粒子物理学的自旋极化
Lecture 2

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2.1 . 自旋和螺旋度

- 自旋-$\frac{1}{2}$ 粒子

  - 1925-1927, formulated by Dutch physicists
  - 1927, Pauli introduced spin as operator in QM
  - Particles classified into Bosons and Fermions in spin

$$\hat{s} = \frac{1}{2}\hat{\sigma}$$

Pauli operator:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Two component spinor χ:

$$\chi = \begin{pmatrix} \cos\frac{1}{2}\theta\ e^{-\frac{i\phi}{2}} \\ \sin\frac{1}{2}\theta\ e^{\frac{i\phi}{2}} \end{pmatrix},\ \chi^{\dagger}\chi = 1$$

  - Polarization vector

$$P_{\chi} = \chi^{+}\hat{s}\chi = (sin\theta cos\phi, sin\theta sin\phi, cos\theta),\ \text{极化度：} p = \sqrt{P_{\chi}^{2}} = 1$$

Wolfgang Ernst Pauli, 1900-1958

2.2 . 单粒子态

- 非相对论的单粒子态

Angular momentum operator (J_x, J_y, J_z) or (J_1, J_2, J_3) :

$$[J_i, J_j] = i\, \epsilon_{ijk}\, J_k \ ,$$

Define $J^2 = J_x^2 + J_y^2 + J_z^2, J_+ = J_x + iJ_y,\ J_- = J_x - iJ_y$

$$\begin{aligned} J^2|jm\rangle &= j(j+1)|jm\rangle \\ J_z|jm\rangle &= m|jm\rangle \\ J_\pm|jm\rangle &= [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}}|jm \pm 1\rangle \end{aligned}$$

Completeness relation

$$\langle j'm'|jm\rangle = \delta_{j'j}\, \delta_{m'm}$$

$$\sum_{jm} |jm\rangle\langle jm| = I \ ,$$

非相对论的单粒子态(续)

- finite rotation $R(\alpha,\beta,\gamma)$

$$U[R(\alpha,\beta,\gamma)] = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\gamma J_z}$$

- representation of rotation matrix

$$\begin{aligned} D^j_{m'm}(R) = D^j_{m'm}(\alpha,\beta,\gamma) &= \langle jm'|\, U[R(\alpha,\beta,\gamma)]\, |jm\rangle \\ &= e^{-i\,m'\alpha}\, d^j_{m'm}(\beta)\, e^{-i\,m\,\gamma} \end{aligned}$$

with $d^{\,j}_{m'm}(\beta) = \langle jm'|e^{-i\,\beta J_y}|jm\rangle$

- rotation of state $|jm\rangle$

$$U[R(\alpha,\beta,\gamma)]\,|jm\rangle = \sum_{m'} |jm'\rangle\, D^j_{m'm}(\alpha,\beta,\gamma)\ ,$$

相对论的单粒子态

- Lorentz boost along z-axis: $L_z(\beta)$

One particle state: mass W, energy E, momentum $\vec{p}$, spin state $|jm\rangle$

Proper homogeneous Lorentz transformation:

$$p'^{\mu} = \Lambda^{\mu}_{\nu}\, p^{\nu}$$

with $p^{\nu} = (E, \vec{p})$ and $p'^{\mu} = (E', \overrightarrow{p'})$, Λ^{μ}_{ν} defined by $g_{\alpha\beta} = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu} = g_{\mu\nu}$, $\det\Lambda = 1$, $\Lambda^{0}_{0} > 0$.

$L_z(\beta)\ matrix$:

$$L_z(\beta) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}$$

With velocity $\beta = \tanh\alpha$,

$$\tanh\alpha = \frac{P}{E}, \sinh\alpha = \frac{P}{W}, \cosh\alpha = \frac{E}{W}$$

相对论的单粒子态(续)

- Lorentz boos along arbitrary direction ($\vec{\beta}$)

$$L(\vec{\beta}\,) = R(\phi,\theta,0)L_z(\beta)R^{-1}(\phi,\theta,0)\ ,$$

$$U[L(\vec{p}\,)] = U[\mathring{R}(\phi,\theta,0)]U[L_z(p)]U^{-1}[\mathring{R}(\phi,\theta,0)]\ ,$$

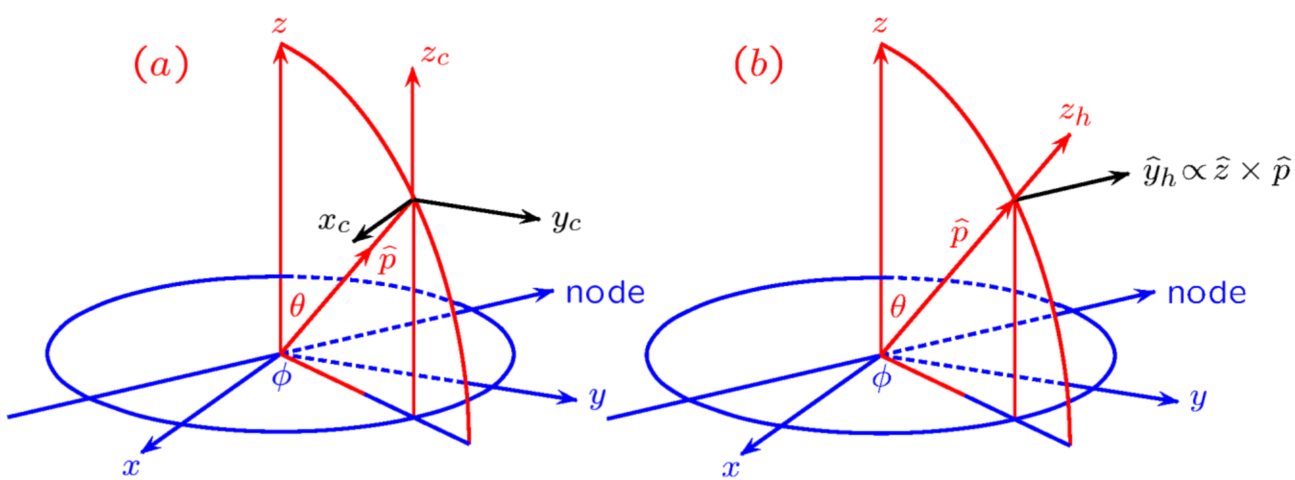
multiplication law

Canonical State:

$$|\vec{p},jm\rangle = |\phi,\theta,p,jm\rangle = U[L(\vec{p}\,)]\,|jm\rangle$$
$$= U[\mathring{R}(\phi,\theta,0)]\,U[L_z(p)]\,U^{-1}[\mathring{R}(\phi,\theta,0)]\,|jm\rangle\ ,$$

Helicity state

$$|\vec{p},j\lambda\rangle = |\phi,\theta,p,j\lambda\rangle = U[L(\vec{p}\,)]\,U[\mathring{R}(\phi,\theta,0)]\,|j\lambda\rangle$$
$$= U[\mathring{R}(\phi,\theta,0)]\,U[L_z(p)]\,|j\lambda\rangle\ .$$



相对论的单粒子态(续)

- Rotations on the canonical state and helicity state

Canonical State:

$$U[R]\,|\vec{p}, jm\rangle = U[R\mathring{R}]\,U[L_z(p)]\,U^{-1}[R\mathring{R}]\,U[R]\,|jm\rangle$$
$$= \sum_{m'} D^{j}_{m'm}(R)\,|R\vec{p}, jm'\rangle\ ,$$

Helicity state

- Rotaion R around $\vec{p}$, helcity invariant:

$$U[R]\,|\vec{p}, j\lambda\rangle \ = U[R\mathring{R}]\,U[L_z]\,|j\lambda\rangle$$
$$= |R\vec{p}, j\lambda\rangle\ .$$

- Boost along $\vec{p}$ direction, i.e., $\vec{p}'//\,p$, helcity invariant:

$$U[L']\,|\vec{p}, j\lambda\rangle \ = \ U[L']U[L(\vec{p}\,)]U[\mathring{R}]\,|j\lambda\rangle$$
$$= \ U[L(\vec{p}\,')]U[\mathring{R}]\,|j\lambda\rangle$$
$$= \ |\vec{p}\,', j\lambda\rangle\ .$$

相对论的单粒子态(续)

- Relationship between canonical and helicity state

$$|\vec{p}, j\lambda\rangle = U[\mathring{R}]U[L_z]U^{-1}[\mathring{R}]U[\mathring{R}]\,|j\lambda\rangle$$
$$= \sum_m D^j_{m\lambda}(\mathring{R})\,|\vec{p}, jm\rangle\ .$$

- Normalization of helicity state

$$\langle\vec{p}'\,j'm'|\vec{p}jm\rangle = \tilde{\delta}(\vec{p}'\,-\vec{p}\,)\delta_{jj'}\,\delta_{mm'}$$
$$\langle\vec{p}'\,j'\lambda'|\vec{p}j\lambda\rangle = \tilde{\delta}(\vec{p}'\,-\vec{p}\,)\delta_{jj'}\,\delta_{\lambda\lambda'}\ ,$$

where $\tilde{\delta}(\vec{p}\,' - \vec{p}\,) = (2\pi)^3(2E)\,\delta^{(3)}(\vec{p}\,' - \vec{p}\,)\ .$

- Completeness relation

$$\sum_{jm}\int|\vec{p}jm\rangle\,\tilde{d}p\,\langle\vec{p}jm| = I$$
$$\sum_{j\lambda}\int|\vec{p}j\lambda\rangle\,\tilde{d}p\,\langle\vec{p}j\lambda| = I\ ,$$

with $\tilde{d}p = \dfrac{d^3\vec{p}}{(2\pi)^3(2E)}\ ,$

单粒子态的宇称和时间反演操作

- Particle with position $\vec{x}$, momentum $\vec{p}$, and angular momentum J

$$\boldsymbol{P}: \quad \vec{x} \to -\vec{x}, \quad \vec{p} \to -\vec{p}, \quad \vec{J} \to \vec{J}$$

$$\boldsymbol{T}: \quad \vec{x} \to \vec{x}, \quad \vec{p} \to -\vec{p}, \quad \vec{J} \to -\vec{J}$$

Parity: $\Pi|jm\rangle = \eta|jm\rangle$, Time reversal: $\mathbb{T}\,|jm\rangle = \sum_k T_{km}|jk\rangle$.

Canonical basis:

$$\Pi\,|\vec{p}, jm\rangle = \eta|-\vec{p}, jm\rangle$$

$$\Pi\,|\phi, \theta, p, jm\rangle = \eta|\pi + \phi, \pi - \theta, p, jm\rangle$$

$$\mathbb{T}\,|\vec{p}, jm\rangle = (-)^{j-m}|-\vec{p}, j-m\rangle$$

$$\mathbb{T}\,|\phi, \theta, p, jm\rangle = (-)^{j-m}|\pi + \phi, \pi - \theta, p, j-m\rangle$$

Helicity basis:

$$\Pi|\phi, \theta, p, j\lambda\rangle = \eta\, e^{-i\pi j}|\pi + \phi, \pi - \theta, p, j\, -\lambda\rangle$$

$$\mathbb{T}\,|\phi, \theta, p, j\lambda\rangle = e^{-i\pi\lambda}|\pi + \phi, \pi - \theta, p, j\lambda\rangle$$

双粒子态

- Construction of two-particle canonical states

Particle	Spin	mass	momentum
1	s_1	w_1	$\vec{p}$
2	s_2	w_2	$-\vec{p}$

- spin states

$$|\phi\theta m_1 m_2\rangle = a\, U[L(\vec{p}\,)]\, |s_1 m_1\rangle U[L(-\vec{p}\,)]\, |s_2 m_2\rangle$$

$$L(\pm\vec{p}\,) = \overset{\circ}{R}(\phi,\theta,0) L_{\pm z}(p) \overset{\circ}{R}^{-1}(\phi,\theta,0)\ ,$$

Total spin:

$$|\phi\theta s m_s\rangle = \sum_{m_1 m_2} (s_1 m_1\, s_2 m_2 | s m_s) |\phi\theta m_1 m_2\rangle$$

Rotation property

$$U[R]\, |\Omega\, s m_s\rangle = \sum_{m'_s} D^s_{m'_s m_s}(R) |R'\, s m'_s\rangle$$

$$R' = R\Omega$$

双粒子态

- orbital momentum

$$|\ell m s m_s\rangle = \int d\Omega\, Y^\ell_m(\Omega)|\Omega s m_s\rangle$$

$$U[R]\,|\ell m s m_s\rangle = \int d\Omega\, Y^\ell_m(\Omega) D^s_{m'_s m_s}(R)|R' s m'_s\rangle$$ Rotation property

$$\begin{aligned} Y^\ell_m(\Omega) &= \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell *}_{m0}(R^{-1}R') \\ &= \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'} D^{\ell *}_{mm'}(R^{-1})\, D^{\ell *}_{m'0}(R') \\ &= \sum_{m'} D^\ell_{m'm}(R) Y^\ell_{m'}(\beta', \alpha')\ , \end{aligned}$$

With:

$$R' = R'(\alpha', \beta', \gamma') = R\,\Omega,$$

$$d\Omega = d\alpha'\, d\cos\beta',$$

$$U[R]\,|\ell m s m_s\rangle = \sum_{m' m'_s} D^\ell_{m'm}(R) D^s_{m'_s m_s}(R)|\ell m' s m'_s\rangle$$

Total spin *J*:

$$\begin{aligned} |JM\ell s\rangle &= \sum_{m m_s} (\ell m s m_s|JM)|\ell m s m_s\rangle \\ &= \sum_{\substack{m m_s \\ m_1 m_2}} (\ell m\, s m_s|JM)(s_1 m_1\, s_2 m_2|s m_s) \int d\Omega\, Y^\ell_m(\Omega)|\Omega m_1 m_2\rangle \end{aligned}$$

双粒子态

- Construction of two-particle helicity states

$$
\begin{aligned}
|\phi\theta\lambda_1\lambda_2\rangle \; &= a\, U[\overset{\circ}{R}]\Big\{U[L_z(p)]\,|s_1\lambda_1\rangle U[L_{-z}(p)]\,|s_2\; -\lambda_2\rangle\Big\} \\
&\equiv U[\overset{\circ}{R}(\phi,\theta,0)]\,|00\lambda_1\lambda_2\rangle \;,
\end{aligned}
$$

$$
|JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi}\int dR\, D^{J\,*}_{M\mu}(R) U[R]\,|00\lambda_1\lambda_2\rangle
$$

$$
U[R'\,]\,|JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi}\int dR\, D^{J\,*}_{M\mu}(R) U[R''\,]\,|00\lambda_1\lambda_2\rangle
$$

Use:

$$
\begin{aligned}
D^{J\,*}_{M\mu}(R) &= D^{J\,*}_{M\mu}(R'^{\,-1}R''\,) \\
&= \sum_{M'} D^{J\,*}_{MM'}(R'^{\,-1}) D^{J\,*}_{M'\mu}(R''\,) \\
&= \sum_{M'} D^{J}_{M'M}(R')\, D^{J\,*}_{M'\mu}(R''\,)\;.
\end{aligned}
$$

双粒子态(续)

$$U[R'\,]\,|JM\lambda_1\lambda_2\rangle = \sum_{M'} D^J_{M'M}(R'\,)|JM'\lambda_1\lambda_2\rangle$$

Specify R:

$$\begin{aligned} & U[R(\phi,\theta,\gamma)]\,|00\lambda_1\lambda_2\rangle \\ = \ & U[R(\phi,\theta,0)]U[R(0,0,\gamma)]\,|00\lambda_1\lambda_2\rangle \\ = \ & e^{-i(\lambda_1-\lambda_2)\gamma}\,U[R(\phi,\theta,0)]\,|00\lambda_1\lambda_2\rangle\ . \end{aligned}$$

Target:

$$|JM\lambda_1\lambda_2\rangle = N_J \int d\Omega D^{J*}_{M,\lambda_1-\lambda_2}(\phi,\theta,0)|\phi\theta\lambda_1\lambda_2\rangle$$

双粒子态(续)

- Normalization

$$\langle \Omega' m_1' m_2' | \Omega m_1 m_2 \rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{m_1 m_1'}\delta_{m_2 m_2'}$$

$$\langle \Omega' \lambda_1' \lambda_2' | \Omega \lambda_1 \lambda_2 \rangle = \delta^{(2)}(\Omega' - \Omega)\delta_{\lambda_1 \lambda_1'}\delta_{\lambda_2 \lambda_2'} \ .$$

$$\langle J' M' \ell' s' | J M \ell s \rangle = \delta_{JJ'}\delta_{MM'}\delta_{\ell\ell'}\delta_{ss'} \ . \qquad \sum_{\substack{JM \\ \ell s}} |JM\ell s\rangle\langle JM\ell s| = I$$

$$\langle J' M' \lambda_1' \lambda_2' | J M \lambda_1 \lambda_2 \rangle = \delta_{JJ'}\delta_{MM'}\delta_{\lambda_1 \lambda_1'}\delta_{\lambda_2 \lambda_2'} \ , \qquad \sum_{\substack{JM \\ \lambda_1 \lambda_2}} |JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2| = I$$

$$\langle \Omega \lambda_1' \lambda_2' | J M \lambda_1 \lambda_2 \rangle = N_J D^{J\,*}_{M\lambda}(\phi, \theta, 0)\, \delta_{\lambda_1 \lambda_1'}\delta_{\lambda_2 \lambda_2'}$$

双粒子态(续)

- relationship between canonical and helicity states

$$\begin{aligned}
|\phi\theta\lambda_1\lambda_2\rangle &= a\,U[\overset{\circ}{R}]\Big\{U[L_z(p)]\,|s_1\lambda_1\rangle U[L_{-z}(p)]\,|s_2\,{-\lambda_2}\rangle\Big\} \\
&= a\,U[L(\vec{p}\,)]U[\overset{\circ}{R}]\,|s_1\lambda_1\rangle U[L(-\vec{p}\,)]U[\overset{\circ}{R}]\,|s_2\,{-\lambda_2}\rangle \\
&= \sum_{m_1 m_2} D^{s_1}_{m_1\lambda_1}(\phi,\theta,0)D^{s_2}_{m_2-\lambda_2}(\phi,\theta,0)|\phi\theta m_1 m_2\rangle\ ,
\end{aligned}$$

$$|JM\lambda_1\lambda_2\rangle = N_J\sum_{m_1 m_2}\int d\Omega\, D^{J\,*}_{M\lambda}(\phi,\theta,0)D^{s_1}_{m_1\lambda_1}(\phi,\theta,0)D^{s_2}_{m_2-\lambda_2}(\phi,\theta,0)|\phi\theta m_1 m_2\rangle$$

Use:

$$D^{s_1}_{m_1\lambda_1}D^{s_2}_{m_2-\lambda_2} = \sum_{s m_s}(s_1 m_1 s_2 m_2|s m_s)(s_1\lambda_1 s_2 - \lambda_2|s\lambda)D^{s}_{m_s\lambda}$$

$$D^{J\,*}_{M\lambda}D^{s}_{m_s\lambda} = \sum_{\ell m}\sqrt{\frac{4\pi}{2\ell+1}}\left(\frac{2\ell+1}{2J+1}\right)(\ell m s m_s|JM)(\ell 0 s\lambda|J\lambda)Y^{\ell}_{m}\ .$$

$$|JM\lambda_1\lambda_2\rangle = \sum_{\ell s}\left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}}(\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2\,{-\lambda_2}|s\lambda)|JM\ell s\rangle$$

双粒子态(续)

$$\langle J'M'\ell s|JM\lambda_1\lambda_2\rangle = \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}}(\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2 -\lambda_2|s\lambda)\delta_{JJ'}\delta_{MM'}$$

$$\begin{aligned}|JM\ell s\rangle &= \sum_{\lambda_1\lambda_2}|JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2|JM\ell s\rangle \\ &= \sum_{\lambda_1\lambda_2}\left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}}(\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2 -\lambda_2|s\lambda)|JM\lambda_1\lambda_2\rangle\ .\end{aligned}$$

- Symmetry relations

Parity operation: $\Pi|\phi\theta m_1 m_2\rangle = \eta_1\eta_2|\pi+\phi,\pi-\theta,m_1 m_2\rangle\ ,$

l-s coupled state: $\Pi|JM\ell s\rangle = \eta_1\eta_2(-)^{\ell}|JM\ell s\rangle\ ,$

Helicity state: $\Pi\,|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-)^{J-s_1-s_2}|JM\,-\lambda_1\,-\lambda_2\rangle$

双粒子态(续)

- time reversal operation

$$\mathbb{T}\,|\phi\theta m_1 m_2\rangle = (-)^{s_1-m_1}(-)^{s_2-m_2}|\pi+\phi,\pi-\theta,-m_1-m_2\rangle$$

l-s coupled state: $\mathbb{T}\,|JM\ell s\rangle = (-)^{J-M}|J-M\ell s\rangle$

Helicity state: $\mathbb{T}\,|JM\lambda_1\lambda_2\rangle = (-)^{J-M}|J-M\lambda_1\lambda_2\rangle$

- identical particle symmetry

$$|JM\ell s\rangle_s = a_s[1+(-)^{2s_1}\,\mathbb{P}_{12}]\,|JM\ell s\rangle$$

l-s coupled state:

$$\mathbb{P}_{12}|JM\ell s\rangle = (-)^{\ell+s-2s_1}|JM\ell s\rangle$$

or

$$|JM\ell s\rangle_s = a_s[1+(-)^{\ell+s}]\,|JM\ell s\rangle$$

双粒子态(续)

- identical particle symmetry (continued)

Helicity state: $|JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2)[1+(-)^{2s_1}\,\mathbb{P}_{12}]\,|JM\lambda_1\lambda_2\rangle$

or $|JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2)\left\{|JM\lambda_1\lambda_2\rangle + (-)^J|JM\lambda_2\lambda_1\rangle\right\}$

- Two body decays $J \to 1 + 2$

particle	J	1	2
spin	J	s_1	s_2
helicity	M	λ_1	λ_2
parity	η	η_1	η_2
momentum	$(\omega,\vec{0})$	$(E_1,\vec{p})$	$(E_2,-\vec{p})$

双粒子态(续)

- Two body decays $J \to 1 + 2$ (cont.)

$$
\begin{aligned}
A &= \langle \vec{p}\lambda_1; -\vec{p}\lambda_2 | \mathcal{M} | JM \rangle \\
&= 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle \phi\theta\lambda_1\lambda_2 | JM\lambda_1\lambda_2 \rangle \langle JM\lambda_1\lambda_2 | \mathcal{M} | JM \rangle \\
&= N_J F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0), \qquad \lambda = \lambda_1 - \lambda_2 \ ,
\end{aligned}
$$

Helicity decay amplitude:

$$
F^J_{\lambda_1\lambda_2} = 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle JM\lambda_1\lambda_2 | \mathcal{M} | JM \rangle
$$

Expansion in l-s coupling

$$
\begin{aligned}
\langle JM\lambda_1\lambda_2 | \mathcal{M} | JM \rangle &= \sum_{\ell s} \langle JM\lambda_1\lambda_2 | JM\ell s \rangle \langle JM\ell s | \mathcal{M} | JM \rangle \\
&= \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0 s\lambda | J\lambda)(s_1\lambda_1 s_2 -\lambda_2 | s\lambda) \langle JM\ell s | \mathcal{M} | JM \rangle
\end{aligned}
$$

双粒子态(续)

- Two body decays $J \rightarrow 1+2$ (cont.)

$$F^J_{\lambda_1\lambda_2} = \sum_{\ell s}\left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_{\ell s}\,(\ell 0 s\lambda|J\lambda)\,(s_1\lambda_2 s_2\,-\lambda_2|s\lambda)$$

with partial-wave amplitude a_{ls}

$$a^J_{\ell s} = 4\pi\left(\frac{w}{p}\right)^{\frac{1}{2}}\langle JM\ell s|\mathcal{M}|JM\rangle$$

$$\sum_{\lambda_1\,\lambda_2}|F^J_{\lambda_1\lambda_2}|^2 = \sum_{\ell\,s}|a^J_{\ell s}|^2$$

- If decay conserves parity

$$F^J_{\lambda_1\lambda_2} = \eta\eta_1\eta_2(-)^{J-s_1-s_2}F^J_{-\lambda_1-\lambda_2}$$

- If 1 and 2 are identical particle

$$F^J_{\lambda_1\lambda_2} = (-)^J F^J_{\lambda_2\lambda_1}$$

双粒子态(续)

- Example

Give a process $e^+e^- \to J/\psi \to e^+e^-$, calculate the electron angular distribution.

Helicity: $J/\psi(M)$, $e^+(\lambda_1)$, $e^-(\lambda_2)$, decay amp. F_{λ_1,λ_2}

$$A = D^{1*}_{M,\lambda_1-\lambda_2}(\phi,\theta,0)F_{\lambda_1,\lambda_2} \Rightarrow$$

$$|A|^2 = \sum_{\substack{M=\pm 1,\\ \lambda_1,\lambda_2=\pm 1/2}} D^{1*}_{M,\lambda_1-\lambda_2}(\phi,\theta,0)D^{1}_{M,\lambda_1-\lambda_2}(\phi,\theta,0)\left|F_{\lambda_1,\lambda_2}\right|^2$$

$\propto (1+\cos^2\theta)$ (assuming helicity conservation)

三粒子态

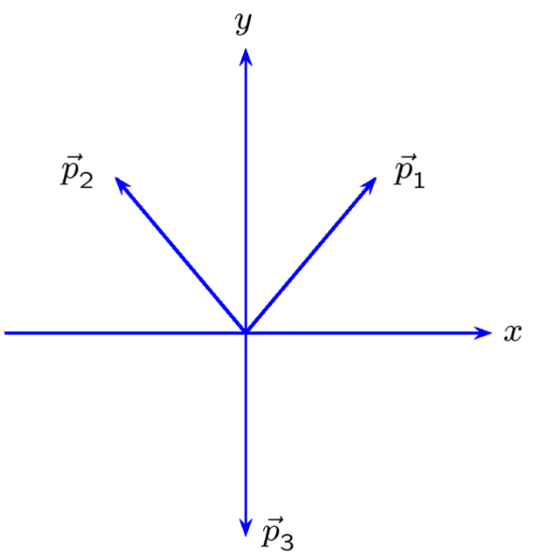
- Three particle state

In standard orientation:

$$|000, E_i\lambda_i\rangle = b\prod_{i=1}^{3} |\vec{p}_i\ s_i\lambda_i\rangle$$

with $|\vec{p}\ s_i\lambda_i\rangle = U[R_i L_z(p_i)]\ |s_i\lambda_i\rangle$

$$R_i = R(\phi_i, \pi/2, 0)$$



Standard orientation system

In arbitrary orientation

$$|\alpha\beta\gamma, E_i\lambda_i\rangle = U[R(\alpha, \beta, \gamma)]\ |000, E_i\lambda_i\rangle$$

Normalization: ➔ $b = 8\pi^2\sqrt{4\pi}$

$$\langle\alpha'\beta'\gamma', E_i'\lambda_i'|\alpha\beta\gamma, E_i\lambda_i\rangle = \delta^{(3)}(R' - R)\,\delta(E_1' - E_1)\,\delta(E_2' - E_2)\prod_i \delta_{\lambda_i\lambda_i'}$$

三粒子态(续)

- State with momentum *J*

$$|JM\mu, E_i\lambda_i\rangle = \frac{N_J}{\sqrt{2\pi}} \int dR\, D^{J\,*}_{M\mu}(\alpha,\beta,\gamma)|\alpha\beta\gamma, E_i\lambda_i\rangle$$

Rotation transformation:

$$U[R']\,|JM\mu, E_i\lambda_i\rangle = \sum_{M'} D^J_{M'M}(R')|JM'\mu, E_i\lambda_i\rangle$$

it shows that μ is invariant, identified as the z-component of angular momentum.

- parity operation

$$\Pi|JM\mu, E_i\lambda_i\rangle = \eta_1\eta_2\eta_3(-)^{s_1+s_2+s_3-\mu}|JM\mu, E_i\ -\lambda_i\rangle$$

- identical particle symmetry

$$\mathbb{P}_{12}|\alpha\beta\gamma, E_1\lambda_1, E_2\lambda_2, E_3\lambda_3\rangle = |\pi+\alpha, \pi-\beta, \pi-\gamma, E_2\lambda_2, E_1\lambda_1, E_3\lambda_3\rangle$$

三粒子态(续)

- identical particle symmetry (cont.)

$$\mathbb{P}_{12}|JM\mu, E_1\lambda_1, E_2\lambda_2, E_3\lambda_3\rangle = (-)^{J+\mu}|JM\ -\mu, E_2\lambda_2, E_1\lambda_1, E_3\lambda_3\rangle$$

- normalization

$$\langle J'M'\mu'E_i'\lambda_i'|J\ M\mu E_i\lambda_i\rangle$$

$$= \delta_{JJ'}\delta_{MM'}\delta_{\mu\mu'}\delta(E_1 - E_1')\delta(E_2 - E_2')\prod_i \delta_{\lambda_i\lambda_i'}$$

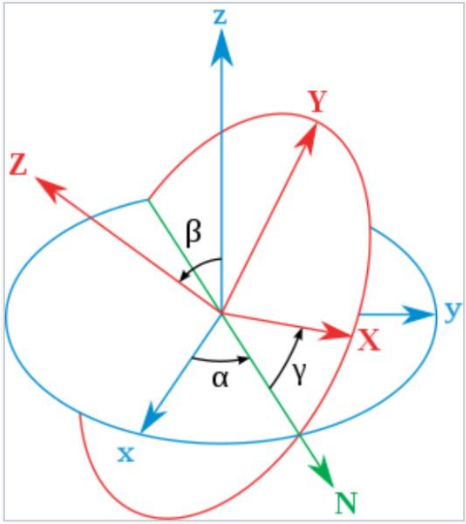
- completeness relation

$$\sum_{\substack{JM \\ \mu\lambda_i}} \int |JM\mu E_i\lambda_i\rangle dE_1 dE_2 \langle JM\mu E_i\lambda_i| = I$$

三粒子态(续)

- decay amplitude

Decay $J \to s_1 + s_2 + s_3$, with orientation (α, β, γ)



$$\begin{aligned} A &= \langle \alpha\beta\gamma, E_i\lambda_i | \mathcal{M} | JM \rangle \\ &= \langle \alpha\beta\gamma, E_i\lambda_i | JM\mu E_i\lambda_i \rangle \langle JM\mu E_i\lambda_i | \mathcal{M} | JM \rangle \\ &= \frac{N_J}{\sqrt{2\pi}} F^J_\mu (E_i\lambda_i) D^{J\,*}_{M\mu}(\alpha\beta\gamma) \end{aligned}$$

Helicity amplitude:

$$F^j_\mu(E_i\lambda_i) = \langle JM\mu E_i\lambda_i | \mathcal{M} | JM \rangle$$

Parity conservation:

$$F^J_\mu(E_i\lambda_i) = \eta\eta_1\eta_2\eta_3(-)^{s_1+s_2+s_3+\mu} F^j_\mu(E_i\,-\lambda_i)$$

Identical particle symmetry:

$$F^J_\mu(E_1\lambda_1, E_2\lambda_2, E_3\lambda_3) = \pm(-)^{J+\mu} F^J_{-\mu}(E_2\lambda_2, E_1\lambda_1, E_3\lambda_3)$$

e.g. $e^+e^- \to \gamma^* \to \pi^+\pi^-\pi^0$

两体→两体过程的S矩阵

- 2→ 2 process: $a+b\rightarrow c+d$

particle	a	b	c	d
spin	s_a	s_b	s_c	s_d
helicity	λ_a	λ_b	λ_c	λ_d
momentum	$\vec{p}_a=\vec{p}_i$	$\vec{p}_b=-\vec{p}_i$	$\vec{p}_c=\vec{p}_f$	$\vec{p}_d=-\vec{p}_f$
Angles	(0,0)		$\Omega_0(\theta,\phi)$	

$$
\begin{aligned}
\langle\vec{p}_c\lambda_c;\vec{p}_d\lambda_d|S|\vec{p}_a\lambda_a;\vec{p}_b\lambda_b\rangle &= \langle\vec{p}_f\lambda_c;-\vec{p}_f\lambda_d|S|\vec{p}_i\lambda_a;-\vec{p}_i\lambda_b\rangle \\
&= (4\pi)^2\frac{w_0}{\sqrt{p_f p_i}}\langle\Omega_0\lambda_c\lambda_d|S|00\lambda_a\lambda_b\rangle
\end{aligned}
$$

$$
\langle\Omega_0\lambda_c\lambda_d|S|00\lambda_a\lambda_b\rangle=(2\pi)^4\delta^{(4)}(p_c+p_d-p_a-p_b)\,\langle\Omega_0\lambda_c\lambda_d|S(w_0)|00\lambda_a\lambda_b\rangle
$$

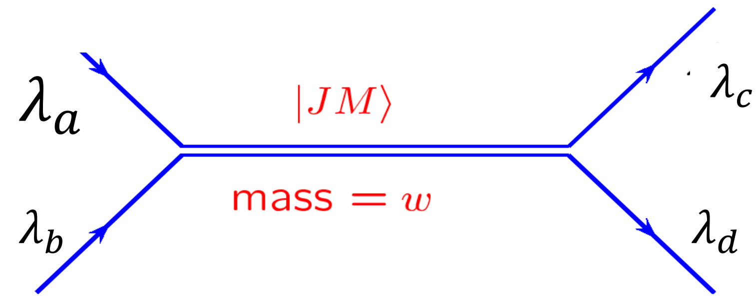
两体→两体过程的S矩阵(续)

T-matrix: $S = 1 + iT$

$$(2\pi)^4\delta^{(4)}(p_c + p_d - p_a - p_b)\,\mathcal{M}_{fi} = \langle p_c\lambda_c; p_d\lambda_d|T|p_a\lambda_a; p_b\lambda_b\rangle$$

$$\mathcal{M}_{fi} = (4\pi)^2\frac{w_0}{\sqrt{p_f p_i}}\langle\Omega_0\lambda_c\lambda_d|T(w_0)|00\lambda_a\lambda_b\rangle$$

$$\frac{d\sigma}{d\Omega_0} = \frac{p_f}{p_i}\left|\frac{\mathcal{M}_{fi}}{8\pi w_0}\right|^2$$



$$\begin{aligned}\langle\Omega_0\lambda_c\lambda_d|T(w_0)|00\lambda_a\lambda_b\rangle &= \sum_{JM}\langle\Omega_0\lambda_c\lambda_d|JM\lambda_c\lambda_d\rangle\langle JM\lambda_c\lambda_d|T(w_0)|JM\lambda_a\lambda_b\rangle \\ &\quad\times\langle JM\lambda_a\lambda_b|00\lambda_a\lambda_b\rangle \\ &= \frac{1}{4\pi}\sum_J(2J+1)\langle\lambda_c\lambda_d|T^J(w_0)|\lambda_a\lambda_b\rangle D^{J*}_{\lambda\lambda'}(\phi_0,\theta_0,0)\end{aligned}$$

两体→两体过程的S矩阵(续)

- Scattering amplitude

$$\frac{d\sigma}{d\Omega_0} = |f(\Omega_0)|^2 \qquad f(\Omega_0) = \frac{(p_f/p_i)^{\frac{1}{2}}}{8\pi w_0}\mathcal{M}_{fi} \ .$$

$$f(\Omega_0) = \frac{1}{p_i}\sum_J \left(J+\frac{1}{2}\right)\langle\lambda_c\lambda_d|T^J(w_0)|\lambda_a\lambda_b\rangle D^{J\,*}_{\lambda\lambda'}(\phi_0,\theta_0,0) \ .$$

- relationship of S-T matrix

$$\langle\lambda_c\lambda_d|S^J(w_0)|\lambda_a\lambda_b\rangle = \delta_{fi}\delta_{\lambda_c\lambda_a}\delta_{\lambda_d\lambda_b} + i\langle\lambda_c\lambda_d|T^J(w)|\lambda_a\lambda_b\rangle$$

- parity conservation

$$\langle-\lambda_c-\lambda_d|S^J(w_0)|-\lambda_a-\lambda_b\rangle = \eta\,\langle\lambda_c\lambda_d|S^J(w_0)|\lambda_a\lambda_b\rangle$$

with $$\eta = \frac{\eta_c\eta_d}{\eta_a\eta_b}(-)^{s_c+s_d-s_a-s_b}$$

两体→两体过程的S矩阵(续)

- Example

Give a process $e^+e^- \to \gamma^* \to \mu^+\mu^-$, calculate the muon angular distribution.

Helicity: $\gamma^*(\lambda)$, $e^+(\lambda_1)$, $e^-(\lambda_2)$, decay amp. F_{λ_1,λ_2}

$$A = D^{1*}_{M,\lambda_1-\lambda_2}(\phi,\theta,0)F_{\lambda_1,\lambda_2} \Rightarrow$$

$$|A|^2 = \sum_{\substack{M=\pm 1,\\ \lambda_1,\lambda_2=\pm 1/2}} D^{1*}_{M,\lambda_1-\lambda_2}(\phi,\theta,0)D^{1}_{M,\lambda_1-\lambda_2}(\phi,\theta,0)\left|F_{\lambda_1,\lambda_2}\right|^2$$

$\propto (1+\cos^2\theta)$ (assuming helicity conservation)