# Quarkonium Processes @ NNLO

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# Outline:

- Basic Procedures in *Automated* Higher-Order Computation
- Some *Results* in our Recent Works
- Summary

# Automated
# Higher-Order Computation

# Basic Procedures:

1. Feynman Diagrams & Amplitudes

2. Trace & Contraction

3. Partial Fragmentation & IBP Reduction

4. Master Integrals – *Numerical*

5. Other Processing – Plots *etc.*

# Basic Procedures:

1. Feynman Diagrams & Amplitudes

- FeynArts - Mathematica

  **http://www.feynarts.de**

- Qgraf - Fortran

  **http://cfif.ist.utl.pt/~paulo/qgraf.html**

- FeynRules - Mathematica

  **http://feynrules.irmp.ucl.ac.be**

- . . .

# Basic Procedures:

## 1. Feynman Diagrams & Amplitudes

### Load Package

```
<< Qgraf`

LoadFeynRules["sm"];
```

### QgRun

```
HAmp = QgRun["
output='out';
style='math.sty';
model='sm.mod';
in=C[p1],Cbar[p2];
out=g[k1],g[k2];
loops=2;
loop_momentum=q;
options=notadpole,onshell;
true=vsum[QCD,6,6];
true=vsum[QED,0,0];
"];
```

→

```
--------------------------------------------------------------------
            qgraf-3.1.4
--------------------------------------------------------------------

output='out';
style='math.sty';
model='sm.mod';
in=C[p1],Cbar[p2];
out=g[k1],g[k2];
loops=2;
loop_momentum=q;
options=notadpole,onshell;
true=vsum[QCD,6,6];
true=vsum[QED,0,0];

--------------------------------------------------------------------

 24P --- 7+ 17- --- 5N+ 2C+ 17C-

 169V --- 3^142 4^27

--------------------------------------------------------------------

 -  4^3 --- 0 diagrams
 3^2 4^2 --- 23 diagrams
 3^4 4^1 --- 244 diagrams
 3^6  -  --- 1122 diagrams

 total = 1389 diagrams
```

# Basic Procedures:

## 1. Feynman Diagrams & Amplitudes

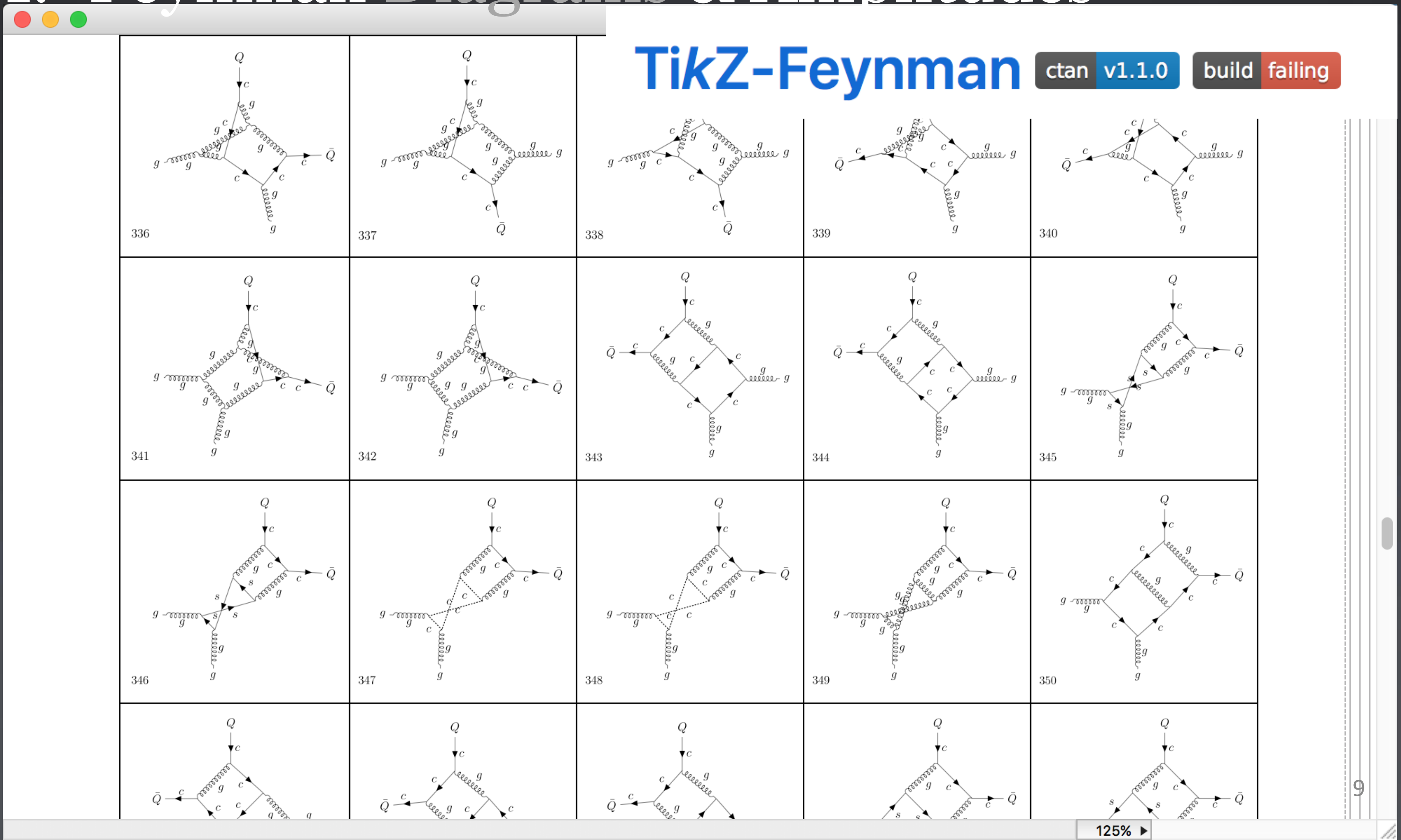
math.sty

math.sty 〉 No Selection

```
<prologue>
(* QGraf Output in Mathematica Style By F.Feng *)

<diagram>

Amp[[<diagram_index>]]:= Hold[[(<sign><symmetry_factor>)*
<in_loop>InField[[<field>,<field_index>,<momentum>]]*
<end>
<back><out_loop>OutField[[<field>,<field_index>,<momentum>]]*
<end>
<back><propagator_loop>Propagator[[
<back>Field[[<field>,<field_index>]],
<back>Field[[<dual-field>,<dual-field_index>]],
<back><momentum>]]*
<end>
<back><vertex_loop>Vertex[[
<back><ray_loop>Field[[<field>,<field_index>,<momentum>]],
<back><end><back>]]*
<end><back><back>]];

<epilogue>

<exit>
```

# Basic Procedures:

1. Feynman Diagrams & Amplitudes



# Basic Procedures:

## 1. Feynman Diagrams & Amplitudes

```
HAmp[[342]]

Hold[+1 InField(C, −1, p1) InField(Cbar, −3, p2)
    OutField(g, −2, k1) OutField(g, −4, k2) Propagator(Field(g, 1), Field(g, 2), −q1)
    Propagator(Field(g, 7), Field(g, 8), −q2) Propagator(Field(g, 3), Field(g, 4), k1 + q1)
    Vertex(Field(g, −2, −k1), Field(g, 1, −q1), Field(g, 3, k1 + q1))
    Propagator(Field(g, 5), Field(g, 6), k2 − q1)
    Vertex(Field(g, −4, −k2), Field(g, 2, q1), Field(g, 5, k2 − q1))
    Propagator(Field(C, 9), Field(Cbar, 10), p1 − q2)
    Propagator(Field(g, 11), Field(g, 12), −k1 − q1 + q2)
    Vertex(Field(g, 4, −k1 − q1), Field(g, 8, q2), Field(g, 12, k1 + q1 − q2))
    Vertex(Field(Cbar, 10, q2 − p1), Field(C, −1, p1), Field(g, 7, −q2))
    Propagator(Field(C, 13), Field(Cbar, 14), k1 − p2 + q1 − q2)
    Vertex(Field(Cbar, −3, p2), Field(C, 13, k1 − p2 + q1 − q2), Field(g, 11, −k1 − q1 + q2))
    Vertex(Field(Cbar, 14, −k1 + p2 − q1 + q2), Field(C, 9, p1 − q2), Field(g, 6, q1 − k2))]
```

# Basic Procedures:

## 1. Feynman Diagrams & Amplitudes

### Propagator

```
Propagator[Field[q_, fi1_], Field[qbar_, fi2_], mom_] := I MatDelta[TI[fi1], TI[fi2]] Mat[GSD[mom] + Mass[q], {DI[fi1], DI[fi2]}] / (SPD[mom] - Mass[q]^2);

Propagator[Field[g, fi1_], Field[g, fi2_], mom_] :=
  -I SUNDelta[CI[fi1], CI[fi2]] (MTD[LI[fi1], LI[fi2]] / SPD[mom] - (1 - ξ) FVD[mom, LI[fi1]] FVD[mom, LI[fi2]] / (SPD[mom]^2));

Propagator[Field[gh, fi1_], Field[ghbar, fi2_], mom_] := I SUNDelta[CI[fi1], CI[fi2]] / (SPD[mom]);
```

### Vertex

```
Vertex[Field[qbar, fi1_, mom1_], Field[q, fi2_, mom2_], Field[g, fi3_, mom3_]] :=
  I Gstrong Mat[GAD[LI[fi3]], {DI[fi1], DI[fi2]}] Mat[SUNT[CI[fi3]], {TI[fi1], TI[fi2]}];

Vertex[Field[g, fi1_, mom1_], Field[g, fi2_, mom2_], Field[g, fi3_, mom3_]] :=
  Gstrong SUNF[CI[fi1], CI[fi2], CI[fi3]]
   (MTD[LI[fi1], LI[fi2]] FVD[mom1 - mom2, LI[fi3]] + MTD[LI[fi2], LI[fi3]] FVD[mom2 - mom3, LI[fi1]] + MTD[LI[fi3], LI[fi1]] FVD[mom3 - mom1, LI[fi2]]);

Vertex[Field[g, fi1_, mom1_], Field[g, fi2_, mom2_], Field[g, fi3_, mom3_], Field[g, fi4_, mom4_]] :=
  -I Gstrong^2 (SUNF4[CI[fi1], CI[fi2], CI[fi3], CI[fi4]] (MTD[LI[fi1], LI[fi3]] MTD[LI[fi2], LI[fi4]] - MTD[LI[fi1], LI[fi4]] MTD[LI[fi2], LI[fi3]]) +
     SUNF4[CI[fi1], CI[fi3], CI[fi2], CI[fi4]] (MTD[LI[fi1], LI[fi2]] MTD[LI[fi3], LI[fi4]] - MTD[LI[fi1], LI[fi4]] MTD[LI[fi3], LI[fi2]]) +
     SUNF4[CI[fi1], CI[fi4], CI[fi2], CI[fi3]] (MTD[LI[fi1], LI[fi2]] MTD[LI[fi4], LI[fi3]] - MTD[LI[fi1], LI[fi3]] MTD[LI[fi4], LI[fi2]]));

Vertex[Field[ghbar, fi1_, mom1_], Field[gh, fi2_, mom2_], Field[g, fi3_, mom3_]] := -Gstrong SUNF[CI[fi1], CI[fi2], CI[fi3]] FVD[mom1, LI[fi3]];
```

# Basic Procedures:

1. Feynman Diagrams & Amplitudes

```
HAmp[[342]] // ReleaseHold
```

$$\big(\delta_{\mathrm{CI}(1)\,\mathrm{CI}(2)}\,\delta_{\mathrm{CI}(3)\,\mathrm{CI}(4)}\,\delta_{\mathrm{CI}(5)\,\mathrm{CI}(6)}\,\delta_{\mathrm{CI}(7)\,\mathrm{CI}(8)}\,\delta_{\mathrm{CI}(11)\,\mathrm{CI}(12)}\,f_{\mathrm{CI}(-4)\,\mathrm{CI}(2)\,\mathrm{CI}(5)}\,f_{\mathrm{CI}(-2)\,\mathrm{CI}(1)\,\mathrm{CI}(3)}\,f_{\mathrm{CI}(4)\,\mathrm{CI}(8)\,\mathrm{CI}(12)}\,g^{\mathrm{LI}(1)\,\mathrm{LI}(2)}$$
$$g^{\mathrm{LI}(3)\,\mathrm{LI}(4)}\,g^{\mathrm{LI}(5)\,\mathrm{LI}(6)}\,g^{\mathrm{LI}(7)\,\mathrm{LI}(8)}\,g^{\mathrm{LI}(11)\,\mathrm{LI}(12)}\,g_s^6\,\mathrm{OutField}(g,-2,\mathrm{k1})\,\mathrm{OutField}(g,-4,\mathrm{k2})$$
$$\big(g^{\mathrm{LI}(-2)\,\mathrm{LI}(1)}\big(\mathrm{q1}-\mathrm{k1}^{\mathrm{LI}(3)}\big)+g^{\mathrm{LI}(1)\,\mathrm{LI}(3)}\big(-\mathrm{k1}-2\,\mathrm{q1}^{\mathrm{LI}(-2)}\big)+g^{\mathrm{LI}(3)\,\mathrm{LI}(-2)}\big(2\,\mathrm{k1}+\mathrm{q1}^{\mathrm{LI}(1)}\big)\big)$$
$$\big(g^{\mathrm{LI}(-4)\,\mathrm{LI}(2)}\big(-\mathrm{k2}-\mathrm{q1}^{\mathrm{LI}(5)}\big)+g^{\mathrm{LI}(2)\,\mathrm{LI}(5)}\big(2\,\mathrm{q1}-\mathrm{k2}^{\mathrm{LI}(-4)}\big)+g^{\mathrm{LI}(5)\,\mathrm{LI}(-4)}\big(2\,\mathrm{k2}-\mathrm{q1}^{\mathrm{LI}(2)}\big)\big)$$
$$\big(g^{\mathrm{LI}(4)\,\mathrm{LI}(8)}\big(-\mathrm{k1}-\mathrm{q1}-\mathrm{q2}^{\mathrm{LI}(12)}\big)+g^{\mathrm{LI}(8)\,\mathrm{LI}(12)}\big(-\mathrm{k1}-\mathrm{q1}+2\,\mathrm{q2}^{\mathrm{LI}(4)}\big)+$$
$$g^{\mathrm{LI}(12)\,\mathrm{LI}(4)}\big(2\,\mathrm{k1}+2\,\mathrm{q1}-\mathrm{q2}^{\mathrm{LI}(8)}\big)\big)\,\mathrm{ColorLine}\big(T_{\mathrm{CI}(11)}.T_{\mathrm{CI}(6)}.T_{\mathrm{CI}(7)},\{\mathrm{p2},\mathrm{p1}\}\big)\,\mathrm{SpinLine}\big($$
$$\gamma^{\mathrm{LI}(11)}.(\mathrm{Mass}(C)+\gamma\cdot(\mathrm{k1}-\mathrm{p2}+\mathrm{q1}-\mathrm{q2})).\gamma^{\mathrm{LI}(6)}.(\mathrm{Mass}(C)+\gamma\cdot(\mathrm{p1}-\mathrm{q2})).\gamma^{\mathrm{LI}(7)},\{\mathrm{p2},\mathrm{p1}\}\big)\big)\big/$$
$$\big(\mathrm{q1}^2\,\mathrm{q2}^2\,(\mathrm{k1}+\mathrm{q1})^2\,(\mathrm{k2}-\mathrm{q1})^2\,(-\mathrm{k1}-\mathrm{q1}+\mathrm{q2})^2\,\big((\mathrm{p1}-\mathrm{q2})^2-\mathrm{Mass}(C)^2\big)$$
$$\big((\mathrm{k1}-\mathrm{p2}+\mathrm{q1}-\mathrm{q2})^2-\mathrm{Mass}(C)^2\big)\big)$$

# Basic Procedures:

2. Trace & Contraction

- FeynCalc - Mathematica Package

  **https://github.com/FeynCalc**

- Form - C Program (tForm, parForm)

  **https://github.com/vermaseren/form**

- FeynCalc/FormLink - Combined

  **https://github.com/FormLink**

- ...

# Basic Procedures:

## 2. Trace & Contraction

Untitled-1

FeynCalc/FormLink

```
In[1]:= << FeynCalcFormLink`

In[8]:= exp = DiracTrace[GAD[μ].(GSD[p1 + p2] + m).GAD[ν].(GSD[k1 + k2] + m) GSD[p1, p2, k1, k2]]
          (FVD[p1 + k1, μ] FVD[p2 + k2, ν] + MTD[μ, ν])
```

Out[8]= $(g^{\mu\nu} + (\mathrm{k1} + \mathrm{p1}^{\mu})(\mathrm{k2} + \mathrm{p2}^{\nu}))\,\mathrm{tr}((\gamma\cdot\mathrm{p1}).(\gamma\cdot\mathrm{p2}).(\gamma\cdot\mathrm{k1}).(\gamma\cdot\mathrm{k2})\,\gamma^{\mu}.(m+\gamma\cdot(\mathrm{p1}+\mathrm{p2})).\gamma^{\nu}.(\gamma\cdot(\mathrm{k1}+\mathrm{k2})+m))$

```
In[9]:= FeynCalcFormLink[exp] // Factor
```

```
Symbols D,m;
Dimension D;
Vectors k1,k2,p1,p2;
AutoDeclare Index lor;
Format Mathematica;
AutoDeclare Symbol str;
L ltr1 =
(g_(1,lor1)*(m*gi_(1)+g_(1,p1)+g_(1,p2))*g_(1,lor2)*(m*gi_(1)+g_(1,k1)+g_(1,k2))*g_(1,p1)*g_(1,p2)*g_(1,k1)*g
_(1,k2));
tracen,1;
.sort;
L resFL = (str1*(d_(lor1,lor2)+(k1(lor1)+p1(lor1))*(k2(lor2)+p2(lor2))));
id str1=ltr1;
.sort;
contract 0;
.sort;
#call put("%E", resFL)
.end
```



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# Basic Procedures:

## 2. Trace & Contraction

Untitled-1

FEYNCALC/FORMLINK

Piping the script to FORM and running FORM

Time needed by FORM : 0.006 seconds. FORM finished. Got the result back to Mathematica as a string.

Start translation to Mathematica / FeynCalc syntax

Total wall clock time used: 0.15 seconds. Translation to Mathematica and FeynCalc finished.

Out[9]=

$$4\big(2\,k1\cdot p2\,k2\cdot p1^2\,m^2 + 2\,k1\cdot k2\,p1\cdot p2^2\,m^2 + D\,k1\cdot p2\,k2\cdot p1\,m^2 + 2\,k1\cdot k2\,k1\cdot p2\,k2\cdot p1\,m^2 - D\,k1\cdot p1\,k2\cdot p2\,m^2 - 2\,k1\cdot k2\,k1\cdot p1\,k2\cdot p2\,m^2 -$$

$$2\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2\,m^2 - k1\cdot p2\,k2^2\,p1^2\,m^2 + 2\,k1\cdot k2^2\,p1\cdot p2\,m^2 + D\,k1\cdot k2\,p1\cdot p2\,m^2 + 4\,k1\cdot k2\,k1\cdot p2\,p1\cdot p2\,m^2 - k1^2\,k2^2\,p1\cdot p2\,m^2 +$$

$$2\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2\,m^2 + 2\,k1\cdot p2\,k2\cdot p1\,p1\cdot p2\,m^2 - 2\,k1^2\,k2\cdot p2\,p1\cdot p2\,m^2 - 2\,k1\cdot p1\,k2\cdot p2\,p1\cdot p2\,m^2 - 2\,k1\cdot k2\,k1\cdot p1\,p2^2\,m^2 +$$

$$k1^2\,k2\cdot p1\,p2^2\,m^2 - k1\cdot k2\,p1^2\,p2^2\,m^2 - 4\,k1\cdot p2^2\,k2\cdot p1^2 - 2\,D\,k1\cdot p2\,k2\cdot p1^2 + 2\,k1^2\,k1\cdot p2\,k2\cdot p1^2 + 4\,k1\cdot p2\,k2\cdot p1^2 - 4\,k1\cdot p1^2\,k2\cdot p2^2 +$$

$$2\,D\,k1\cdot p1\,k2\cdot p2^2 - 2\,k1^2\,k1\cdot p1\,k2\cdot p2^2 - 4\,k1\cdot p1\,k2\cdot p2^2 + 4\,k1\cdot k2^2\,p1\cdot p2^2 + 2\,k1^2\,k1\cdot k2\,p1\cdot p2^2 + 2\,k1\cdot k2\,k2^2\,p1\cdot p2^2 -$$

$$2\,D\,k1\cdot p2^2\,k2\cdot p1 - 4\,k1\cdot k2\,k1\cdot p2^2\,k2\cdot p1 + 4\,k1\cdot p2^2\,k2\cdot p1 - 2\,D\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1 - 4\,k1\cdot k2\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1 + 4\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1 +$$

$$2\,D\,k1\cdot p1^2\,k2\cdot p2 + 4\,k1\cdot k2\,k1\cdot p1^2\,k2\cdot p2 - 4\,k1\cdot p1^2\,k2\cdot p2 + 2\,D\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2 + 4\,k1\cdot k2\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2 -$$

$$4\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2 + 2\,D\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2 - 2\,k1^2\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2 - 4\,k1\cdot p1\,k2\cdot p1\,k2\cdot p2 - 2\,D\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 +$$

$$2\,k1^2\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 + 8\,k1\cdot p1\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 + 4\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2 + 4\,k1\cdot k2\,k1\cdot p2^2\,p1^2 - 2\,k1\cdot p1\,k2\cdot p2^2\,p1^2 +$$

$$4\,k1\cdot k2^2\,k1\cdot p2\,p1^2 + 2\,D\,k1\cdot k2\,k1\cdot p2\,p1^2 - 4\,k1\cdot k2\,k1\cdot p2\,p1^2 + 2\,k1\cdot p2^2\,k2^2\,p1^2 + D\,k1\cdot p2\,k2^2\,p1^2 - k1^2\,k1\cdot p2\,k2^2\,p1^2 +$$

$$2\,k1\cdot k2\,k1\cdot p2\,k2^2\,p1^2 - 2\,k1\cdot p2\,k2^2\,p1^2 + 2\,k1\cdot p2^2\,k2\cdot p1\,p1^2 + k1\cdot p2\,k2^2\,k2\cdot p1\,p1^2 - D\,k1^2\,k2\cdot p2\,p1^2 + 2\,k1^2\,k2\cdot p2\,p1^2 -$$

$$2\,k1^2\,k1\cdot k2\,k2\cdot p2\,p1^2 - 2\,k1^2\,k1\cdot p2\,k2\cdot p2\,p1^2 - 2\,k1\cdot p1\,k1\cdot p2\,k2\cdot p2\,p1^2 - k1^2\,k2^2\,k2\cdot p2\,p1^2 - k1\cdot p1\,k2^2\,k2\cdot p2\,p1^2 +$$

$$2\,k1\cdot p2\,k2\cdot p1\,k2\cdot p2\,p1^2 - 4\,k1\cdot k2^2\,k1\cdot p1\,p1\cdot p2 - 2\,D\,k1\cdot k2\,k1\cdot p1\,p1\cdot p2 + 4\,k1\cdot k2\,k1\cdot p1\,p1\cdot p2 + 4\,k1\cdot k2^2\,k1\cdot p2\,p1\cdot p2 +$$

$$2\,D\,k1\cdot k2\,k1\cdot p2\,p1\cdot p2 - 4\,k1\cdot k2\,k1\cdot p2\,p1\cdot p2 - 8\,k1\cdot k2\,k1\cdot p1\,k1\cdot p2\,p1\cdot p2 + 2\,k1^2\,k1\cdot p1\,k2^2\,p1\cdot p2 + 2\,D\,k1\cdot p2\,k2^2\,p1\cdot p2 +$$

$$4\,k1\cdot k2\,k1\cdot p2\,k2^2\,p1\cdot p2 - 4\,k1\cdot p1\,k1\cdot p2\,k2^2\,p1\cdot p2 - 4\,k1\cdot p2\,k2^2\,p1\cdot p2 - 2\,D\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2 + 2\,k1^2\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2 +$$

$$4\,k1\cdot k2\,k2\cdot p1\,p1\cdot p2 + 2\,k1^2\,k1\cdot p2\,k2\cdot p1\,p1\cdot p2 + 2\,k1\cdot p2\,k2^2\,k2\cdot p1\,p1\cdot p2 - 2\,D\,k1^2\,k2\cdot p2\,p1\cdot p2 + 4\,k1^2\,k2\cdot p2\,p1\cdot p2 -$$

$$2\,D\,k1\cdot k2\,k2\cdot p2\,p1\cdot p2 - 2\,k1^2\,k1\cdot k2\,k2\cdot p2\,p1\cdot p2 + 4\,k1\cdot k2\,k2\cdot p2\,p1\cdot p2 + 2\,k1^2\,k1\cdot p1\,k2\cdot p2\,p1\cdot p2 - 2\,k1^2\,k2^2\,k2\cdot p2\,p1\cdot p2 -$$

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# Basic Procedures:

3. Partial Fragmentation & IBP Reduction

- APART - MATHEMATICA

  **https://github.com/F-Feng**

  FIRE - MATHEMATICA Program & C++

  **http://science.sander.su**

- AIR - MAPLE Program

  **https://www.phys.ethz.ch/~pheno/air/**

- REDUZE - C (MPI supported)

  **https://reduze.hepforge.org**

- . . .

# Basic Procedures:

## 3. Partial Fragmentation & IBP Reduction

- Integrate By Part (IBP)

JHEP **0810**, 107 (2008)

$$F(a_1, \cdots, a_n) = \int \cdots \int \frac{d^d k_1 \cdots d^d k_h}{E_1^{a_1} \cdots E_n^{a_n}}$$

where $k_i$, $i = 1, \cdots, h$, are loop momenta and the denominators $E_r$ are either quadratic or linear with respect to the loop momenta $k_i$ of the graph. Irreducible polynomials in the numerator can be represented as denominators raised to negative powers.

- Basic idea:

$$\int \cdots \int d^d k_1 d^d k_2 \cdots \frac{\partial}{\partial k_i} \left[ \frac{p_j}{E_1^{a_1} \cdots E_n^{a_n}} \right] = 0$$

- List of equations:

$$\sum \alpha_i F(a_1 + b_{i,1}, \cdots, a_n + b_{i,n}) = 0$$

# Basic Procedures:

## 3. Partial Fragmentation & IBP Reduction

Apart function in MATHEMAITCA

In[1]:= $\text{Apart}\left[\frac{1}{(x-a)\,(x-b)\,(x-c)}\right]$

Out[1]= $-\frac{1}{(a-b)(b-c)(x-b)}-\frac{1}{(a-c)(c-b)(x-c)}+\frac{1}{(a-b)(a-c)(x-a)}$

In[2]:= $\text{Apart}\left[\frac{1}{(x+a)\,(y+2\,b)\,(3\,x+4\,y)}\right]$

Out[2]= $\frac{1}{(a+x)(3\,x-8\,b)(2\,b+y)}-\frac{4}{(a+x)(3\,x-8\,b)(3\,x+4\,y)}$

# Basic Procedures:

## 3. Partial Fragmentation & IBP Reduction

Apart Package:

In[1]:= `<< CalcExt`Apart``

In[2]:= $\$Apart\left[\frac{1}{(x-a)\,(x-b)\,(x-c)},\ \{x\}\right]$

Out[2]= $-\frac{\left\|\frac{1}{a-x}\right\|}{(a-b)\,(a-c)}+\frac{\left\|\frac{1}{b-x}\right\|}{(a-b)\,(b-c)}+\frac{\left\|\frac{1}{c-x}\right\|}{(a-c)\,(c-b)}$

In[3]:= $\$Apart\left[\frac{1}{(x+a)\,(y+2\,b)\,(3\,x+4\,y)},\ \{x,\ y\}\right]$

Out[3]= $-\frac{\left\|\frac{1}{(a+x)\,(2\,b+y)}\right\|}{3\,a+8\,b}+\frac{4\left\|\frac{1}{(a+x)\,(3\,x+4\,y)}\right\|}{3\,a+8\,b}+\frac{3\left\|\frac{1}{(2\,b+y)\,(3\,x+4\,y)}\right\|}{3\,a+8\,b}$

# Basic Procedures:

## 3. Partial Fragmentation & IBP Reduction

$$F[\{l,m,n\}] = \int \frac{d^4k}{(2\pi)^4} \frac{(k\cdot p_2)^{-l}}{(m^2-k^2-2k\cdot p_1-p_1^2)^m(-m^2+k^2+2k\cdot p_2+p_2^2)^n}$$

```
In[1]:= << HighEnergyPhysics`fc`

In[2]:= << FIRE`

In[3]:= Replacement = {p1^2 → m^2, p2^2 → m^2, p1 p2 → SP[p1, p2]};
        Internal = {k};
        External = {p1, p2};
        Propagators = {k p2, -2 k p1 - k^2 + m^2 - p1^2, 2 k p2 + k^2 - m^2 + p2^2};
        PrepareIBP[];
        startinglist = {IBP[k, k], IBP[k, p1], IBP[k, p2]} /. Replacement;
        Prepare[];

In[10]:= Burn[];

In[11]:= F[{-1, 1, 2}]
```

$$G[\{l,m,n\}] = F[\{1,m,n\}]$$

Out[11]= $\frac{(d-2)\,G(\{0,0,1\})}{8\,(m^2-\text{p1}\cdot\text{p2})} + \frac{(d-2)\,G(\{0,1,0\})}{8\,(m^2-\text{p1}\cdot\text{p2})} + \frac{1}{4}(4-d)\,G(\{0,1,1\})$

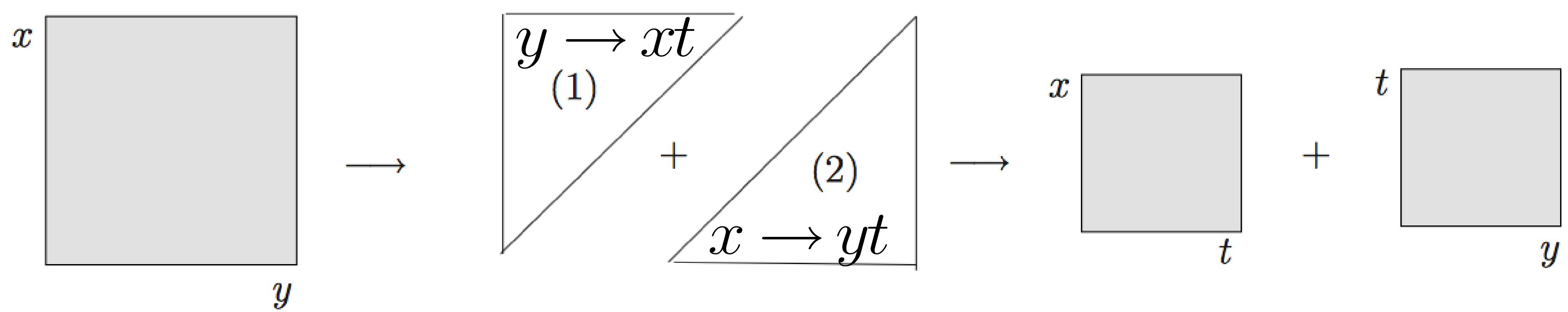
# Basic Procedures:

## 4. Master Integrals - *Numerical*

Sector Decomposition

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x + (1-x)y)^{-1}$$

$$= \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x + (1-x)y)^{-1} \left[\theta(x-y) + \theta(y-x)\right]$$



# Basic Procedures:

## 4. Master Integrals - *Numerical*

Sector Decomposition

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x + (1-x)y)^{-1}$$

$$= \int_0^1 dx \int_0^1 dy\, x^{-1-a\varepsilon} y^{-b\varepsilon} (x + (1-x)y)^{-1} \Big[\theta(x-y) + \theta(y-x)\Big]$$

$$= \int_0^1 dx x^{-1-(a+b)\varepsilon} \int_0^1 t^{-b\varepsilon} \Big(1 + (1-x)t\Big)^{-1}$$

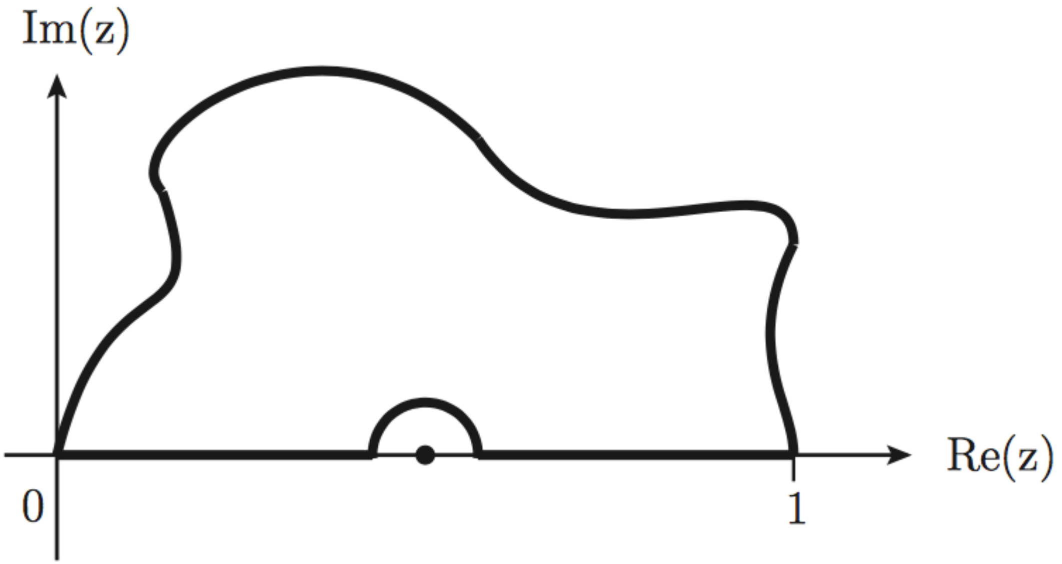
$$+ \int_0^1 dy y^{-1-(a+b)\varepsilon} \int_0^1 t^{-1-a\varepsilon} \Big(1 + (1-y)t\Big)^{-1}$$

# Basic Procedures:

4. Master Integrals - *Numerical*

Contour Deformation

*Deformation of the integration contour*



# Basic Procedures:

4. Master Integrals - *Numerical*

Contour Deformation

$$z_k = x_k - i\lambda_k x_k(1 - x_k)\frac{\partial \mathcal{F}}{\partial x_k}$$

$$\mathcal{F}[\vec{z}(\vec{x})] = \mathcal{F}(\vec{x}) - i\sum_k \lambda_k x_k(1 - x_k)\left(\frac{\partial \mathcal{F}}{\partial x_k}\right)^2 + \mathcal{O}(\lambda^2)$$

# Basic Procedures:

4. Master Integrals - *Numerical*
   - FIESTA - MATHEMATICA

     **https://bitbucket.org/feynmanIntegrals/**

   - SECDEC - MATHEMATICA/PYTHON

     **http://secdec.hepforge.org**

   - SECTOR_DECOMPOSITION - C++

     **http://wwwthep.physik.uni-mainz.de/~stefanw/sector_decomposition/**

   - CSECTORS - MATHEMATCA/C++

     **http://prac.us.edu.pl/~gluza/csectors/**

   - . . .

# Basic Procedures:

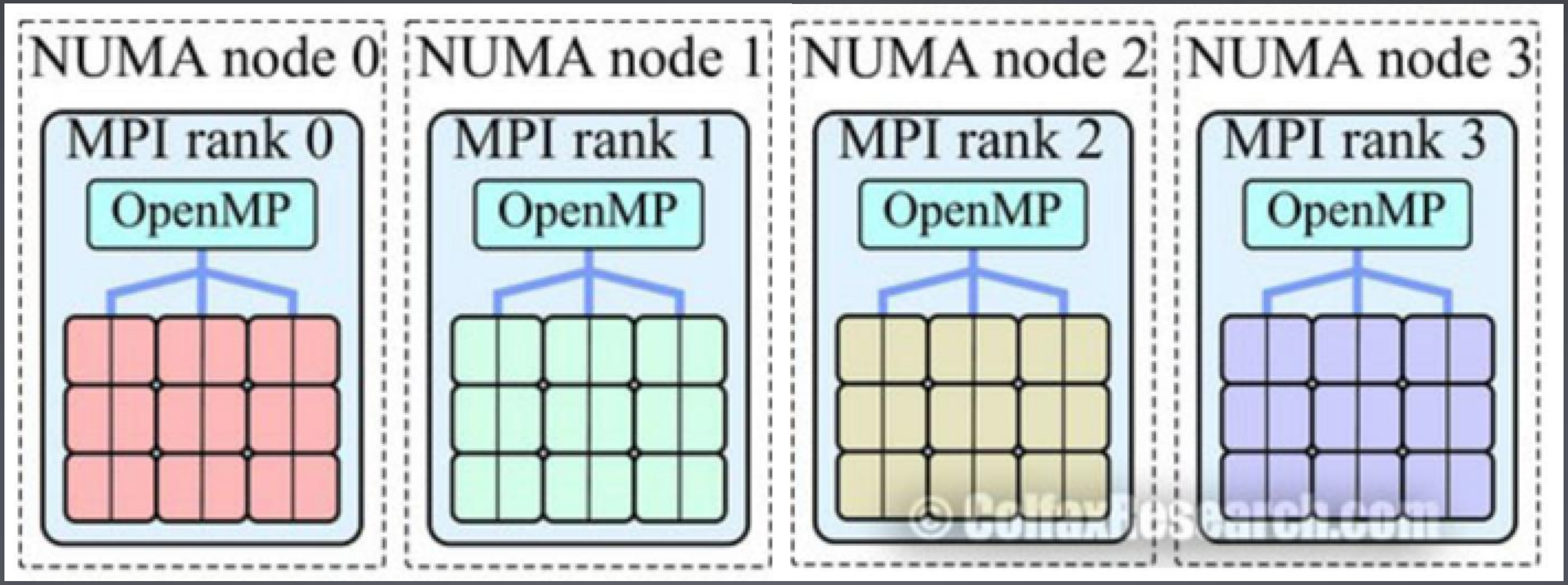
4. Master Integrals - *Numerical*
   - CubPack - Fortran
     **http://nines.cs.kuleuven.be/software/cubpack/**
   - Cuba - C & Fortran
     **http://www.feynarts.de/cuba/**
   - ParInt - C & Fortran
     **https://cs.wmich.edu/parint/**
   - Hcubature - C & Fortran
     **https://github.com/stevengj/cubature**
   - ...

# Basic Procedures:

4. Master Integrals - *Numerical*

Parallelization

- Shared-Memory: `OpenMP`
- Distributed: Message Passing Interface(`MPI`)
- Hybrid `OpenMP` + `MPI`



# Basic Procedures:

## 4. Master Integrals - *Numerical*

Float Precision

- Long-Double Precision ($18 \sim 19$ digits)
  **Fortran native:** `REAL(KIND=10)`

- Quadruple Precision ($33 \sim 36$ digits)
  **Fortran native:** `REAL(KIND=16)`

- Double-Quadruple Precision ($\sim 66$ digits)
  **http://www.davidhbailey.com/dhbsoftware/**

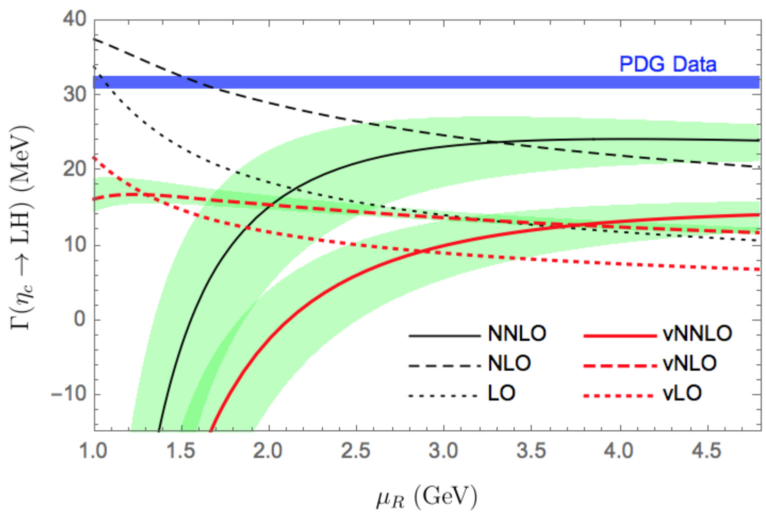
- Arbitrarily High Numeric Precision
  **http://www.davidhbailey.com/dhbsoftware/**

# Basic Procedures:

## 5. Other Processing – Plots etc.

Plots-v2.nb

```
    Text[Style["vLO"], {1 + r1 + 0.05, -10}, {-1, 0}]
   }]]
Export[FileNameJoin[{NotebookDirectory[], "TeX", "draft", "Gmc.eps"}], %];
```




### Bottom-Gamma

```
Show[bGammaPlot, Graphics[{
   Text[Style["PDG Data", Blue], {10.4, 17}, {-1, 0}],
   , {Thickness[0.003], Line[{{6.3, -5}, {7.3, -5}}]}, Text[Style["NNLO"], {7.5, -5}, {-1, 0}]
   , {Dashed, Thickness[0.003], Line[{{6.3, -10}, {7.3, -10}}]},
```



100%

# Basic Procedures:

1. Feynman Diagrams & Amplitudes
2. Trace & Contraction
3. Partial Fragmentation & IBP Reduction
4. Master Integrals – *Numerical*
5. Other Processing – Plots *etc*.

# Some Results in our Recent Works

# Recent Works:

@NNLO Quarkonium Process

- 1st *Production* Process: $\gamma + \gamma^* \to \eta_c$

  F. Feng, Y. Jia and W. L. Sang, Phys. Rev. Lett. **115**, no. 22, 222001 (2015)

- 1st *P-Wave Decay* Process: $\chi_{c0,2} \to \gamma + \gamma$

  W. L. Sang, F. Feng, Y. Jia and S. R. Liang, Phys. Rev. D **94**, no. 11, 111501 (2016)

- 1st *Inclusive Decay* Process: $\eta_{c,b} \to$ Light Hadrons

  F. Feng, Y. Jia and W. L. Sang, Phys. Rev. Lett. **119**, no. 25, 252001 (2017)

- 1st *Double Charmonia Production* Process:

  F. Feng, Y. Jia and W. L. Sang, arXiv:1901.08447 (2019) $e^+e^- \to J/\psi + \eta_c$

- $\cdots$

# Inclusive Decay Process

# $\eta_c \rightarrow \text{Light Hadrons}$ :

$$\Gamma(\eta_c \rightarrow \text{LH}) = \frac{F_1(^1S_0)}{m^2}\langle \eta_c | \mathcal{O}_1(^1S_0) | \eta_c \rangle$$

$$+\frac{G_1(^1S_0)}{m^4}\langle \eta_c | \mathcal{P}_1(^1S_0) | \eta_c \rangle + \mathcal{O}(v^3\Gamma)$$

# $\eta_c \to$ Light Hadrons :

$$\Gamma(\eta_c \to \mathrm{LH}) = \frac{F_1(^1S_0)}{m^2}\langle\eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle + \frac{G_1(^1S_0)}{m^4}\langle\eta_c|\mathcal{P}_1(^1S_0)|\eta_c\rangle + \mathcal{O}(v^3\Gamma),$$

$$F_1(^1S_0) = \frac{\pi C_F \alpha_s^2}{N_c}\left\{1 + \frac{\alpha_s}{\pi}f_1 + \frac{\alpha_s^2}{\pi^2}f_2 + \cdots\right\}$$

$$G_1(^1S_0) = -\frac{4\pi C_F \alpha_s^2}{3N_c}\left\{1 + \frac{\alpha_s}{\pi}g_1 + \cdots\right\}$$

# $\eta_c \to$ Light Hadrons :

$$\Gamma(\eta_c \to \mathrm{LH}) = \frac{F_1(^1S_0)}{m^2}\langle\eta_c|\mathcal{O}_1(^1S_0)|\eta_c\rangle + \frac{G_1(^1S_0)}{m^4}\langle\eta_c|\mathcal{P}_1(^1S_0)|\eta_c\rangle + \mathcal{O}(v^3\Gamma),$$

$$F_1(^1S_0) = \frac{\pi C_F \alpha_s^2}{N_c}\left\{1 + \frac{\alpha_s}{\pi} f_1 + \frac{\alpha_s^2}{\pi^2} f_2 + \cdots\right\},$$

$$G_1(^1S_0) = -\frac{4\pi C_F \alpha_s^2}{3N_c}\left\{1 + \frac{\alpha_s}{\pi} g_1 + \cdots\right\}.$$

$$f_1 = \frac{\beta_0}{2}\ln\frac{\mu_R^2}{4m^2} + \left(\frac{\pi^2}{4} - 5\right)C_F + \left(\frac{199}{18} - \frac{13\pi^2}{24}\right)C_A - \frac{8}{9}n_L - \frac{2n_H}{3}\ln 2,$$

Barbieri et al.,1979, Hagiwara et al.,1980

$$g_1 = \frac{\beta_0}{2}\ln\frac{\mu_R^2}{4m^2} - C_F\ln\frac{\mu_\Lambda^2}{m^2} - \left(\frac{49}{12} - \frac{5\pi^2}{16} - 2\ln 2\right)C_F + \left(\frac{479}{36} - \frac{11\pi^2}{16}\right)C_A - \frac{41}{36}n_L - \frac{2n_H}{3}\ln 2.$$

Guo,Ma,Chao, 2011

# $\eta_c \to$ Light Hadrons :

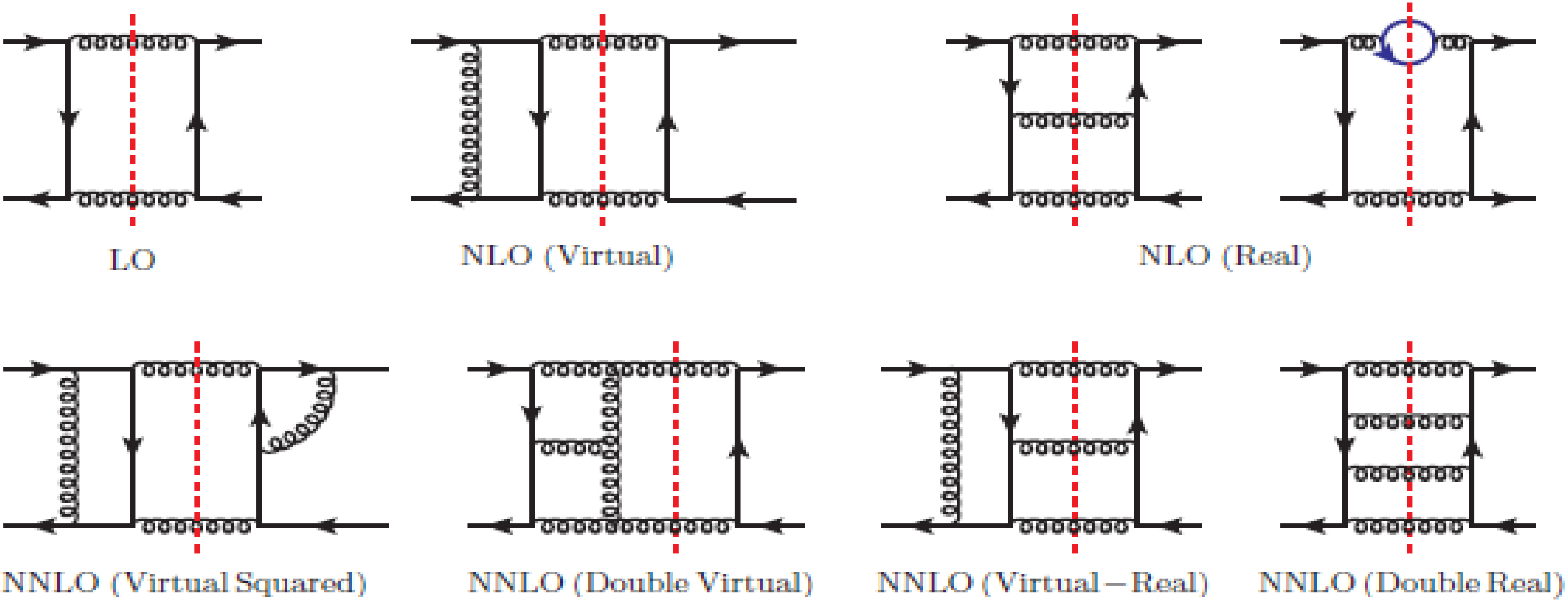



FIG. 1: Representative cut Feynman diagrams responsible for the quark reaction $c\bar{c}(^1S_0^{(1)}) \to c\bar{c}(^1S_0^{(1)})$ through NNLO in $\alpha_s$. The vertical dashed line denotes the Cutkosky cut.

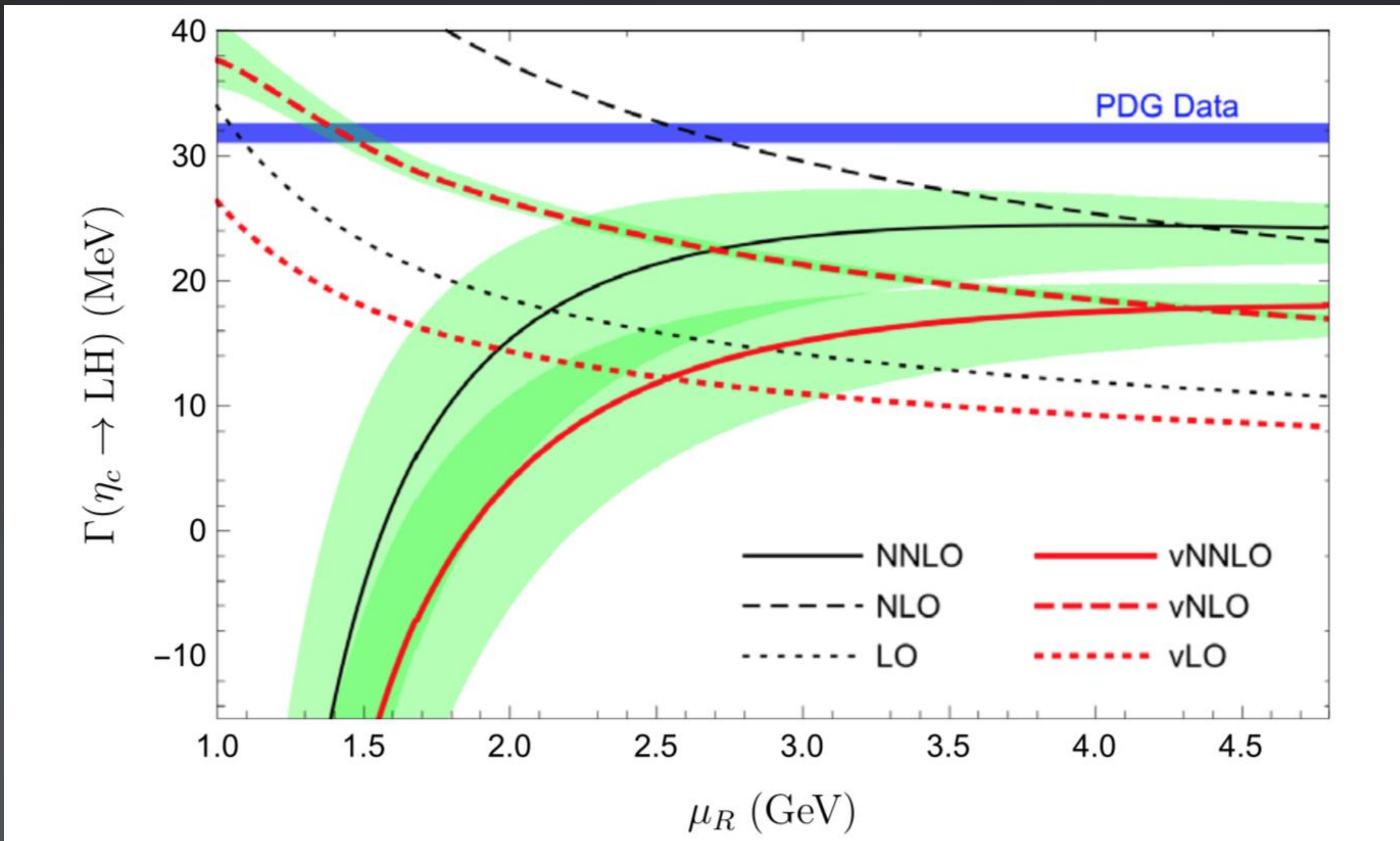
Roughly 1700 3-loop forward-scattering diagrams
Cutkosky rule is imposed

# $\eta_c \to$ Light Hadrons :

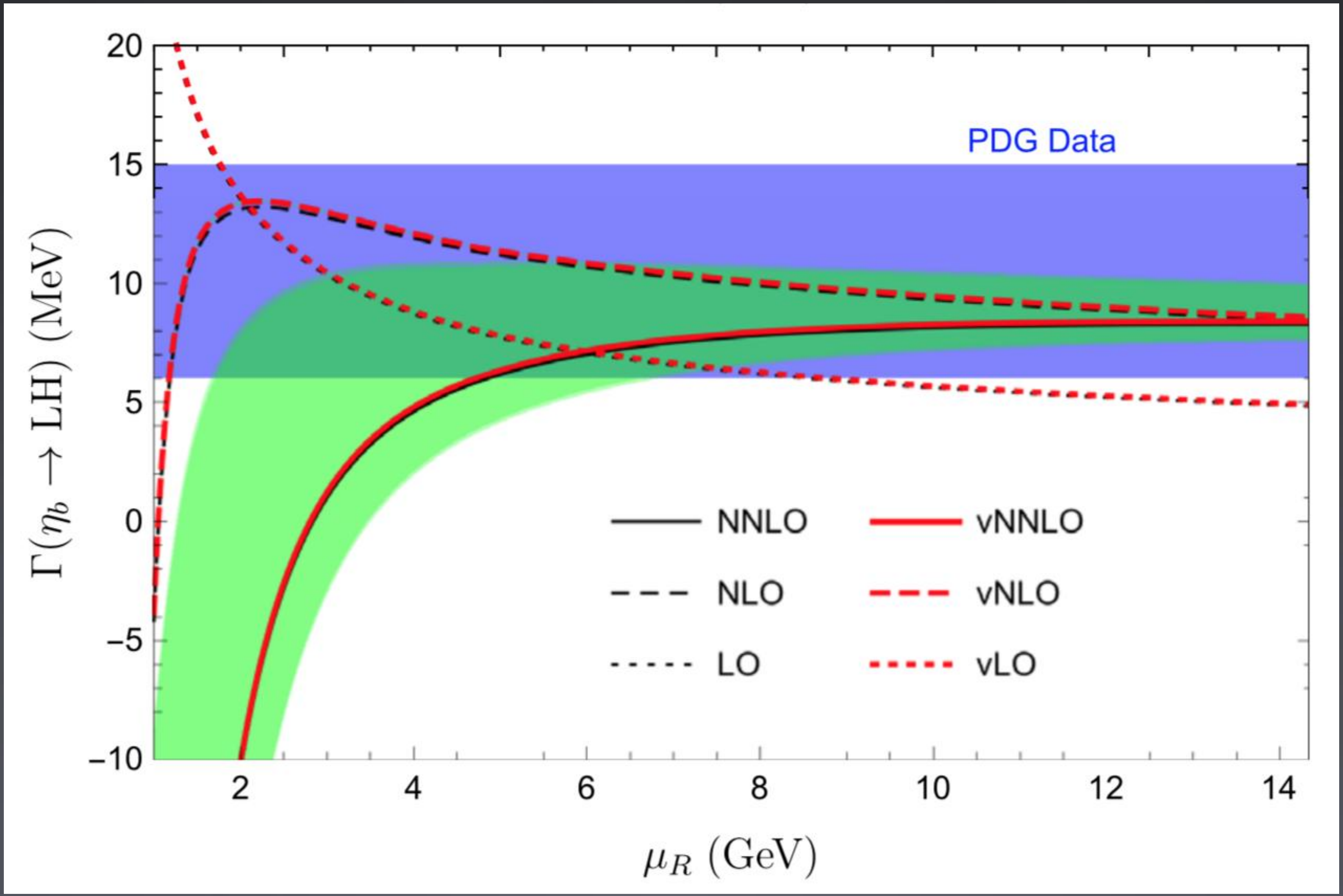
$$f_2 = \hat{f}_2 + \frac{3\beta_0^2}{16}\ln^2\frac{\mu_R^2}{4m^2} + \left(\frac{\beta_1}{8} + \frac{3}{4}\beta_0\hat{f}_1\right)\ln\frac{\mu_R^2}{4m^2}$$
$$-\pi^2\left(C_F^2 + \frac{C_A C_F}{2}\right)\ln\frac{\mu_\Lambda^2}{m^2}$$

$$\begin{aligned}\hat{f}_2 \;=\;& -0.799(13)N_c^2 - 7.4412(5)n_L N_c - 3.6482(2)N_c \\ & +0.37581(3)n_L^2 + 0.56165(5)n_L + 32.131(5) \\ & -0.8248(3)\frac{n_L}{N_c} - \frac{0.67105(3)}{N_c} - \frac{9.9475(2)}{N_c^2}\end{aligned}$$

# $\eta_c \to$ Light Hadrons :



# $\eta_c \to$ Light Hadrons :

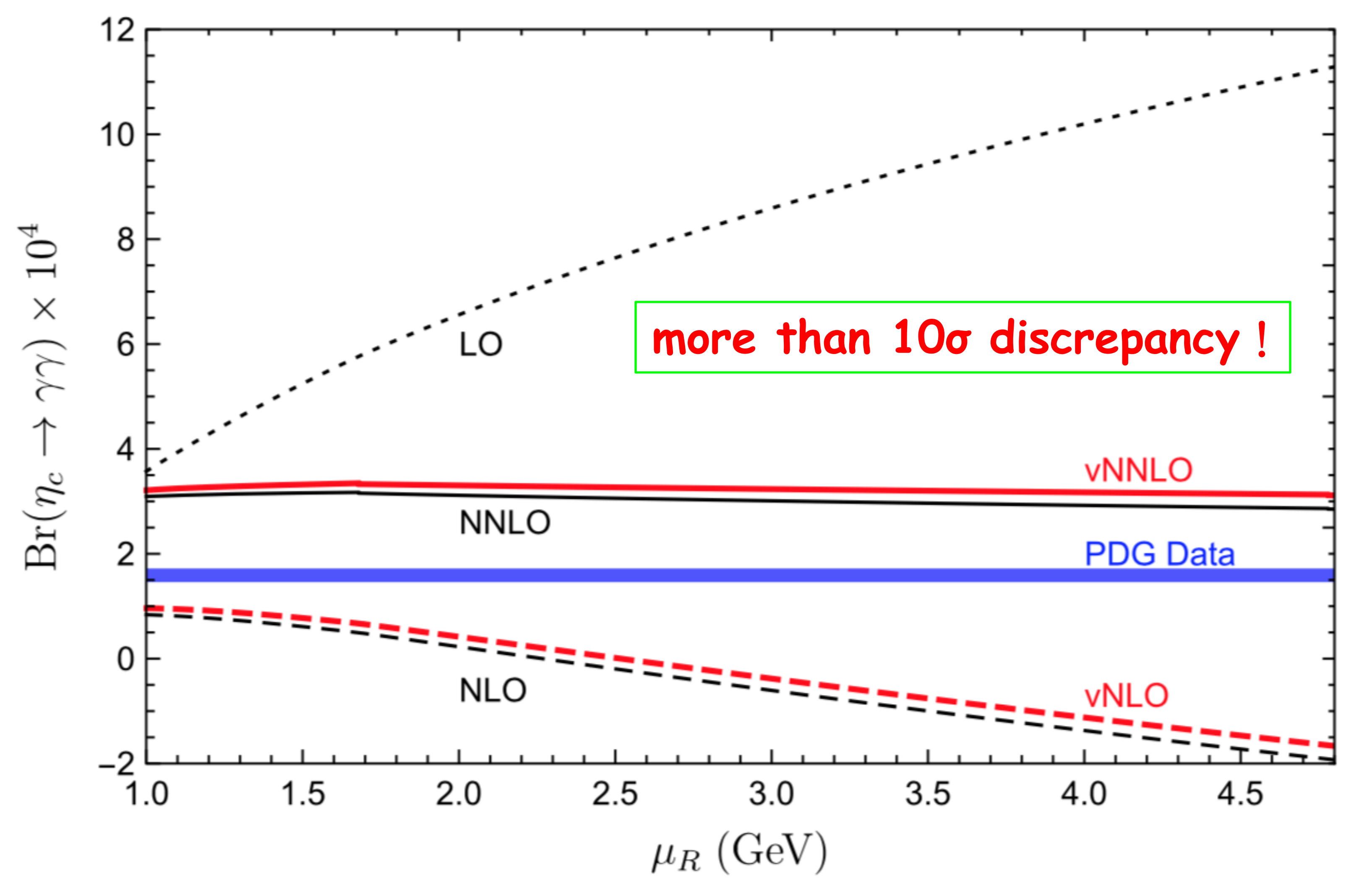


# $\eta_c \to$ Light Hadrons :

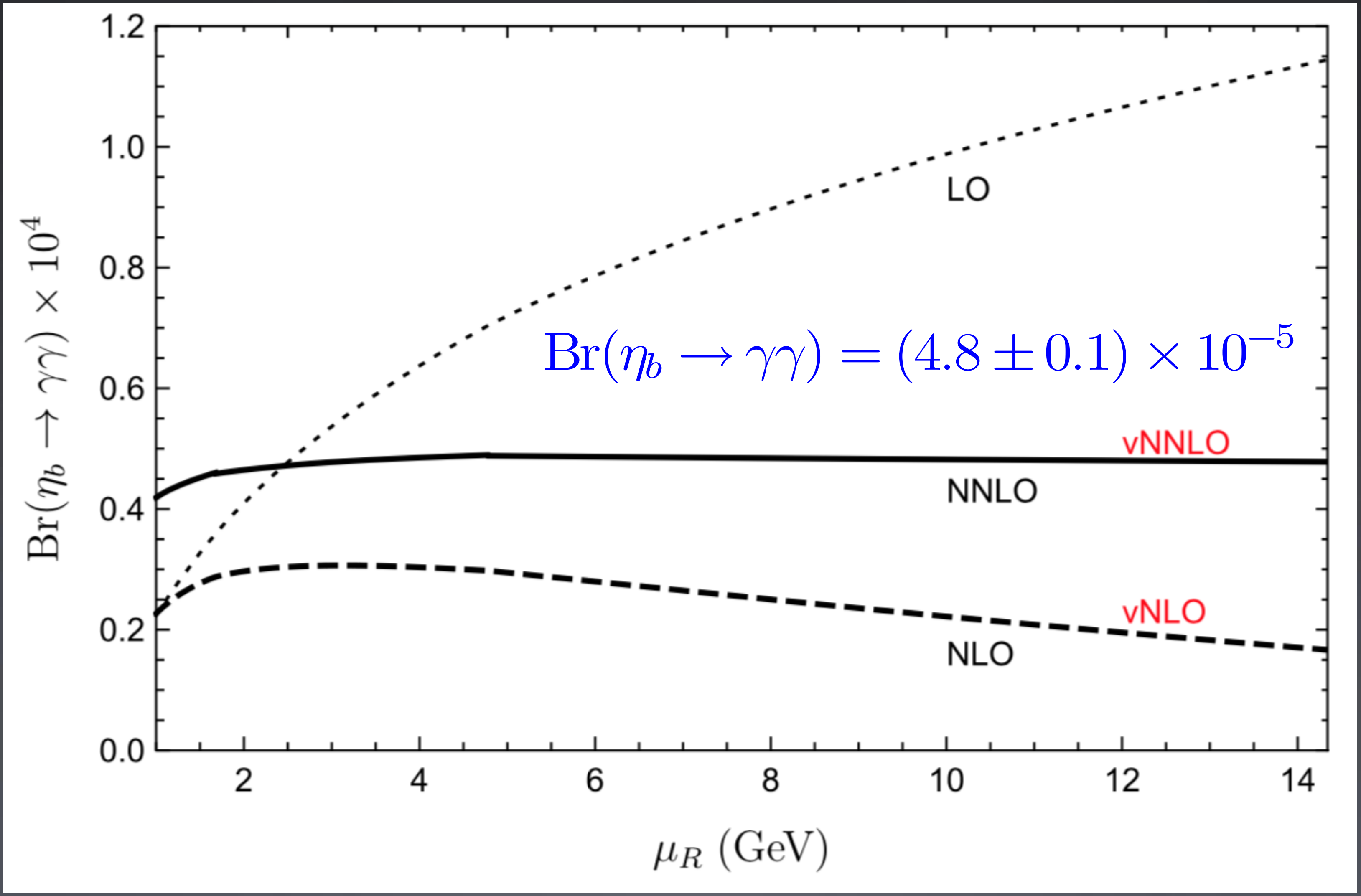
$$\mathrm{Br}(\eta_c \to \gamma\gamma) = \frac{8\alpha^2}{9\alpha_s^2}\left\{1 - \frac{\alpha_s}{\pi}\left[4.17\ln\frac{\mu_R^2}{4m_c^2} + 14.00\right]\right.$$
$$\left. + \frac{\alpha_s^2}{\pi^2}\left[4.34\ln^2\frac{\mu_R^2}{4m_c^2} + 22.75\ln\frac{\mu_R^2}{4m_c^2} + 78.8\right] + 2.24\langle v^2\rangle_{\eta_c}\frac{\alpha_s}{\pi}\right\}$$

$$\mathrm{Br}(\eta_b \to \gamma\gamma) = \frac{\alpha^2}{18\alpha_s^2}\left\{1 - \frac{\alpha_s}{\pi}\left[3.83\ln\frac{\mu_R^2}{4m_b^2} + 13.11\right]\right.$$
$$\left. + \frac{\alpha_s^2}{\pi^2}\left[3.67\ln^2\frac{\mu_R^2}{4m_b^2} + 20.30\ln\frac{\mu_R^2}{4m_b^2} + 85.5\right] + 1.91\langle v^2\rangle_{\eta_b}\frac{\alpha_s}{\pi}\right\}$$

# $\eta_c \to$ Light Hadrons :



# $\eta_c \to$ Light Hadrons :



# Double Charmonia Production Process

$e^+e^- \to J/\psi + \eta_c$

# Motivation

## *Experimental:*

$$\sigma[e^+e^- \to J/\psi + \eta_c] \times \mathcal{B}_{\geq 4} = 33^{+7}_{-6} \pm 9 \text{ fb}$$ BELLE @ 2002

$$\sigma[e^+e^- \to J/\psi + \eta_c] \times \mathcal{B}_{>2} = 25.6 \pm 2.8 \pm 3.4 \text{ fb}$$ BELLE @ 2004

$$\sigma[e^+e^- \to J/\psi + \eta_c] \times \mathcal{B}_{>2} = 17.6 \pm 2.8^{+1.5}_{-2.1} \text{ fb.}$$ BABAR @ 2005

$\mathcal{B}_{>n}$ denotes the branching fraction for the $\eta_c$ into $n$ charged tracks.

## *Theoretical:*

$$\sigma[e^+e^- \to J/\psi + \eta_c] = 3.78 \pm 1.26 \text{ fb}$$ E. Braaten & J. Lee @ 2003

$$\sigma[e^+e^- \to J/\psi + \eta_c] = 5.5 \text{ fb}$$ K. Y. Liu, Z. G. He& K. T. Chao @ 2003

# Motivation

$e^+e^- \to J/\psi + \eta_c$

*Large NLO correction:*

- Y. J. Zhang, Y. J. Gao, and K. T. Chao, Phys. Rev. Lett. **96**, 092001 (2006)
- B. Gong and J. X. Wang, Phys. Rev. D 77, 054028 (2008)

K-factor $1.8 \sim 2.1$

*One may naturally wonders:*

*How about the NNLO QCD corrections?*

# Results $^{\text{@NNLO}}$

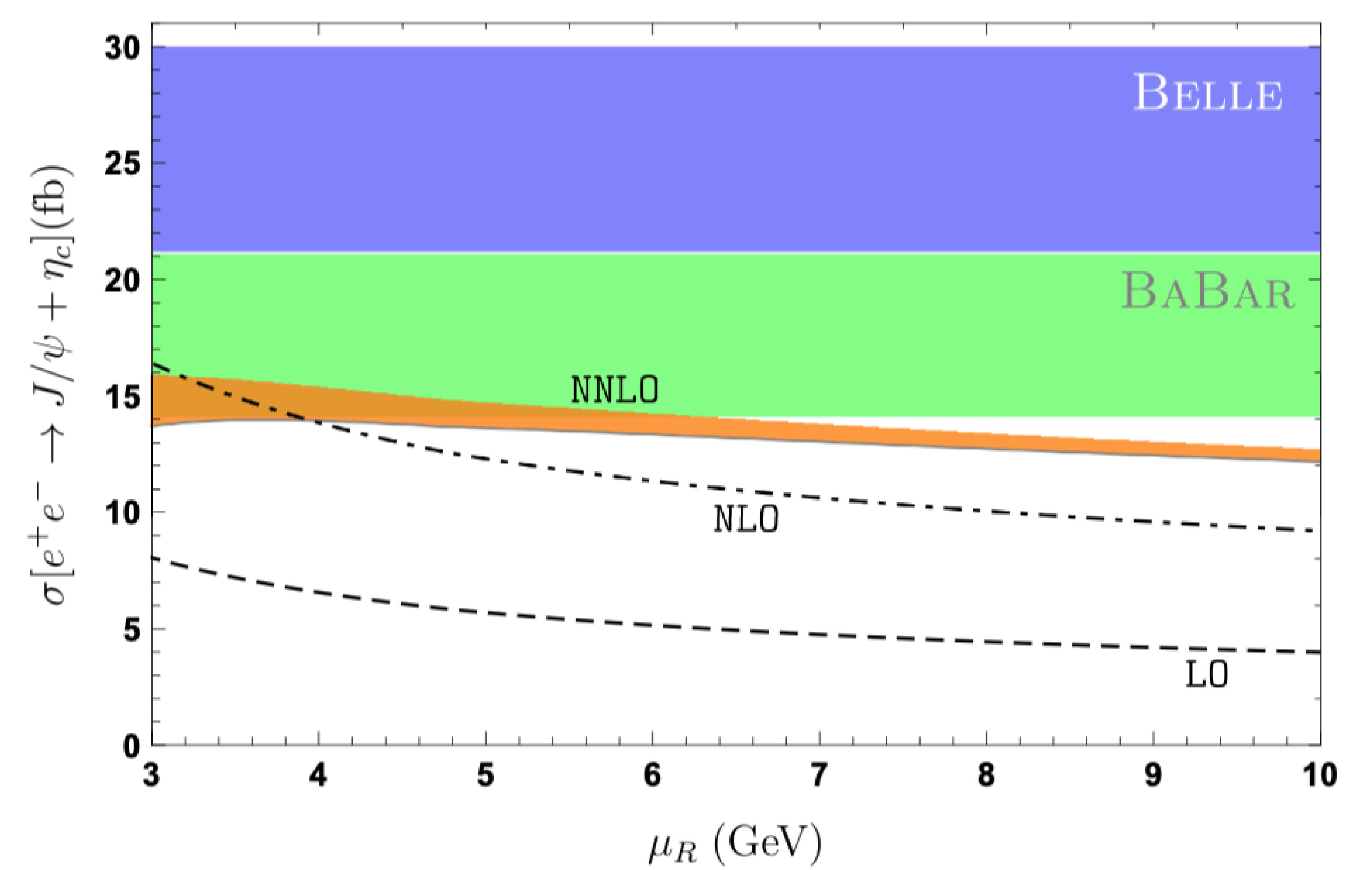
$e^+e^- \to J/\psi + \eta_c$

| $\mu_R$ | LO | $\mathcal{O}(v^2)$ | $\mathcal{O}(\alpha_s)$ | $\mathcal{O}(\alpha_s v^2)$ | $\mathcal{O}(\alpha_s^2)$ | Total |
|---|---|---|---|---|---|---|
| $2m$ | 8.48 | 4.36 | 8.64 | 0.34 | $-3.7(5)$ | 18.1(5) |
| $\frac{\sqrt{s}}{2}$ | 5.52 | 2.84 | 6.48 | 1.18 | 1.6(2) | 17.6(2) |
| $2m$ | 5.59 | 1.44 | 4.71 | $-0.33$ | $-1.4(4)$ | 10.0(4) |
| $\frac{\sqrt{s}}{2}$ | 4.16 | 1.07 | 4.08 | 0.06 | 0.7(2) | 10.1(2) |

$$\sigma = 8.48\,\text{fb}\,[1 + 0.51 + 1.02 + 0.04 - 0.44(6)]$$
$$\sigma = 5.52\,\text{fb}\,[1 + 0.51 + 1.17 + 0.21 + 0.28(4)]$$
$$\sigma = 5.59\,\text{fb}\,[1 + 0.26 + 0.84 - 0.06 - 0.25(6)]$$
$$\sigma = 4.16\,\text{fb}\,[1 + 0.26 + 0.98 + 0.01 + 0.16(5)]$$

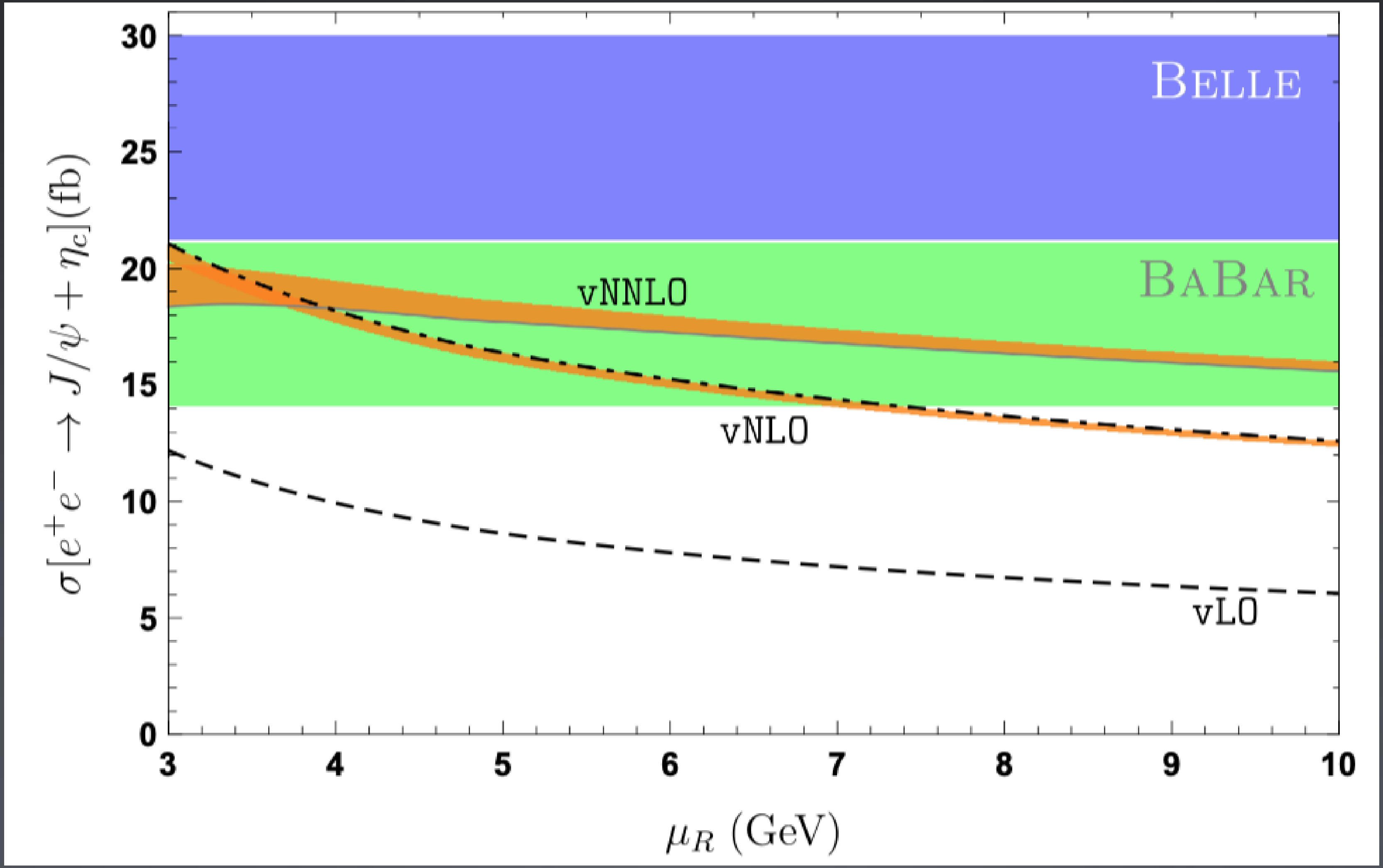
# Results @NNLO

$e^+e^- \to J/\psi + \eta_c$



# Results @NNLO

$$e^+e^- \to J/\psi + \eta_c$$



# Summary

- Basic Procedures in *Automated* Higher-Order Computation

- Some *Results* in our Recent Work
  - 1st Inclusive Decay Process
  - 1st Double Charmonia Production Process

Thanks for your attention!