粒子物理学的自旋极化

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教学内容

- **第一讲：概要和基础知识 （2学时）**
  介绍矢量和张量的基础知识，笛卡尔和球张量的关系，自旋的SU（2）表示的基础知识
- **第二讲：螺旋度振幅1 （2学时）**
  介绍单粒子的螺旋度态，两体衰变螺旋度振幅和对称性变换
- **第三讲：螺旋度振幅2 （2学时）**
  介绍螺旋度振幅在粒子物理中的具体应用。
- **第四讲：自旋密度矩阵1 （2学时）**
  介绍自旋密度矩阵的性质，和几种常见自旋密度矩阵的构造和计算。
- **第五讲：自旋密度矩阵2（2学时）**
  介绍几个自旋密度矩阵的计算和应用实例
- **第六讲：粒子物理学中的自旋极化的应用特例 (2学时)**
  介绍粒子物理学中自旋极化物理的应用：包括EPR佯谬的检验，CP破坏的寻找。

第一讲：概要和基础知识

- 第一讲：概要和基础知识
- **1 .**张量基础知识
- **2.** 笛卡尔坐标系和球坐标系中的张量变换关系
- **3.** 角动量矢量的耦合和有限转动的表达

张量基础知识

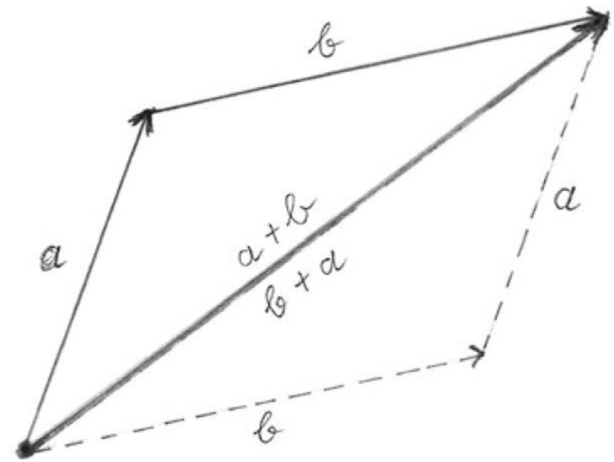
- **Vector space**

Vector: $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$$

$$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$$

$$\boldsymbol{a}+\boldsymbol{x}=\boldsymbol{b} \rightarrow \quad \boldsymbol{x}=\boldsymbol{b}-\boldsymbol{a}$$



Real number: k, l

$$(k+l)\boldsymbol{a}=k\boldsymbol{a}+l\boldsymbol{a}$$

$$k(\boldsymbol{a}+\boldsymbol{b})=k\boldsymbol{a}+k\boldsymbol{b}$$

Norm $||\boldsymbol{a}||$: length of vector $\boldsymbol{a}$

Distance $\boldsymbol{a}-\boldsymbol{b}$:

$$d(\boldsymbol{a},\boldsymbol{b}) \coloneqq ||\boldsymbol{a}-\boldsymbol{b}||$$

张量基础知识(续)

- Basis
  - Cartesian components

Vector $\boldsymbol{S}$: (S_1, S_2, S_3), $\boldsymbol{r}$: (r_1, r_2, r_3)

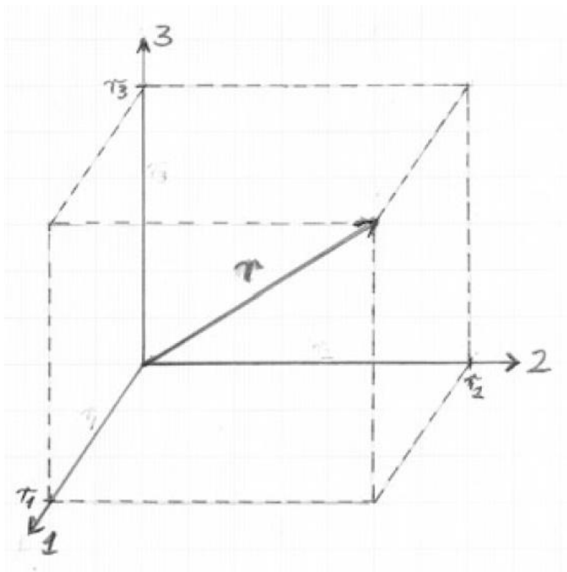
$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{r}: \quad (S_1 + r, S_2 + r_2, S_3 + r_3)$$

Or $A_\mu = S_\mu + r_\mu$

Multiplication: $\boldsymbol{R} = k\boldsymbol{r} \rightarrow \quad R_\mu = k r_\mu$

Scalar product:

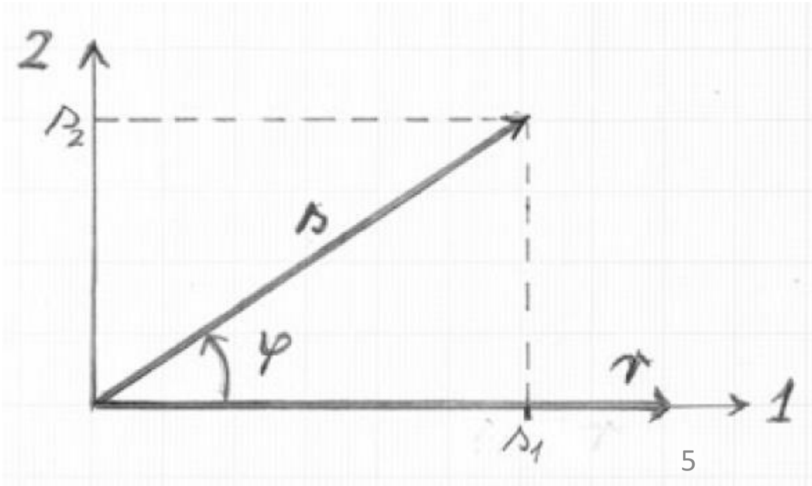
$$S \cdot r = S_1 r_1 + S_2 r_2 + S_3 r_3 = S_\mu r_\mu$$
$$= r * s \cos\phi$$



- Spherical polar coordinates

Vector $\boldsymbol{r}(r, \theta, \phi)$

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi,$$
$$z = r\, cos\theta$$



张量基础知识(续)

- Orthogonal basis

$$\boldsymbol{e}^{(i)}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \quad \boldsymbol{and} \quad \boldsymbol{e}^{(i)} \cdot \boldsymbol{e}^{(j)} = \boldsymbol{\delta_{ij}}$$

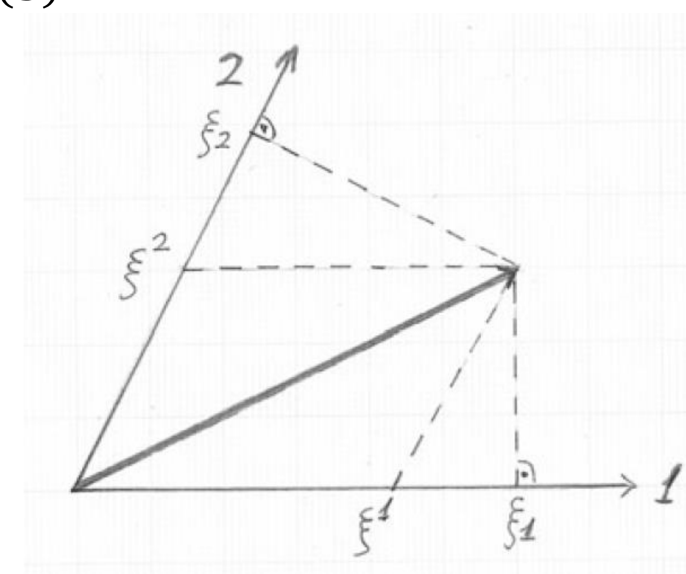
Three vector: $\boldsymbol{r} = r_1\boldsymbol{e}^{(\mathbf{1})} + r_2\boldsymbol{e}^{(\mathbf{2})} + r_3\boldsymbol{e}^{(\mathbf{3})}$

- Non-orthogonal basis

$$\boldsymbol{a}^{(i)}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \quad not \ \ within \ one \ plane$$

$$\boldsymbol{r} = \xi_1\boldsymbol{a}^{(\mathbf{1})} + \xi_2\boldsymbol{a}^{(\mathbf{2})} + \xi_3\boldsymbol{a}^{(\mathbf{3})}$$

$$\boldsymbol{\xi_i} = \boldsymbol{a}^{(i)}.\boldsymbol{r} = \sum_{j=1}^{3} \boldsymbol{g_{ij}\xi^j}$$



With $g_{ij} = \boldsymbol{a}^{(i)}.\boldsymbol{a}^{(j)} = g^{ij}$

ξ^i: contra-component

ξ_i: covariant component

g_{ij}: metric of coordinate system

二阶张量

- For any tensor A, symmetric part: A^{sym}, antisymmetric: A^{asm}

$$A_{\mu\nu}^{sym} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}), \qquad A_{\mu\nu}^{asy} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}),$$

So $A_{\mu\nu}^{sym} = A_{\nu\mu}^{sym}$, $\qquad A_{\mu\nu}^{asy} = -A_{\nu\mu}^{asy}$

- symmtric part decomposition: isotropic $\mathrm{tr}(A)\delta_{\mu\nu}$, traceless part: $\overline{\mathbf{A}}$

$$\overline{A_{\mu\nu}} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) - \frac{1}{3} A_{\lambda\lambda}\,\delta_{\mu\nu}.$$

➔ $$A_{\mu\nu} = \frac{1}{3} A_{\lambda\lambda}\,\delta_{\mu\nu} + A_{\mu\nu}^{\mathrm{asy}} + \overline{A_{\mu\nu}}.$$

二阶张量

- Trace tr $A = A_{11} + A_{22} + A_{33} = A_{\lambda\lambda}$

  tr A= $A_{\lambda\lambda}^{sym}$ = tr(A^{sym})

  Note: tr(A^{asy})=0.

  e.g. $D = 3,\ tr(\delta) = \delta_{\nu\nu}\ = 3.$
- Multiplication $A_{\mu\nu},\ B_{\lambda\kappa}$

  dot product :

  $$A\,.\,B = A_{\mu\nu}B_{\nu\kappa}$$

  total contraction:

  $$A\!:\!B = A_{\mu\nu}B_{\nu\mu}$$

$$= \frac{1}{3}A_{\lambda\lambda}B_{\kappa\kappa} + A_{\mu\nu}^{\mathrm{asy}}B_{\nu\mu}^{\mathrm{asy}} + \overline{A_{\mu\nu}}\ \overline{B_{\nu\mu}}\ .$$

二阶张量

- Norm or magnitude

$$|\mathbf{A}|^2 = A_{\mu\nu} A^T_{\nu\mu} = A_{\mu\nu} A_{\mu\nu}.$$

- Second rank tensor decomposition using projection tensor

$$A^{(\mathrm{i})}_{\mu\nu} = P^{(\mathrm{i})}_{\mu\nu\mu'\nu'} A_{\mu'\nu'}.$$

$$P^{(0)}_{\mu\nu,\mu'\nu'} := \frac{1}{3}\,\delta_{\mu\nu}\delta_{\mu'\nu'}, \quad P^{(1)}_{\mu\nu\mu'\nu'} := \frac{1}{2}\,(\delta_{\mu\mu'}\delta_{\nu\nu'} - \delta_{\mu\nu'}\delta_{\nu\mu'}),$$

$$P^{(2)}_{\mu\nu\mu'\nu'} \equiv \Delta_{\mu\nu,\mu'\nu'} := \frac{1}{2}\,(\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'}) - \frac{1}{3}\,\delta_{\mu\nu}\delta_{\mu'\nu'}.$$

二阶张量

- Dyadic tensor: vector $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\mathbf{a}\,\mathbf{b} := \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$$

Decomposition of $A_{\mu\nu} = a_\mu b_\nu$

$$a_\mu b_\nu = \frac{1}{3} a_\lambda b_\lambda \delta_{\mu\nu} + \frac{1}{2}\left(a_\mu b_\nu - a_\nu b_\mu\right) + \overline{a_\mu b_\nu}\,.$$

$$\overline{a_\mu b_\nu} = \frac{1}{2}\left(a_\mu b_\nu + a_\nu b_\mu\right) - \frac{1}{3} a_\lambda b_\lambda \delta_{\mu\nu}.$$

全反对称张量(ϵ, Levi-Civita tensor)

$$\varepsilon_{\mu\nu\lambda} := \begin{vmatrix} \delta_{1\mu} & \delta_{1\nu} & \delta_{1\lambda} \\ \delta_{2\mu} & \delta_{2\nu} & \delta_{2\lambda} \\ \delta_{3\mu} & \delta_{3\nu} & \delta_{3\lambda} \end{vmatrix}.$$

$$\varepsilon_{\mu\nu\lambda} = \begin{matrix} 1, \mu\,\nu\,\lambda = 123, 231, 312 \\ -1, \mu\,\nu\,\lambda = 213, 132, 321 \\ 0, \mu\,\nu\,\lambda = \text{else}, \end{matrix}$$

Dual relation: $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ ➔ $\boldsymbol{c_\mu} = \boldsymbol{\epsilon_{\mu\nu\lambda}}\,\boldsymbol{a_\nu b_\lambda}$

Spate product: $d \cdot (a \times b) = \epsilon_{\mu\nu\lambda} d_\mu a_\nu b_\lambda$

$$\varepsilon_{\mu\nu\lambda} A_{\nu\lambda} = \frac{1}{2}\varepsilon_{\mu\nu\lambda}(A_{\nu\lambda} - A_{\lambda\nu})$$

$$\varepsilon_{\mu\nu\lambda}\, A_{\nu\lambda} = \varepsilon_{\mu\nu\lambda}\, A^{\text{asy}}_{\nu\lambda}\,.$$

全反对称张量(ϵ, Levi-Civita tensor)

$$\varepsilon_{\mu\nu\lambda}\,\varepsilon_{\mu'\nu'\lambda'} = \begin{vmatrix} \delta_{\mu\mu'} & \delta_{\mu\nu'} & \delta_{\mu\lambda'} \\ \delta_{\nu\mu'} & \delta_{\nu\nu'} & \delta_{\nu\lambda'} \\ \delta_{\lambda\mu'} & \delta_{\lambda\nu'} & \delta_{\lambda\lambda'} \end{vmatrix}.$$

$$= \delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{\lambda\lambda'} + \delta_{\mu\nu'}\delta_{\nu\lambda'}\delta_{\lambda\mu'} + \delta_{\mu\lambda'}\delta_{\nu\mu'}\delta_{\lambda\nu'}$$
$$- \delta_{\mu\mu'}\delta_{\nu\lambda'}\delta_{\lambda\nu'} - \delta_{\mu\nu'}\delta_{\nu\mu'}\delta_{\lambda\lambda'} - \delta_{\mu\lambda'}\delta_{\nu\nu'}\delta_{\lambda\mu'}.$$

$$\varepsilon_{\mu\nu\lambda}\,\varepsilon_{\mu'\nu'\lambda} = \begin{vmatrix} \delta_{\mu\mu'} & \delta_{\mu\nu'} \\ \delta_{\nu\mu'} & \delta_{\nu\nu'} \end{vmatrix} = \delta_{\mu\mu'}\delta_{\nu\nu'} - \delta_{\mu\nu'}\delta_{\nu\mu'}.$$

$$\varepsilon_{\mu\nu\lambda}\,\varepsilon_{\mu'\nu\lambda} = 2\,\delta_{\mu\mu'}.$$

$$\varepsilon_{\mu\nu\lambda}\,\varepsilon_{\mu\nu\lambda} = 6.$$

常用的矢量积

$$\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$$

$$\boldsymbol{a}\times\boldsymbol{a}=\boldsymbol{0}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{0}$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c})=[\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})]\boldsymbol{a}$$

矢量标量积（内积）：$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\boldsymbol{cos\theta}$

矢量外积： $\boldsymbol{a}\times\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\boldsymbol{sin\theta}\,\widehat{\boldsymbol{n}},\widehat{\boldsymbol{n}}$是$\boldsymbol{a}\times\boldsymbol{b}$平面的单位法线

张量对空间的平均

单位矢量的球坐标 (θ,ϕ), $ie.\ \hat{n}=(sin\theta cos\phi, sin\theta sin\phi, cos\theta)$, 定义：

$$<n^k>=\frac{1}{4\pi}\int n^k sin\theta d\theta d\phi$$

二维情况：二维正交单位张量l_i, m_i

$$<l_i>=0,\qquad <l_i l_j>=\frac{1}{2}\delta_{ij},\qquad <l_i m_j>=\frac{1}{2}\epsilon_{ij},$$

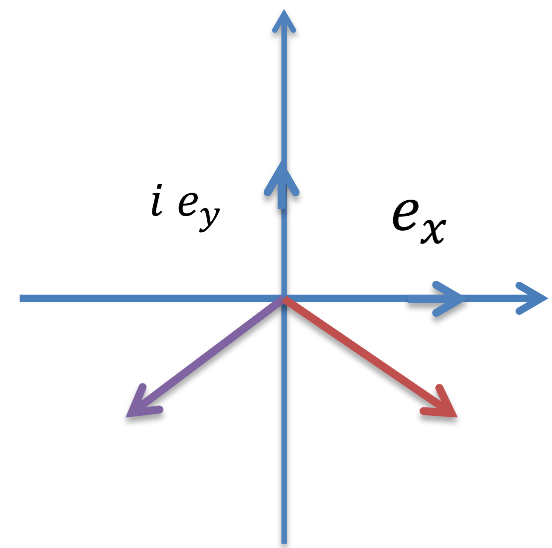
$$<l_{i_1}l_{i_2}\dots l_{i_n}>=\frac{1}{(2n)!!}\delta_{i_1 i_2 \dots i_n}$$

三维情况：正交的单位张量l_i, m_i, n_i

$$<l_i>=0,\qquad <l_i l_j>=\frac{1}{3}\delta_{ij},\qquad <l_i m_j n_k>=\frac{1}{6}\epsilon_{ijk},$$

$$<l_{i_1}l_{i_2}\dots l_{i_n}>=\frac{1}{(2n+1)!!}\delta_{i_1 i_2 \dots i_n}$$

笛卡尔坐标系和球坐标系中的张量变换关系



- Spherical tensor: rank 1

Cartesian basis: $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$

Vector $\boldsymbol{A} = \boldsymbol{A}_x\boldsymbol{e}_x + \boldsymbol{A}_y\boldsymbol{e}_y + \boldsymbol{A}_z\boldsymbol{e}_z$

covariant spherical basis

$$\epsilon_1^1 = -\frac{e_x + ie_y}{\sqrt{2}}$$
$$\epsilon_{-1}^1 = \frac{e_x - ie_y}{\sqrt{2}}$$
$$\epsilon_0^1 = e_z$$

covariant spherical component

$$\begin{cases} A_{1+1} = -\dfrac{1}{\sqrt{2}}\left(A_x + iA_y\right) \\ A_{10} = A_z \\ A_{1-1} = \dfrac{1}{\sqrt{2}}\left(A_x - iA_y\right) \end{cases}$$

$$(\epsilon_s^1) *= (-1)^s \epsilon_{-s}^1$$

$$A = \sum_{s=0,\pm 1} (-1)^s A_{1s} \epsilon_{-s}^1$$

Spherical tensor: rank 1

- position vector

$$\boldsymbol{r} = -r_1^1\,\epsilon_1^1 - r_{-1}^1\,\epsilon_{-1}^1 + r_0^1\,\epsilon_0^1$$

where $r_1^1 = -(e_x + ie_y)/\sqrt{2}, r_{-1}^1 = (e_x - ie_y)/\sqrt{2}, r_0^1 = e_z$.

$Y_m^L(\theta,\phi)$: sphere harmonics

$$r_1^1 = -\frac{r}{\sqrt{2}}\sin\theta\, e^{i\phi} = \sqrt{\frac{4\pi}{3}}\, rY_1^1(\theta,\phi)$$

$$r_{-1}^1 = \frac{r}{\sqrt{2}}\sin\theta\, e^{-i\phi} = \sqrt{\frac{4\pi}{3}}\, rY_{-1}^1(\theta,\phi)$$

$$r_0^1 = r\cos\theta = \sqrt{\frac{4\pi}{3}}\, rY_0^1(\theta,\phi)$$

with

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

$$Y_{-1}^1 = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$$

$$Y_0^1 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

➔ $\boldsymbol{r} = \sqrt{\frac{4\pi}{3}}\, r\sum_\mu (-1)^\mu Y_\mu^1 \epsilon_\mu^1$

Spherical tensor: rank 2

Covariant spherical bases:

$$t_{rs} = \sum_{p,q=\pm,0} \langle 1p1q|rs\rangle(\epsilon_p^1 \otimes \epsilon_q^1) \rightarrow \left(\mathbf{t}_{rs}\right)^* = (-1)^{r+s}\mathbf{t}_{r-s}.$$

e.g. $t_{00} = \frac{1}{\sqrt{3}}(\epsilon_1^1 \otimes \epsilon_{-1}^1 - \epsilon_0^1 \otimes \epsilon_0^1 + \epsilon_{-1}^1 \otimes \epsilon_1^1)$ Spherical basis

$= -\frac{1}{\sqrt{3}}(e_x \otimes e_y + e_y \otimes e_y + e_z \otimes e_z)$ Cartesian basis

M matrix:

$$M_{00} = -\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical tensor: rank 2

Expansion : $$\mathbf{T}=\sum_{r=0}^{2}\sum_{s=-r}^{+r} T_{rs}(\mathbf{t}_{rs})^{*}=\sum_{r=0}^{2}\sum_{s=-r}^{+r}(T_{rs})^{*}\mathbf{t}_{rs}.$$

$$(T_{rs})^{*}=(-1)^{r-s}T_{r-s}.$$

$$T_{rs}=\mathrm{Tr}[\sum_{i,j=x,y,z}(e_i\otimes e_j)T_{ij}\,M_{rs}]$$

$$T_{00}=-\frac{1}{\sqrt{3}}\left(T_{xx}+T_{yy}+T_{zz}\right),\qquad T_{20}=\frac{1}{\sqrt{6}}\left\{3T_{zz}-\left(T_{xx}+T_{yy}+T_{zz}\right)\right\}$$

$$T_{10}=\frac{1}{\sqrt{2}}i\left(T_{xy}-T_{yx}\right)\ ,\qquad T_{2\pm1}=\mp\frac{1}{2}\{T_{xz}+T_{zx}\pm i(T_{yz}+T_{zy})\}$$

$$T_{1\pm1}=\frac{1}{2}\left\{T_{zx}-T_{xz}\pm i\left(T_{zy}-T_{yz}\right)\right\},\qquad T_{2\pm2}=\frac{1}{2}\left\{T_{xx}-T_{yy}\pm i\left(T_{xy}+T_{yx}\right)\right\}$$

$$T_{lm}=\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\langle l_1 0\, l_2 0|l0\rangle r^{l_1+l_2}\,Y_m^L(\theta,\phi)$$

角动量矢量的耦合和有限转动的表达

- Quantum mechanics definition

Classical :$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$

Uncertainty principle: $[x, p_x] = i\hbar,\ [y, p_y] = i\hbar, [z, p_z] = i\hbar$

$$[L_x, L_y] = i\hbar L_z, [L_y, L_z] = i\hbar L_x, [L_z, L_x] = i\hbar L_y$$

$[J_x, J_y] = i\hbar J_z, [J_y, J_z] = i\hbar J_x, [J_z, J_x] = i\hbar J_y$ or $\boldsymbol{J} \times \boldsymbol{J} = \boldsymbol{iJ}$

Commutators: $[J^2, J_x] = 0,\ [J^2, J_y] = 0,\ [J^2, J_z] = 0$

角动量矢量的耦合和有限转动的表达

- Raising and lowering operators:

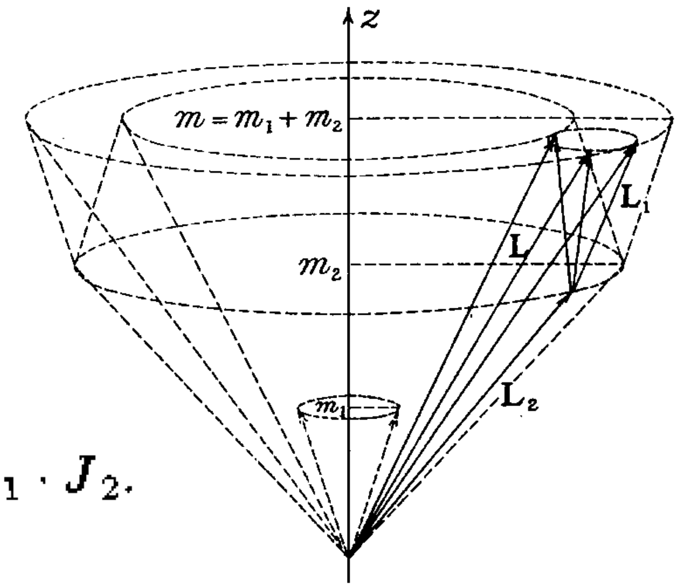
$$J_{\pm} = J_x \pm iJ_y,\ [J^2, J_{\pm}] = 0, [J_z, J_{\pm}] = \pm J_{\pm}$$

Matrix elements

$$\begin{aligned}
\langle j'm' \,|\, \boldsymbol{J}^2 \,|\, jm \rangle &= j(j+1)\,\delta_{jj'}\,\delta_{mm'}. \\
\langle j'm' \,|\, J_z \,|\, jm \rangle &= m\,\delta_{jj'}\,\delta_{mm'}. \\
\langle j'm' \,|\, J_+ \,|\, jm \rangle &= [(j-m)(j+m+1)]^{1/2}\,\delta_{jj'}\,\delta_{m'\,m+1}. \\
\langle j'm' \,|\, J_- \,|\, jm \rangle &= [(j+m)(j-m+1)]^{1/2}\,\delta_{jj'}\,\delta_{m'\,m-1}.
\end{aligned}$$

角动量矢量的耦合和有限转动的表达

- Addition of angular momenta



$$\boldsymbol{J} = \boldsymbol{J_1} + \boldsymbol{J_2}$$

$$\begin{aligned} J_x &= J_{1x} + J_{2x}, \\ J_y &= J_{1y} + J_{2y}, \\ J_z &= J_{1z} + J_{2z}. \end{aligned} \quad \Rightarrow \quad \boldsymbol{J}^2 = \boldsymbol{J}_1^2 + \boldsymbol{J}_2^2 + 2\boldsymbol{J}_1 \cdot \boldsymbol{J}_2.$$

$$[\boldsymbol{J}^2, J_z] = 0; \quad [J_x, J_y] = iJ_z. \quad [\boldsymbol{J}^2, J_{1z}] = -[\boldsymbol{J}^2, J_{2z}] \neq 0.$$

Uncoupled representation: $J_1^2, J_2^2, J_{1z}, J_{2z}$

Coupled representation: J_1^2, J_2^2, J^2, J_z

角动量矢量的耦合和有限转动的表达(续)

- Addition of angular momenta

$$|j_1 j_2 j m\rangle = \sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 j_2 m_1 m_2\rangle.$$

C-G coefficient: $\langle j_1 j_2 m_1 m_2 | j_1 J_2 J m\rangle = \langle j_1 m_1; j_2 m_2 | J m\rangle = \begin{bmatrix} j_1 & j_2 & J \\ m_1 & m_2 & m \end{bmatrix}$

$$|j_1 j_2 m_1 m_2\rangle = \sum_{jm} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 j_2 j m\rangle.$$

- ➢ Real number : $\langle j_1 j_2 m_1 m_2 | j_1 J_2 J m\rangle = \langle j_1 j_2 J m | j_1 J_2 m_1 m_2\rangle$

- properties of C-G coefficients

$$m|j_1 j_2 j m\rangle = \sum_{m_1 m_2} (m_1 + m_2) \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j_1 j_2 m_1 m_2\rangle.$$

角动量矢量的耦合和有限转动的表达(续)

- properties of C-G coefficients

$$\sum_{j}(2j+1)=(2j_1+1)(2j_2+1).$$

$$\begin{bmatrix} j & 0 & j \\ m & 0 & m \end{bmatrix} = \begin{bmatrix} 0 & j & j \\ 0 & m & m \end{bmatrix} = 1.$$

$$\begin{aligned} \langle j_1 j_2 m_1' m_2' | j_1 j_2 m_1 m_2 \rangle &= \delta_{m_1 m_1'}\ \delta_{m_2 m_2'}. \\ \langle j_1 j_2 j' m' | j_1 j_2 j m \rangle &= \delta_{jj'}\ \delta_{mm'}. \end{aligned}$$

$$\begin{aligned} \langle j_1 j_2 j' m' | j_1 j_2 j m \rangle &= \sum_{m_1 m_2} \sum_{m_1' m_2'} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ m_1' & m_2' & m' \end{bmatrix} \\ &\quad \times \langle j_1 j_2 m_1' m_2' | j_1 j_2 m_1 m_2 \rangle, \qquad (\end{aligned}$$

角动量矢量的耦合和有限转动的表达(续)

- properties of C-G coefficients

$$\sum_{m_1 m_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix} = \delta_{jj'}\, \delta_{mm'}.$$

$$\sum_{j} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix} = \delta_{m_1 m_1'}\, \delta_{m_2 m_2'}.$$

- General expression

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} = \delta_{m\, m_1+m_2} \left[(2j+1)AB\right]^{\frac{1}{2}} \sum_{\nu} \frac{(-1)^{\nu}}{\nu!} C_{\nu}^{-1},$$

$$\begin{aligned}
A &= \frac{(j_1+j_2-j)!\,(j+j_1-j_2)!\,(j_2+j-j_1)!}{(j_1+j_2+j+1)!}, \\
B &= (j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)!, \\
C_{\nu} &= (j_1+j_2-j-\nu)!\,(j_1-m_1-\nu)!\,(j_2+m_2-\nu)! \\
&\quad \times (j-j_2+m_1+\nu)!\,(j-j_1-m_2+\nu)!.
\end{aligned}$$

角动量矢量的耦合和有限转动的表达(续)

- symmetry properties

$$\begin{aligned}
\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} &= (-1)^{j_1+j_2-j} \begin{bmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{bmatrix} \\
&= (-1)^{j_1+j_2-j} \begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix} \\
&= (-1)^{j_1-m_1} \frac{[j]}{[j_2]} \begin{bmatrix} j_1 & j & j_2 \\ m_1 & -m & -m_2 \end{bmatrix} \\
&= (-1)^{j_2+m_2} \frac{[j]}{[j_1]} \begin{bmatrix} j & j_2 & j_1 \\ -m & m_2 & -m_1 \end{bmatrix},
\end{aligned}$$

with $[j] = \sqrt{2j+1}$

especially: $j_1 + j_2 - j$ is odd, $\begin{bmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{bmatrix}$=0

角动量矢量的耦合和有限转动的表达(续)

- other notations

C-G coefficient: $C^{jm}_{j_1 m_1 j_2 m_2}$, $C(j_1 j_2 j; m_1 m_2 m)$

Wigner $3j$ symbol $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$:

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \frac{(-1)^{j_1-j_2-m}}{[j]} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{bmatrix}$$

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j_1+j_2+j} \begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{pmatrix}$$
$$= (-1)^{j_1+j_2+j} \begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix}$$

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \begin{pmatrix} j & j_1 & j_2 \\ m & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j & j_1 \\ m_2 & m & m_1 \end{pmatrix}$$

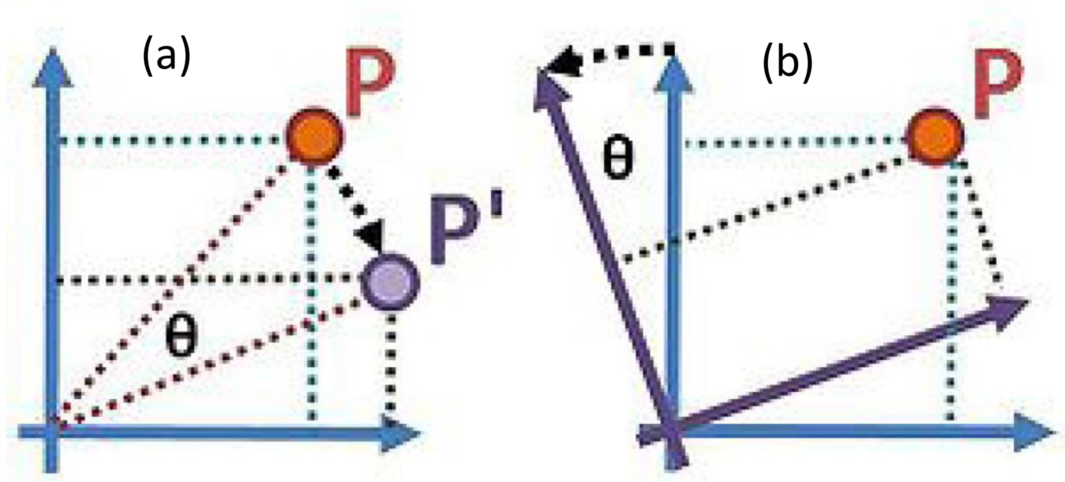
角动量矢量的耦合和有限转动的表达(续)

- Active rotation and passive rotation

Passive rotation

$$R_z(\theta):\ (x,y) \to (x',y')$$
$$\Psi(x,y) \to \Psi(x',y')$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



(a) active rotation (b) passive rotation

Active rotation

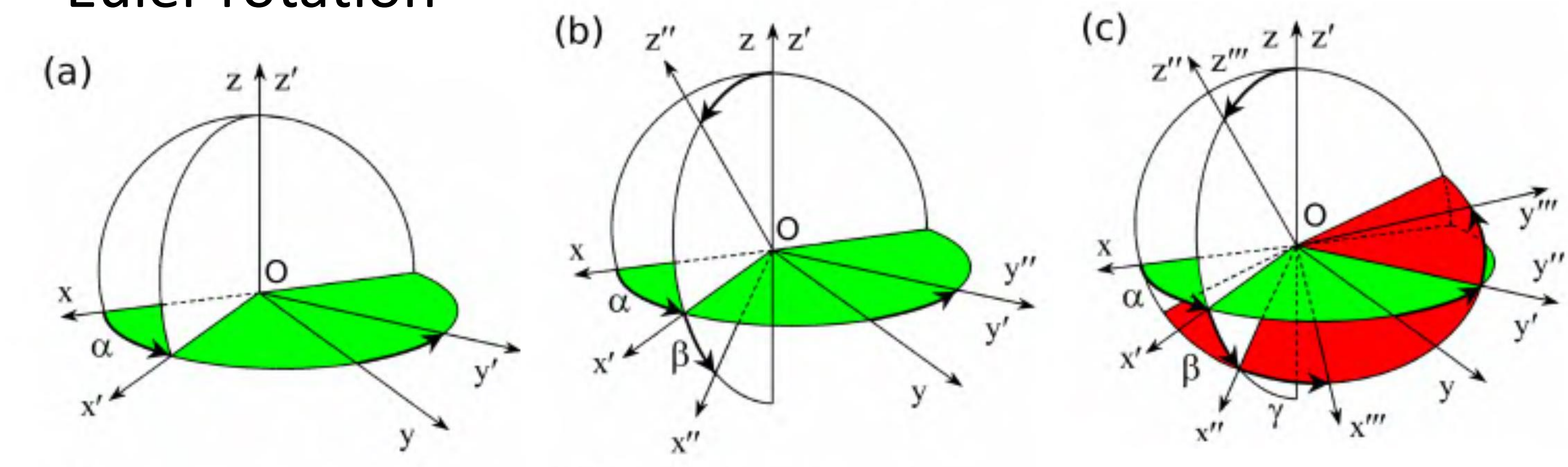
$$R_z(\theta):\ \Psi(x,y) \to \Psi'(x',y')$$
$$P(x,y) \to P'(x',y')$$

$$\Psi'(x',y') = \Psi(x\ ,y)$$
$$P' = R_z(\theta)P \Rightarrow P = R_z(-\theta)P'$$

$$\Psi'(x',y') = \Psi(R_z(-\theta)P')$$

角动量矢量的耦合和有限转动的表达(续)

- Euler rotation



$$XYZ \xrightarrow[Z-axis]{\alpha} X'Y'Z' \xrightarrow[Y'-axis]{\beta} X''Y''Z'' \xrightarrow[Z''-axis]{\gamma} X'''Y'''Z'''$$

$$R(\alpha,\beta,\gamma) = R_{Z''}(\gamma)R_{Y'}(\beta)R_Z(\alpha)$$

Matrix: $$\psi_{jm}(\boldsymbol{r}') = \sum_{m'} D^j_{m'm}(\alpha\beta\gamma)\,\psi_{jm'}(\boldsymbol{r}),$$

角动量矢量的耦合和有限转动的表达(续)

- Transformation of spherical vector

  Define rotation matrix : ${A^1}'_\mu = M^1_{\mu,\nu}(\alpha,\beta,\gamma)A^1_\nu$

Cartesian coordinate:

$$
\begin{aligned}
A_{X_1} &= A_X \cos\alpha + A_Y \sin\alpha, \\
A_{Y_1} &= A_Y \cos\alpha - A_X \sin\alpha, \\
A_{Z_1} &= A_Z.
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_X \\ A_Y \\ A_Z \end{bmatrix}
$$

$$
\begin{aligned}
A_{X2} &= A_{X1} \cos\beta - A_{Z1} \sin\beta. \\
A_{Y2} &= A_{Y1}. \\
A_{Z2} &= A_{Z1} \cos\beta + A_{X1} \sin\beta.
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} A_{X2} \\ A_{Y2} \\ A_{Z2} \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} A_{X1} \\ A_{Y1} \\ A_{Z1} \end{bmatrix}
$$

角动量矢量的耦合和有限转动的表达(续)

- Derive matrix $D^1(\alpha,\beta,\gamma)$

Sphere component transform

$$\begin{bmatrix} (A_1^1)_1 \\ (A_1^0)_1 \\ (A_1^{-1})_1 \end{bmatrix} = \begin{bmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{bmatrix} \begin{bmatrix} A_1^1 \\ A_1^0 \\ A_1^{-1} \end{bmatrix} \rightarrow \boldsymbol{A}_1 = M_Z(\alpha)\,\boldsymbol{A},$$

$$\begin{bmatrix} (A_1^1)_2 \\ (A_1^0)_2 \\ (A_1^{-1})_2 \end{bmatrix} = \begin{bmatrix} 1/2\,(1+\cos\beta) & \sqrt{1/2}\sin\beta & 1/2(1-\cos\beta) \\ -\sqrt{1/2}\sin\beta & \cos\beta & \sqrt{1/2}\sin\beta \\ 1/2(1-\cos\beta) & -\sqrt{1/2}\sin\beta & 1/2(1+\cos\beta) \end{bmatrix} \begin{bmatrix} (A_1^1)_1 \\ (A_1^0)_1 \\ (A_1^{-1})_1 \end{bmatrix}$$

$$\rightarrow \boldsymbol{A_2} = M_Y(\beta)\boldsymbol{A_1}$$

$$M(\alpha,\beta,\gamma) = M_{Z_2}(\gamma)\,M_{Y_1}(\beta)\,M_Z(\alpha),$$

角动量矢量的耦合和有限转动的表达(续)

- Derive matrix $D^1(\alpha,\beta,\gamma)$

  $D^1(\alpha,\beta,\gamma)$ defined by $A'_\mu = \sum_\nu D^1_{\nu\mu}(\alpha,\beta,\gamma)\,A_\nu$

  different from that in Mathematic

$$D^1(\alpha,\beta,\gamma) = \begin{bmatrix} e^{-i\gamma}\frac{1+\cos\beta}{2}e^{-i\alpha} & -\frac{\sin\beta}{\sqrt{2}}e^{-i\alpha} & e^{i\gamma}\frac{1-\cos\beta}{2}e^{-i\alpha} \\ e^{-i\gamma}\frac{\sin\beta}{\sqrt{2}} & \cos\beta & -e^{i\gamma}\frac{\sin\beta}{\sqrt{2}} \\ e^{-i\gamma}\frac{1-\cos\beta}{2}e^{i\alpha} & \frac{\sin\beta}{\sqrt{2}}e^{i\alpha} & e^{i\gamma}\frac{1+\cos\beta}{2}e^{i\alpha} \end{bmatrix}$$

- same procedure applicable to rank 2 case

角动量矢量的耦合和有限转动的表达(续)

- Rotation operator

Proof:

$$R(\alpha,\beta,\gamma) = R_{Z_2}(\gamma)R_{Y_1}(\beta)R_Z(\alpha)$$

$$= e^{-i\gamma J_{Z_2}} e^{-i\beta J_{Y_1}} e^{-i\alpha J_Z}$$

$$XYZ \xrightarrow[R_Z(\alpha)]{} X_1Y_1Z_1 \xrightarrow[R_{Y_1}(\beta)]{} X_2Y_2Z_2 \xrightarrow[R_{Z_2}(\gamma)]{} X'Y'Z'.$$

Rotation carried out in the original system

$$R(\alpha,\beta,\gamma) = e^{-i\alpha J_Z}\ e^{-i\beta J_Y}\ e^{-i\gamma J_Z}.$$

角动量矢量的耦合和有限转动的表达(续)

- $d^j_{m',m}(\beta)$ matrix

$$\begin{aligned}\Psi_{jm}(\boldsymbol{r}') &= R(\alpha,\beta,\gamma)\,\Psi_{jm}(\boldsymbol{r}) \\ &= \sum_{m'} D^j_{m'm}(\alpha,\beta,\gamma)\,\Psi_{jm'}(\boldsymbol{r}),\end{aligned}$$

$$\begin{aligned}D^j_{m'm}(\alpha,\beta,\gamma) &= \langle \Psi_{jm'}(\boldsymbol{r})\,|\,R(\alpha,\beta,\gamma)\,|\,\Psi_{jm}(\boldsymbol{r})\rangle \\ &= \langle jm'\,|\,R(\alpha,\beta,\gamma)\,|\,jm\rangle.\end{aligned}$$

$$\begin{aligned}D^j_{m'm}(\alpha,\beta,\gamma) &= \langle jm'\,|\,e^{-i\alpha J_Z}\;e^{-i\beta J_Y}\;e^{-i\gamma J_Z}\,|\,jm\rangle \\ &= e^{-i\alpha m'}\,\langle jm'\,|\,e^{-i\beta J_Y}\,|\,jm\rangle\,e^{-i\gamma m}.\end{aligned}$$

$$= e^{-i\alpha m'}\,d^j_{m'm}(\beta)\,e^{-i\gamma m}.$$

$$\begin{aligned}d^j_{m'm}(\beta) &= d^j_{mm'}(-\beta) \\ &= (-1)^{m'-m}\,d^j_{mm'}(\beta) \\ &= d^j_{-m,-m'}(\beta).\end{aligned}$$

角动量矢量的耦合和有限转动的表达(续)

- Rotation matrix for spinor

$$R_Y(\beta) = e^{-i\beta S_Y} = e^{-i\beta/2\,\sigma_y} = \cos\frac{\beta}{2} - i\,\sigma_y \sin\beta$$

$$\Rightarrow \quad d^{1/2}(\beta) = R_Y(\beta) = \cos\frac{\beta}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i\sin\frac{\beta}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}.$$

Similarly :

$$R_X(\theta) = \cos\frac{\theta}{2} - i\sigma_x \sin\frac{\theta}{2} \doteq \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_Z(\theta) = \cos\frac{\theta}{2} - i\sigma_z \sin\frac{\theta}{2} \doteq \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix}$$

角动量矢量的耦合和有限转动的表达(续)

- Clebsch-Gordan series

$$D^{j_1}_{\mu_1 m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega) = \sum_j \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^j_{\mu m}(\omega).$$

- inverse Clebsch-Gordan series

$$\sum_{m_1 \mu_1} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j_1}_{\mu_1 m_1}(\omega) D^{j_2}_{\mu_2 m_2}(\omega) = D^{j'}_{\mu m}(\omega).$$

- symmetry and properties of wigner-D matrix

$$\delta_{\mu m} = \sum_{\mu'} D^{j\,*}_{\mu' \mu}(\omega)\, D^{j}_{\mu' m}(\omega).$$ unitary

角动量矢量的耦合和有限转动的表达(续)

- symmetry and properties of wigner-D matrix

$$\delta_{\mu m} = \sum_{m'} D^{j\,*}_{mm'}(\omega)\, D^{j}_{\mu m'}(\omega). \qquad \text{unitary}$$

If $\omega = \omega_1\omega_2$, then $R(\omega) = R(\omega_1\omega_2)$=$R(\omega_1)R(\omega_2)$,

$$D^{j}_{m''m}(\omega) = \sum_{m'} D^{j}_{m''m'}(\omega_1)\, D^{j}_{m'm}(\omega_2).$$

Symmetry relations:

$$\begin{aligned} D^{j\,*}_{\mu m}(\alpha,\beta,\gamma) &= (-1)^{\mu-m} D^{j}_{-\mu-m}(\alpha,\beta,\gamma). \\ D^{j}_{\mu m}(-\gamma,-\beta,-\alpha) &= D^{j\,*}_{m\mu}(\alpha,\beta,\gamma). \end{aligned}$$

角动量矢量的耦合和有限转动的表达(续)

- Orthoganality and normalization

$$I = \int D^{j_1 *}_{\mu_1 m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega)\, d\omega ,$$

$$= \frac{8\pi^2}{2j_2+1}\, \delta_{j_1 j_2}\, \delta_{\mu_1 \mu_2}\, \delta_{m_1 m_2} .$$

where $\int d\omega = \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma .$

$$I_2 = \int D^{j_1}_{\mu_1, m_1}(\omega) D^{j_2}_{\mu_2, m_2}(\omega)\, D^{j_3 *}_{\mu_3, m_3}(\omega) d\omega$$
$$= \frac{8\pi^2}{2j_3+1} \begin{bmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}$$

角动量矢量的耦合和有限转动的表达(续)

- General expression

$$
\begin{aligned}
D^j_{\mu m}(\alpha,\beta,\gamma) &= e^{-i\mu\alpha}\, e^{-im\gamma}\left[(j+m)!(j-m)!(j+\mu)!(j-\mu)!\right]^{\frac{1}{2}} \\
&\quad \times \sum_x \frac{(-1)^x}{x!(j-x-\mu)!(j+m-x)!(\mu+x-m)!} \\
&\quad \times \left(\cos\frac{\beta}{2}\right)^{2j+m-\mu-2x} \left(-\sin\frac{\beta}{2}\right)^{\mu-m+2x} .
\end{aligned}
$$

The sum over x is over all integers for which the factorial arguments are greater than or equal to 0

Exercise

- 1. 用全反对称张量证明
  $$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c})=[\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})]\boldsymbol{a}$$
- 2.对于正交的单位张量l_i, m_i, n_i，证明：
  $$< l_i > = 0, \qquad < l_i l_j > = \frac{1}{3}\delta_{ij},$$
  $$< l_i m_j n_k > = \frac{1}{6}\epsilon_{ijk},$$
- 3. 用球张量变换的方法证明，绕Y 轴转动β 角的2阶张量T_2^μ 的变换矩阵为

Exercise

$$M(\beta) =$$

$$\begin{bmatrix}
\frac{1}{4}(1+\cos\beta)^2 & \frac{1}{2}(1+\cos\beta)\sin\beta & \sqrt{\frac{3}{8}}\sin^2\beta & \frac{1}{2}(1-\cos\beta)\sin\beta & \frac{1}{4}(1-\cos\beta)^2 \\
-\frac{1}{2}(1+\cos\beta)\sin\beta & \frac{1}{2}(\cos\beta+\cos2\beta) & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{1}{2}(\cos\beta-\cos2\beta) & \frac{1}{2}(1-\cos\beta)\sin\beta \\
\sqrt{\frac{3}{8}}\sin^2\beta & -\sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{1}{2}(3\cos^2\beta-1) & \sqrt{\frac{3}{2}}\sin\beta\cos\beta & \sqrt{\frac{3}{8}}\sin^2\beta \\
-\frac{1}{2}(1-\cos\beta)\sin\beta & \frac{1}{2}(\cos\beta-\cos2\beta) & -\sqrt{\frac{3}{2}}\sin\beta\cos\beta & \frac{1}{2}(\cos\beta+\cos2\beta) & \frac{1}{2}(1+\cos\beta)\sin\beta \\
\frac{1}{4}(1-\cos\beta)^2 & -\frac{1}{2}(1-\cos\beta)\sin\beta & \sqrt{\frac{3}{8}}\sin^2\beta & -\frac{1}{2}(1+\cos\beta)\sin\beta & \frac{1}{4}(1+\cos\beta)^2
\end{bmatrix}$$

- 4. 利用反C-G级数， 构造转动矩阵$D^1(\alpha,\beta,\gamma)$, 已知：

$$d^{1/2}(\beta) = \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}. \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \pm\frac{1}{2} & \pm\frac{1}{2} & \pm1 \end{bmatrix} = 1, \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \pm\frac{1}{2} & \mp\frac{1}{2} & 0 \end{bmatrix} = \sqrt{\frac{1}{2}}.$$