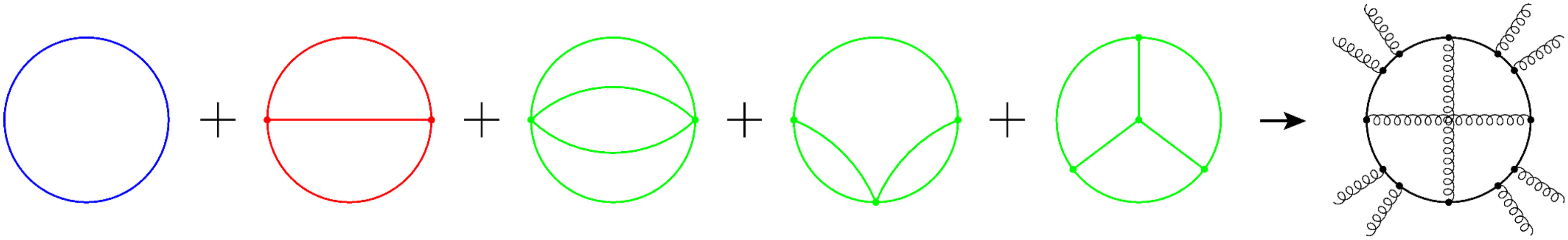
Evaluation of multiloop integrals



Yan-Qing Ma

Peking University

Based on works done with Xiao Liu(刘霄) and Chen-Yu Wang(王辰宇)
1711.09572, 1801.10523 and works in preparation

Mini-workshop on precision physics and future colliders
2019/10/14, IHEP, Beijing

Outline

I. Introduction

II. A New Representation

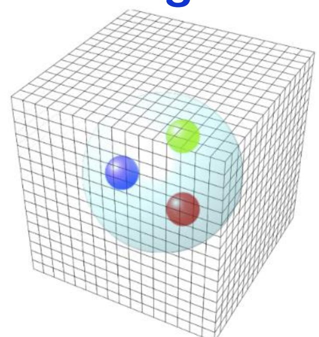
III. Reduction

IV. Outlook

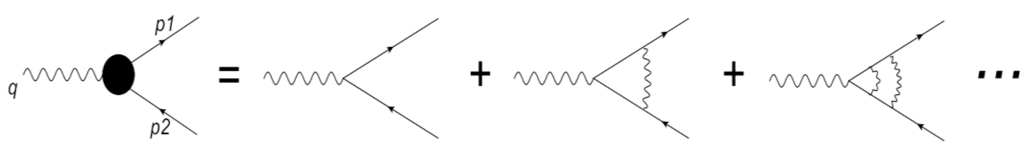
Quantum field theory

- **QFT: the underlying theory of modern physics**
  - Solving QFT is important for testing the SM and discovering NP

- **How to solve QFT:**



- Nonperturbatively (e.g. lattice field theory):
  discretize spacetime, numerical simulation
  complicated, application limited



- Perturbatively (small coupling constant):
  generate and calculate Feynman amplitudes,
  relatively simpler, the primary method

Super computer

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules
- New developments: unitarity, recurrence relation

2. Calculate Feynman loop integrals

3. Calculate phase-space integrals

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals

$$\int \frac{d^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \lim_{\eta\to 0^+}\int \frac{d^D p}{(2\pi)^D}\left(\frac{i}{p^2+i\eta}+\frac{-i}{p^2-i\eta}\right)$$

Feynman loop integrals

➢ The key to apply pQFT

$$\lim_{\eta\to 0^+}\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

- q_α: linear combination of loop momenta and external momenta
- Taking $\eta \to 0^+$ before taking $D \to 4$

➢ Theorem:

Smirnov, Petukhov, 1004.4199

For a given set of propagators, Feynman integrals form a **finite**-dimensional linear space

Multi-loop: a challenge for intelligence

- One-loop calculation: (up to 4 legs) satisfactory approaches existed as early as 1970s

’t Hooft, Veltman, NPB (1979); Passarino, Veltman, NPB (1979); Oldenborgh, Vermaseren (1990)

Developments of unitarity-based method in the past decade made the calculation efficient for multi-leg problems

Britto, Cachazo, Feng, 0412103; Ossola, Papadopoulos, Pittau, 0609007; Giele, Kunszt, Melnikov, 0801.2237

- About 40 years later, a satisfactory method for multi-loop calculation is still missing

Main strategy

1) Reduce loop integrals to basis (Master Integrals)

- Integration-by-parts (IBP) reduction: Chetyrkin, Tkachov, NPB (1981) Laporta, 0102033

  main bottleneck

  extremely time consuming for multi-scale problems

  unitarity-based reduction is efficient but cannot give complete reduction

2) Calculate MIs/original integrals

- Differential equations (depends on reduction and BCs) Kotikov, PLB (1991)
- Difference equations (depends on reduction and BCs) Laporta, 0102033
- Sector decomposition (extremely time-consuming) Binoth, Heinrich, 0004013
- Mellin-Barnes representation (nonplanar, time) Usyukina (1975) Smirnov, 9905323

IBP redution

➤ A result of dimensional regularization

Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(v_k^\mu \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}} \right) = 0\,, \quad \forall j, k$$

$$\Downarrow$$

- **Linear equations:** $$\sum_{i=1} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$
- M_i **scalar integrals,** Q_i **polynomials in** $D, \vec{s}, \eta$

➤ For each problem, the number of MIs is FINITE

Smirnov, Petukhov, 1004.4199

- **Feynman integrals form a finite dimensional linear space**
- **Reduce thousands of loop integrals to much less MIs**

Difficulty of IBP reduction

- **Solve IBP equations**

Laporta's algorithm, 0102033

$$\sum_{i=1} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

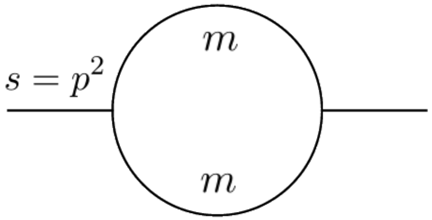
- Very large scale of linear equations (can be billions of) E.g., Laporta 1910.01248
- Coupled, it is hard to solve
- Hard to do analytic Gaussian elimination for many variables $D, \vec{s}, \eta$
- Too slow if solving it numerically for each phase space point

- **Cutting-edge problems**

- Hundreds GB RAM
- Months of runtime using super computer

Differential Equation Method

➢ Differential Equation + Boundary Condition



$$I(D;\{1,1\}) = \int \frac{\mathrm{d}^D \ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)[(\ell+p)^2 - m^2]}$$

$$\begin{aligned}
\frac{\partial}{\partial m^2} I(D;\{1,1\}) &= I(D;\{2,1\}) + I(D;\{1,2\}) \\
&\overset{\text{IBP}}{=} \frac{2(D-3)}{4m^2 - s} I(D;\{1,1\}) - \frac{D-2}{m^2(4m^2-s)} I(D;\{1,0\})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial m^2} I(D;\{1,0\}) &= I(D;\{2,0\}) \\
&\overset{\text{IBP}}{=} \frac{D-2}{2m^2} I(D;\{1,0\})
\end{aligned}$$

$$I(D;\{1,1\})|_{m^2=0} = \Gamma(2 - D/2)(-s)^{D/2-2} \frac{\Gamma(D/2-1)^2}{\Gamma(D-2)}, \quad I(D;\{1,0\})|_{m^2=0} = \cdots$$

Differential Equation Method

➢ Step1: Set up the differential equation

- Differentiate w.r.t. invariants, such as m^2, p^2
- IBP relations $$\frac{\partial}{\partial x}\vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$$

Kotikov, 1991

➢ Step2: Calculate boundary condition

- Calculate integrals at special value of m^2, p^2
- General method?

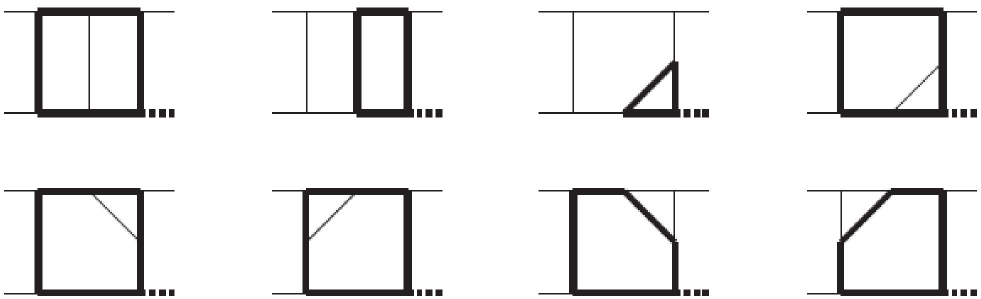
➢ Step3: Solve the differential equation

- Analytically (if possible) $\partial_x\vec{I}(x;\epsilon) = \epsilon A(x)\vec{I}(x;\epsilon)$ Henn 2013
- Numerically

Difficulty of MIs calculation

- **Analytical: Higgs → 3 partons (Euclidean Region)**

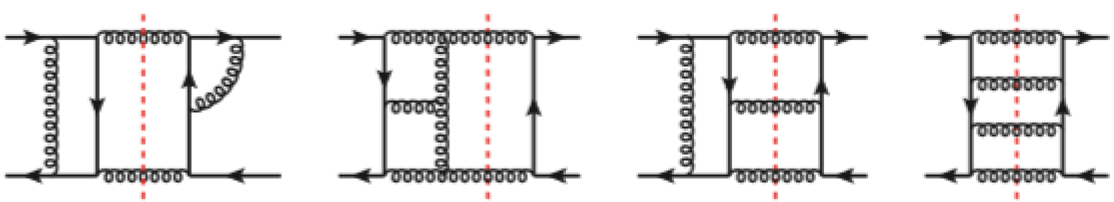
R. Bonciani, et.al 2016



200MB, 10 min

- **Numerical: Quarkonium decay at NNLO**

Feng, Jia, Sang, 1707.05758



NNLO (Virtual Squared) NNLO (Double Virtual) NNLO (Virtual – Real) NNLO (Double Real)

10^5 **CPU core-hour**

Recent developments

➢ Improvements for IBP reduction

- Finite field method Manteuffel, Schabinger, 1406.4513
- Direct solution Kosower, 1804.00131
- Syzygies method Böhm, Georgoudis, Larsen, Schönemann, Zhang, 1805.01873
- Obtain one coefficient at each step Chawdhry, Lim, Mitov, 1805.09182
- Expansion of small parameters Xu, Yang, 1810.12002; Mishima, 1812.04373
- Intersection Numbers Frellesvig, et. al., 1901.11510

➢ Improvements for evaluating scalar integrals

- Quasi-Monte Carlo method Li, Wang, Yan, Zhao, 1508.02512
- Finite basis Manteuffel, Panzer, Schabinger, 1510.06758
- Uniform-transcendental basis Boels, Huber, Yang, 1705.03444
- Loop-tree duality Capatti, Hirschi, Kermanschah, Ruijl, 1906.06138

State-of-the-art computation

- 2→2 process with massive particles at two-loop order: almost done $g+g\to t+\bar{t},\quad g+g\to H+H(g)$
- Very time-consuming
  - **Two-loop $g+g\to H+H\ (g)$: complete IBP reduction cannot be achieved within tolerable time**

    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349
  - **Two-loop decay $Q+\bar{Q}\to g+g$, MIs cost $O(10^5)$ CPU core-hour**

    Feng, Jia, Sang, 1707.05758
- Current frontier: 2→3 processes at two loop

  5-gluon scattering may be feasible; hard for massive particles

New ideas are badly needed

Outline

Modified FIs

➢ **Modify Feynman loop integral by keeping finite η**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}} \qquad \mathcal{D}_\alpha \equiv q_\alpha^2 - m_\alpha^2$$

- Take it as an analytical function of η
- Physical result is defined by

$$\mathcal{M}(D,\vec{s},0) \equiv \lim_{\eta\to 0^+} \mathcal{M}(D,\vec{s},\eta)$$

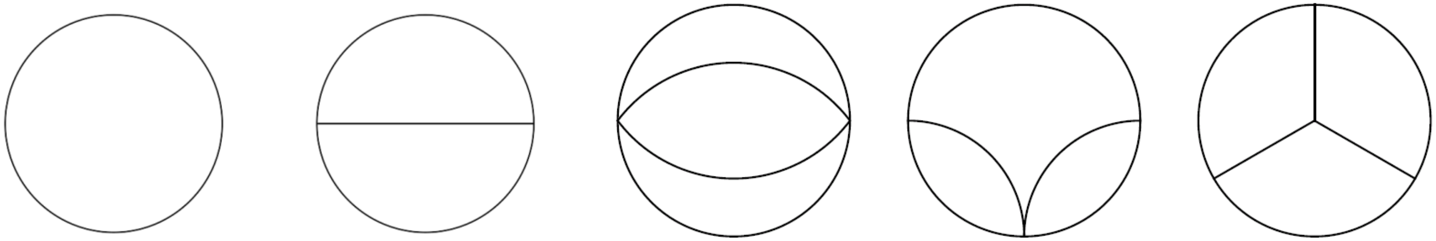
Expansion at infinity

- **Expansion of propagators around $\eta = \infty$**

$$\frac{1}{[(\ell+p)^2 - m^2 + \mathrm{i}\eta]^\nu} = \frac{1}{(\ell^2 + \mathrm{i}\eta)^\nu} \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left(\frac{-2\ell \cdot p - p^2 + m^2}{\ell^2 + \mathrm{i}\eta} \right)^n$$

  - Only one region in the method of region: $l^\mu \sim |\eta|^{1/2}$
  - No external momenta in denominator, vacuum integrals
  - Simple enough to deal with

- **Vacuum MIs with equal internal masses**



  - Analytical results are known up to 3-loop
  - Numerical results are known up to 5-loop

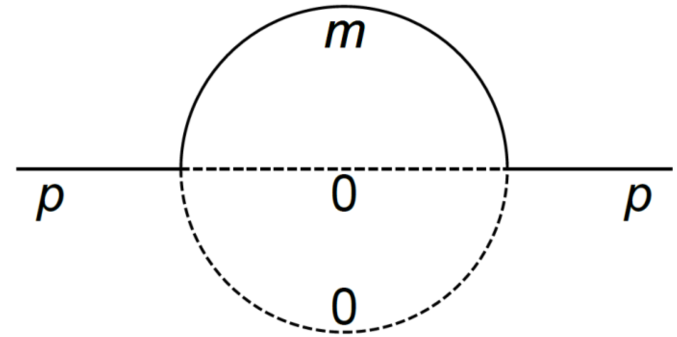
Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

Example

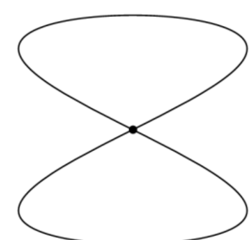
- **Sunrise integral**

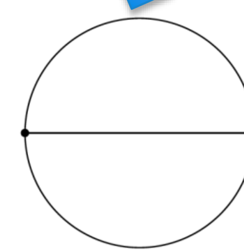
$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



$$\mathcal{D}_1 = (\ell_1+p)^2 - m^2,\ \mathcal{D}_2 = \ell_2^2,\ \mathcal{D}_3 = (\ell_1+\ell_2)^2$$

$$I_{111} = \eta^{D-3}\left\{\left[1 - \frac{D-3}{3}\frac{m^2}{\mathrm{i}\eta} + \frac{(D+4)(D-3)}{9D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,2}^{\mathrm{bub}} - \mathrm{i}\left[\frac{(D-2)^2}{3D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,1}^{\mathrm{bub}} + \mathcal{O}(\eta^{-2})\right\}$$





A new representation

➢ Asymptotic expansion

$$\mathcal{M}(D,\vec{s},\eta)=\eta^{LD/2-\sum_\alpha \nu_\alpha}\sum_{\mu_0=0}^{\infty}\eta^{-\mu_0}\mathcal{M}^{\text{bub}}_{\mu_0}(D,\vec{s})$$

$$\mathcal{M}^{\text{bub}}_{\mu_0}(D,\vec{s})=\sum_{k=1}^{B_L}I^{\text{bub}}_{L,k}(D)\sum_{\vec{\mu}\in\Omega^r_{\mu_0}}C_k^{\mu_0\cdots\mu_r}(D)s_1^{\mu_1}\cdots s_r^{\mu_r}$$

- $I^{\text{bub}}_{L,k}(D)$: k-th master vacuum integral at L-loop order
- $C_k^{\mu_0\dots\mu_r}(D)$: rational functions of D

➢ A new representation

- Uniqueness theorem of analytical functions: physical FI is uniquely determined by this asymptotic series via analytical continuation
- A new series representation of FIs
- All FIs (therefore scattering amplitudes) are determined by equal-mass vacuum integrals

Outline

I. Introduction

II. A New Representation

III. Reduction

IV. Outlook

What is reduction

➢ Reduction

- Find relations between loop integrals
- Use them to express all loop integrals as linear combinations of MIs

➢ Relations among $G \equiv \{M_1, M_2, \dots, M_n\}$

$$\sum_{i=1}^{n} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

- $Q_i(D, \vec{s}, \eta)$: homogeneous polynomials of $\vec{s}, \eta$ of degree d_i

➢ Constraints from mass dimension

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_n + \mathrm{Dim}(\mathcal{M}_n)$$

- Only 1 degree of freedom in $\{d_i\}$, chosen as $d_{\max} \equiv \mathrm{Max}\,\{d_i\}$

Find relations

$$\sum_{i=1}^{n} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta)=0$$

- **Decomposition of $Q_i(D,\vec{s},\eta)$**

$$Q_i(D,\vec{s},\eta)=\sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\cdots\lambda_r}(D)\,\eta^{\lambda_0}s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

$$\Rightarrow \sum_{k,\rho_0,\vec{\rho}} f_k^{\rho_0\cdots\rho_r}\,\mathcal{I}_{L,k}^{\mathrm{bub}}(D)\,\eta^{\rho_0}s_1^{\rho_1}\cdots s_r^{\rho_r}=0$$

- **Linear equations:** $f_k^{\rho_0\cdots\rho_r}=0$
  - With enough constraints $\Rightarrow Q_i^{\lambda_0\ldots\lambda_r}(D)$
  - With finite field technique, only integers in a finite field are involved, equations can be efficiently solved

- **Relations among $G\equiv\{M_1,M_2,\ldots,M_n\}$ with fixed $d_{\max}$ are fully determined**

Reduction

- **With $G = G_1 \cup G_2$, satisfy**
  - G_1 is more complicated than G_2
  - G_1 can be reduced to G_2
- **Algorithm** ***Search for simplest relations***
  1. Set $d_{\max} = 0$
  2. Find out all reduction relations among G with fixed $d_{\max}$
  3. If obtained relations are enough to determine G_1 by G_2, stop;

     else, $d_{\max} = d_{\max} + 1$ and go to step 2
- **Conditions for G_1 and G_2**
  1. Relations among G_1 and G_2 are not too complicated: easy to find
  2. $\#G_1$ is not too large: numerically diagonalize relations easily

Reduction scheme with only dots

- **FIs:** $\vec{\nu} = (\nu_1, \dots, \nu_N), \nu_i \geq 0$
  - $\mathbf{0}^{\pm} \equiv \textbf{Identity}, \ \mathbf{m}^{\pm} \equiv (\mathbf{m}-\mathbf{1})^{\pm}\mathbf{1}^{\pm}$
  - $\mathbf{1}^{+}(5,1,0,3) = \{(6,1,0,3), (5,2,0,3), (5,1,0,4)\}$
  - $\mathbf{1}^{-}(5,1,0,3) = \{(4,1,0,3), (5,0,0,3), (5,1,0,2)\}$
- **1-loop:** $G_1 = \mathbf{1}^{+}\vec{\nu}, G_2 = \mathbf{1}^{-}\mathbf{1}^{+}\vec{\nu}$ Duplancic and Nizic, 0303184
- **Multi-loop:**

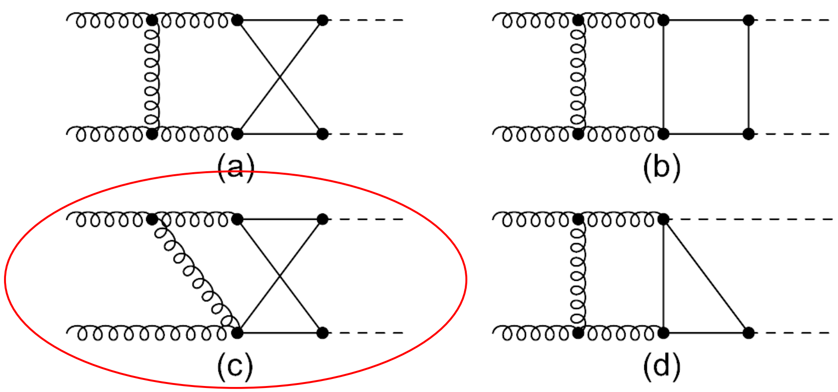
$$G_1 = \mathbf{m}^{+}\vec{\nu}, G_2 = \{\mathbf{1}^{-}\mathbf{m}^{+}, \mathbf{1}^{-}(\mathbf{m}-\mathbf{1})^{+}, \dots, \mathbf{1}^{-}\mathbf{1}^{+}\}\vec{\nu}$$

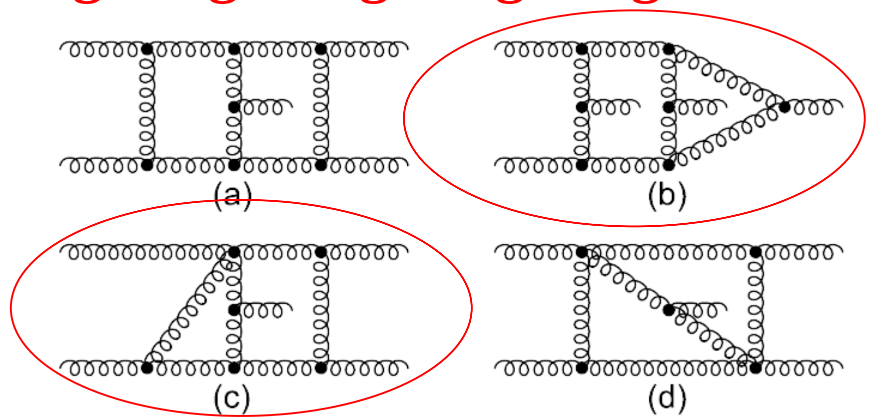
  - $m = 2,3$ in examples, $\#G_1$ is not too large, include dozens of integrals
  - Relations among G_1 and G_2 are not too complicated, see examples

A step-by-step reduction is realized!

Examples

- 2-loop $g+g\to H+H$ and $g+g\to g+g+g$





$g+g\to H+H$				$g+g\to g+g+g$			
Sector	Type	$d_{\max}$	$\mathbf{m}^+$	Sector	Type	$d_{\max}$	$\mathbf{m}^+$
1(a)	7-NP	1	$\mathbf{3}^+$	2(a)	8-NP	1	$\mathbf{3}^+$
1(b)	7-P	1	$\mathbf{3}^+$	2(b)	8-NP	3	$\mathbf{3}^+$
1(c)	6-NP	5	$\mathbf{3}^+$	2(c)	7-NP	4	$\mathbf{3}^+$
1(d)	6-P	4	$\mathbf{2}^+$	2(d)	6-NP	2	$\mathbf{3}^+$

Difficulty:

- More legs > less legs
- Nonplanar > Planar
- $\mathbf{m}^+\vec{e} > \mathbf{m}^+\vec{\nu}$

- Relations can be obtained by a single-core laptop in a few hours
- Diagonalizing at each phase space point (floating number): 0.01 second
- Results checked numerically by FIRE

Reduction of numerators

- Method similar to the reduction of denominators (paper in preparation)

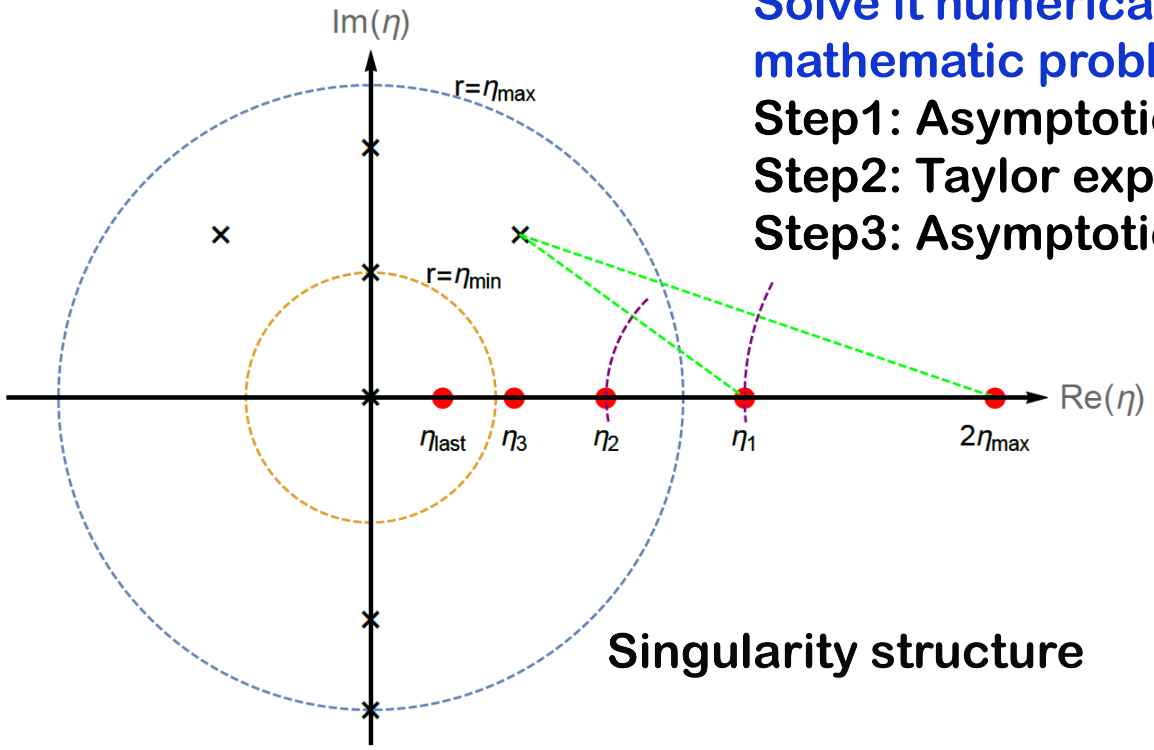
- Use η expansion to directly reduce amplitudes

  Wang, Li, Basat, 1901.09390

Analytical continuation

- **Set up and solve DEs of MIs**

$$\frac{\partial}{\partial \eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$ **with known** $\vec{I}(D;\infty)$



Solve it numerically: a well-studied mathematic problem

Step1: Asymptotic expansion at $\eta = \infty$

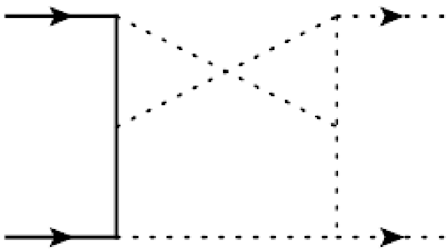
Step2: Taylor expansion at analytical points

Step3: Asymptotic expansion at $\eta = 0$

Singularity structure

Example

- **2-loop non-planar sector for $\mathrm{Q} + \bar{\mathrm{Q}} \to g + g$**



- 168 master integrals
- Traditional method sector decomposition: $O(10^4)$ CPU core-hour
- Our method: a few minutes

Feng, Jia, Sang, 1707.05758

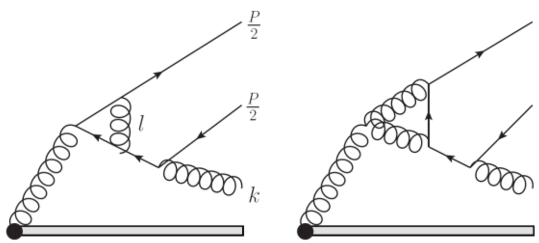
MIs can be thought as special functions, and DEs tell us how to evaluate these special functions

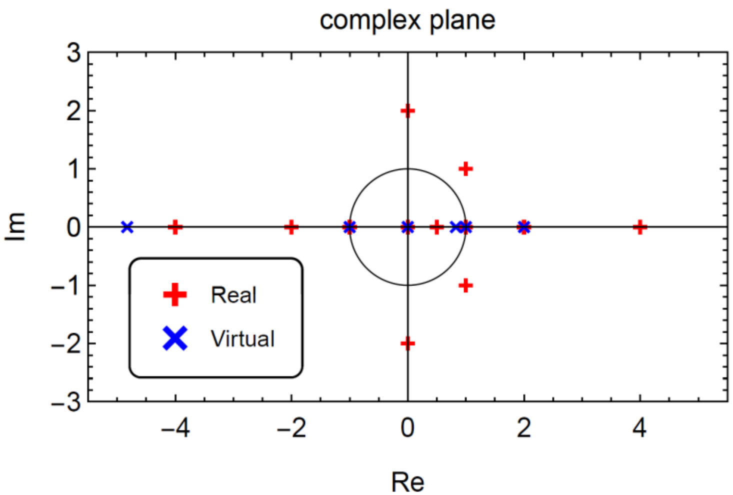
Practical use

Use η expansion to determine MIs

FFs of $g \to Q\bar{Q}({}^1S_0^{[1,8]}) + X$

Zhang, Wang, Liu, Ma, Meng, Chao, 1810.07656





$$\frac{\mathrm{d}\boldsymbol{I}(\epsilon, z)}{\mathrm{d}z} = A(\epsilon, z)\boldsymbol{I}(\epsilon, z)$$

$$d_{\mathrm{NLO}}^{[1]}(z) = \frac{\alpha_s^3}{2\pi N_c m_Q^3} \times \left(d^{[1]}(z) + \ln\left(\frac{\mu_r^2}{4m_Q^2}\right) b_0\, d_{\mathrm{LO}}^{(0)}(z) + \ln\left(\frac{\mu_f^2}{4m_Q^2}\right) f(z) \right),$$

$$d_{\mathrm{NLO}}^{[8]}(z) = \frac{\alpha_s^3 (N_c^2-4)}{4\pi N_c (N_c^2-1) m_Q^3} \times \left(d^{[8]}(z) + \ln\left(\frac{\mu_r^2}{4m_Q^2}\right) b_0\, d_{\mathrm{LO}}^{(0)}(z) + \ln\left(\frac{\mu_f^2}{4m_Q^2}\right) f(z) \right)$$

$$d^{[1/8]}(z) = \begin{cases} -\dfrac{N_c}{2z} + \displaystyle\sum_{i=0}^{2}\sum_{j=0}^{\infty} \ln^i z\,(2z)^j \left(A_{ij}^f\, n_f + A_{ij}^{[1/8]}\, N_c + \frac{A_{ij}^N}{N_c} \right), & \text{for } 0 < z < \frac{1}{4} \\ \displaystyle\sum_{j=0}^{\infty} (2z-1)^j \left(B_j^f\, n_f + B_j^{[1/8]}\, N_c + \frac{B_j^N}{N_c} \right), & \text{for } \frac{1}{4} \le z \le \frac{3}{4} \\ \displaystyle\sum_{i=0}^{3}\sum_{j=0}^{\infty} \ln^i (1-z)\,(2-2z)^j \left(C_{ij}^f\, n_f + C_{ij}^{[1/8]}\, N_c + \frac{C_{ij}^N}{N_c} \right), & \text{for } \frac{3}{4} < z < 1 \end{cases}$$

of z. To obtain about 150-digit precision for any value of z, we will attach an ancillary file for the arXiv preprint in future, in which these coefficients will be calculated up to $j = 500$ with 150 digits for each coefficient.

- Use η expansion at $z = \frac{1}{4}, \frac{3}{4}$ to obtain 200-digit precision
- Combine η expansion and numerical Des w.r.t. kinematic variables

Outline

I. Introduction

II. A New Representation

III. Reduction

IV. Outlook

Summary

- A new (series) representation: Feynman integrals are fully determined by vacuum integrals
- A general strategy to do reduction
- A general strategy to evaluate MIs
- Two–loop examples: our method is correct and efficient
- Application to fragmentation function: correct and helpful

Future plan: practical application

- A package to do systematic reduction
  - Express all FIs as linear combinations of MIs
- A package to calculate MIs
  - Can be thought as a multi-loop version of "looptools"

Thank you!