A Gentle Introduction to modern Spinor Helicity Amplitudes

$$A(--++) = \boxed{\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}}$$

Kirtimaan Mohan

References: There are many good references for this topic

Massless spinor helicity:

1.) A brief introduction to modern amplitude methods - Lance Dixon (TASI 2013)

2.) Constructing Scattering amplitudes - Ruth Britto

3.) Scattering Amplitudes - Henriette Elvang, Yu-tin Huang (arXiv : 1308.1697)

4.) TASI lectures on scattering Amplitudes - Clifford Cheung (arXiv : 1708.03872)

Massive Spinor helicity:

1.) Scattering Amplitudes for all masses and Spins - N. Arkani-Hamed, TC Huang
YT Huang
arXiv 1709.04891

There are many more references, I have only listed the ones that I think are pedagogically good.

What I hope to cover in these lectures:

1.) Introduction to spinors

2.) Massless spinor helicity formalism:

   a) Traditional spinor helicity methods

   b.) Modern Approach (Amplitude Bootstrap)

3.) Massive spinor helicity methods.:

4.) Loop Amplitudes:

Hopefully, by the end of these lectures you should have a broad overview of the subject as well as a foundation to build on further.

Motivation + Few Introductory remarks

Motivation :-

Traditional Techniques developed because

- To glean information about spin

Simplify Calculation

- Dirac algebra too complicated in some cases
- Include spin information in decays of particles
- Unitarity calculations require helicity amplitudes
  ↳ Partial wave analysis.

Modern Techniques :

- Simplification of Calculations
- Understand the Structure of a theory by observing how its amplitudes behave.
- "Lagrangian-less" formalism?
  ↳ We are not there yet.

- Cumberse machinery of QFT makes understanding some results difficult
- This provides an alternate path to understanding - simpler.

$g+g \to g+g$ 4 diagrams

$g+g \to g+g+g$ 25 diagrams

$g+g \to g+g+g+g$ 220 diagrams

$g+g \to 8g$ $> 10^6$ diagrams.

$\boxed{A(1^{\pm}, \ldots n^{+}) = 0}$?

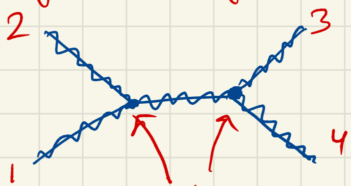
Compare with DeWitt's calculation of graviton scattering as another example:

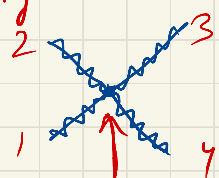
Quantum Theory of Gravity. III. Applications of the Covariant Theory

Bryce S. DeWitt
Phys. Rev. 162, 1239 – Published 25 October 1967

https://journals.aps.org/pr/abstract/10.1103/PhysRev.162.1239

graviton - graviton scattering





170 terms
(11 terms)
group as symmetric combinations

O(2500) terms in vertex

28 terms if you group as symmetric combinations.

- In spite of the complexity of the vertices and propagators:

$$\boxed{A(1^{--}, 2^{++}, 3^{--}, 4^{++}) = \frac{\langle 13\rangle^4 [24]^4}{s\,t\,u}}$$

→ Simple form of amplitude.

- The Spinor Helicity method :→ represent all objects with Spinors and perform Explicit calculations with it.

- Why is this advantageous?

- Short answer :– No redundant degrees of freedom (aka gauge freedom)

  - For Example Photons have 2 d.o.f. however they are represented by Four Vectors
    - Additional constraints needed to remove additional d.of.
    - Quantization also complicated – Gupta-Bleuler formalism.

  - On the other hand Photons have a representation in terms of spinors with no redundant d.of.

- The Lagrangian formalism introduces redundant degrees of freedom, ⇒ cumbersome calculations

Fields are integration variables for action S and S can be invariant under field redefinitions

$A_\mu \to A_\mu + \partial_\mu \theta$ → Gauge invari

$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \theta_\nu + \partial_\nu \theta_\mu$ – Diffeomorphism invariance

Spinors :

The Lorentz Group:

$SO(3,1)$

- Let us recall how $SO(3,1)$ is isomorphic to $SL(2,\mathbb{C})$

$SL(2,\mathbb{C})$: Group of 2×2 matrices with $\mathrm{Det}(M) = 1$

$SL(2,\mathbb{C})$: Can also be thought of as the complexified direct product of

$SU(2) \otimes SU(2)$

- Lorentz group leaves $x_\mu x^\mu = g_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2$ invariant.

  - Transformations that leave the inner product invariant are

a) rotations $(SO(3))$: $x^2 + y^2 = \text{const}$

b) Boosts : $t^2 - z^2 = \text{const}$

A lie group is a group whose elements depend on a set of parameters θ^a in a continuous and differentiable way.

Lorentz transformation $\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}} = e^{\frac{-i}{2}\vec{\theta}\cdot\vec{J} - \vec{\eta}\cdot\vec{K}}$

$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Pauli matrices

$J^i = \frac{1}{2}\epsilon^{ijk} J^{jk}$, $K^i = J^{i0}$
(Rotations) (Boosts)

For spinors we will consider one particular representation
$J^i = \sigma^i/2$; where σ = Pauli matrices.

Algebra:

$[J_a, J_b] = i\epsilon_{abc} J_c$, $[J_a, K_b] = i\epsilon_{abc} K_c$, $[K_a, K_b] = -i\epsilon_{abc} K_c$

Let $J_a^{\pm} = J_a \pm i K_a$; Then algebra in new basis

$[J_a^+, J_b^+] = i\epsilon_{abc} J_c^+$, $[J_a^+, J_b^-] = 0$, $[J_a^-, J_b^-] = i\epsilon_{abc} J_c^-$

- Algebra $so(3,1)$ is isomorphic to complexified $su(2) \otimes su(2)$
- Hence representations of $SO(3,1)$ can be labelled by casimirs of $su(2)$, i.e. ½ integers (j^-, j^+).

(j^-, j^+)

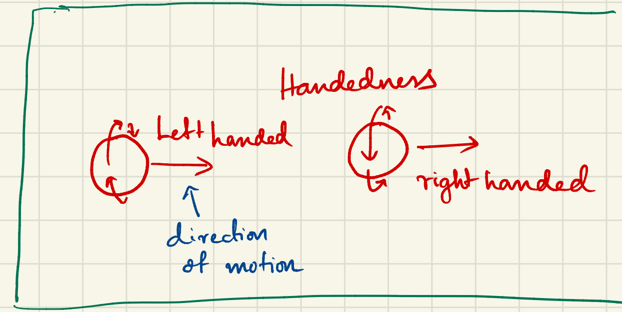
$(0, 0)$: Lorentz Scalar

$(\frac{1}{2}, 0)$: ψ_L (Left handed spinor) ; dimension of rep. $= 2j+1 = 2$

$(0, \frac{1}{2})$: ψ_R (Right handed spinor) : dimension of rep $= 2j+1 = 2$

We are going to focus on these.

$(\frac{1}{2}, \frac{1}{2})$: Lorentz Scalar $\oplus$ Lorentz vector



ψ_L : $\vec{J}^+ = 0$, $\vec{J}^- = \vec{\sigma}/2$ (2x2) matrices

$\vec{J} = \vec{J}^+ + \vec{J}^- = \frac{\vec{\sigma}}{2}$

$\vec{K} = -(\vec{J}^+ - \vec{J}^-) = i\frac{\vec{\sigma}}{2}$ → Not hermitian (Non compact group - No finite dimensional unitary rep.)

$$\psi_L \to \Lambda_L \psi_L = \exp\left\{(-i\vec{\theta} - \vec{\eta})\cdot\frac{\vec{\sigma}}{2}\right\} \psi_L$$

$$\psi_R \to \Lambda_R \psi_R = \exp\left\{(-i\vec{\theta} + \vec{\eta})\cdot\frac{\vec{\sigma}}{2}\right\} \psi_R$$

Notice signs on η parallel and antiparallel to boost direction.

J : SO(3) transformations

K : Boosts

We will focus on Weyl Spinors

$\;\;j^- \; j^+$

$(0,0)$: Scalar

$(\frac{1}{2},0)$: ψ_L (left handed spinor) : Dimension of representation $2j+1 = 2$

$(0,\frac{1}{2})$: ψ_R (Right handed spinor) : $2j+1 = 2$

$\therefore (\psi_L)_\alpha$ $\quad (\psi_R)_{\dot{\alpha}}$: Dot stresses that the index belongs to a different representation.

Representation of

ψ_L : $\vec{J}^+ = 0$, $\vec{J}^- = \vec{\sigma}/2$ (2×2) matrices

$$\vec{J} = \vec{J}^+ + \vec{J}^- = \frac{\vec{\sigma}}{2}$$

$$\vec{K} = -(\vec{J}^+ - \vec{J}^-) = \frac{i\vec{\sigma}}{2}$$ $\rightarrow$ Not hermitian (Non compact group – No finite dimensional unitary rep.)

$$\psi_L \rightarrow \Lambda_L \psi_L = \exp\left\{\left(-i\vec{\theta} - \vec{\eta}\right)\frac{\vec{\sigma}}{2}\right\} \psi_L$$

$$\psi_R \rightarrow \Lambda_R \psi_R = \exp\left\{\left(-i\vec{\theta} + \vec{\eta}\right)\cdot\frac{\vec{\sigma}}{2}\right\} \psi_R$$

Notice signs on η parallel and antiparallel to boost direction.

NOTATION: (dotted & undotted)

$$\boxed{\begin{array}{cc} \psi_L{}^{\alpha} \to \text{spinor} & \tilde{\psi}_R{}^{\dot\beta} \to \text{antispinor} \\ \tilde{\psi}_{L\alpha} \to \text{antispinor} & \psi_{R\dot\beta} \to \text{spinor} \end{array}}$$

$$i\sigma_2{}^{\alpha\beta} = \varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \psi\chi &= \varepsilon^{\alpha\beta}\,\psi_\alpha(x)\chi_\beta(x) \\ \psi\chi &= \chi\psi \end{aligned}}$$ — Inner product.

- Spinors can be fields or numbers
  - i.e real, complex or grassmann numbers
- For our purpose it is enough to take them to be complex.

- Using the dotted and undotted notation can be messy so we make our notations simpler.

NOTATION v2: (BRA KET)

$\lambda^\alpha = p\rangle \equiv \frac{1+\gamma_5}{2} u(p)$, | $\lambda_\alpha = \langle p \equiv \bar{u}(p) \frac{1+\gamma_5}{2}$ | $\tilde{\lambda}_{\dot\alpha} = p] \equiv \frac{1-\gamma_5}{2} u(p)$ | , $\tilde{\lambda}^{\dot\alpha} = [p \equiv \bar{u}(p) \frac{(1-\gamma_5)}{2}$

Q: What is p here?

$\lambda^\alpha \tilde{\lambda}^{\dot\alpha} = p^{\alpha\dot\alpha}$ is equivalent to a Lorentz vector, and is a direct product of two spinors (bi-spinor).

– $p^{\alpha\dot\alpha}$ transforms under $\Lambda_L \otimes \Lambda_R \longrightarrow su(2) \otimes su(2) \Rightarrow SL(2,\mathbb{C})$

Let us write down explicitly a representation for $p^{\alpha\dot\alpha}$.

$p^\mu = (p_0, \vec{p})$ $\quad \sigma^\mu = (1, \vec{\sigma})$ $\quad$ Writing p^μ in $SL(2,\mathbb{C})$ rep

$\bar{\sigma}^\mu = (1, -\vec{\sigma})$

$$p^\mu \sigma_\mu = \begin{pmatrix} p_0+p_3 & p_1 - ip_2 \\ p_1+ip_2 & p_0-p_3 \end{pmatrix} \longrightarrow \text{Hermitean } 2\times 2$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(p^\mu \sigma_\mu)_{\alpha\dot\alpha} = p_{\alpha\dot\alpha} = (\ \text{"}\ )$$

- We want λs to be solutions of the dirac equation

$$\not{p}\,\psi = 0 \quad \text{or in}$$

$$p^{\alpha\dot{\alpha}}\,\tilde{\lambda}_{\dot{\alpha}} = 0 \qquad p^{\alpha\dot{\alpha}}\,\lambda_{\alpha} = 0$$

$SL(2,\mathbb{C})$

$\mathbb{L} \to$ unitary $\to$ rotation $\to$ compact

$\mathbb{L} = \begin{pmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{pmatrix} \to$ Boosts.

non-compact

$p_{\alpha\dot\alpha} = \mathbb{P}$ then $\mathbb{L}^{+} \mathbb{P} \mathbb{L}$

$\because \det \mathbb{L} = 1$ $\det \mathbb{P}$ is lorentz invariant.

— We will only be concerned with massless spinors and therefore massless momenta.

$\therefore \det \mathbb{P} = 0$

— Linear algebra: 2×2 matrix with det 0 can be written as a product of two vectors.

let $\lambda_\alpha = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\tilde\lambda_{\dot\alpha} = (b_1 \; b_2)$

$$p_{\alpha\dot\alpha} = \det(\lambda_\alpha \tilde\lambda_{\dot\alpha}) = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix} = a_1 b_1 a_2 b - a_1 b_2 a_2 b_1 = 0$$

Explicit decomposition of λ:

$$\lambda^\alpha = \frac{z}{\sqrt{p^0 - p^3}} \begin{pmatrix} p_0 - p_3 \\ -p_1 - ip_2 \end{pmatrix} \qquad \tilde{\lambda}^{\dot\alpha} = \frac{z^{-1}}{\sqrt{p^0 - p^3}} \left(p^0 - p^3 , \; -p^1 + ip^2 \right)$$

$$p_0 \equiv \sqrt{p_1^2 + p_2^2 + p_3^2}$$

$z = e^{i\phi}$ pure phase for real momenta.
(phase of the little group)

For real momentum: $\lambda^\alpha = (\tilde{\lambda}^{\dot\alpha})^\dagger$

For complex momentum: $\lambda^\alpha \neq (\tilde{\lambda}^{\dot\alpha})^\dagger$;

Important we will come back to this.

Lorentz Invariants:

$p^{\alpha\dot\alpha} = \lambda^\alpha \tilde{\lambda}^{\dot\alpha} \qquad q^{\alpha\dot\alpha} = \chi^\alpha \tilde{\chi}^{\dot\alpha}$

$$p \cdot q = \frac{1}{4} g_{\mu\nu} \sigma^\mu_{\alpha\dot\alpha} \sigma^\nu_{\beta\dot\beta} \lambda^\alpha \tilde{\lambda}^{\dot\alpha} \chi^\alpha \tilde{\chi}^{\dot\alpha} = \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\dot\alpha\dot\beta} \lambda^\alpha \tilde{\lambda}^{\dot\alpha} \chi^\beta \tilde{\chi}^{\dot\beta} = \frac{1}{2} \langle \lambda \chi \rangle [\chi \lambda]$$

$$= \frac{1}{2} \langle q p \rangle [q p]$$

$p^{\alpha\dot\alpha} = p\rangle[p \qquad p_{\dot\alpha\alpha} = p]\langle p$

$$\langle \lambda \chi \rangle = \sqrt{2 p \cdot q}\, e^{i\phi}$$

$$[\chi \lambda] = \sqrt{2 p \cdot q}\, e^{-i\phi}$$

Notation:

$$p^\mu = (p_0, \vec{p}) \qquad (p^\mu \sigma^\mu)_{\alpha\dot{\alpha}} = \begin{pmatrix} p_0+p_3 & p_1 - ip_2 \\ p_1+ip_2 & p_0-p_3 \end{pmatrix}$$

This transforms as

FYI: in this ref p is alway $\not{p}$ so no need of $\not{p}$ notation

$\mathbb{L}\, \mathbb{P}\, \mathbb{L}^\dagger$; Note $\mathbb{L} \to \det \mathbb{L} = 1$ $\therefore$ $\mathbb{L}$ is a representation of $SL(2,\mathbb{C})$

$\mathbb{L}$: Lorentz transformation.

$\det P = p^0 - p_3^2 - p_1^2 - p_2^2 = \cancel{m^2} \to 0$ (We will only be talking about massless momenta)

For a 2x2 matrix with det=0 we can decompose it as an outer product of two vectors

Proof: $\lambda = (a_1 \ \ b_1)$ $\chi = (a_2 \ \ b_2)$

$$\therefore \lambda^T \chi = \begin{pmatrix} a_1 a_2 & a_1 b_2 \\ b_1 a_2 & b_1 b_2 \end{pmatrix} \qquad \therefore \det \lambda^T \chi = a_1 a_2 b_1 b_2 - a_1 b_2 a_2 b_1 = 0$$

Explicit decomposition of Spinors

$$\therefore \; p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} \; ; \quad \lambda_\alpha = \frac{e^{i\theta}}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - i p_2 \\ p_0 - p_3 \end{pmatrix} \qquad \tilde\lambda_{\dot\alpha} = \frac{e^{-i\theta}}{\sqrt{p_0 - p_3}} \left(p_1 + i p_2 , \; p_0 - p_3 \right)$$

→ Check that this gives you $p_{\alpha\dot\alpha}$ $\qquad p_0 = \sqrt{p_1^2 + p_2^2 + p_3^2}$

$e^{i\theta}$ → phase (related to little group phase);

for real valued momentum $\lambda_\alpha = (\tilde\lambda_{\dot\alpha})^{+}$; This is not true for complex momenta.

Weyl	Spinor (weyl)	Dirac Spinor	Dirac anti spinor	Helicity (all outgoing)
$\vert p\rangle$	λ^α	$P_L\, u(p)$	$P_L\, v(p)$	$-$
$\vert p]$	$\tilde\lambda_{\dot\alpha}$	$P_R\, u(p)$	$P_R\, v(p)$	$+$
$\langle p \vert$	λ_α	$\bar u\, P_R$	$\bar v\, P_R$	$-$
$[p\vert$	$\tilde\lambda^{\dot\alpha}$	$\bar u\, P_L$	$\bar v\, P_L$	$+$

$$P_{L/R} = \frac{1}{2}(1 \pm \gamma_5)$$

spinor products as

$$\epsilon^{ab}\lambda_a\mu_b \equiv \langle\lambda\mu\rangle = -\langle\mu\lambda\rangle\,, \tag{10}$$

$$\epsilon^{\dot a\dot b}\tilde\lambda_{\dot a}\tilde\mu_{\dot b} \equiv [\tilde\mu\tilde\lambda] = -[\tilde\lambda\tilde\mu]. \tag{11}$$

Just as we had $p\cdot p = \det(p_{a\dot a})$, it is easy to see that $p\cdot q = \frac{1}{2}\epsilon^{ab}\epsilon^{\dot a\dot b}p_{a\dot a}q_{b\dot b}$. If both these vectors are null, we find

$$p\cdot q = \frac{1}{2}\langle\lambda\mu\rangle\,[\tilde\mu\tilde\lambda]. \tag{12}$$

In the literature, it is very common to find 4-component Dirac spinors rather than the 2-component Weyl spinors. We summarize the correspondence and shorthand below. The subscripts on the Dirac spinors indicate helicity.

Weyl shorthand	Weyl spinor	Dirac shorthand	Dirac spinor pos. energy	Dirac spinor neg. energy
$\vert i\rangle$	$\lambda_a(p_i)$	$\vert i^+\rangle$	$u_+(p_i)$	$v_-(p_i)$
$\vert i]$	$\tilde\lambda^{\dot a}(p_i)$	$\vert i^-\rangle$	$u_-(p_i)$	$v_+(p_i)$
$\langle i\vert$	$\lambda^a(p_i)$	$\langle i^-\vert$	$\overline{u_-}(p_i)$	$\overline{v_+}(p_i)$
$[i\vert$	$\tilde\lambda_{\dot a}(p_i)$	$\langle i^+\vert$	$\overline{u_+}(p_i)$	$\overline{v_-}(p_i)$

It is also important to be aware that there are two commonly used conventions for the square-bracket spinor product (for negative chiralities), which differ by a sign. Here we do our best to follow the more traditional "QCD" conventions, where

$$\langle ij\rangle\,[ji] = 2p_i\cdot p_j = (p_i+p_j)^2 \equiv s_{ij}. \tag{13}$$

The opposite sign is used in [2] and many "twistor-inspired" papers that followed.

More complicated contractions of spinor indices are expressed by expanded spinor products. Notice that

$$\not{p}_i = |i\rangle\,[i| + |i]\,\langle i|\,. \tag{14}$$

This is an alternate scheme.

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Imp! You may use Weyl or Dirac spinors easily in $|p\rangle [p|$ notation.

Relation to dirac spinor

$$\psi = \begin{pmatrix} \psi_L^{\alpha} \\ \psi_{R\dot{\beta}} \end{pmatrix} \qquad \bar{\psi} = \begin{pmatrix} \bar{\psi}_{L\alpha} & \bar{\psi}_R^{\dot{\beta}} \end{pmatrix}$$

; Simplify notation remove $-$ and L, R,

Inner product:

$$\epsilon^{\alpha\beta} \lambda_\alpha \chi_\beta = \langle \lambda \chi \rangle = -\langle \chi \lambda \rangle \longrightarrow \text{antisymmetry}$$

$$\epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{\dot{\alpha}} \tilde{\chi}_{\dot{\beta}} = [\tilde{\mu} \tilde{\lambda}] = -[\tilde{\lambda} \tilde{\mu}]$$

Note: For calculations, one does not need dotted and undotted notation in most cases,
- Nice to make connection with theory
- Also for some complicated situations may be useful.

$\Rightarrow \langle \lambda \lambda \rangle = 0$ or $[\lambda \lambda] = 0$ (inner product 0 if ∥)

Relation to 4 momenta:

$$\langle \lambda \chi \rangle = \sqrt{2p \cdot q}\, e^{i\phi} \qquad [\chi \lambda] = \sqrt{2p \cdot q}\, e^{-i\phi}$$

$$\langle \lambda \chi \rangle [\chi \lambda] = 2p \cdot q$$

$$\langle \lambda \chi] = 0 = [\chi \lambda \rangle$$

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = |p\rangle [p|$$

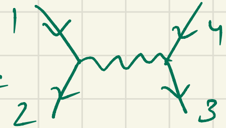
Schouten identity

$$\boxed{0 = \langle ij \rangle \langle kl \rangle + \langle ik \rangle \langle lj \rangle + \langle il \rangle \langle jk \rangle \quad \text{cyclic in } jkl}$$

$$\sum p_i = 0$$

$$\sum_j \langle ij \rangle [jk] = 0$$

Example 1 $e^+e^- \longrightarrow \mu^+\mu^-$ (All outgoing momenta)

$$iM(1^-2^+3^-4^+) = $$  $$= (-ie^2)\,\langle 1\,\gamma^\mu\,2]\,\frac{-ig^{\mu\nu}}{s}\,\langle 3\,\gamma_\nu\,4]$$

$$1_\alpha\,\sigma^{\mu\alpha\dot\alpha}\,2_{\dot\alpha} \qquad 3_\beta\,\sigma_\nu^{\beta\dot\beta}\,4_{\dot\beta}$$

$$[k_1\,\gamma^\mu\,k_2] = 0$$

$$= \underline{2ie^2}\;\underline{\langle 13\rangle[24]}$$

use

$$g^{\mu\nu}\,\sigma_\mu^{\alpha\dot\alpha}\,\sigma_\nu^{\beta\dot\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}$$

$$M^2 = \frac{4e^4}{s^2}\,\overbrace{\langle 13\rangle[31]}^{s\ \ t}\,\overbrace{[24]\langle 42\rangle}^{t}$$

$$= \frac{4e^4}{s^2}\,t^2$$

$$M^2_{tot} = \frac{4e^4}{s^2}\left(t^2+u^2\right)$$

- Dirac algebra look simpler
- But nothing new other than notation
- We have used Fierz transformations