}_{\nu\sigma} + \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} - \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} + \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$$

Ricci Curvature:

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}$$

Ricci Scalar:

$$R_{4D} \equiv \tilde{g}^{\mu\nu}R_{\mu\nu}$$

$\Lambda$: Cosmological constant
For Simplicity we will set the 4D cosmological to zero.
In reality, evidence suggests otherwise.

$$\kappa_{4D} = 2/\overline{M}_{Pl}$$

$$\overline{M}_{Pl} \equiv \frac{M_{Pl}}{\sqrt{8\pi}} \equiv \frac{1}{\sqrt{8\pi G}} = 2.435 \times 10^{15}\ \text{TeV}$$

$$\kappa_{4D} = 8.214 \times 10^{-16}\ \text{TeV}^{-1}$$

# The weak field expansion

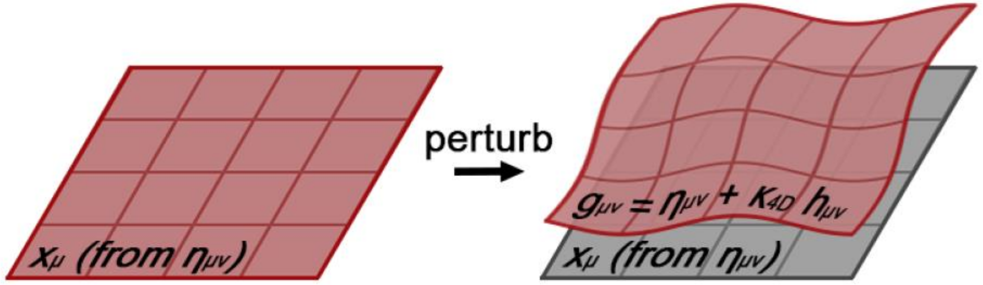
$$\kappa_{4D} = 8.214 \times 10^{-16}\ \text{TeV}^{-1}$$

- Expand the curved metric (g) about a flat background metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\eta = \text{Diag}(+1, -1, -1, -1)$$

- We assume $\kappa\, h_{\mu\nu}$ is small - a perturbative expansion in $\kappa$ is called the weak field expansion.
- We substitute this into the Lagrangian




$$\eta_{\mu\nu} = \text{Diag}(+,-,-,-) \quad \Big| \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa_{4\text{D}}\, h_{\mu\nu}$$

- Expansion of the inverse metric: $g^{\mu\nu}\tilde{g}_{\nu\rho} = \delta^{\mu}_{\rho}$

$$[\tilde{g}]^{\mu\nu} = \sum_{n=0}^{+\infty}(-\kappa)^n[\eta(\eta h)^n]^{\mu\nu}$$

$$\tilde{g}^{\alpha\beta} = \eta^{\alpha\beta} - \kappa h^{\alpha\beta} + \kappa^2[hh]^{\alpha\beta} - \kappa^3[hhh]^{\alpha\beta} + +\mathcal{O}(\kappa^4)$$

- Expansion of determinant of metric:

$$\begin{aligned}\sqrt{-g} &= \sqrt{-\det g} = \sqrt{-\det(\bar{g}+h)} \\ &= \sqrt{-\det(\bar{g})\det(1+\bar{g}^{-1}h)} \\ &= \sqrt{-\det(\bar{g})}\sqrt{\det(1+\bar{g}^{-1}h)} \\ &= \sqrt{-\det(\bar{g})}\exp\left[\frac{1}{2}\ln\left(\det\left(1+\bar{g}^{-1}h\right)\right)\right].\end{aligned}$$

$$\begin{aligned}\sqrt{-g} &= \sqrt{-\det\bar{g}}\exp\left[\frac{1}{2}\operatorname{Tr}\ln\left(1+\bar{g}^{-1}h\right)\right] \\ &= \sqrt{-\det\bar{g}}\exp\left[\frac{1}{2}\operatorname{Tr}\left(\bar{g}^{-1}h - \frac{1}{2}(\bar{g}^{-1}h)^2\right)\right] \\ &= \sqrt{-\det\bar{g}}\left[1+\frac{1}{2}\operatorname{Tr}(\bar{g}^{-1}h) - \frac{1}{4}\operatorname{Tr}(\bar{g}^{-1}h)^2 + \frac{1}{8}\operatorname{Tr}^2(\bar{g}^{-1}h)\right] \\ &= \sqrt{-\det\bar{g}}\left[1+\frac{1}{2}h^{\mu}_{\mu} - \frac{1}{4}\operatorname{Tr}(\bar{g}^{-1}h)^2 + \frac{1}{8}(h^{\mu}_{\mu})^2\right].\end{aligned}$$

$$\begin{aligned}\operatorname{Tr}(\bar{g}^{-1}h)^2 &= \operatorname{Tr}(g^{\alpha\beta}h_{\alpha\beta}g_{\beta\gamma}h^{\beta\gamma}) = \operatorname{Tr}(g^{\alpha\beta}g_{\beta\gamma}h_{\alpha\beta}h^{\beta\gamma}) \\ &= \operatorname{Tr}(\delta^{\alpha}_{\gamma}h_{\alpha\beta}h^{\beta\gamma}) = \operatorname{Tr}(h_{\alpha\beta}h^{\alpha\beta}).\end{aligned}$$

$$\sqrt{-g} = \sqrt{-\det\bar{g}}\left[1+\frac{1}{2}h^{\mu}_{\mu} - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta} + \frac{1}{8}(h^{\mu}_{\mu})^2\right]$$

$$= \prod_{n=1}^{+\infty}\sum_{m_n=0}^{+\infty}\frac{(-1)^{m_n}}{m_n!(2n)^{m_n}}(-\kappa)^{n\cdot m_n}\operatorname{tr}\left[(\eta h)^n\right]^{m_n}$$

# Degrees of freedom

- How many degrees of freedom does a symmetric rank -2 Lorentz tensor have? i.e. how many polarization states are possible for $h_{\mu\nu}$?
- We know that for a massless particle there can be only two polarization states.
- So we have too many degrees of freedom – diffeomorphism invariance (general coordinate invariance)$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \theta_\nu + \partial_\nu \theta_\mu$
- What should we do to get back the correct degrees of freedom?
- DeDonder (Harmonic) Gauge condition $\partial_\alpha h^\alpha_\mu = \frac{1}{2}\partial_\mu h$
- We end up with D(D-3)/2 degrees of freedom in D-dimensions.

# Connecting to Newtonian gravity

- Vary the metric and determine the equations of motion – Alternatively use the Euler-Lagrange equations.

$$R_{4D,\mu\nu} - \frac{g_{4D,\mu\nu}}{2}\left[R_{4D} - 2\Lambda_{4D}\right] = \frac{\kappa_{4D}^2}{4} T_{\mu\nu}$$

$$T_{\mu\nu} \equiv \left[-g_{4D,\mu\nu} + 2\frac{\delta}{\delta(g_{4D}^{-1})^{\mu\nu}}\right]\mathcal{L}_{matter}$$

Stress Energy Tensor –
the conserved current to
Which gravitons must couple

- Expanding in lowest order to the DeDonder Gauge

$$\frac{1}{2}\kappa_{4D}\,\partial^2 \underbrace{\left[h_{4D,\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(h_{4D})\right]}_{\equiv \bar{h}_{\mu\nu}} = \frac{\kappa_{4D}^2}{4} T_{\mu\nu}$$

- Acceleration of a point mass in the presence of a mass distribution:

$$\vec{a} = -\vec{\nabla}\phi \qquad\qquad \vec{\nabla}^2\phi = 4\pi G\rho$$

Supposing $T_{00} = \rho$, all other components of $T_{\mu\nu}$ vanish, and that the field is slow-varying in time, then

$$\vec{\nabla}^2\left[\frac{1}{4}\,\kappa_{4D}\,\overline{h}_{00}\right] = \frac{\kappa_{4D}^2}{8}\rho \qquad\qquad \vec{\nabla}^2\overline{h}_{ij} = \vec{\nabla}^2\overline{h}_{0j} = \vec{\nabla}^2\overline{h}_{i0} = 0$$

- From the geodesic equation we get

$$\vec{a} = -\Gamma^i{}_{00} = -(-\vec{\nabla})\left[-\frac{1}{2}\kappa_{4D}h_{4D,00}\right] = -\vec{\nabla}\left[\frac{1}{4}\kappa_{4D}\overline{h}_{00}\right]$$

- Comparing $\phi = \frac{1}{4}\kappa_{4D}\overline{h}_{00}$

$$4\pi G = \frac{\kappa_{4D}^2}{8} \qquad \Longrightarrow \qquad \kappa_{4D} = \sqrt{32\pi G}$$

Consistent with Newtonian Gravity.

The expanded lagrangian

$$\mathcal{L}_g = \mathcal{L}_g^0 + \kappa \mathcal{L}_g^1 + \kappa^2 \mathcal{L}_g^2 + \cdots,$$

$$\mathcal{L}_g^0 = -\frac{1}{4}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu},$$

$$\mathcal{L}_g^1 = \frac{1}{2}h^\alpha_\beta \partial^\mu h^\beta_\alpha \partial_\mu h - \frac{1}{2}h^\alpha_\beta \partial_\alpha h^\mu_\nu \partial^\beta h^\nu_\mu - h^\alpha_\beta \partial_\mu h^\nu_\alpha \partial^\mu h^\beta_\nu$$
$$+\frac{1}{4}h\partial^\beta h^\mu_\nu \partial_\beta h^\nu_\mu + h^\beta_\mu \partial_\nu h^\alpha_\beta \partial^\mu h^\nu_\alpha - \frac{1}{8}h\partial^\nu h \partial_\nu h,$$

$$\mathcal{L}_g^2 = \frac{1}{16}h^2\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + h^\lambda_\mu h^\nu_\beta \partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\nu} - \frac{1}{8}h^{\mu\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\alpha\beta}$$
$$-2h^{\lambda\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial_\alpha h^\mu_\beta + \frac{1}{2}hh^\lambda_\mu \partial_\lambda h^{\alpha\beta}\partial_\alpha h^\mu_\beta - \frac{1}{2}hh^\mu_\beta \partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\alpha\mu}$$
$$+h^\nu_\beta h^\mu_\nu \partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\alpha\mu} - \frac{1}{2}h_{\alpha\beta}h^{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial^\lambda h_{\mu\nu} + \frac{1}{2}h^\mu_\alpha h^\nu_\beta \partial_\lambda h^{\alpha\beta}\partial^\lambda h_\mu$$
$$-\frac{1}{4}hh^\lambda_\mu \partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + \frac{1}{2}h^{\lambda\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - h^\lambda_\beta h^\nu_\mu \partial_\lambda h^{\alpha\beta}\partial^\mu h_{\alpha\nu}$$
$$+\frac{1}{4}hh^\mu_\beta \partial_\lambda h\partial^\lambda h^\beta_\mu - \frac{1}{2}h^{\mu\nu}h_{\nu\beta}\partial_\lambda h\partial^\lambda h^\beta_\mu + \frac{1}{2}h^{\mu\nu}h_{\nu\beta}\partial_\lambda h\partial^\beta h^\lambda_\mu$$
$$-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial_\alpha h^\lambda_\beta - \frac{1}{32}h^2\partial_\lambda h\partial^\lambda h + \frac{1}{8}h^{\mu\nu}h_{\mu\nu}\partial_\lambda h\partial^\lambda h.$$

- The graviton propagator

$$iD_{\mu\nu\alpha\beta} = \frac{i}{q^2 + i\epsilon} P_{\mu\nu,\alpha\beta}$$

$$P_{\mu\nu,\alpha\beta} \equiv \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right]$$

- Coupling of matter to a graviton $\tau_{\mu\nu} = -\frac{i\kappa}{2}\left(p_\mu p'_\nu + p'_\mu p_\nu - g_{\mu\nu}[p \cdot p' - m^2]\right)$
- Calculate the matrix element for the scattering of two massive particles

$$M = \frac{1}{2}\tau_{\mu\nu}(q) D^{\mu\nu,\alpha\beta}(q)\tau_{\alpha\beta}(q)$$

- Take the non-relativistic limit

$$\frac{1}{2m_1}\frac{1}{2m_2}M = 4\pi G\frac{m_1 m_2}{q^2} \quad \Rightarrow \quad V(r) = -G\frac{m_1 m_2}{r}$$

- Consistency with Newtonian Gravity.

# Unitarity

- Linearized gravity is an effective theory of gravity with operators of dimension > 4. The obvious question to ask is : What is the range of validity of the theory?
- To answer this question formally we ask at what point does the theory stop being unitary?

$$S^\dagger S = S S^\dagger = 1.$$

$$S \equiv 1 + iT.$$

$$T^\dagger T = 2\mathrm{Im}T$$

Taking the matrix element for identical 2 body initial and final states and Inserting a complete set of states (including an integration over phase space)

$$\int \sum_n d\Pi_n \langle 2|T|n\rangle \langle n|T^\dagger|2\rangle = 2\mathrm{Im}\langle 2|T|2\rangle$$

$$\int \sum_n d\Pi_n \langle 2|T|n\rangle\langle n|T^\dagger|2\rangle = 2\mathrm{Im}\langle 2|T|2\rangle$$

$$\langle a|T|b\rangle \equiv (2\pi)^4\delta(p_{in} - p_{out})\mathcal{M}(a \to b)$$

$$\int d\Pi_{2'}|\mathcal{M}_{\mathrm{el}}(2 \to 2')|^2 + \sum_n \int d\Pi_n|\mathcal{M}_{\mathrm{inel}}(2 \to n)|^2 = 2\mathrm{Im}\left[\mathcal{M}_{\mathrm{el}}(2 \to 2)\right]$$

Sum is over all possible inelastic channels

Now perform a partial wave decomposition with Wigner-D functions which obey orthogonality relations

$$\mathcal{M}_{\mathrm{el}}(2 \to 2') = 16\pi e^{i(\lambda'-\lambda)\varphi} \sum_i (2j+1) d^j_{\lambda'\lambda}(\cos\theta)\, a^{\mathrm{el}}_j$$

$$\int_{-1}^{1} dx\, d^j_{\lambda'\lambda}(x) d^{j'}_{\lambda'\lambda}(x) = \frac{2\delta_{jj'}}{2j+1}$$

Invert above relations to obtain partial waves $a_j$

$$a^{\mathrm{el}}_j = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, d^j_{\lambda'\lambda}(\cos\theta)\, \mathcal{M}_{\mathrm{el}}(2 \to 2').$$

$$\int d\Pi_{2'}|\mathcal{M}_{\mathrm{el}}(2 \to 2')|^2 = \frac{32\pi}{\rho}\sum_j (2j+1)\, d^j_{\lambda'\lambda}\, |a^{\mathrm{el}}_j|^2,$$

$\rho$ is an identical particle facctor

- Substituting

$$\int d\Pi_{2'}|\mathcal{M}_{\rm el}(2\to 2')|^2 = \frac{32\pi}{\rho}\sum_j (2j+1)\, d^j_{\lambda'\lambda}\, |a_j^{\rm el}|^2,$$

$$a_j^{\rm el} = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, d^j_{\lambda'\lambda}(\cos\theta)\, \mathcal{M}_{\rm el}(2\to 2').$$

- Into

$$\int d\Pi_{2'}|\mathcal{M}_{\rm el}(2\to 2')|^2 + \sum_n \int d\Pi_n |\mathcal{M}_{\rm inel}(2\to n)|^2 = 2{\rm Im}\,[\mathcal{M}_{\rm el}(2\to 2)]$$

- We get

$$\sum_j (2j+1)\frac{1}{\rho}\left[\frac{\rho^2}{4} - \left({\rm Re}\, a_j^{\rm el}\right)^2 - \left({\rm Im}\, a_j^{\rm el} - \frac{\rho}{2}\right)^2\right] = \frac{1}{32\pi}\sum_n \int d\Pi_n |\mathcal{M}_{\rm inel}(2\to n)|^2.$$

RHS is positive definite

$LHS \geq 0$

$$\left({\rm Re}\, a_j^{\rm el}\right)^2 - \left({\rm Im}\, a_j^{\rm el} + \frac{\rho}{2}\right)^2 \leq \frac{\rho^2}{4}.$$

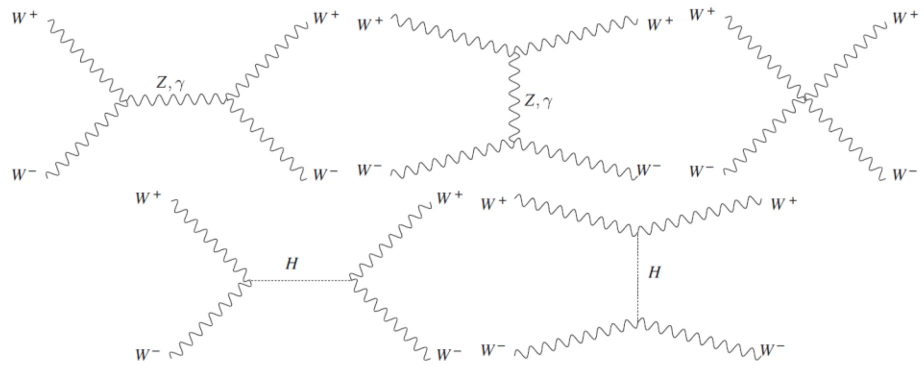
For a superposition of states

$$|{\rm Re}(a_{\phi,j})| \leq \frac{1}{n} \qquad |{\rm Im}(a_{\phi,j})| \leq \frac{2}{n}.$$

$$\left|{\rm Re}\, a_j^{\rm el}\right| \leq \frac{\rho}{2}, \qquad \left|{\rm Im}\, a_j^{\rm el}\right| \leq \rho.$$

Unitarity condition for $2 \to 2$ scattering

# Example: Higgs mass




- Look at j=0 partial wave of WW scattering

$$a_0^{(\text{gauge})} = -\frac{g^2\, s}{128\pi m_W^2} + \mathcal{O}\left(\left(\frac{m_W}{s}\right)^0\right),$$

$$|\text{Re}(a_0)| \le 1/2,$$

$$E_\star = 1.7 \text{ TeV}.$$

$$a_0^{(\text{higgs})} = \frac{g^2\, s}{128\pi m_W^2} + \mathcal{O}\left(\frac{m_W^0}{s^0}\right),$$

Higgs, unitarizes WW scattering

After cancellation of above two terms, the leading order term is

$$a_0^{(\text{total})} = -\frac{g^2\, m_H^2}{32\pi\, m_W^2},$$ ➡ $$m_H^2 \le \frac{32\pi\, m_W^2}{g^2},$$ ➡ $$m_H \lesssim 1.2 \text{ TeV}.$$

Note stronger constraints obtained when we include scattering of other coupled channels (like ZZ, Zh …)

$$m_H \le \sqrt{\frac{32\pi m_W^2}{3g^2}} \simeq 700 \text{ GeV}.$$

# Unitarity of Linearized Gravity

- Scattering amplitudes for S-channel exchange of graviton

| | $s'\bar{s}'$ | $f'_+\bar{f}'_-$ | $f'_-\bar{f}'_+$ | $V'_+V'_-$ | $V'_-V'_+$ |
|---|---|---|---|---|---|
| $s\bar{s}\rightarrow$ | $2/3d^2_{0,0}-2/3(1+12a)^2d^0_{0,0}$ | $\sqrt{2/3}\ d^2_{0,1}$ | $\sqrt{2/3}\ d^2_{0,-1}$ | $2\sqrt{2/3}\ d^2_{0,2}$ | $2\sqrt{2/3}\ d^2_{0,-2}$ |
| $f_+\bar{f}_-\rightarrow$ | $\sqrt{2/3}\ d^2_{1,0}$ | $d^2_{1,1}$ | $d^2_{1,-1}$ | $2\ d^2_{1,2}$ | $2\ d^2_{1,-2}$ |
| $f_-\bar{f}_+\rightarrow$ | $\sqrt{2/3}\ d^2_{-1,0}$ | $d^2_{-1,1}$ | $d^2_{-1,-1}$ | $2\ d^2_{-1,2}$ | $2\ d^2_{-1,-2}$ |
| $V_+V_-\rightarrow$ | $2\sqrt{2/3}\ d^2_{2,0}$ | $2\ d^2_{2,1}$ | $2\ d^2_{2,-1}$ | $4\ d^2_{2,2}$ | $4\ d^2_{2,-2}$ |
| $V_-V_+\rightarrow$ | $2\sqrt{2/3}\ d^2_{-2,0}$ | $2\ d^2_{-2,1}$ | $2\ d^2_{-2,-1}$ | $4\ d^2_{-2,2}$ | $4\ d^2_{-2,-2}$ |

TABLE I: Scattering amplitudes for scalars, fermions, and vector bosons via $s$-channel graviton exchange in terms of the Wigner $d$ functions. The subscripts on the particles indicate their helicities. All particle masses have been neglected. An overall factor $-2\pi G_N E^2_{CM}$ has been extracted from the amplitudes.

T. Han, S. Willenbrock 2005

$$a_2^{(1)}=-\frac{1}{40}G_N E^2_{CM}\left(\frac{2}{3}N_s+N_f+4N_V\right)$$

$$E^2_{CM}=20(G_N N)^{-1},\qquad N\equiv\frac{2}{3}N_s+N_f+4N_V\ .$$

$$E_{CM}=\sqrt{60/283}\ G_N^{-1/2}\approx 6\times 10^{18}\ \text{GeV}.$$

$$
\begin{aligned}
&\frac{\delta S^3}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \to 2\eta^{\mu\tau}\eta^{\nu\sigma}k_1{}^{\lambda}k_1{}^{\rho} + 2\eta^{\mu\sigma}\eta^{\nu\tau}k_1{}^{\lambda}k_1{}^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}k_1{}^{\lambda}k_1{}^{\rho} + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}k_1{}^{\sigma}k_1{}^{\rho} + 2\eta^{\lambda\sigma}\eta^{\mu\nu}k_1{}^{\tau}k_1{}^{\rho} + \eta^{\mu\tau}\eta^{\nu\sigma}k_2{}^{\lambda}k_1{}^{\rho} + \eta^{\mu\sigma}\eta^{\nu\tau}k_2{}^{\lambda}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_2{}^{\mu}k_1{}^{\rho} + \\
&\eta^{\lambda\sigma}\eta^{\nu\tau}k_2{}^{\mu}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_2{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_2{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}k_3{}^{\mu}k_1{}^{\rho} + \eta^{\lambda\sigma}\eta^{\nu\tau}k_3{}^{\mu}k_1{}^{\rho} - \\
&\eta^{\lambda\nu}\eta^{\sigma\tau}k_3{}^{\mu}k_1{}^{\rho} + \eta^{\lambda\tau}\eta^{\mu\sigma}k_3{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\sigma}\eta^{\mu\tau}k_3{}^{\nu}k_1{}^{\rho} - \eta^{\lambda\mu}\eta^{\sigma\tau}k_3{}^{\nu}k_1{}^{\rho} + \eta^{\lambda\nu}\eta^{\mu\tau}k_3{}^{\sigma}k_1{}^{\rho} + \\
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&2\eta^{\lambda\rho}\eta^{\mu\nu}k_1{}^{\sigma}k_1{}^{\tau} + 2\eta^{\lambda\nu}\eta^{\mu\rho}k_1{}^{\sigma}k_1{}^{\tau} + 2\eta^{\lambda\mu}\eta^{\nu\rho}k_1{}^{\sigma}k_1{}^{\tau} + \eta^{\mu\tau}\eta^{\nu\rho}k_1{}^{\sigma}k_2{}^{\lambda} + \eta^{\mu\rho}\eta^{\nu\tau}k_1{}^{\sigma}k_2{}^{\lambda} + \\
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&\eta^{\lambda\tau}\eta^{\mu\rho}k_1{}^{\sigma}k_3{}^{\nu} + \eta^{\lambda\mu}\eta^{\rho\tau}k_1{}^{\sigma}k_3{}^{\nu} + \eta^{\lambda\sigma}\eta^{\mu\rho}k_1{}^{\tau}k_3{}^{\nu} + \eta^{\lambda\mu}\eta^{\rho\sigma}k_1{}^{\tau}k_3{}^{\nu} + \eta^{\mu\tau}\eta^{\rho\sigma}k_2{}^{\lambda}k_3{}^{\nu} + \\
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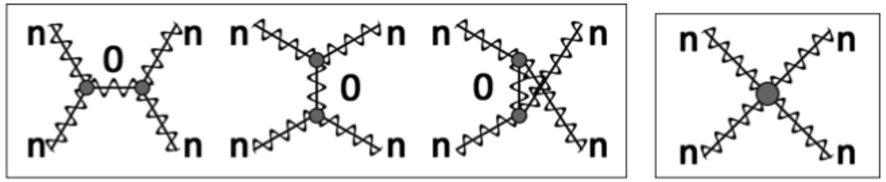
&2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_2\cdot k_3
\end{aligned}
$$

Graviton triple vertex
Source:arXiv 1506.00974

Of course for massless gravitons
We can simply use Recursion Relations

B. S. DeWitt, Phys. Rev. **162**, 1239 (1967)

# Graviton-Graviton scattering




- Energy dependence for the amplitude

$$M(h_1^-, h_2^-, h_3^+, h_4^+) = \frac{\langle 12\rangle^4[34]^4}{stu} \qquad \sim \frac{s}{M_{Pl}^2}$$

- Linearized gravity is a valid EFT as long as we stick to energies below $M_{Pl}$

# Massive Spin-2

# Massive 4D Gravitons

- Can gravitons have a mass term? (Fierz/Pauli (1939), van Dam/Veltman/Zakharov (1970), Boulware/Deser (1972))

$$S_G = \int d^4x \sqrt{g} R + m^2((h_{\mu\nu})^2 - h^2)$$

- Other mass terms are possible – this one avoids Ostrogradsky instability of the Hamiltonian

# Uniqueness of the spin-2 lagrangian

- Recall how we got the kinetic term of massless spin-2 particle by expanding the metric in the Einstein-Hilbert action.

$$\mathcal{L}_g^0 = -\frac{1}{4}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu},$$

- We will see from pure QFT arguments, why a spin-2 field must have kinetic terms like this.

- Consider the spin-1 lagrangian – the most general Lorentz-invariant and local action one can write down is.

$$\mathcal{L}_{\rm kin}^{\rm spin-1} = a_1\mathcal{L}_1 + a_2\mathcal{L}_2 + a_3\mathcal{L}_3\,,$$

$$\begin{aligned}\mathcal{L}_1 &= \partial_\mu A^\nu \partial^\mu A_\nu \\ \mathcal{L}_2 &= \partial_\mu A^\mu \partial_\nu A^\nu \\ \mathcal{L}_3 &= \partial_\mu A^\nu \partial_\nu A^\mu\,,\end{aligned}$$

- Decomposing into Longitudinal and transverse modes $A_\mu = A_\mu^\perp + \partial_\mu \chi.$ $\quad \partial^\mu A_\mu^\perp = 0$

$$\mathcal{L}_{\rm kin}^{\chi} = (a_1+a_2)\partial_\mu\partial_\nu\chi\partial^\mu\partial^\nu\chi = (a_1+a_2)(\Box\chi)^2\,,$$

- To avoid Ostrogradsky Instabilities (i.e. higher order derivative terms): $a_1 = -a_2$ . Note $a_3 = 0$, since it is equivalent to $a_1$ term up to some boundary term.
- We recover the Maxwell lagrangian!

- Similarly for Spin-2 fields we may write

$$\mathcal{L}_{\text{kin}}^{\text{spin}-2} = \frac{1}{2}\partial^\alpha h^{\mu\nu}\left(b_1\partial_\alpha h_{\mu\nu} + 2b_2\partial_{(\mu}h_{\nu)\alpha} + b_3\partial_\alpha h\eta_{\mu\nu} + 2b_4\partial_{(\mu}h\eta_{\nu)\alpha}\right) ,$$

$$h_{\mu\nu} = h^T_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)} .$$

$$\begin{aligned}\mathcal{L}_{\text{kin}}^{\text{spin}-2} \supset\ & (b_1+b_2)\chi^\mu\Box^2\chi_\mu + (b_1+3b_2+2b_3+4b_4)\chi^\mu\Box\partial_\mu\partial_\nu\chi^\nu \\ & -2h^{T\mu\nu}\big((b_2+b_4)\partial_\mu\partial_\nu\partial_\alpha\chi^\alpha + (b_1+b_2)\partial_\mu\Box\chi_\mu \\ & \qquad + (b_3+b_4)\Box\partial_\alpha\chi^\alpha\,\eta_{\mu\nu}\big) .\end{aligned}$$

- In order to avoid higher derivative terms we must have $b_4 = -b_3 = -b_2 = b_1 ,$

$$\mathcal{L}_{\text{kin}}^{\text{spin}-2} = -\frac{1}{4}h^{\mu\nu}\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}$$

$$\hat{\mathcal{E}}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = -\frac{1}{2}\left(\Box h_{\mu\nu} - 2\partial_{(\mu}\partial_\alpha h^\alpha_{\nu)} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta})\right)$$

Lichnerowizc operator

- Kinetic term is invariant under

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} .$$

- Using the DeDonder gauge condition gives us back $\mathcal{L}^0_g = -\frac{1}{4}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu},$
- And also reduced 10 dofs to 2 dofs, required for a massless graviton.

- The general form a mass term is $\mathcal{L}_{\text{mass}} = -\frac{1}{8}m^2\left(h_{\mu\nu}^2 - Ah^2\right)$ ,
- This breaks diffeomorphism invariance.
- Just like we do for gauge theories, one can restore diffeomorphism invariance by introducing Stuckelberg fields

$$\mathcal{L}_{\text{mass}} = -\frac{1}{8}m^2\left((h_{\mu\nu} + 2\partial_{(\mu}\chi_{\nu)})^2 - A(h + 2\partial_\alpha\chi^\alpha)^2\right) ,$$

$$\begin{aligned} h_{\mu\nu} &\rightarrow h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)} , \\ \chi_\mu &\rightarrow \chi_\mu - \frac{1}{2}\xi_\mu . \end{aligned}$$

- In order to avoid higher derivative terms for $\chi$ we must set A=1 and we recover the Fierz-Pauli lagrangian.
- Higgs mechanism for gravity?

# Massive graviton : Polarization and Propagator

$$G^{\text{massive}}_{\mu\nu\alpha\beta}(x,x') = \frac{f^{\text{massive}}_{\mu\nu\alpha\beta}}{\Box - m^2}\,,$$

$$\begin{aligned} f^{\text{massive}}_{\mu\nu\alpha\beta}(p_\mu, m) &= \frac{2}{3m^4} p_\mu p_\nu p_\alpha p_\beta + \eta_{\mu(\alpha}\eta_{\nu\beta)} - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta} \\ &+ \frac{1}{m^2}\left( p_\alpha p_{(\mu}\eta_{\nu)\beta} + p_\beta p_{(\mu}\eta_{\nu)\alpha} - \frac{1}{3} p_\mu p_\nu \eta_{\alpha\beta} - \frac{1}{3} p_\alpha p_\beta \eta_{\mu\nu} \right) . \end{aligned}$$

Notice that the propagator is singular in the limit $m \to 0$. Unlike massive spin-1, there is no smooth high energy limit for massive spin-2. This is known as the vDVZ discontinuity and is related to the violation of unitarity in theories of Massive gravity.

$$\begin{aligned} \epsilon^{++}_{\mu\nu} &= \epsilon^+_\mu \epsilon^+_\nu \\ \epsilon^{+}_{\mu\nu} &= \frac{1}{\sqrt{2}}[\epsilon^+_\mu \epsilon^0_\nu + \epsilon^0_\mu \epsilon^+_\nu] \\ \epsilon^{0}_{\mu\nu} &= \frac{1}{\sqrt{6}}[\epsilon^+_\mu \epsilon^-_\nu + \epsilon^+_\mu \epsilon^-_\nu - 2\epsilon^0_\mu \epsilon^0_\nu] \\ \epsilon^{+}_{\mu\nu} &= \frac{1}{\sqrt{2}}[\epsilon^-_\mu \epsilon^0_\nu + \epsilon^0_\mu \epsilon^-_\nu] \\ \epsilon^{--}_{\mu\nu} &= \epsilon^-_\mu \epsilon^-_\nu \end{aligned}$$

$$\epsilon^\mu(k,\pm 1) = \frac{1}{2}(0; \mp\cos\theta\cos\phi + i\sin\phi, \mp\cos\theta\sin\phi - i\sin\phi, \sin\theta)$$

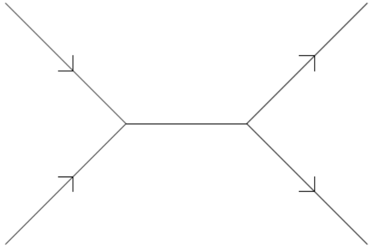
$$\epsilon^\mu(k,0) = (\frac{|\vec{k}|}{m}; \frac{k^0}{m}\hat{k})$$

Massive spin-s states must have 2s+1 dofs. Hence 5 polarization states

# High energy behavior of massive gravitons scattering

- Naïve power counting:

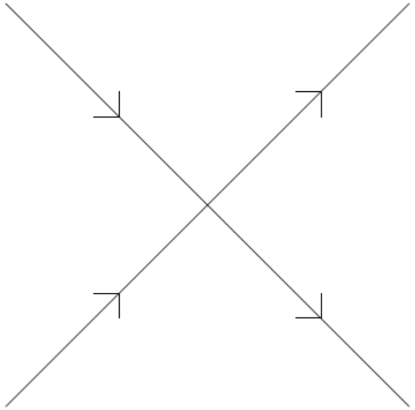
$$e^{(5)}_{\mu\nu} = -\sqrt{\frac{2}{3}}\,\frac{k_\mu k_\nu}{m^2} + \mathcal{O}(m^0) \sim \frac{s}{m^2}$$



$$\sim \frac{s^6}{M_{Pl}^2 m^{10}}$$

Propagator $\sim \frac{s}{m^2} + \mathcal{O}(m^0)$

Vertex $\sim \frac{\sqrt{s}}{M_{Pl}}$



$$\sim \frac{s^5}{M_{Pl}^2 m^8}$$

Vertex $\sim \frac{s}{M_{Pl}^2}$

- If we do the full calculation we find

$$
\begin{aligned}
\mathcal{M}_4 &= -\frac{2\pi}{9M_{Pl}^2 m^8} stu(s^2+t^2+u^2) \\
\mathcal{M}_3 &= \frac{7\pi}{54M_{Pl}^2 m^8} stu(s^2+t^2+u^2) \\
\mathcal{M}_{TOT} &= -\frac{5\pi}{54M_{Pl}^2 m^8} stu(s^2+t^2+u^2)
\end{aligned}
$$

$$
\mathcal{M} \sim \frac{E^{10}}{\Lambda^{10}}
$$

Unitarity violated at the scale $\Lambda_5 = (M_{Pl}\mathrm{m}^4)^{\frac{1}{5}}$

So for a 1 GeV graviton unitarity is violated at a scale ~6 TeV !

Is it possible to improve the high energy behavior of massive spin-2 particles?

- In order to answer that question let us simplify calculations by using the goldstone boson equivalence theorem for gravity. i.e. introduce Stuckelberg fields that restore diffeomorphism invariance and look at the high energy behavior of longitudinal mode scattering (scattering of the scalar modes)

$$h_{\mu\nu} \rightsquigarrow A_{\mu,\nu} + A_{\nu,\mu} + A_{\alpha,\mu}A_{\alpha,\nu}$$

$$A_\mu \to A_\mu + \partial_\mu \phi,$$

$\phi$ are the goldstone modes and we need only look at their scattering.

$$\mathcal{L} = M_{\mathrm{Pl}}^2 h \Box h + M_{\mathrm{Pl}}^2 m_g^2 (A \Box A + h \Box \phi) + \cdots$$

A generic interaction term can be written as

$$\sim m^2 M_P^2 (\partial A')^{n_A} (\partial^2 \phi')^{n_\phi} (h'_{\mu\nu})^{n_h} = (\Lambda_\lambda)^{4-n_h-2n_A-3n_\phi} h'^{n_h} (\partial A')^{n_A} (\partial^2 \phi')^{n_\phi}$$

$\Lambda_\lambda$ is the scale of the EFT and is given by

$$\Lambda_\lambda = (M_P m^{\lambda-1})^{1/\lambda}, \quad \lambda = \frac{3n_\phi + 2n_A + n_h - 4}{n_\phi + n_A + n_h - 2}$$

For interaction terms we need $n_\phi + n_A + n_h \geq 3.$

The smallest term is the scalar cubic $n_\phi = 3, n_A = n_h = 0$ → $\frac{(\partial^2 \phi)^3}{\Lambda_5^5}, \ \Lambda_5 = (M_P m^4)^{1/5}$

We recover our previous result

Arkani-Hamed, Georgi, Schwartz,2002

## Is it possible to improve the high energy behavior of massive spin-2 particles?

Consider some non-linear extension of spin-2 lagrangian ($q_i$ are some arbitrary coeffs.)

$$\Delta\mathcal{L} = c_1[h^3] + c_2[h^2][h] + c_3[h]^3 + q_1[h^4] + q_2[h^2][h^2] + q_3[h^3][h] + q_4[h^2][h]^2 + q_5[h]^4$$

Substitute Stuckelberg fields  and calculate amplitude in terms of $q_i$. One can show that it is possible to chose $q_i$ so that amplitudes cancel all the way down to $\Lambda_3$

See Schwartz (2003)

It is not possible to improve further than $\Lambda_3$, even by including new scalar or vector fields to our theory (Hinterbichler 2018)

It is however possible to  bring $\Lambda_3 \rightarrow \Lambda_1$ by adding in additional massive spin-2 particles (Chivukula, Foren, KM, Sengupta, 2019)

# Unitarity in Extra Dimension models

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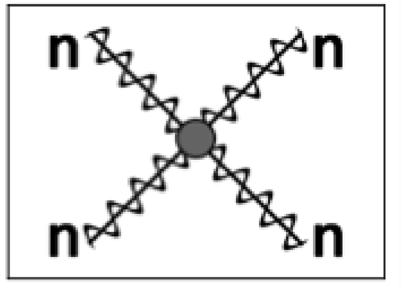



Work with

Dennis Foren R. S. Chivukula D. Sengupta E. H. Simmons

arXiv: 1906.11098, 19XX.XXXX

# Motivation

- Hierarchy problem
- Flavor
- Dark Matter
- Unification ?
- Our motivation : study phenomenology of extra dimension models in a region of parameter space that has been neglected
- KK excitations of gravitons are studied only for the first massive mode in an effective field theory description - we would like to explore phenomenology of energy scales where probing at least two or more all the way to the cutoff of the theory
- Obvious question we must first answer - where is the cutoff of the theory, i.e. up to what scale is the theory unitary?

# Graviton dofs

- Massless graviton represented by a symmetric traceless tensor with redundant degrees of freedom (local diffeomorphism invariance) $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$

- In general a massless graviton in d dimensions has d(d-3)/2 physical degrees of freedom

- 4D graviton has 2 degrees of freedom and 5D has 5 degrees of freedom. $m^2((h_{\mu\nu})^2 - h^2)$

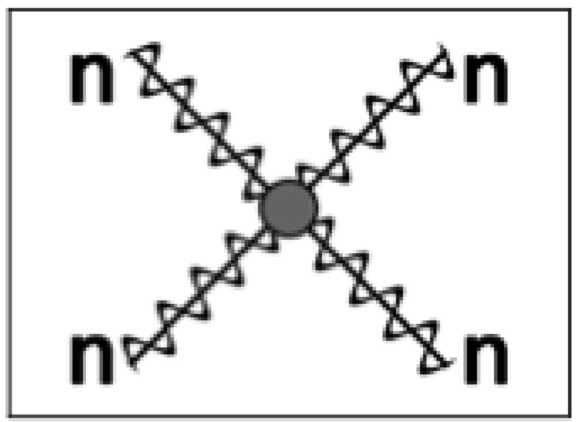
- Mass term breaks diffeomorphism invariance - A 4D massive graviton has 5 degrees of freedom (2 spin helicity states, 2 spin-1 helicity states and 1 spin-0 helicity states )

- A massless 5D graviton has the same dofs as a massive 4D

# Unitarity in 2→2 graviton scattering

- Determine feynman rules of 3 point (~180 terms) and 4 point (~2800 terms) vertices from lagrangian
- Plug in polarization vectors of gravitons
- //Simplify //Simplify //FullSimplify !
- Alternative: gravity amplitudes are constructible entirely from lower point amplitudes. Use recursion relations to determine the scattering amplitudes — can be easily done with pencil and paper!
- Unfortunately we don't have a proper formalism for using recursion relations with massive gravitons as yet.

# Naive expectation




$$\rightarrow \frac{s^5}{m^8 M_{pl}^2}$$

$$\epsilon^0_{\mu\nu} \rightarrow \frac{k_\mu k_\nu}{m^2}$$

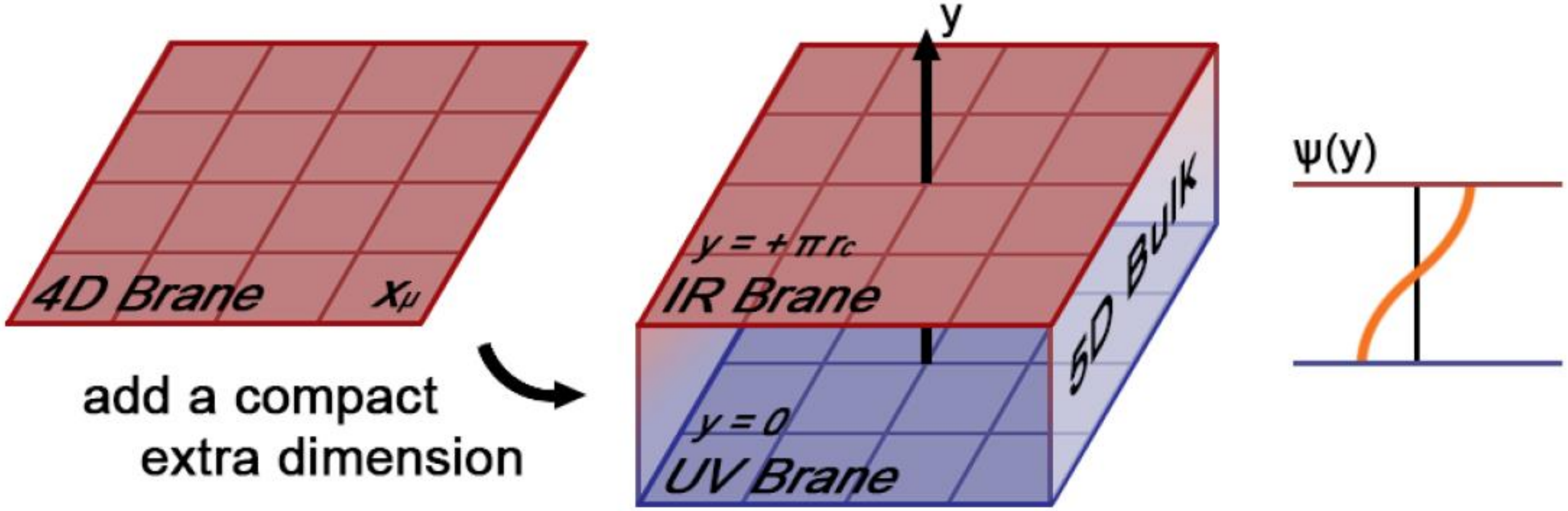
**Unitarity is violated at a scale**

$$\Lambda_5 = (M_{pl} m^4)^{1/5} \ll M_{pl}$$

# A Toy model – The orbifolded torus

## ⋆ Step 1: Build a 5D Torus ⋆

- **Parameter 1:** Add a compact spatial dimension $y \in [0, \pi r_c]$ to 4D spacetime ($x^\mu$), where $r_c \equiv$ the **compactification radius**




$$\eta^{(5DOT)}_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \implies \boxed{ds^2 = (dt^2 - d\vec{x}^{\,2}) - dy^2}$$

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 e^{+2\kappa r}\rho_\mu\rho_\nu & \kappa\, e^{+2\kappa r}\rho_\mu \\ \kappa\, e^{+2\kappa r}\rho_\nu & -e^{+2\kappa r} \end{pmatrix} \qquad \boxed{S = M_5^3 \int d^4x dy \sqrt{G} R^{(5)}.} \qquad M_5^3 = M_{Pl}^2 / L$$

Compactification allows us to expand
The fields in fourier modes over the 5D coordinates

$$h_{\mu\nu}(x,y) = \sum_{n=-\infty}^{\infty} h_{\mu\nu,n}(x) e^{i\omega_n y}$$

$$h_{\mu 5}(x,y) = \sum_{n=-\infty}^{\infty} \rho_\mu(x,n) e^{i\omega_n y}$$

$$h_{55}(x,y) = \sum_{n=-\infty}^{\infty} r(x,n) e^{i\omega_n y}$$

$$\omega_n \equiv \frac{2\pi n}{L}.$$

Orthogonality of the fourier expansion imposes

$$\int_0^L dy e^{(i\omega_m y)^*} e^{i\omega_n y} = L\delta_{mn}.$$

5D diffeomorphism invariance implies we must choose a gauge condition.
We can choose a gauge condition such that we are left with only
one massless radion
One massless graviphoton
One massless graviton
And a tower of massive gravitons

$$\tilde{G}^{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu}/w(x,y) & 0 \\ 0 & -1/v(x,y) \end{pmatrix}$$

**Christoffel Symbols:** $\Gamma^{P}_{MN} = \frac{1}{2}\tilde{G}^{PQ}\left(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN}\right)$

**Ricci Curvature:** $R_{MN} = \partial_N \Gamma^{P}_{MP} - \partial_P \Gamma^{P}_{MN} + \Gamma^{P}_{NQ}\Gamma^{Q}_{MP} - \Gamma^{P}_{PQ}\Gamma^{Q}_{MN}$

**Scalar Curvature:** $R = \tilde{G}^{MN} R_{MN}$

$$\sqrt{\det G}\; d^4x\, dy = \left[w^2\sqrt{-\det g}\; d^4x\right] \cdot \left(\sqrt{v}\; dy\right)$$

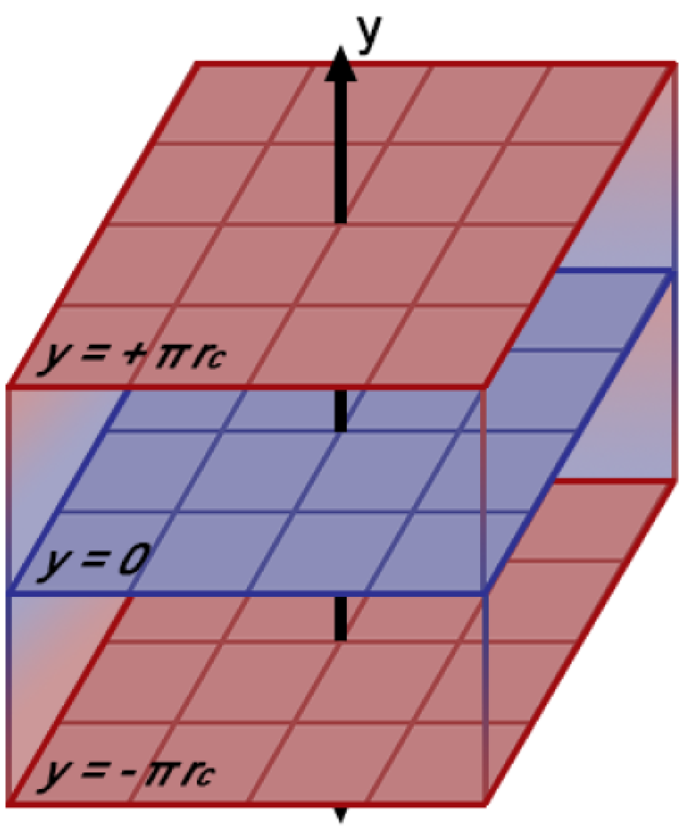
$$\mathcal{L}_{5D} = \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{CC}}$$

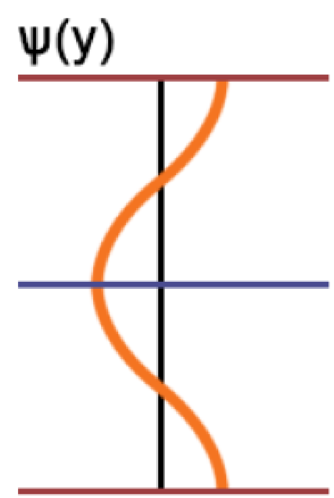
$$\mathcal{L}_{\mathrm{EH}} \equiv \frac{2}{\kappa^2}\sqrt{\det G}\, R \qquad \mathcal{L}_{\mathrm{CC}} \equiv \frac{12}{\kappa^2} k r_c \left\{ 2\sqrt{\det G}\,(\partial_\phi|\phi|)^2 - \left[w^2\sqrt{-\det g}\right]\,(\partial_\phi^2|\phi|) \right\}$$

**Cosmological constant terms are necessary to ensure that the equations of motion are satisfied**

## ⋆ Step 2: Orbifold ⋆

**Orbifold Symmetry:** Reflect extra dimension about $y = 0$, demand $ds^2$ be invariant under $y \leftrightarrow -y$, so $y \in [-\pi r_c, +\pi r_c]$.

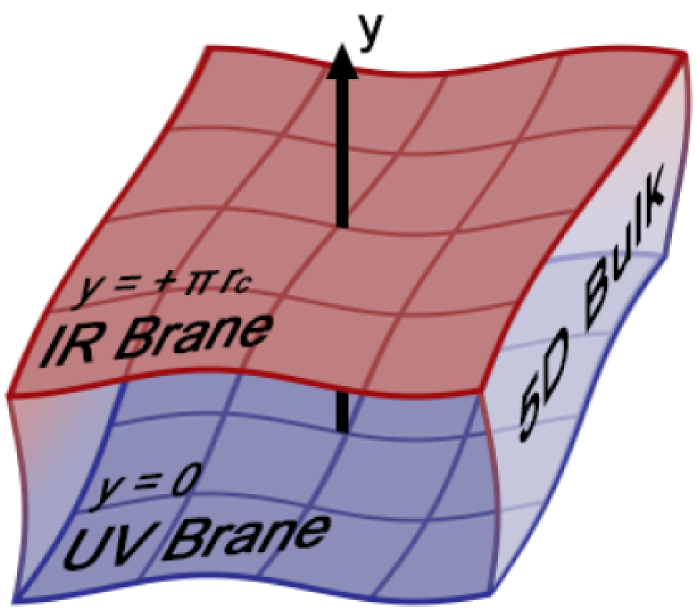



**Why Orbifold the Torus?**
The 5DOT is a nice limit of the RS model. Also, relative to 5D Torus, 5DOT eliminates the spin-1 **graviphoton** $\rho_\mu$.

*Schematically...*

$$\delta G_{MN} \approx \kappa \begin{pmatrix} h_{\mu\nu} & \cancel{\rho_\mu} \\ \cancel{\rho_\nu} & r \end{pmatrix}$$




- **Parameter 2:** 5D exp. parameter $\kappa$

$$G_{MN} = \begin{pmatrix} e^{-\kappa\hat{r}/\sqrt{6}}(\eta_{\mu\nu}+\kappa\hat{h}_{\mu\nu}) & 0 \\ 0 & -(1+\hat{r}/\sqrt{6})^2 \end{pmatrix}$$

where $[\kappa] = [\text{Energy}]^{-3/2}$ (implies $\kappa_{4D}$)

**Particles in 5D Matter-Free Orbifolded Torus:**

- **5D Graviton** $= \hat{h}_{\mu\nu}$, a massless spin-2 5D particle
  - ▷ **Origin:** local coordinate invariance of constant $y$ sheets
- **5D Radion** $= \hat{r}$, a massless spin-0 5D particle
  - ▷ **Origin:** locally perturbing distance between branes

  *Caution:* In realistic theories, radion requires additional external stabilization, e.g. Goldberger-Wise mechanism; radion stabilization plays no role in the present analysis.

# Organizing the 5D Lagrangian

Define $\mathcal{L}_{h^H r^R} \equiv$ all $\mathcal{L}_{\text{5D}}$ terms with $H$ gravitons and $R$ radions:

$$\mathcal{L}_{\text{5D}} = \sum_{H,R} \mathcal{L}_{h^H r^R} \equiv \sum_{H,R} (\cdots)^{\vec{\mu}} \, \hat{h}^H_{\vec{\mu}} \, \hat{r}^R$$

By construction, each term in this set is either

- **A-Type:** has two 4D derivatives $\partial_\mu \partial_\nu$, or
- **B-Type:** has two extra-dimensional derivatives $\partial_y^2$

$$\boxed{\mathcal{L}_{h^H r^R} = \kappa^{(H+R-2)} \left[ \lambda_A(R) \, \overline{\mathcal{L}}_{A:h^H r^R} + \lambda_B(R) \, \overline{\mathcal{L}}_{B:h^H r^R} \right]}$$

<u>**5D to 4D:**</u> Once we have a 5D WFE theory, we convert it into an effective 4D theory via $y$-integration

$$S = \int d^4x \left( \int dy \;\; \mathcal{L}_{\text{5D}} \right) \quad \Longrightarrow \quad \boxed{\mathcal{L}^{(\text{eff})}_{\text{4D}} \equiv \int_{-\pi r_c}^{+\pi r_c} dy \;\; \mathcal{L}_{\text{5D}}}$$

$$\boxed{\underbrace{\hat{f}_{\vec{\mu}}(x,y)}_{\text{5D field}} = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \underbrace{\hat{f}_{\vec{\mu}}^{(n)}(x)}_{\text{4D fields}} \; \underbrace{\psi^{(n)}(y)}_{\text{wfxns}}}$$

**KK Decomposition**

We utilize extra-dimensional wavefunctions $\psi^{(n)}(y)$ that satisfy

$$\begin{cases} \dfrac{1}{\pi r_c}\displaystyle\int_{-\pi r_c}^{+\pi r_c} dy \;\; \lambda_A(0)\; \psi^{(m)}\psi^{(n)} = \delta_{m,n} \\ \dfrac{1}{\pi r_c}\displaystyle\int_{-\pi r_c}^{+\pi r_c} dy \;\; \lambda_B(0)\; (\partial_y\psi^{(m)})(\partial_y\psi^{(n)}) = m_n^2\; \delta_{m,n} \end{cases}$$

where $n \equiv$ **KK number**. Given $R$ radion fields, we define...

<u>**5DOT:**</u> $\begin{cases} \lambda_A(R) = 1 \\ \lambda_B(R) = 1 \end{cases}$ $\left( \underline{\textbf{RS:}} \begin{cases} \lambda_A(R) = e^{k[2(R-1)|y| - R\pi r_c]} \\ \lambda_B(R) = e^{-2k|y|}\lambda_A(R) \end{cases} \right)$

$$\boxed{\mathcal{L}_{hh} = \lambda_A(0)\,\overline{\mathcal{L}}_{A:hh} + \lambda_B(0)\,\overline{\mathcal{L}}_{B:hh}}$$

$$\overline{\mathcal{L}}_{A:hh} = -\hat{h}_{\mu\nu}(\partial^\mu\partial^\nu\hat{h}) + \hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) - \frac{1}{2}\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) + \frac{1}{2}\hat{h}(\Box\hat{h})$$

$$\overline{\mathcal{L}}_{B:hh} = -\frac{1}{2}(\partial_y\hat{h}_{\mu\nu})(\partial_y\hat{h}^{\mu\nu}) + \frac{1}{2}(\partial_y\hat{h})^2$$

such that the 4D effective spin-2 Lagrangian is canonical:

$$\boxed{\mathcal{L}^{(\text{eff})}_{hh} = \mathcal{L}^{(S=2)}_{\text{Kin}}(\hat{h}^{(0)}) + \sum_{n=1}^{+\infty}\mathcal{L}_{\text{FP}}(m_n, \hat{h}^{(n)})}$$

<u>**5D Graviton**</u> $\hat{h}_{\mu\nu}$ becomes many 4D particles:

- **4D Graviton:** a massless spin-2 4D particle $\hat{h}^{(0)}_{\mu\nu}$
- **KK Modes:** massive spin-2 4D particles $\hat{h}^{(n)}_{\mu\nu}$ for $n > 0$

Similarly, we may decompose the 5D radion:

$$\boxed{\mathcal{L}_{rr} = \lambda_A(2)\, \overline{\mathcal{L}}_{A:rr}} \qquad \text{where} \qquad \overline{\mathcal{L}}_{A:rr} = \frac{1}{2}(\partial_\mu \hat{r})(\partial^\mu \hat{r})$$

5D diffeomorphism invariance may used to ensure **the 5D radion is flat in the extra dimension** (e.g. $\hat{r}(x,y) \to \hat{r}(x)$). Thus, its KK decomposition has only a zero mode:

$$\hat{r}(x) = \frac{1}{\sqrt{\pi r_c}} \hat{r}^{(0)}_{\mu\nu}(x)\psi^{(0)} \qquad \Longrightarrow \qquad \boxed{\mathcal{L}^{(\text{eff})}_{rr} = \mathcal{L}^{(S=0)}_{\text{Kin}}(\hat{r}^{(0)})}$$

**<u>5D Radion</u>** $\hat{r}$ becomes a single 4D particle:

- **4D Radion:** a massless spin-0 4D particle $\hat{r}^{(0)}$

Per field content, our 4D effective Lagrangian equals...

$$\mathcal{L}^{(\text{eff})}_{h^H r^R} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{(H+R-2)} \sum_{\vec{n}=\vec{0}}^{+\infty} \Big\{ \overline{a}_{(R|\vec{n})} \cdot \mathcal{K}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{A:h^H r^R}\Big]$$

$$+\,\overline{b}_{(R|\vec{n})} \cdot \mathcal{K}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{B:h^H r^R}\Big] \Big\}$$

where $\mathcal{K}$ is an operator that maps 5D fields to 4D fields, and

$$\overline{a}_{(R|n_1\cdots n_H)} \equiv \frac{1}{\pi r_c}\int_{-\pi r_c}^{+\pi r_c} dy\ \ \lambda_A(R)\ \psi^{(n_1)}\ldots\psi^{(n_H)}\left[\psi^{(0)}\right]^R$$

$$\overline{b}_{(R|n_3\cdots n_H|n_1 n_2)} \equiv \frac{r_c}{\pi}\int_{-\pi r_c}^{+\pi r_c} dy\ \ \lambda_B(R)$$

$$\times(\partial_y\psi^{(n_1)})(\partial_y\psi^{(n_2)})\psi^{(n_3)}\ldots\psi^{(n_H)}\left[\psi^{(0)}\right]^R$$

These couplings *embody all nontrivial model dependence*; they are evaluated analytically for 5DOT & numerically for RS.

By choosing an appropriate gauge (**unitary gauge**), we have...

$$h_{\mu\nu}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} h_{\mu\nu}^{(n)}(x)\, \psi_n(y)$$

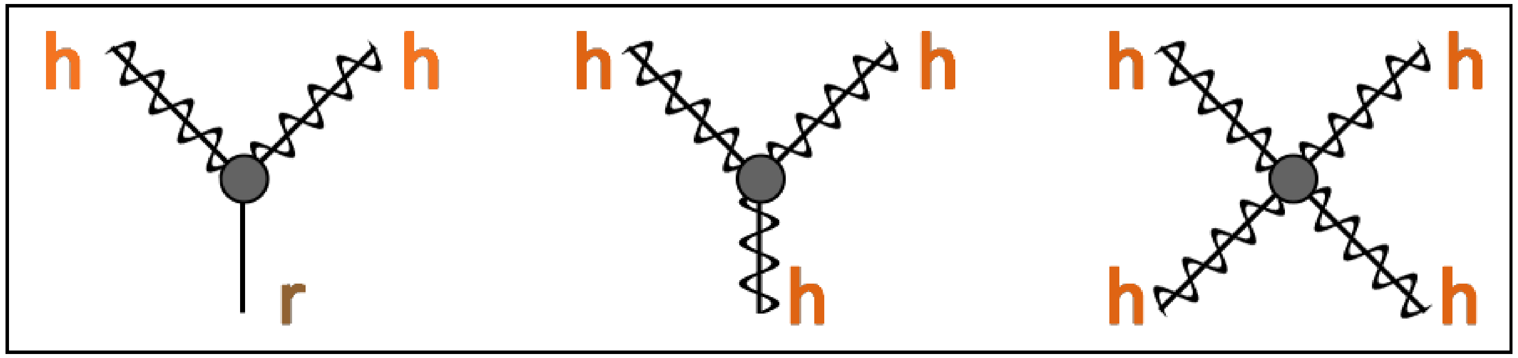
$$\rho_\mu(x) = \frac{1}{\sqrt{2\pi r_c}} \rho_\mu^{(0)}(x) \qquad r(x) = \frac{1}{\sqrt{2\pi r_c}} r^{(0)}(x)$$

The EOM from $\mathcal{L}_{5D}$ admit 5D wavefunctions equaling...

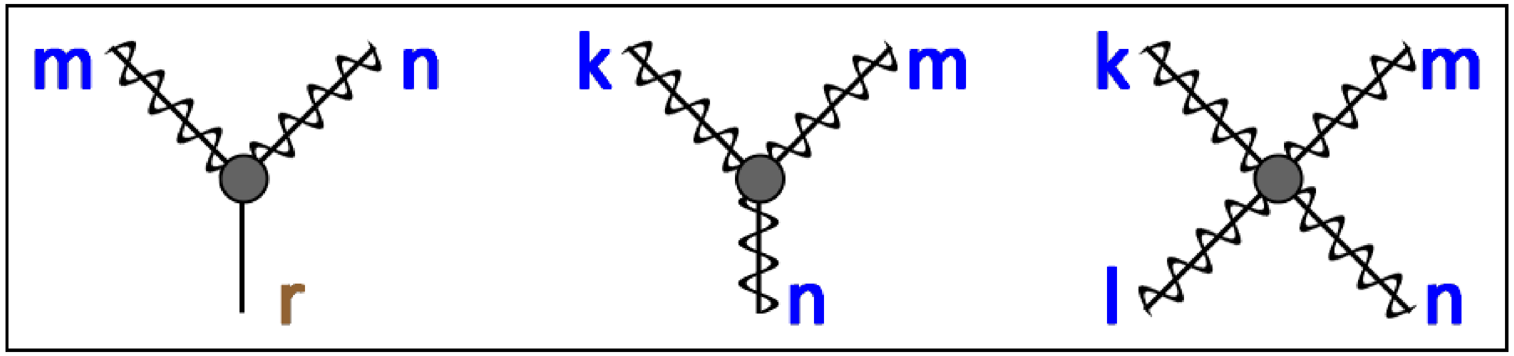
$$\psi^{(n)}(y) = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \quad \leftarrow \textbf{flat in } y! \\ \cos\left(\frac{n\,y}{r_c}\right) & n > 0 \end{cases}$$

And so, the orbifolded torus' KK number conservation is apparent from **orthogonality of Fourier modes!**

$\mathcal{L}_{5\mathrm{D}}$ contains the following important vertices:

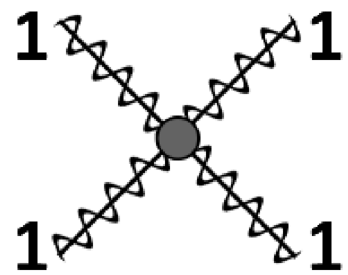



This implies that (after KK decomposition) $\mathcal{L}_{4D}^{(\mathrm{eff})}$ contains:



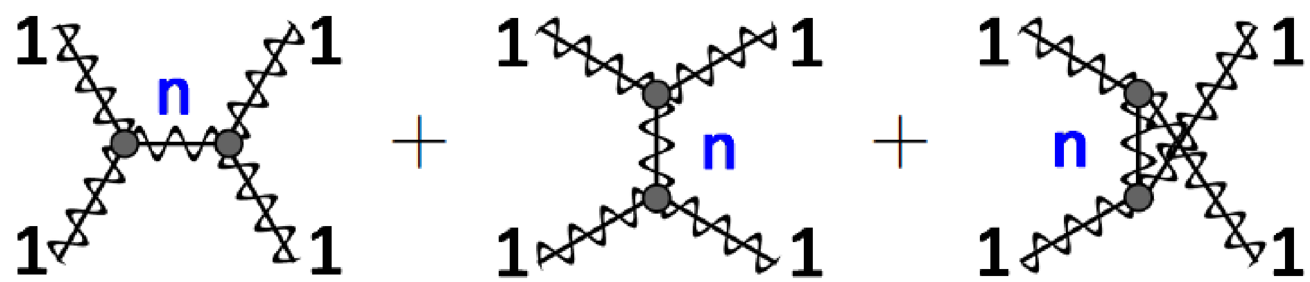

The explicit vertex rules are complicated, but once we have them, we can calculate the desired matrix elements...

$\mathcal{M}_{sg} =$



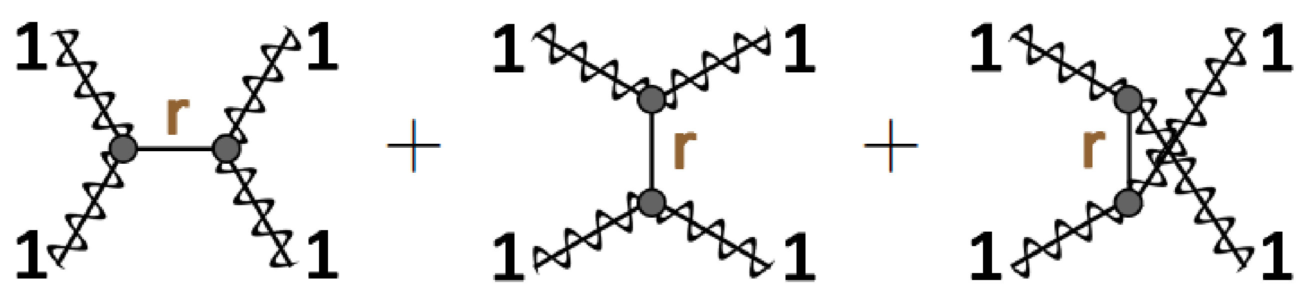

$\sim \mathcal{O}(s^5)$

$\mathcal{M}_h(n) =$




$\sim \mathcal{O}(s^5)$

$\mathcal{M}_r =$

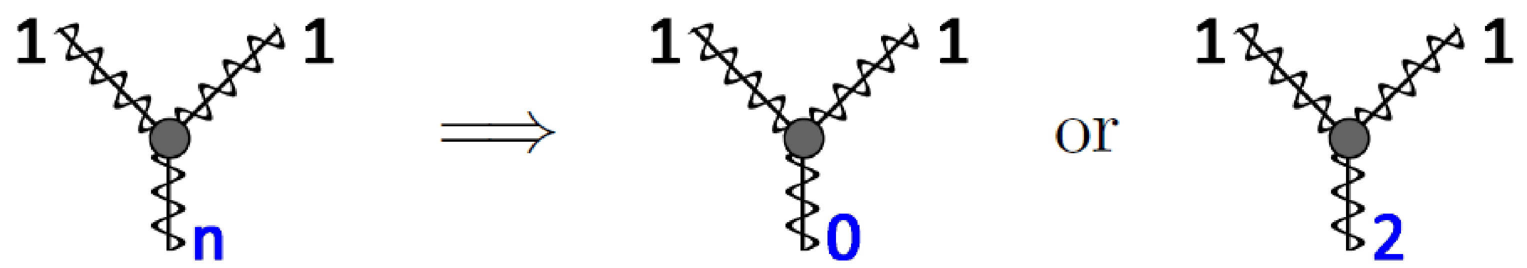



$\sim \mathcal{O}(s^3)$

We've directly confirmed these behaviors, which are consistent with the arguments of Arkani-Hamed, et. al. Define...

$$\mathcal{M}(n_{\max}) = \mathcal{M}_{sg} + \mathcal{M}_r + \sum_{n=0}^{n_{\max}} \mathcal{M}_h(n) = \sum_k \overline{\mathcal{M}}^{(k)}(n_{\max}) \cdot s^k$$

The 5D Orbifolded Torus has **KK number conservation:**




Therefore, we may calculate $\mathcal{M}$ exactly with a finite sum:

$$\mathcal{M} = \mathcal{M}_{sg} + \mathcal{M}_r + \mathcal{M}_h(0) + \mathcal{M}_h(2) = \sum_{k=-\infty}^{+5} \overline{\mathcal{M}}^{(k)} \cdot s^k$$

$$\overline{\mathcal{M}}^{(5)} = \overline{\mathcal{M}}^{(4)} = \overline{\mathcal{M}}^{(3)} = \overline{\mathcal{M}}^{(2)} = 0$$

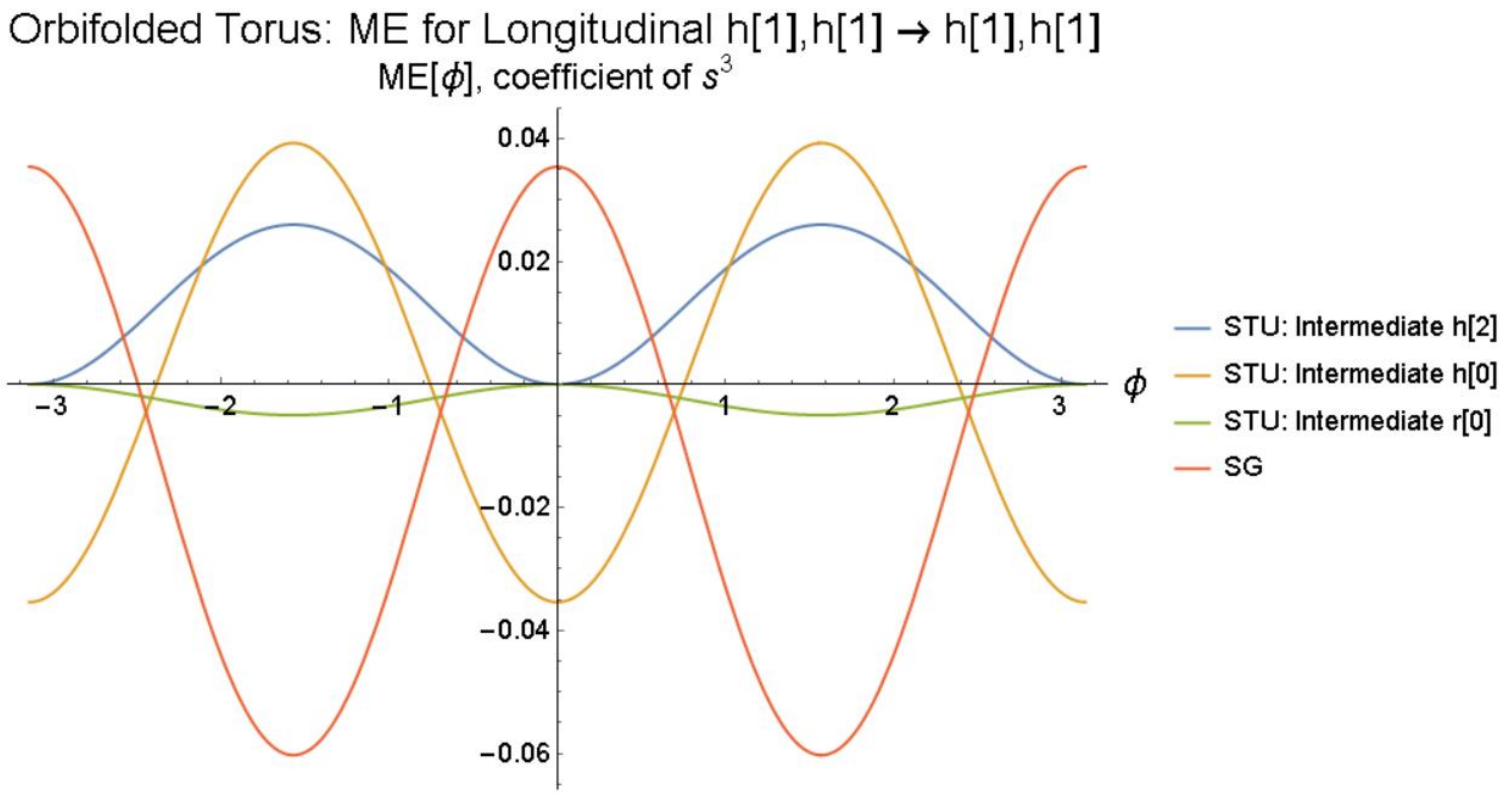
$$\overline{\mathcal{M}}^{(1)} = \frac{3\kappa^2}{256\pi r_c}\left[7 + \cos(2\theta)\right]\csc^2\theta$$

With warping, we instead have an infinite sum...

| | $s^5$ | $s^4$ | $s^3$ | $s^2$ |
|---|---|---|---|---|
| $2n$ | $\frac{r_c^7\kappa^2(7+\cos(2\theta))\sin^2\theta}{9216n^8\pi}$ | $\frac{r_c^5\kappa^2(-13+\cos(2\theta))\sin^2\theta}{1152n^6\pi}$ | $\frac{r_c^3\kappa^2(97+3\cos(2\theta))\sin^2\theta}{1152n^4\pi}$ | $-\frac{r_c\kappa^2(179-116\cos(2\theta)+\cos(4\theta))}{1152n^2\pi}$ |
| 0 | . $\frac{r_c^7\kappa^2(7+\cos(2\theta))\sin^2\theta}{4608n^8\pi}$ | $\frac{r_c^5\kappa^2(9-140\cos(2\theta)+3\cos(4\theta))}{9216n^6\pi}$ | $\frac{r_c^3\kappa^2(-15+270\cos(2\theta)+\cos(4\theta))}{2304n^4\pi}$ | $-\frac{r_c\kappa^2(175-624\cos(2\theta)+\cos(4\theta))}{1152n^2\pi}$ |
| Seagull | $-\frac{r_c^7\kappa^2(7+\cos(2\theta))\sin^2\theta}{3072n^8\pi}$ | $\frac{r_c^5\kappa^2(63-196\cos(2\theta)+5\cos(4\theta))}{9216n^6\pi}$ | $\frac{r_c^3\kappa^2(692\cos(2\theta)+5(-37+\cos(4\theta)))}{4608n^4\pi}$ | $\frac{r_c\kappa^2(7+\cos(2\theta))}{4608n^4\pi}$ |
| Radion | – | – | $\frac{r_c^3\kappa^2\sin^2\theta}{64n^4\pi}$ | $\frac{r_c\kappa^2(7+\cos(2\theta))}{96n^2\pi}$ |
| Sum | 0 | 0 | 0 | 0 |

**Amplitude is 0 and everything cancels up to order $s^2$**

**Radion diagrams start contributing at order $s^3$**



# Unitarity analysis

$$a^J_{\lambda_a\lambda_b\to\lambda_c\lambda_d} = \frac{1}{32\pi^2}\int d\Omega \quad D^J_{\lambda_i\lambda_f}(\theta,\phi)\mathcal{M}_{a\lambda_b\to\lambda_c\lambda_d}(s,\theta,\phi)\ ,$$

$$a^{J=0}_{00\to 00}(14\to 23) = \frac{s}{M^2_{\rm Pl}}\ln\left(s r_c^2\right) + \ldots\ .$$

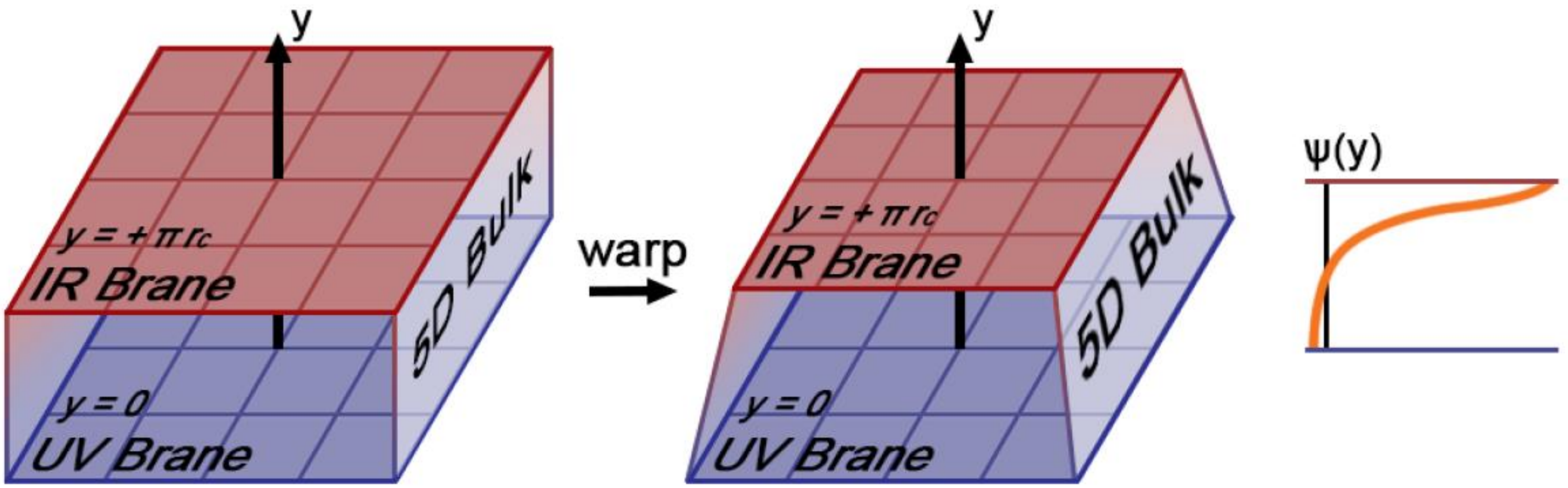
$$s \simeq M^2_{Pl}.$$

**For n coupled channels** $\frac{\Lambda^2 n m_1}{M_{pl}^2 m_1} = \frac{\Lambda^3}{M_5^3}$ $\Lambda = M_5 < M_{pl}$

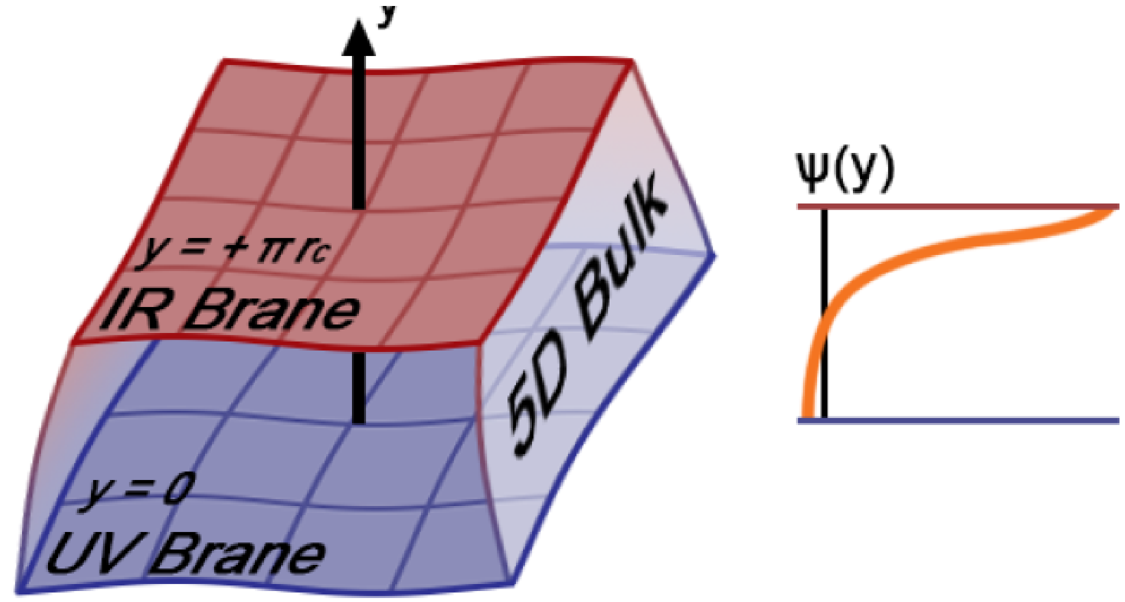
# The Randall-Sundaram Model

⋆ **Step 3: Warp the Extra Dimension** ⋆

- **Parameter 3:** Add a **warping parameter** $k$




$$\eta^{(\mathrm{RS})}_{MN} = \begin{pmatrix} e^{-2k|y|}\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \implies \boxed{ds^2 = e^{-2k|y|}(dt^2 - d\vec{x}^{\,2}) - dy^2}$$




Many options for perturbing the vacuum. The **Einstein frame parameterization** is automatically canonical in 4D:

$$G_{MN} = \begin{pmatrix} e^{-2(k|y|+\hat{u})}(\eta_{\mu\nu}+\kappa\hat{h}_{\mu\nu}) & 0 \\ 0 & -(1+2\hat{u})^2 \end{pmatrix} \qquad \hat{u} \equiv \frac{\kappa\,\hat{r}}{2\sqrt{6}} e^{+k(2|y|-\pi r_c)}$$

**Eliminate 4D cosmological constant** via 5D CC & tensions:

$$\star \qquad \boxed{\mathcal{L}_{5\text{D}}^{(\text{RS})} = \frac{2}{\kappa^2}\sqrt{\det G}\; R + \Big[\text{bulk CC} + \text{brane tensions}\Big]} \qquad \star$$

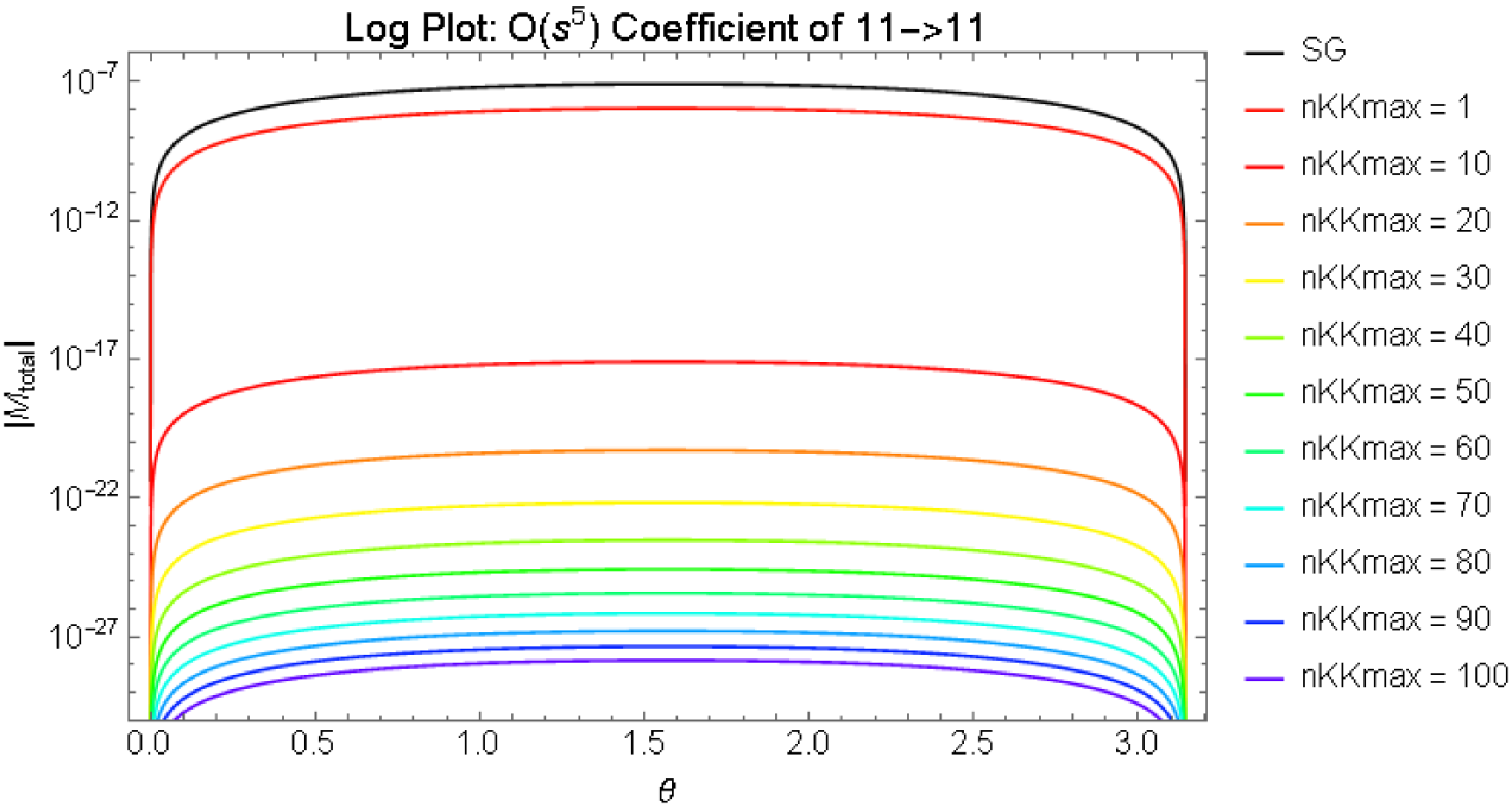
We then weak field expand $\mathcal{L}_{5D}^{(\mathrm{RS})}$, integrate the extra dimension, and KK decompose according to the procedure described earlier.

... except now we calculate couplings numerically, because of

$$\underline{\textbf{RS:}} \quad \begin{cases} \lambda_A(R) = e^{k[2(R-1)|y|-R\pi r_c]} \\ \lambda_B(R) = e^{-2k|y|}\lambda_A(R) \end{cases}$$

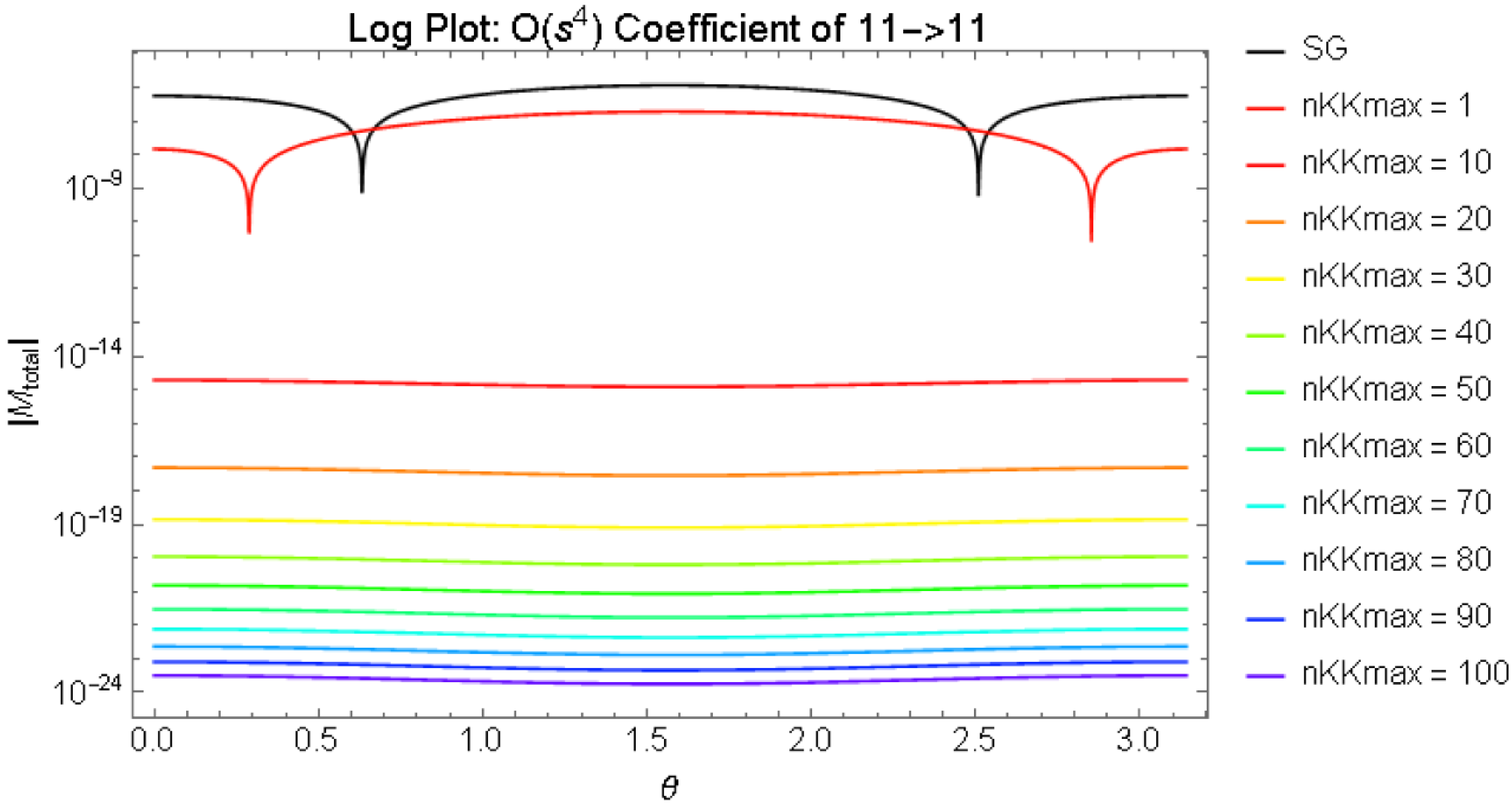
In order to show you RS plots, there are 3 parameters ($\kappa$,$r_c$,$k$) we need to fix. I set...

| | |
|---|---|
| **Lightest Spin-2 Mass:** | $m_1 = 1$ TeV |
| **Unitless Parameter:** | $kr_c = 9.5$ |
| **4D Planck Mass:** | $M_{\mathrm{Pl}} = 2.435 \times 10^{15}$ TeV |

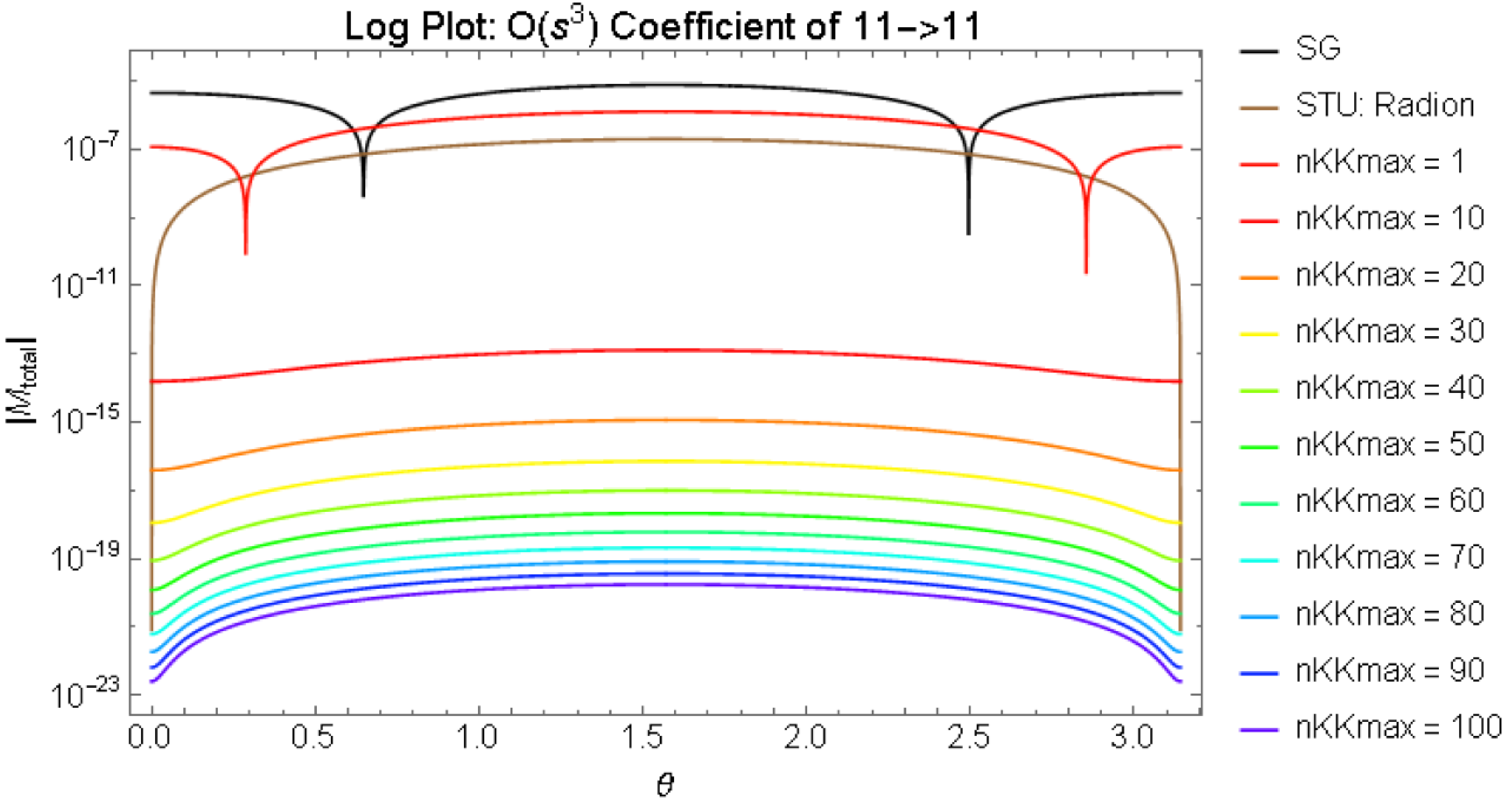



⋆ **As we include more modes, $\mathcal{O}(s^5)$ tends to 0...** ⋆

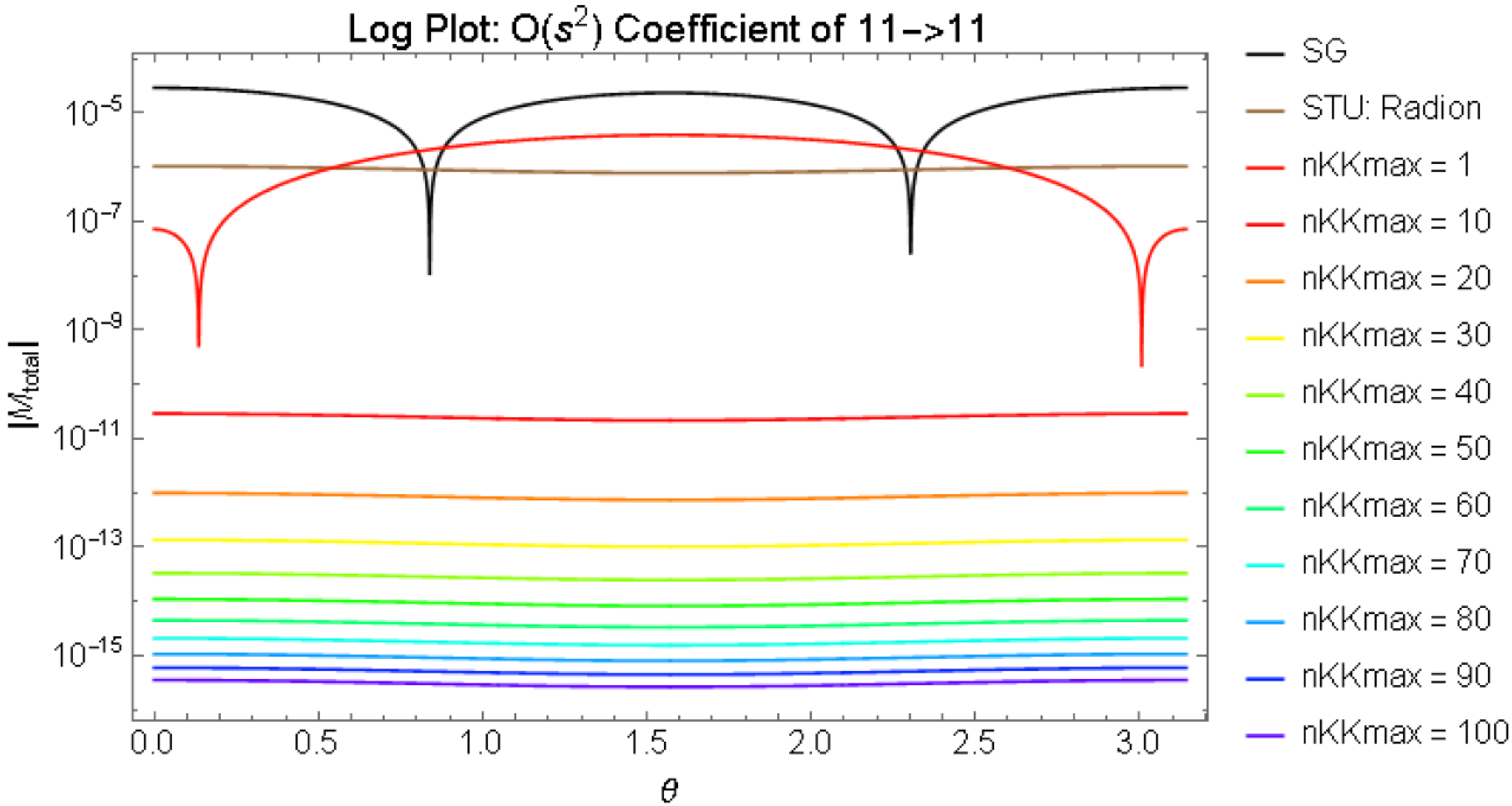
(All plots are in powers of TeV: $\mathcal{O}(s^k) \rightarrow \text{TeV}^{-2k}$)



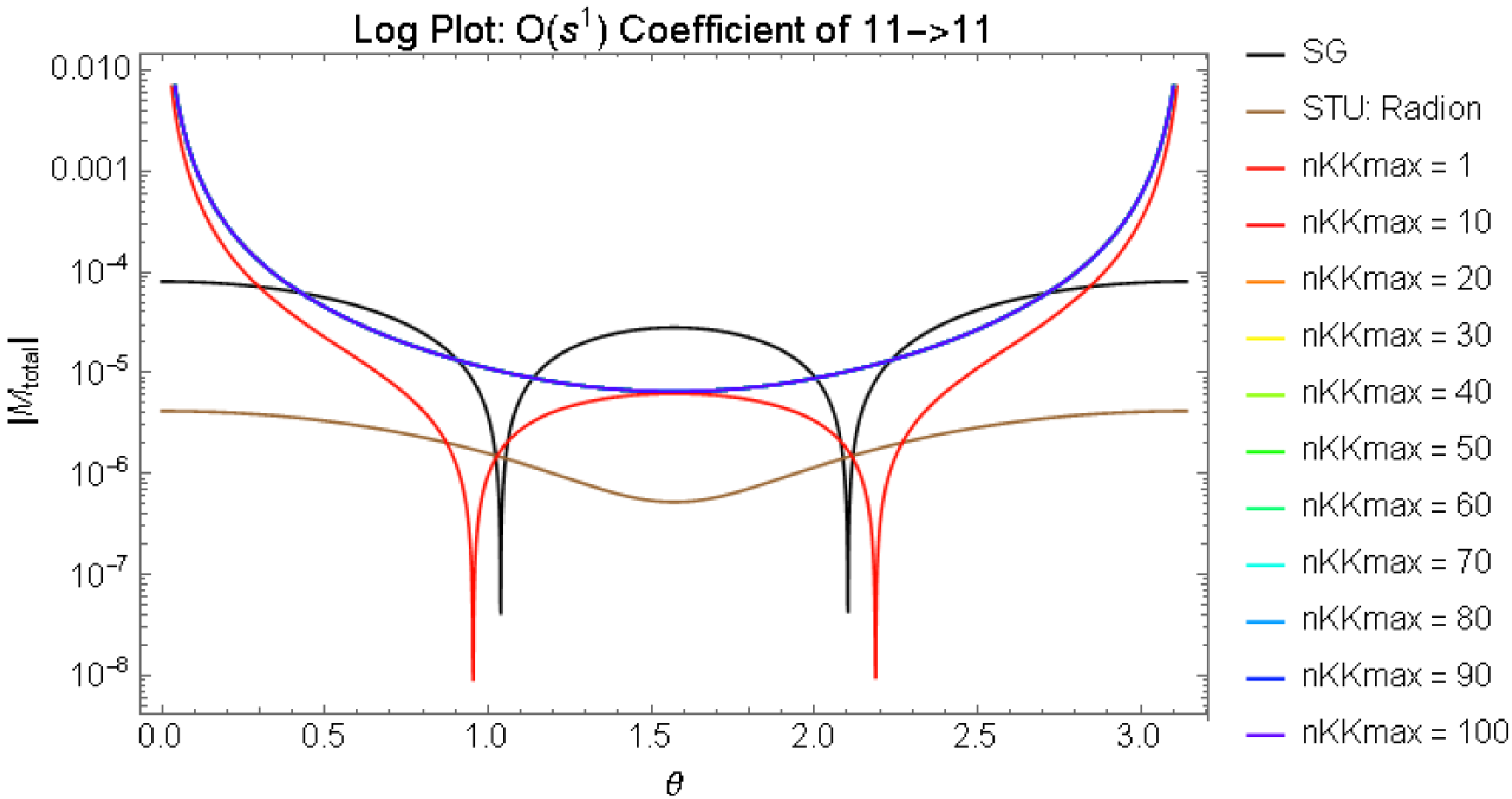

⋆ **.. and so does $\mathcal{O}(s^4)$...** ⋆




⋆ **.. and (with the radion's help) so does $\mathcal{O}(s^3)$...** ⋆




⋆ **.. and so does $\mathcal{O}(s^2)$...** ⋆




⋆ **.. until finally $\mathcal{O}(s)$ converges!** ⋆

Same behavior found in other KK combos (inc. $14 \to 23$) and for other values of warping ($k \neq 9.5$) too!

- For the RS model there is a new intermediate scale where the theory becomes strongly coupled

$$\Lambda_\pi \equiv M_{Pl} e^{-kr_c\pi}$$

- This also controls the radion and KK mode couplings

# Outlook

- Strong scale of massless 4D gravity is at $M_{pl}$
- Strong scale of massive 4D gravitons is $(m^4 M_{pl})^{(1/5)}$ much smaller than $M_{pl}$
- Compactification of a higher dimensional theory leads to a tower of massive gravitons in the lower dimension theory - what is the strong scale ?
- Strong scale simply $M_{D.}$ For torus
- Strong scale $\Lambda_\pi = M_{Pl} e^{-k\, r_c \pi}$
- It is possible to improve the bad high energy behavior of massive spin-2 theories.
- We also know the cutoff for the EFT.
- We can start exploring phenomenological consequences.