粒子物理学的自旋极化

Lecture 4

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第四讲：自旋密度矩阵1

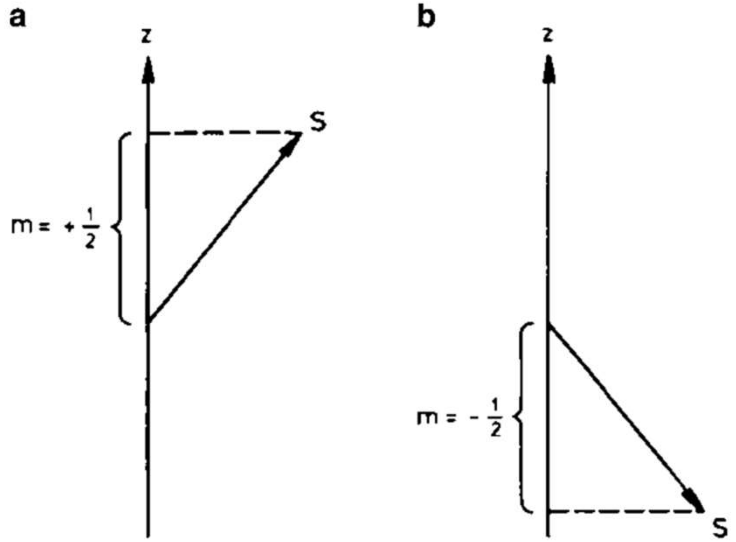
- **4.1. Density matrix of spin-1/2 particles**
- **4.2. Density matrix of spin-1 particles**
- **4.3 . General properties of spin density matrix**
- **4.4. Multipole parameters**
- **4.5. Other choice of basis matrices**
- **4.6. Relativistic case**

4.1. Density matrix of spin-1/2 particles

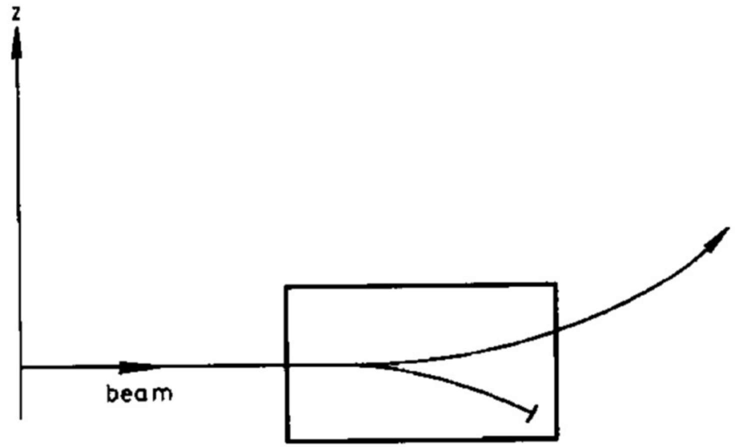
- pure spin state

旋转Stern-Gerlach装置的磁场，如果能够找到一个方向，使得束流能完全通过该装置，我们说：束流处于纯的自旋态。

Stern-Gerlach experiment



$$\left|+\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}; \qquad \left|-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$



4.1. Density matrix of spin-1/2 particles (cont.)

More general spin state:

$$|\chi\rangle = a_1 \left|+\frac{1}{2}\right\rangle + a_2 \left|-\frac{1}{2}\right\rangle$$

or $$|\chi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$ → adjoin state → $$\langle\chi| = (a_1^*, a_2^*)$$

pure spin state: specify direction of spin vector , or specify a_1 and a_2

Stern-Gerlach apparatus act as a spin filter, earlier way to prepare the pure spin state (or polarized beam)

4.1. Density matrix of spin-1/2 particles (cont.)

- polarization vector

Definition: $P_i = \langle \sigma_i \rangle = \langle \chi | \sigma_i | \chi \rangle$

Pauli-matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

e.g. beam in a pure state $|+\frac{1}{2}\rangle$ has

$$P_x = (1,\ 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$P_y = (1,\ 0) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$P_z = (1,\ 0) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

If beam in $|-1/2\rangle$, then $\vec{P} =?$

4.1. Density matrix of spin-1/2 particles (cont.)

A more general pure state $|\chi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\delta}\sin\frac{\theta}{2} \end{pmatrix}$

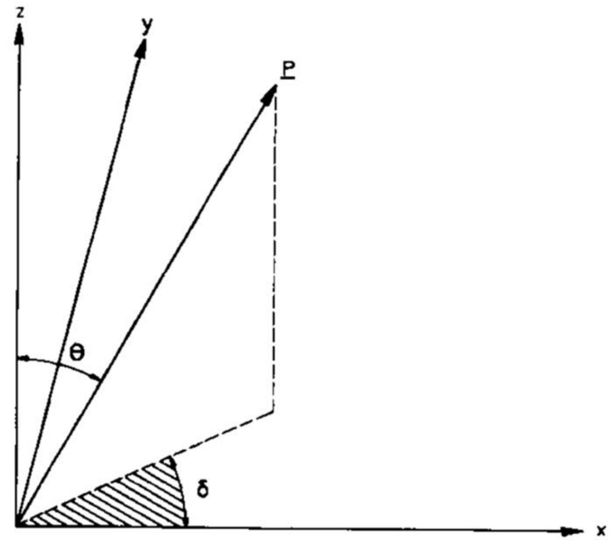
$$P_x = \left(\cos\frac{\theta}{2}, e^{-i\delta}\sin\frac{\theta}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\delta}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \sin\theta\cos\delta$$

$$P_y = \sin\theta\sin\delta$$

$$P_z = \cos\theta$$

$$|\mathbf{P}| = \left(P_x^2 + P_y^2 + P_z^2\right)^{1/2} = 1$$



4.1. Density matrix of spin-1/2 particles (cont.)

- specify a pure state with polarization vector
  - Stern-Gerlach orientation parallel to $\vec{P}$

$$P_x = P_y = 0, \qquad P_z = \pm 1$$

  - Stern-Gerlach orientation parallel to x

$$\theta = 90°,\ \delta = 0. \qquad \left| +\frac{1}{2}\ ,\ x \right\rangle = \frac{1}{2^{1/2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\theta = 90°, \delta = 180° \qquad \left| -\frac{1}{2}\ ,\ x \right\rangle = \frac{1}{2^{1/2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

  - Stern-Gerlach orientation parallel to y

$$\left| +\frac{1}{2}\ ,\ y \right\rangle = \frac{1}{2^{1/2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \left| -\frac{1}{2}\ ,\ y \right\rangle = \frac{1}{2^{1/2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

4.1. Density matrix of spin-1/2 particles (cont.)

- mixed spin state

States which are not pure are called mixed state or mixtures

Beam A : N_1 particle in $|+1/2\rangle$,
Beam B : N_2 particle in $|-1/2\rangle$,

- ➢ Can not described with polarization vector
- ➢ Can not described with superposition of two spin states
- ➢ described by specifying the way how the beams are prepared
- ➢ Polarization of this example

$$P_i = W_1 \left\langle \frac{1}{2} \left| \sigma_i \right| \frac{1}{2} \right\rangle + W_2 \left\langle -\frac{1}{2} \left| \sigma_i \right| -\frac{1}{2} \right\rangle$$

$$P_x = 0, \quad P_y = 0, \quad P_z = W_1 - W_2 = \frac{N_1 - N_2}{N}$$

4.1. Density matrix of spin-1/2 particles (cont.)

More general case

$$P_i = W_a \langle \chi_a | \sigma_i | \chi_a \rangle + W_b \langle \chi_b | \sigma_i | \chi_b \rangle$$
$$= W_a P_i^{(a)} + W_b P_i^{(b)}$$

with $W_a = N_a / N,\ W_b = N_b / N,\ N = N_a + N_b$

or $$\mathbf{P} = W_a \mathbf{P}^{(a)} + W_b \mathbf{P}^{(b)}$$

$$P^2 = (W_a \mathbf{P}^{(a)} + W_b \mathbf{P}^{(b)})^2$$
$$= W_a^2 (P^{(a)})^2 + W_b (P^{(b)})^2 + 2 W_a W_b \mathbf{P}^{(a)} \cdot \mathbf{P}^{(b)}$$
$$\leq W_a^2 + W_b^2 + 2 W_a W_b$$
$$= (W_a + W_b)^2 = 1$$

➔ $\boxed{0 \leq |\mathbf{P}| \leq 1}$

4.1. Density matrix of spin-1/2 particles (cont.)

$$|P| = 0, \qquad \text{unpolarized state}$$
$$|P| = 1, \qquad \text{pure state}$$
$$|P| > 0, \qquad \text{polarized state}$$

Comparison of two cases:

Case I: pure state

$$\left|+\frac{1}{2}, y\right\rangle = \frac{1}{2^{1/2}}\left(\left|+\frac{1}{2}\right\rangle + i\left|-\frac{1}{2}\right\rangle\right)$$

Case II: mixture

$$N_1 = N/2 \text{ particles in the state } |+1/2\rangle$$

$$N_2 = N/2 \text{ particles in the state } |-1/2\rangle$$

4.1. Density matrix of spin-1/2 particles (cont.)

- Spin density matrix defined by

$$\rho = W_a|\chi_a\rangle\langle\chi_a| + W_b|\chi_b\rangle\langle\chi_b|$$

with $W_a = N_a/N$, $W_b = N_b/N$, and $N = N_a + N_b$.

$\rho = |\chi\rangle\langle\chi|$: spin density operator, or statistical operator

$$|\chi_a\rangle = a_1^{(a)}\left|+\frac{1}{2}\right\rangle + a_2^{(a)}\left|-\frac{1}{2}\right\rangle$$

$$|\chi_b\rangle = a_1^{(b)}\left|+\frac{1}{2}\right\rangle + a_2^{(b)}\left|-\frac{1}{2}\right\rangle$$

➔

$$|\chi_a\rangle = \begin{pmatrix} a_1^{(a)} \\ a_2^{(a)} \end{pmatrix},$$

$$|\chi_b\rangle = \begin{pmatrix} a_1^{(b)} \\ a_2^{(b)} \end{pmatrix}$$

4.1. Density matrix of spin-1/2 particles (cont.)

- Spin density matrix

  with adjoint state

$$\langle \chi_a | = \left(a_1^{(a)*}, a_2^{(a)*} \right), \qquad \langle \chi_b | = \left(a_1^{(b)*}, a_2^{(b)*} \right)$$

$$\rho = \begin{pmatrix} W_a \left| a_1^{(a)*} \right|^2 + W_b \left| a_1^{(b)*} \right|^2 & W_a a_1^{(a)} a_2^{(a)*} + W_b a_1^{(b)} a_2^{(b)*} \\ W_a a_1^{(a)*} a_2^{(a)} + W_b a_1^{(b)*} a_2^{(b)} & W_a \left| a_2^{(a)*} \right|^2 + W_b \left| a_2^{(b)*} \right|^2 \end{pmatrix}$$

or $\langle \chi_i | \rho | \chi_j \rangle = W_a a_1^{(a)} a_j^{(a)*} + W_b a_i^{(b)} a_j^{(b)*}$

with $i, j = 1, 2.$

$$\langle \chi_i | \chi_j \rangle = \delta_{ij} \qquad \mathrm{tr}\,\rho = W_a + W_b = 1$$

4.1. Density matrix of spin-1/2 particles (cont.)

- significance of spin density matrix

$$\langle \chi_i | \rho | \chi_i \rangle = W_a \left| a_i^{(a)} \right|^2 + W_b \left| a_i^{(b)} \right|^2 \qquad (i = 1,\ 2)$$

gives the total probability of finding a particle in the corresponding basis state $|\chi_i\rangle$.

- the number of independent parameters

Hermitian matrix $\langle \chi_i | \rho | \chi_j \rangle = \langle \chi_j | \rho | \chi_i \rangle^*$

➔

$$\mathrm{Re}\left\langle +\frac{1}{2} \left| \rho \right| -\frac{1}{2} \right\rangle = \mathrm{Re}\left\langle -\frac{1}{2} \left| \rho \right| +\frac{1}{2} \right\rangle$$

$$\mathrm{Im}\left\langle +\frac{1}{2} \left| \rho \right| -\frac{1}{2} \right\rangle = -\mathrm{Im}\left\langle -\frac{1}{2} \left| \rho \right| +\frac{1}{2} \right\rangle$$

4.1. Density matrix of spin-1/2 particles (cont.)

- The number of independent parameters (cont.)

  Hermitian matrix reduces the number of parameters to four, and normalization further reduce on parameter.

  Three independent measurements must be performed in order to completely specify the spin density matrix for any spin-1/2 particles.

  - ➢ Check the number of parameters in the spin density matrix for mixture

- parametrization of spin density matrix

$$\begin{aligned}\mathrm{tr}\,\rho\sigma_i &= W_a \mathrm{tr}\,(|\chi_a\rangle\langle\chi_a|\sigma_i) + W_b \mathrm{tr}\,(|\chi_b\rangle\langle\chi_b|\sigma_i) \\ &= W_a\langle\chi_a|\sigma_i|\chi_a\rangle + W_b\langle\chi_b|\sigma_i|\chi_b\rangle\end{aligned}$$

4.1. Density matrix of spin-1/2 particles (cont.)

Using $\mathrm{tr}(\,|\chi\rangle\langle\chi|\sigma_i) = \langle\chi|\sigma_i|\chi\rangle$, and $\mathrm{tr}(\rho\sigma_i) = P_i$, one has

$$\rho = \frac{1}{2}\begin{pmatrix} 1+P_z & P_x - iP_y \\ P_x + iP_y & 1-P_z \end{pmatrix}$$

- when a filter oriented in the z-direction

$$\left\langle +\frac{1}{2}\right|\rho\left|+\frac{1}{2}\right\rangle = \frac{1}{2}(1+P_z)$$

similarly

$$\left\langle +\frac{1}{2}, x\right|\rho\left|+\frac{1}{2}, x\right\rangle = \frac{1}{2}(1+P_x)$$

$$\left\langle +\frac{1}{2}, y\right|\rho\left|+\frac{1}{2}, y\right\rangle = \frac{1}{2}(1+P_y)$$

4.1. Density matrix of spin-1/2 particles (cont.)

➢ when quantization axis along polarization vector P

$$\rho = \frac{1}{2}\begin{pmatrix} 1+|\mathbf{P}| & 0 \\ 0 & 1-|P| \end{pmatrix}$$

- Identification of pure spin state

$$\mathrm{tr}(\rho)^2 = (1/2)(1 + P_x^2 + P_y^2 + P_z^2)$$

$$= (1/2)(1 + |\mathbf{P}|^2)$$

$\mathrm{tr}\,(\rho)^2 = 1$ 是束流处于自旋纯态的充要条件.

这个方程进一步减少一个纯态密度矩阵的自由度，即初态只需要两个参数描述（比如θ, ϕ）.

4.1. Density matrix of spin-1/2 particles (cont.)

- ρ representation with Pauli matrices

$$\rho = a\mathbf{1} + \sum_i b_i \sigma_i$$

$$a = 1/2 \qquad \text{tr}\, \rho\sigma_j = 2\sum_i b_i \delta_{ij} = 2b_j$$

$$\boxed{\rho = \frac{1}{2}\left(\mathbf{1} + \sum_i P_i \sigma_i\right)}$$

- The pure state of probability measured in spin density matrix

$$\langle \chi | \rho | \chi \rangle = \text{tr} | \chi \rangle \langle \chi | \rho$$

4.1. Density matrix of spin-1/2 particles (cont.)

$$\rho^{(x)} = |\chi\rangle\langle\chi| = \frac{1}{2}\left(1 + \sum_i P_i^{(x)}\sigma_i\right)$$

$$\begin{aligned}\langle\chi|\rho|\chi\rangle &= \frac{1}{4}\,\mathrm{tr}\left[\left(1 + \sum_i P_i^{(x)}\sigma_i\right)\left(1 + \sum_j P_j\sigma_i\right)\right] \\ &= \frac{1}{4}\,\mathrm{tr}\left(1 + \sum_i P_i^{(x)}\sigma_i + \sum_j P_j\sigma_j + \sum_{ij} P_i^{(x)}P_j\sigma_i\sigma_j\right) \\ &= \frac{1}{2}(1 + \mathbf{P}^{(x)}\cdot\mathbf{P})\end{aligned}$$

4.2. Density matrix of spin-1 particles

- Classical concept of the electromagnetic wave polarization

$$\mathbf{E} = A\mathbf{e}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \mathbf{e} = a_1\mathbf{e}_x + a_2 e^{i\delta}\mathbf{e}_y$$

normalization $\quad a_1^2 + a_2^2 = 1$

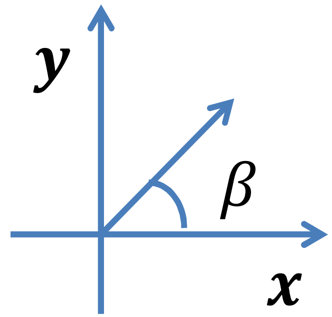
$$\mathbf{E} = A_1\mathbf{e}_x e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + A_2\mathbf{e}_y e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)}$$

$$= A(a_1\mathbf{e}_x + a_2 e^{i\delta}\mathbf{e}_y)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$\boldsymbol{e} = a_1\boldsymbol{e_x} + a_2 e^{i\delta}\boldsymbol{e_y} \rightarrow \mathbf{e} = \cos\beta\,\mathbf{e}_x + e^{i\delta}\sin\beta\,\mathbf{e}_y$$

- Linear polarization

$$\mathbf{e} = \cos\beta\,\mathbf{e}_x + \sin\beta\,\mathbf{e}_y$$



4.2. Density matrix of spin-1 particles(cont.)

- circular polarization

$$\mathbf{e} \sim \mathbf{e}_x \pm i\,\mathbf{e}_y \qquad (a_1 = a_2, \delta = 90^{\circ})$$

- elliptical polarization

$$\boldsymbol{e} = a_1 \boldsymbol{e_x} + a_2 e^{i\delta} \boldsymbol{e_y} \ (a_1 \neq a_2,\ \delta \neq 0)$$

- polarization filter, e.g. Nicol prism

- pure and mixed polarization

如果一束光的极化矢量不能写成单一的态矢量，就说它是极化的混合态

4.2. Density matrix of spin-1 particles(cont.)

- Helicity state of photon

Two helicity state: $\lambda = \pm 1$, to note that $\boldsymbol{J} \cdot \boldsymbol{n} = (\boldsymbol{L} + \boldsymbol{S}) \cdot \boldsymbol{n}$ $= \boldsymbol{S} \cdot \boldsymbol{n} = \boldsymbol{\lambda}$. The total angular momentum in the direction of propagation $\boldsymbol{n}$.

$\lambda = +1$, e_{+1}: right hand circularly polarized
$\lambda = -1$, e_{-1}: left hand circularly polarized

Polarization vector: $e_{\pm} = \mp \frac{1}{\sqrt{2}}(e_x \pm i e_y)$

Polarization state: $|\pm 1\rangle = \mp \frac{1}{\sqrt{2}}(|e_x\rangle \pm i|e_y\rangle)$

General polarization sate: $|e\rangle = a_1|+1\rangle + a_2|-1\rangle$, with

$$|+1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4.2. Density matrix of spin-1 particles(cont.)

- Helicity state of photon (cont.)

  Hence the general pure state formulated: $|e\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

  Adjoint state: $|e\rangle = (a_1^*, a_2^*)$

  Reversely,

$$|e_x\rangle = -\frac{1}{2^{1/2}}(|+1\rangle - |-1\rangle)$$

$$|e_y\rangle = -\frac{i}{2^{1/2}}(|+1\rangle + |-1\rangle)$$

as transverse polarization of spin-1/2 particle

- polarization density matrix of photon

$$\rho' = W_a|e_a\rangle\langle e_a| + W_b|e_b\rangle\langle e_b|$$

$$|e_a\rangle = a_1^{(a)}\Big|+1\Big\rangle + a_2^{(a)}\Big|-1\Big\rangle$$

$$|e_b\rangle = a_1^{(b)}\Big|+1\Big\rangle + a_2^{(b)}\Big|-1\Big\rangle$$

4.2. Density matrix of spin-1 particles(cont.)

- polarization density matrix of photon

$$\rho' = \begin{pmatrix} W_a \left|a_1^{(a)}\right|^2 + W_b \left|a_1^{(b)}\right|^2 & W_a a_1^{(a)} a_2^{(a)*} + W_b a_1^{(b)} a_2^{(b)*} \\ W_a a_1^{(a)*} a_2^{(a)} + W_b |a_1^{(b)*} a_2^{(b)} & W_a \left|a_2^{(a)}\right|^2 + W_b \left|a_2^{(b)}\right|^2 \end{pmatrix}$$

Normalization: $\mathrm{tr}\,\rho' = W_a + W_b = 1$

ρ defined in terms of intensity

$$\rho = I_a |e_a\rangle\langle e_a| + I_b |e_b\rangle\langle e_b|$$

$$\mathrm{tr}\,\rho = I_a + I_b = I$$

➢ properties:

4.2. Density matrix of spin-1 particles(cont.)

- Stokes parameter description
  - 1. total intensity I
  - 2. degree of linear polarization

    $\eta_3 = \frac{I(0)-I(90^\circ)}{I}$, $\quad I(\beta)$: intensity in the direction of β with respect to x-axis.
  - 3. degree of linear polarization

    $\eta_1 = \frac{I(45^\circ)-I(135^\circ)}{I}$
  - 4. degree of circular polarization

    $\eta_2 = \frac{I_+-I_-}{I}$

$$\rho = \frac{I}{2}\begin{pmatrix} 1+\eta_2 & -\eta_3 + i\eta_1 \\ -\eta_3 - i\eta_1 & 1-\eta_2 \end{pmatrix}$$

4.2. Density matrix of spin-1 particles(cont.)

- massive particles of spin-1

$$\rho = \sum_{\mu=1}^{n^2} C_\mu S_\mu .$$

$$\mathrm{Tr}\,(S_\mu S_\nu) = n\,\delta_{\mu,\nu} .$$

$$\langle S_\mu \rangle = \frac{\mathrm{Tr}\,(\rho\, S_\mu)}{\mathrm{Tr}\,\rho} = \frac{n\, C_\mu}{\mathrm{Tr}\,\rho} .$$

$$\rho = \frac{\mathrm{Tr}\,\rho}{n} \left(I + \sum_{\mu=1}^{n^2-1} \langle S_\mu \rangle\, S_\mu \right)$$

S_μ: μ-阶张量

Spin operators:

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

4.2. Density matrix of spin-1 particles(cont.)

- massive particles of spin-1

Rank -2 tensor $S_i S_j (i, j = x, y, z)$

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \begin{pmatrix} S_x & S_y & S_z \end{pmatrix} = \begin{pmatrix} S_x S_x & S_x S_y & S_x S_z \\ S_y S_x & S_y S_y & S_y S_z \\ S_z S_x & S_z S_y & S_z S_z \end{pmatrix}.$$

$$S_i S_j = \tfrac{1}{2}(S_i S_j + S_j S_i) + \tfrac{1}{2}(S_i S_j - S_j S_i).$$

$$S_i S_j - S_j S_i = \mathrm{i} S_k \qquad (i, j, k \text{ cyclic}).$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathscr{P}_x = S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathscr{P}_y = S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \qquad \mathscr{P}_z = S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

4.2. Density matrix of spin-1 particles(cont.)

- massive particles of spin-1

$$\mathscr{P}_{xy} = 3S_x S_y = \frac{3i}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathscr{P}_{xx} = 3S_x S_x - 2I = \frac{1}{2}\begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

$$\mathscr{P}_{xz} = 3S_x S_z = \frac{3i}{\sqrt{8}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \mathscr{P}_{yy} = 3S_y S_y - 2I = \frac{1}{2}\begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix}$$

$$\mathscr{P}_{yz} = 3S_y S_z = \frac{1}{\sqrt{8}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \mathscr{P}_{zz} = 3S_z S_z - 2I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One constraint: $\mathscr{P}_{xx} + \mathscr{P}_{yy} + \mathscr{P}_{zz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

We have 8 independent matrices

4.2. Density matrix of spin-1 particles(cont.)

- massive particles of spin-1 (cont.)

  Orthogonal operators are chosen to meet the requirement

$$\mathrm{Tr}\,\Omega_i\,\Omega_j = 3\delta_{ij}$$

$$\rho = \tfrac{1}{3}\{I + \tfrac{3}{2}(p_x\mathscr{P}_x + p_y\mathscr{P}_y + p_z\mathscr{P}_z) + \tfrac{2}{3}(p_{xy}\mathscr{P}_{xy} + p_{yz}\mathscr{P}_{yz} + p_{xz}\mathscr{P}_{xz}) + \tfrac{1}{6}(p_{xx} - p_{yy})(\mathscr{P}_{xx} - \mathscr{P}_{yy}) + \tfrac{1}{2}p_{zz}\mathscr{P}_{zz}\}$$

where $p_i = \mathrm{Tr}\,\rho\mathscr{P}_i$, etc.

p_i, : vector polarization,
p_{ij}: tensor polarization

- special case: e.g., quantization axis in the z-direction

$$\rho = \frac{1}{3}\begin{pmatrix} 1 + \frac{3}{2}p_z + \frac{1}{2}p_{zz} & 0 & 0 \\ 0 & 1 - p_{zz} & 0 \\ 0 & 0 & 1 - \frac{3}{2}p_z + \frac{1}{2}p_{zz} \end{pmatrix}$$

Spin density matrix for high spin particles

- Expansion of spin density matrix in terms of irreducible tensor

For spin-s particle

$$\rho(\mathbf{s}) = \rho_0 \left[1 + \sum_{\ell=1}^{2s} \frac{s^{\ell}(2\ell+1)!!}{\ell!\, S_0^2 S_1^2 \cdots S_{\ell-1}^2} P_{\mu_1\mu_2\cdots\mu_\ell} \overline{s_{\mu_1} s_{\mu_2} \cdots s_{\mu_\ell}} \right]$$

$$\rho_0 = 1/(2s+1), S_k^2 = s(s+1) - \frac{k}{2}\left(\frac{k}{2}+1\right)$$

$\overline{s_{\mu_1} s_{\mu_2} \cdots s_{\mu_\ell}}$ is the rank-s_{μ_l} irreducible tensor representation

especially, for $s = 1/2$, $\rho(\mathbf{s}) = \frac{1}{2}[1 + 2\, P_\mu\, s_\mu].$

For $s = 1$

$$\rho(\mathbf{s}) = \frac{1}{3}\left[1 + \frac{3}{2} P_\mu\, s_\mu + 3\, P_{\mu\nu}\, \overline{s_\mu s_\nu}\right]$$

4.3 . General properties of spin density matrix

- Shows the general properties of QM
- Shows the general kinematical-dynamical properties, eg. Interaction and symmetry
- Shows model dependent –dynamical properties

- Definition

Pure state $|\psi >$

$$|\psi\rangle = \sum_{m=-s}^{s} c_m |s, m\rangle.$$

$$O_{mm'} \equiv \langle sm|\hat{O}|sm'\rangle$$

$$\langle\hat{O}\rangle_\psi \equiv \langle\psi|\hat{O}|\psi\rangle = \sum_{m,m'} c^*_{m'} O_{m'm} c_m.$$

$$\rho_{mm} = c^*_{m'} c_m$$

mixture of pure sates $|\psi^{(i)} >$ possibility $p^{(i)}$

$$\langle\hat{O}\rangle_{\psi^{(i)}} = \sum_{m,m'} c^{(i)*}_{m'} O_{m'm} c^{(i)}_m$$

$$\langle\hat{O}\rangle = \sum_i p^{(i)} \langle\hat{O}\rangle_{\psi^{(i)}}$$

$$= \sum_{m,m'} O_{m'm} \sum_i p^{(i)} c^{(i)*}_{m'} c^{(i)}_m.$$

$$\rho_{mm'} \equiv \sum_i p^{(i)} c^{(i)}_m c^{(i)*}_{m'}$$

$$\langle\hat{O}\rangle = \sum_{m,m'} O_{m'm} \rho_{mm'} = \mathrm{Tr}\,(O\rho)$$

4.3 . General properties of spin density matrix (cont.)

- Normalization

$$\mathrm{Tr}(\rho) = 1$$

- Hermitian

$$\rho^*_{mm'} = \rho_{m'm}.$$

- Positive semi-definite

$$\rho_{mm} \geq 0$$

- Diagonalizible

$$U^{-1}\rho U = \rho^D$$

- Inequality

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}[(\rho^D)^2] = \sum_m \lambda_m^2 \leq \left(\sum_m \lambda_m\right)^2 = [\mathrm{Tr}(\rho)]^2 = 1.$$

$$\mathrm{Tr}(\rho^2) = \sum_{m,m'} |\rho_{mm'}|^2 \leq 1.$$

- Pure state

$$\rho = 1\ or\ 0$$

4.4. Multipole parameters

$$spherical\ tensor\ operators\ \hat{T}^L_M, 0 \le L \le 2s,\ \ -L \le M \le L$$

$$\left(T^L_M\right)_{mm'} \equiv \langle sm|\hat{T}^L_M|sm'\rangle \equiv \langle sm|sm';LM\rangle\ ;\ \ T^L_{-M} = (-1)^M {T^L_M}^\dagger .$$

$$\hat{T}^0_0 = 1.$$

$$\hat{T}^1_0 = \frac{1}{\sqrt{s(s+1)}}\hat{s}_z$$

$$\hat{T}^1_1 = -\frac{1}{\sqrt{2s(s+1)}}(\hat{s}_x + i\hat{s}_y)\,.$$

$$\hat{T}^1_{-1} = \frac{1}{\sqrt{2s(s+1)}}(\hat{s}_x - i\hat{s}_y)$$

Define a complex parameter

$$t^L_M\ (0 \le L \le 2s)$$

$${t^L_M}^* = \sum_{m,m'} \langle sm|sm';LM\rangle\, \rho_{mm'}.$$

$$\rho_{mm'} = \frac{1}{2s+1}\sum_{L,M}(2L+1)\,\langle sm|sm';LM\rangle\, {t^L_M}^*$$

$$\rho_{mm'} = \frac{1}{2s+1}\sum_{L,M}(2L+1){t^L_M}^*(T^L_M)_{mm'}.$$

4.4. Multipole parameters

Spin density matrix:

$$\rho = \frac{1}{2s+1} \sum_{L,M} (2L+1) t_M^{L\,*} T_M^L.$$

$$\text{Tr}\left(T_{M'}^{L'} T_M^{L\,\dagger}\right) = \frac{2s+1}{2L+1} \delta_{LL'} \delta_{MM'}$$

$$\text{Tr}\left(\rho T_M^{L\,\dagger}\right) = t_M^{L\,*} \rightarrow t_M^L = \text{Tr}\,(\rho T_M^L).$$

$$t_0^0 = \text{Tr}\,\rho = 1$$

$$t_0^1 = \mathscr{P}_z \sqrt{\frac{s}{s+1}}$$

$$t_1^1 = -(\mathscr{P}_x + i\mathscr{P}_y) \sqrt{\frac{s}{2(s+1)}}$$

$$t_{-1}^1 = (\mathscr{P}_x - i\mathscr{P}_y) \sqrt{\frac{s}{2(s+1)}} \leftarrow t_{-M}^L = (-1)^M t_M^{L\,*}.$$

4.4. Multipole parameters (cont.)

- umpolarized (isotropic) ensemble of spin-2 particle

$$\rho_{\text{iso}} = \frac{1}{2s+1} I$$

- departure from the isotropy

$$\rho - \rho_{\text{iso}} = \frac{1}{2s+1} \sum_{\substack{L\geq 1 \\ M}} (2L+1) {t^L_M}^* T^L_M$$

- overall degree of polarization

$$d \equiv \frac{1}{\sqrt{2s}} \left[(2s+1) \ \mathrm{Tr}\ \rho^2 - 1 \right]^{1/2}$$

$$\Rightarrow \quad = \left\{ \sum_{L\geq 1} d_L^2 \right\}^{1/2} . \quad \text{with } d_L = \sqrt{\frac{2L+1}{2s}} \left(\sum_M |t^L_M|^2 \right)^{1/2}$$

4.5. Other choice of basis matrices

see Ref. Moncel, M.G., et.al., Nucl.Phys. B38, 477

$$M \geq 1 \qquad Q_M^L = \frac{(-1)^M}{2}\sqrt{\frac{2L+1}{s}}\left\{T_M^L + T_M^{L\,\dagger}\right\}$$

$$M = 0 \qquad Q_0^L = \sqrt{\frac{2L+1}{2s}}\,T_0^L$$

$$M \leq -1 \qquad Q_M^L = \frac{(-1)^M}{2i}\sqrt{\frac{2L+1}{s}}\left\{T_{-M}^L - T_{-M}^{L\,\dagger}\right\}$$

(Note: a symbol typo in “Spin in particle physics ” by Elliot Leader)

$$\rho = \frac{1}{2s+1}\left(I + 2s\sum_{L=1}^{2s}\sum_{M=-L}^{L} r_M^L Q_M^L\right).$$

4.6. Relativistic case

- Using covariant spin operator

  $\hat{s}_i = \frac{1}{m}\widehat{W}^i$, where Pauli-Lubansky operator

  $\widehat{W}_\sigma = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\widehat{M}^{\mu\nu}\widehat{P}^\rho$, $\widehat{M}$ and P are the generators of Lorentz transformation.

Nonrelativistic case: $\mathcal{P}_\chi \equiv \frac{1}{s}\langle\chi|\hat{\mathbf{s}}|\chi\rangle \equiv \frac{\mathbf{s}_\chi}{s},$

Relativistic case: $\mathscr{S}^\mu(\mathbf{p},\lambda) \equiv \frac{1}{s}\langle\mathbf{p};\lambda|\hat{W}^\mu|\mathbf{p};\lambda\rangle$

- For $s = \frac{1}{2}$ case

$$\mathscr{S}^\mu(\mathbf{p},\lambda) = 2\lambda(p, E\hat{\mathbf{p}})$$

- orthe method: see Phys.Rev., D40, 2477 by S.Y. Choi, et. Al.,