Massive Spinor Helicity methods.

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The Little group:

- Particles are irreducible unitary representations of the Poincare group.
- Diagonalize the translational operator by labelling states by their momentum p^μ
- Any other labels/quantum numbers particles can carry are labelled σ.

$$|p,\sigma\rangle$$

- Consider some reference momentum k^μ and a Lorentz transformation

$$L(p;k) \quad \text{s.t.} \quad p \equiv L(p;k)\,k \longrightarrow (1)$$

- Note $L(p;k)$ is not unique and the set of transformations that satisfies (1) forms a group known as the Little group.

- Assume that we have a unitary rep. for the Lorentz group. s.t. for every Lorentz transformation Λ there is an associated unitary operator $U(\Lambda)$ acting on the Hilbert space. s.t. $U(\Lambda_1\Lambda_2) = U(\Lambda_1)\,U(\Lambda_2)$.

Then $\boxed{|p,\sigma\rangle \equiv U(L(p;k))\,|k,\sigma\rangle}$

- How does $|p,\sigma\rangle$ transform under a general Lorentz transformation?

$$U(\Lambda)\,|p,\sigma\rangle = U(\Lambda)\,U(L(p;k))\,|k;\sigma\rangle = U(L(\Lambda p;k))\,U\big(\underbrace{L^{-1}(\Lambda p;k)\,\Lambda\,L(p;k)}_{W(\Lambda,p,k)}\big)\,|k,\sigma\rangle$$

$W(\Lambda,p,k)$ leaves k invariant.

$\therefore\; W(k) = k$

little group transformation.

$$U(W(\Lambda,p;k))\,|k,\sigma\rangle = D_{\sigma'\sigma}(W(\Lambda,p;k))\,|k,\sigma'\rangle$$

- A particle is labelled by its momentum and transforms under some representation of the little group.

∴ Scattering amplitudes are labelled by (p_a, σ_a) for $a = 1, \dots n$.

$$\mathcal{M}(p_a, \sigma_a) = \delta^D(p^\mu_{a_1} + \dots + p^\mu_{a_n}) \, M(p_a, \sigma_a)$$

$$M^\Lambda(p_a, \sigma_a) = \prod_a \left(D_{\sigma_a \sigma_a'}(W)\right) M\left((\Lambda p)_a, \sigma_a'\right)$$

— Little group for massive particles $\rightarrow SO(D-1)$

— In 4D $SO(3) \xrightarrow{iso} SU(2)$

— Fields transform under the Lorentz group; whereas states transform under the little group.

— Instead of using conventional J_z to label spin states of $SU(2)$ little group, in order to preserve manifest Lorentz invariance, it is more convenient to label states of spin S as a symmetric tensor of rank $2S$.

$$M^{\underset{S=1/2}{\{I\}}, \underset{S=2}{\{J_1, J_2, J_3, J_4\}}, \underset{h=3/2}{\{+3/2\}}, \underset{h=-1}{\{-1\}}}(p_1, p_2, p_3, p_4)$$

Massless spinor helicity

$$p^\mu \longrightarrow p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} \longrightarrow \text{Massless } p$$

Massive spinor helicity:

Many SU(2).

$$p_{\alpha\dot\alpha} = \lambda^I_\alpha \tilde\lambda_{\dot\alpha I}$$

; I here is the little group SU(2) index.

; $\alpha, \dot\alpha$ are the SU(2) index for $\vec{J}^+$ and $\vec{J}^-$ of $SL(2,\mathbb{C})$

$$I, \alpha, \dot\alpha = 1, 2$$

$$p^2 = m^2 \Rightarrow \underbrace{\det \lambda}_{= m} \times \underbrace{\det \tilde\lambda}_{= m} = m^2$$

λ^I ; $\tilde\lambda_I$ can't uniquely be associated with a given p ; we can use $SL(2)$ transformations on $\lambda^I \to W^I_J \lambda^J$ and $\tilde\lambda_I \to W^{-1}{}^J_I \tilde\lambda_J$

Note

$$p_{\alpha\dot\alpha} \tilde\lambda^{\dot\alpha I} = m \lambda^I_\alpha , \quad p_{\alpha\dot\alpha} \lambda^{\alpha I} = -m \tilde\lambda^I_{\dot\alpha} \quad \leftarrow \text{Dirac Equation.}$$

$\therefore$ $\dfrac{p_{\alpha\dot{\alpha}}}{m}$ allows conversion between $\tilde{\lambda}^I_{\dot{\alpha}}$ and λ^I_α

$\therefore$ One need only use λ^I_α to describe a given massive particle.

$$M^{\{I_1 \cdots I_{2s}\}} = \lambda^{I}_{\alpha_1} \cdots \lambda^{I_{2s}}_{\alpha_{2s}} M^{\{\alpha_1 \cdots \alpha_{2s}\}}$$

$$[2^{J_1} 3][2^{J_2} 3][2^{J_3} 3]\left(k\langle 1^{I} 2^{J_4}\rangle\langle 4|p_1 p_2|4\rangle + k'\langle 4 1^{I}\rangle\langle 2^{J_4} 4\rangle\right) + \text{symmetrize in } J_{1,2,3,4}$$

— It is not necessary to include explicit little group indices because they are symmetrized.

$$[\underline{2}\, 3]\left(k\langle \underline{1}\, \underline{2}\rangle\langle 4|p_1 p_2|4\rangle + k'\langle 4 \underline{1}\rangle\langle 4 \underline{2}\rangle\right)$$

— Note: for massless amplitudes this looks like an illegal combination.

Traditional approach

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} + \frac{m^2}{\langle\lambda\eta\rangle[\tilde{\lambda}\tilde{\eta}]}\eta_\alpha \tilde{\eta}_{\dot{\alpha}}$$; for some reference spinor $\eta, \tilde{\eta}$

— This corresponds to a particular choice of $(\lambda^I_\alpha, \tilde{\lambda}^I_{\dot{\alpha}})$

Explicit form of λ_α^I

$$\lambda_\alpha = \sqrt{2E} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i\phi} \end{pmatrix} \qquad \tilde{\lambda}_{\dot\alpha} = \sqrt{2E} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{-i\phi} \end{pmatrix}$$

$$\lambda_{\alpha I} = \begin{pmatrix} \sqrt{E+p}\; c & -\sqrt{E-p}\; s^* \\ \sqrt{E+p}\; s & \sqrt{E-p}\; c \end{pmatrix}$$

High energy limit:

Expand λ_α^I in a basis of two dimensional vectors $\zeta^{\pm I}$

$$\lambda_\alpha^I = \lambda_\alpha \zeta^{-I} + \eta_\alpha \zeta^{+I} \qquad \langle \lambda \eta \rangle = [\tilde\lambda \tilde\eta] = m$$

$$\tilde\lambda_{\dot\alpha}^I = \tilde\lambda_{\dot\alpha} \zeta^{+I} + \tilde\eta_{\dot\alpha} \zeta^{-I} \qquad \epsilon_{IJ} \zeta^{+I} \zeta^{-J} = 1$$

Here $\zeta^{\pm I}$ can be for example eigenvectors in the direction of motion.

In the HE limit. $\sqrt{E+p} \to \sqrt{2E}$ $\quad \sqrt{E-p} \to m/\sqrt{2E}$

$$\lambda_\alpha = \sqrt{E+p}\begin{pmatrix} c \\ s \end{pmatrix}, \quad \tilde{\lambda}_{\dot\alpha} = \sqrt{E+p}\left(\quad \right)$$

$$\eta_\alpha = \sqrt{E-p}\left(\quad \right), \quad \tilde{\eta}_{\dot\alpha} = \sqrt{E-p}\left(\quad \right)$$

$\therefore$ both η_α and $\tilde{\eta}_{\dot\alpha}$ are proportional to m.

$\therefore$ to take the H.E. limit, substitute $\eta_\alpha = m\hat{\eta}_\alpha$, $\tilde{\eta}_{\dot\alpha} = m\hat{\tilde{\eta}}_{\dot\alpha}$ $\quad \langle \lambda \hat{\eta} \rangle = [\tilde{\lambda}\,\hat{\tilde{\eta}}] = 1$

$p_{3\alpha\dot\alpha}\tilde{\lambda}_3^{\dot\alpha} = m_3\eta_{3\alpha}$

Example: Substitute $|3\rangle \to \lambda_{3\alpha}\zeta^{-I} + \eta_\alpha \zeta^{+I}$

$$\frac{\langle 3\,1\rangle\langle 3\,2\rangle}{\langle 2\,1\rangle} \to \frac{\langle 3\,1\rangle\langle 3\,2\rangle}{\langle 2\,1\rangle} + \frac{\langle 3\,1\rangle\langle \eta_3\,2\rangle + \langle \eta_3\,1\rangle\langle 3\,2\rangle}{2\langle 2\,1\rangle}\;0$$

$$\frac{\langle \eta_3 1\rangle\langle \eta_3 2\rangle}{\langle 2\,1\rangle} = \frac{[3|p_1+p_2|1\rangle[3|p_1+p_2|2\rangle}{m^2\langle 21\rangle}$$

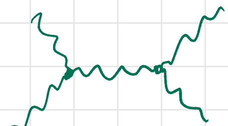
$$= \frac{[3|p_2|1\rangle[3|p_1|2\rangle}{m^2\langle 21\rangle}\;; \quad m^2 = \langle 12\rangle[21]$$

$$= \frac{[32][31]}{[21]}$$

So in the HE limit, only + helicity and − helicity states survive,

- If amplitudes have a continuous $m \to 0$ limit the HE limit is easy to determine

For example



→ compton scattering with a W boson.

$$\left(\langle \underline{1}\, 2\rangle [\underline{4}\, 3] + [\underline{4}\, 2][\underline{1}\, 3]\right)^2 \longrightarrow$$

HE limit : $\left(\langle 1\, 2\rangle [4\, 3]\right)^2$ hel $(-1,+1)$

$\langle 4 2\rangle^2 [3 1]^2$ hel $(+1,-1)$

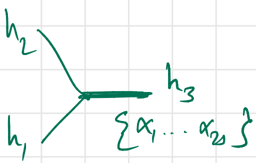
& $\langle 12\rangle [34] \langle 42\rangle [31]$ hel. $(0,0)$

Massive 3-particle amplitudes:

Since λ_{α_I} and $\tilde{\lambda}_{\dot{\alpha}_I}$ are related we can choose to write amplitudes entirely in terms of λ_{α_I}

$$\therefore \; M^{\{I_1 \dots I_{2s}\}} = \lambda^{I_1}_{\alpha_1} \cdots \lambda^{I_{2s}}_{\alpha_{2s}} \, M^{\{\alpha_1 \dots \alpha_{2s}\}}$$

Case I: 1 massive two massless



We want to find two linearly independent λ's that can span the two dimensional space of αs

I can use λ_1 and λ_2 $\qquad \langle 12\rangle[12] = m^2 \Rightarrow$ this λ_1 and λ_2 are independent

So the amplitude

- We can now write down the most general form of this amplitude.

$$\mathcal{M} = \left(\lambda_{1\alpha_1} \cdots \lambda_{1\alpha_A} \lambda_{2\alpha_{A+1}} \cdots \lambda_{2s}\right) [12]^C$$

- Applying the helicity weight condition for the massless particles, we find

$$\left.\begin{aligned} -2h_1 &= A - C \\ -2h_2 &= (2s - A) - C \end{aligned}\right\}$$

Solving these equations gives us

$$A = s + h_2 - h_1 \qquad C = s + h_1 + h_2$$

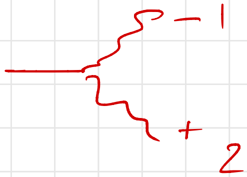
$$\therefore \mathcal{M}^{h_1 h_2}_{\{\alpha_1 \cdots \alpha_{2s}\}} = \frac{g}{m^{(2s + h_1 + h_2 - 1)}} \left(\lambda_1^{s+h_2-h_1} \lambda_2^{s+h_1-h_2}\right)_{\{\alpha_1 \cdots \alpha_{2s}\}} [12]^{s+h_1+h_2}$$

Note: one can always substitute $[12] = \dfrac{m^2}{\langle 12\rangle}$

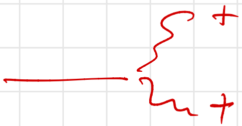
Note 2: There are constraints on $s + h_2 - h_1 \geq 0$

$$s + h_1 - h_2 \geq 0$$

Landau-Yang theorem: Choose $s=1$ $h_1=\pm 1$ $h_2=\pm 1$



$s+h_1-h_2<0$ so this amplitude must be zero.



the $M \sim g\underbrace{(\lambda_1\lambda_2)_{\{\alpha_1\alpha_2\}}}_{\text{symmetric}}\underbrace{[12]}_{\text{anti-symmetric}}^{s+2h_1}$

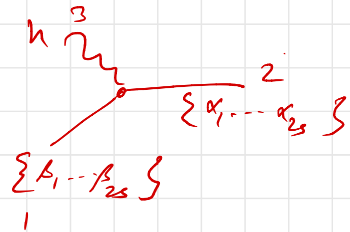
$\therefore$ for odd integer spin particles this amplitude violates bose statistics.

$\Rightarrow$ $\therefore$ A massive spin 1 particle cannot decay to a pair of photons

Ex You can also easily show that a spin 3 particle cannot decay to a pair of gravitons

Case 2: Two massive one massless

a) Unequal masses



Choose $(v_\alpha, u_\alpha) = (\lambda_\alpha, p_{2\alpha\dot\beta}\tilde\lambda^{\dot\beta})$

$$\langle u v\rangle = \langle 3|p_2|3]$$
$$= 2p_2\cdot p_3 = m_1^2 - m_2^2$$

$$\boxed{\begin{aligned} k_1^2 &= (p_2+p_3)^2 = m_2^2 + 2p_2\cdot p_3 \\ m_1^2 - m_2^2 &= 2p_2\cdot p_3 \end{aligned}}$$

– Here the amplitude is unique

$$M^h{}_{\{\alpha_1\cdots\alpha_{2s_1}\}\{\beta_1\cdots\beta_{2s_2}\}} = \sum_{i=1}^{c} g_i \left(u^{s_1+s_2+h}\, v^{s_1+s_2-h}\right)_{\{\alpha_1\cdots\alpha_{2s_1}\}\{\beta_1\cdots\beta_{2s_2}\}}$$

$c = s_1+s_2 - |s_1-s_2|$ possible tensor structures

Equal masses:

if $m_1 = m_2$ then u and v are not linearly indepent.

$\therefore$ v is parallel to u

$\therefore (p_2 \tilde{\lambda}_3) = m\, x\, \lambda_{3\alpha}$

constant of proportionality.

has helicity weigh +2 (converting $\tilde{\lambda}$ to λ)

$$x = \frac{\langle \zeta | p_1 | \tilde{\lambda}]}{m \langle \zeta \lambda \rangle}$$

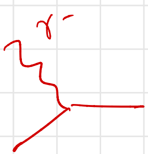
$$x \lambda_\alpha = \frac{p_{2\alpha\dot{\alpha}}}{m} \tilde{\lambda}^{\dot{\alpha}}$$

$$M^h\{\alpha_1 \ldots \alpha_{2s_1}\}\{\beta_1 \ldots \beta_{2s_1}\} = \sum_{i=|s_1-s_2|}^{s_1+s_2} g_i\, x^h \left(\lambda_3^i \left(\frac{p_2 \tilde{\lambda}_3}{m}\right)^i \epsilon^{s_1+s_2-i}\right)_{\{\alpha_1 \ldots \alpha_{2s_1}\}\{\beta_1 \ldots \beta_{2s_2}\}}$$

Example

$e m x$

$\frac{m^2}{M_{pl}} x^2$



$e m x^{-1}$

$\frac{m^2}{M_{pl}} x^{-2}$

electric charge $\qquad\qquad\qquad\qquad$ $(g-2)$

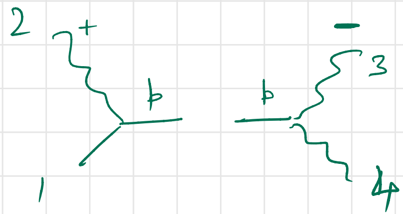
$$e\, m_x\, \varepsilon_{\alpha\beta} + k\, \lambda_{3_\alpha} \lambda_{3_\beta}$$

Case 3 : All massive amplitude

There are no λs available here so the most general form of the amplitude has to be expressed in terms of

$$\varepsilon_{\alpha\beta} \qquad p_{\alpha\dot\alpha}$$

Example: Compton scattering.



$$\sim m^2 \frac{x_{12}}{x_{34}}$$

$$x_{12}\,\lambda_2 = \frac{p_1|2]}{m} \longrightarrow x_{12} = \frac{\langle 3|p_1|2]}{\langle 32\rangle\, m}$$

$$\frac{p_4|3\rangle}{m} = \frac{\tilde{\lambda}_3}{x_{34}} \longrightarrow \frac{1}{x_{34}} \frac{\langle 3|p_4|2]}{[23]\, m}$$

$$\therefore\ m^2 \frac{x_{12}}{x_{34}} = -\frac{\langle 3|p_1|2]^2}{t} \leftarrow \text{residue of the s-channel pole.}$$

$$M^4 = \frac{\langle 3|p_1 - k_4|2]^2}{4(s-m^2)(u-m^2)}$$