Minimal unified field theory from square root metric, fibre bundle theory and Sheaf theory

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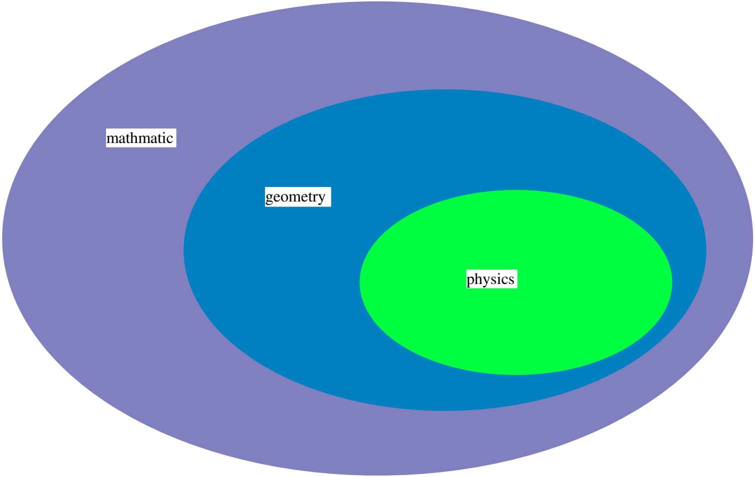
Contents

To grasp the entities by mind, which is physics.

-- Einstein

Pure geometry philosophy



Geometry background of General Relativity (GR) and Standard Model of particle physics (SM)

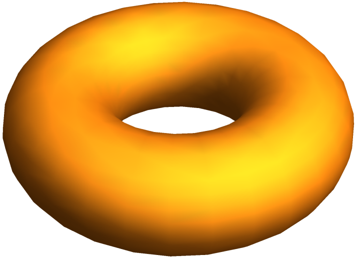


Fig 1: Geometry background of GR is curved, smooth manifold -- pseudo-Riemannian manifold (more precisely, Lorentzian manifold). The gravitational field is determined by the metric of manifold.

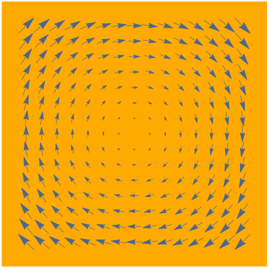


Fig 2: Geometry background of SM is very similar with the flat space-time with G-structure. The electromagnetic field, weak bosons fields, gluon bosons fields are originated from the principal G-bundle connections. Leptons, quarks are originated from the sections of associated bundle.

The curved manifold with G-bundle is a good option for geometry background of unified field theory (UFT)

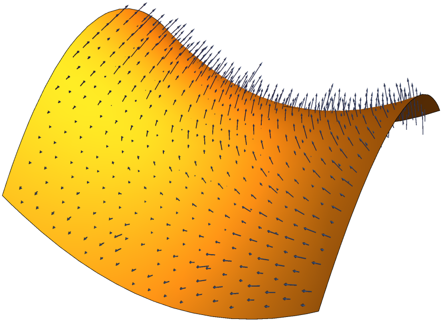


Fig 3: After long time research, we find that "square root metric" manifold not only with metric, but also equipped with U(4)-bundle at the same time. This geometry might give intrinsic geometrical interpretation to all the fields being observed.

Square root something usual lead to unusual

- $$\sqrt{-1} = \mathrm{i}$$
- $$\sqrt{\text{Klein} - \text{Gordon equation}} \quad \Rightarrow \quad \text{Dirac equation}$$
- $$\sqrt{\text{Metric g}} \quad \Rightarrow \quad ?$$

Notations

- $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}$ represent frame indices, and $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} = 0, 1, 2, 3$.
- Spinor indices $\mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{l} = 1, 2, 3, 4$.
- μ, ν, ρ, σ represent coordinates indices, and $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$.
- $\alpha, \beta, \gamma, \tau$ represent group indices, and $\alpha = 0, 1, \cdots, 15$, $\beta = 1, 2, \cdots, 8$, $\gamma = 9, 11, 13$, $\tau = 10, 12, 14$.
- Repeated indices are summed by default.

Metric

- Riemannian manifold is described by metric

$$g(x) = -g_{\mu\nu}(x)dx^{\mu} \otimes dx^{\nu}, g_{\mu\nu}(x) = g_{\nu\mu}(x), \det(g_{\mu\nu}(x)) \neq 0. \tag{2.1}$$

- Here the signature of 4-dimensional Riemannian manifold is

$$(-,-,-,-). \tag{2.2}$$

- The inverse metric is defined

$$g^{-1} = -g^{\mu\nu}(x)\partial_{\mu}\partial_{\nu}, \langle \partial_{\nu}, dx^{\mu} \rangle = \delta^{\mu}_{\nu}. \tag{2.3}$$

- And it can be described by orthonormal frame formalism as

$$g^{-1} = -I^{ab}\theta_a(x)\theta_b(x). \tag{2.4}$$

- Here $\theta_a(x) = \theta^{\mu}_a(x)\partial_{\mu}$ and describe gravitational field, $I^{ab} = \mathrm{diag}(1,1,1,1)$.

Square root metric

- The definition of gamma matrices is

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathrm{I}_{4\times 4}, \tag{2.5}$$

where $\eta^{ab} = \mathrm{diag}(1, -1, -1, -1)$.

- The hermiticity conditions for gamma matrices can be chosen

$$\gamma^a\gamma^{b\dagger} + \gamma^{b\dagger}\gamma^a = 2\mathrm{I}^{ab}\mathrm{I}_{4\times 4}. \tag{2.6}$$

Square root metric

- A new entity l is defined as follows

$$l(x) = i\gamma^0_{ik}\gamma^a_{kj}(x)e_j^\dagger \otimes e_i\theta_a(x), \tag{2.7}$$

where $\mathrm{tr}(e_j^\dagger \otimes e_i) = e_i e_j^\dagger = \delta_{ij}$.

- After using $\gamma^0\gamma^{a\dagger}\gamma^0 = \gamma^a$, we find that

$$g^{-1}(x) = \frac{1}{4}\mathrm{tr}[l(x)l(x)]. \tag{2.8}$$

- Hence l(x) is the square root metric (2.4) in some sense.

Square root metric

- The freedom in $\gamma^a_{ij}(x)$ can be showed as

$$\gamma^0_{ik}\gamma^{a\prime}_{kj}(x) = u^\dagger_{ik}(x)\gamma^0_{kl}\gamma^a_{lm}u_{mj}(x) = \bar{u}_i(x)\gamma^a u_j(x), \qquad (2.9)$$

  where $\bar{u}_i(x) = u^\dagger_i(x)\gamma^0$, $u \in U(4)$.
- So, we define

$$l(x) = i\bar{u}_i(x)\gamma^a u_j(x)e^\dagger_j \otimes e_i\theta_a(x). \qquad (2.10)$$

- Direct calculation show that the definition (2.10) satisfy (2.8) and $l^\dagger(x) = -l(x)$.

Connections and gauge field

- Coefficients of affine connections on coordinates, coefficients of spin connections on orthonormal frame and gauge fields on U(4)-bundle are defined as follows

$$\nabla_\mu \partial_\nu = \Gamma^\rho{}_{\nu\mu}(x)\partial_\rho, \tag{2.11a}$$

$$\nabla_\mu \theta_a(x) = \Gamma^b{}_{a\mu}(x)\theta_b(x), \tag{2.11b}$$

$$\nabla_\mu e_i^\dagger = A_{\mu ij}(x) e_j^\dagger. \tag{2.11c}$$

- $\Gamma^b{}_{a\mu}(x)\theta_b^\rho(x) = \partial_\mu \theta_a^\rho(x) + \theta_a^\nu(x)\Gamma^\rho{}_{\nu\mu}(x)$ is found and

$$A^*_{\mu ij}(x) = -A_{\mu ji}(x). \tag{2.12}$$

- The gauge field $A_{\mu ij}(x)$ can be decomposed to the sum of generators of U(4) group

$$A_{\mu ij}(x) = iA^\alpha_\mu(x)\mathcal{T}^\alpha_{ij}, \tag{2.13}$$

where $\alpha = 0, 1, 2, \cdots, 15$ and $A^{\alpha *}_\mu = A^\alpha_\mu$.

Lagrangian and equation

- A equation satisfy U(4) gauge invariant, locally O(4) invariant and generally covariant is constructed

$$\mathrm{tr}\nabla \mathrm{l}(\mathrm{x}) = 0. \tag{2.14}$$

- This equation describe a manifold parallel transporting itself.
- Eliminate index x, the explicit formula of equation (2.32) is

$$\begin{aligned}\Big[(\mathrm{i}\partial_\mu \bar{\mathrm{u}}_\mathrm{i} + \mathrm{A}_\mu^\alpha \mathcal{T}_{\mathrm{ij}}^\alpha \bar{\mathrm{u}}_\mathrm{j})\gamma^\mathrm{a}\mathrm{u}_\mathrm{i} + \bar{\mathrm{u}}_\mathrm{i}\gamma^\mathrm{a}(\mathrm{i}\partial_\mu \mathrm{u}_\mathrm{i} - \mathrm{u}_\mathrm{j}\mathrm{A}_\mu^\alpha \mathcal{T}_{\mathrm{ji}}^\alpha) \\ + \mathrm{i}\bar{\mathrm{u}}_\mathrm{i}\gamma^\mathrm{b}\mathrm{u}_\mathrm{i}\Gamma^\mathrm{a}{}_{\mathrm{b}\mu}\Big]\,\theta_\mathrm{a}^\mu = 0.\end{aligned} \tag{2.15}$$

- We define a Lagrangian

$$\mathcal{L} = \bar{\mathrm{u}}_\mathrm{i}\gamma^\mathrm{a}(\mathrm{i}\partial_\mu \mathrm{u}_\mathrm{i} - \mathrm{u}_\mathrm{j}\mathrm{A}_\mu^\alpha \mathcal{T}_{\mathrm{ji}}^\alpha)\theta_\mathrm{a}^\mu + \bar{\mathrm{u}}_\mathrm{i}\phi \mathrm{u}_\mathrm{i}, \tag{2.16}$$

$$\phi = \frac{\mathrm{i}}{2}\gamma^\mathrm{b}\Gamma^\mathrm{a}{}_{\mathrm{b}\mu}\theta_\mathrm{a}^\mu, \tag{2.17}$$

  and one find that Lagrangian (2.16) have relation with (2.32)

$$\mathrm{tr}\nabla \mathrm{l}(\mathrm{x}) = \mathcal{L} - \mathcal{L}^\dagger. \tag{2.18}$$

Lagrangian and equation

- If equation (2.32) being satisfied, Lagrangian (2.16) is Hermitian

$$\mathcal{L} = \mathcal{L}^{\dagger}. \tag{2.19}$$

- The equation of motion for Lagrangian (2.16) are

$$\gamma^{a}(\mathrm{i}\partial_{\mu}\mathrm{u}_{\mathrm{i}} - \mathrm{u}_{\mathrm{j}}\mathrm{A}_{\mu}^{\alpha}\mathcal{T}_{\mathrm{ji}}^{\alpha})\theta_{a}^{\mu} + \frac{\mathrm{i}}{2}\gamma^{a}\mathrm{u}_{\mathrm{i}}\Gamma^{\mathrm{b}}{}_{a\mu}\theta_{\mathrm{b}}^{\mu} = 0, \tag{2.20}$$

  and this equation's conjugate transpose. We call equation (2.21) fundamental equation.

- The effective equation of motion for this theory has signature $(1, -1, -1, -1)$

$$\left(\gamma^{a}(\mathrm{i}\partial_{\mu} - \mathrm{A}_{\mu}^{\alpha}\mathcal{T}^{\alpha})\theta_{a}^{\mu} + \frac{\mathrm{i}}{2}\gamma^{a}\Gamma^{\mathrm{b}}{}_{a\mu}\theta_{\mathrm{b}}^{\mu}\right)^{2}\mathrm{u} = 0. \tag{2.21}$$

Curvature, gauge field strength tensor and identity

- Curvature tensor and gauge field strength tensor are defined as follows

$$R^{a}{}_{b\mu\nu} = \partial_\mu \Gamma^{a}{}_{b\nu} - \partial_\nu \Gamma^{a}{}_{b\mu} + \Gamma^{c}{}_{b\nu}\Gamma^{a}{}_{c\mu} - \Gamma^{c}{}_{b\mu}\Gamma^{a}{}_{c\nu}, \tag{2.22a}$$

$$F_{\mu\nu ij} = \partial_\mu A_{\nu ij} - \partial_\nu A_{\mu ij} + A_{\nu ik}A_{\mu kj} - A_{\mu ik}A_{\nu kj}, \tag{2.22b}$$

where $R_{ab\mu\nu} = -R_{ba\mu\nu}$ if $\nabla g = 0$ and $F^{*}_{\mu\nu ij} = -F_{\mu\nu ji}$.

- Gauge field strength can be decomposed by $U(4)$ generators $F_{\mu\nu ij} = iF^{\alpha}_{\mu\nu}\mathcal{T}^{\alpha}_{ij}$. Define torsion $T^{a}{}_{\nu\rho} = 2\Gamma^{a}{}_{[\nu\rho]}$, we have Ricci identity and Bianchi identity on this geometry structure as follows

$$R^{a}{}_{[\rho\mu\nu]} + \nabla_{[\rho}T^{a}{}_{\mu\nu]} = T^{a}{}_{\sigma[\rho}T^{\sigma}{}_{\mu\nu]}, \tag{2.23a}$$

$$\nabla_{[\rho}R^{a}{}_{|b|\mu\nu]} = R^{a}{}_{b\sigma[\rho}T^{\sigma}{}_{\mu\nu]}, \tag{2.23b}$$

$$\partial_{[\mu}F_{\nu\rho]ij} = A_{[\mu|ik|}F_{\nu\rho]kj} - F_{[\mu\nu|ik|}A_{\rho]kj}. \tag{2.23c}$$

Lagrangian of Gravity

- For gravity, Einstein-Hilbert action be showed as follows

$$S = \int R\omega, \tag{2.24}$$

  where R is Ricci scalar curvature, $\omega = \sqrt{-g}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is volume form. The variation of action give us Einstein tensor.
- For Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}, \tag{2.25}$$

- Einstein say: "The reason for the formalism of left hand is to let its divergence identically zero in the meaning of covariant derivative. The right hand of equation are the sum up of all the things still problems in the meaning of field theory."

Lagrangian of Gravity

- And in this geometry framework, the equation can be derived as follows

$$\nabla_{[\mu}\nabla_{\nu]}\mathrm{l} = \frac{\mathrm{i}}{2}\left(\bar{\mathrm{u}}_{\mathrm{i}}\gamma^{\mathrm{a}}\mathrm{u}_{\mathrm{k}}\mathrm{F}_{\mu\nu\mathrm{kj}} + \mathrm{F}^{*}_{\mu\nu\mathrm{ki}}\bar{\mathrm{u}}_{\mathrm{k}}\gamma^{\mathrm{a}}\mathrm{u}_{\mathrm{j}} + \bar{\mathrm{u}}_{\mathrm{i}}\gamma^{\mathrm{b}}\mathrm{u}_{\mathrm{j}}\mathrm{R}^{\mathrm{a}}{}_{\mathrm{b}\mu\nu}\right)\mathrm{e}^{\dagger}_{\mathrm{j}}\otimes\mathrm{e}_{\mathrm{i}}\theta_{\mathrm{a}}. \tag{2.26}$$

  Define $\nabla^2 = \nabla_{[\mu}\nabla_{\nu]}\mathrm{dx}^{\mu}\wedge\mathrm{dx}^{\nu}$, the equation might describe gravity is constructed

$$\mathrm{tr}\nabla^2\mathrm{l}^2 = 0. \tag{2.27}$$

- This equation (2.27) is obviously U(4) gauge invariant, locally O(4) invariant and generally covariant. The explicit formula of equation (2.27) is

$$\mathrm{R} = \frac{1}{2}\mathrm{F}_{\mathrm{abij}}\bar{\mathrm{u}}_{\mathrm{j}}\gamma^{\mathrm{a}}\gamma^{0}\gamma^{\mathrm{b}}\mathrm{u}_{\mathrm{i}}. \tag{2.28}$$

  Here $\partial_{\mu}\mathrm{dx}^{\nu}\otimes\mathrm{dx}^{\rho}\partial_{\sigma} = \delta^{\nu}_{\mu}\delta^{\rho}_{\sigma}$, $\mathrm{dx}^{\mu}\otimes\mathrm{dx}^{\nu}\partial_{\rho}\partial_{\sigma} = \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma}$ are used and $\mathrm{F}_{\mathrm{ab}} = \mathrm{F}_{\mu\nu}\theta^{\mu}_{\mathrm{a}}\theta^{\nu}_{\mathrm{b}}$.

Lagrangian of Gravity

- It is easy to prove each term in (2.28) is Hermitian. So we define a Hermitian Lagrangian

$$\mathcal{L}_g = R\bar{u}_i\gamma^0 u_i - 2F_{abij}\bar{u}_j\gamma^a\gamma^0\gamma^b u_i. \tag{2.29}$$

  The $R\bar{u}_i\gamma^0 u_i$ is Einstein-Hilbert action.
- This Lagrangian can derive (2.28) and Einstein equation.
- Lagrangians (2.16) and (2.29) are U(4) gauge invariant, locally O(4) invariant and generally covariant.

Sheaf quantization

- To avoid the problem of infinity in quantum field theory, we suggest the Sheaf quantization method. We define the entity is Sheaf valued, then

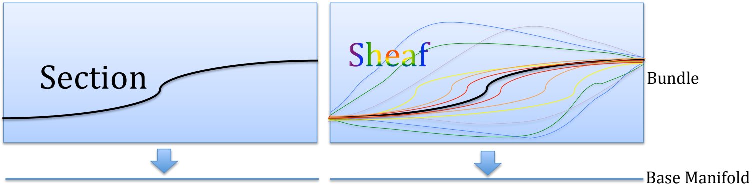
$$\hat{\mathrm{l}}(\mathrm{x}) = \eta_\kappa \mathrm{l}_\kappa(\mathrm{x}), \tag{2.30}$$

where $\eta_\kappa \in [0, 1]$. κ is evaluated in an abelian group and we have possibility complete formula

$$\sum_\kappa \eta_\kappa = 1. \tag{2.31}$$

Sheaf quantization

- Sheaf is a collection of sections, the index of each section κ correspond to a abelian group element.



- For historical reasons, this process is called second quantization because this process switch us from single particle free motion to multi-particles interaction in sometimes.
- The equations of motion after second quantization are

$$\mathrm{tr}\nabla\hat{1} = 0, \tag{2.32}$$

$$\mathrm{tr}\nabla^2\hat{1}^2 = 0. \tag{2.33}$$

Gauge bosons

- A_μ are U(4) gauge fields and can be expanded by generators of U(4).

$$A_\mu = iA_\mu^\alpha \mathcal{T}^\alpha. \tag{3.1}$$

- $\mathcal{T}_0$ correspond to a new particle -- Fiona(芳). Fiona is similar with dark photon.
- SU(4) being decomposed as follows according to the weight diagram Fig. 4.

$$SU(4) = SU(3) \oplus U(1) + U_X + U_{\overline{X}}. \tag{3.2}$$

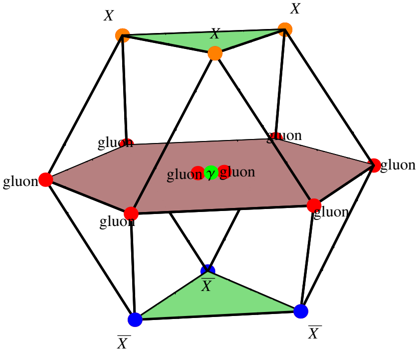


Fig 4: Weight diagram of SU(4) adjoint representation and corresponding gauge bosons.

Electrodynamics, Chromodynamics and Weak interactions

- Electrodynamics like and Chromodynamics like theory appear in U(1) part and SU(3) part of this geometry framework in flat space-time naturally.
- X and $\bar{\mathrm{X}}$ are new particles and we can use it construct 3 color-singlet, composite particles and have nonzero mass. Those 3 particles correspond to $\mathrm{W}^{\pm}$ and Z. The explicit formalism of $\mathrm{W}^{\pm}$ and Z are exhibited as follows

$$\mathrm{W}_{\mu}^{\pm} = \frac{1}{3!}\epsilon_{\mu}{}^{\nu\rho\sigma}\mathrm{A}_{\nu 9}\mathrm{A}_{\rho 11}\mathrm{A}_{\sigma 13}\mathcal{T}_{[9}^{\pm}\otimes\mathcal{T}_{11}^{\pm}\otimes\mathcal{T}_{13]}^{\pm}, \tag{3.3a}$$

$$\partial_{[\mu}\mathrm{Z}_{\nu]} = \mathrm{A}_{\mu\gamma}\mathrm{A}_{\nu(\gamma+1)}\mathcal{T}_{[\gamma}\otimes\mathcal{T}_{\gamma+1]}. \tag{3.3b}$$

- Where $\mathcal{T}_{\gamma}^{\pm} = \mathcal{T}_{\gamma} \pm \mathrm{i}\mathcal{T}_{\gamma+1}$ and $\mathrm{A}_{\mu 9}\mathrm{A}_{\nu 10} = \mathrm{A}_{\mu 11}\mathrm{A}_{\nu 12} = \mathrm{A}_{\mu 13}\mathrm{A}_{\nu 14}$ in (3.3b).

3-boson vertexes

- In flat space-time, the 3-boson vertexes and 4-boson vertexes determined by SU(4) structure constants. The 3-boson vertexes of this geometry framework in flat space-time are showed on Fig. 5.

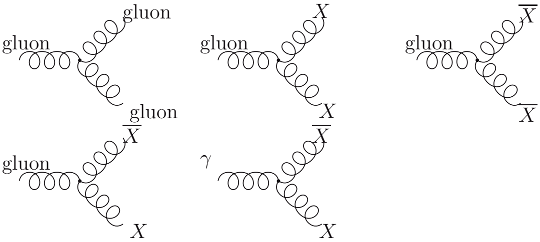


Fig 5: 3-boson vertexes on flat spacetime.

Fermions

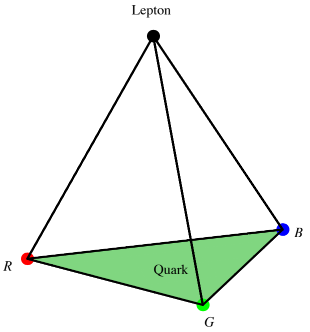
- Fermionic like fields u transfer as U(4) fundamental representation. So, fermions are filled into SU(4) fundamental representation naturally as Table 1.

Table 1: Fermions are filled into SU(4) fundamental representation $\mathbf{4} \otimes \mathbf{6}$.

SU(4)			**6**					
4	Quarks	R G B	u	c	t	d	s	b
	Leptons		e	μ	τ	ν_e	ν_μ	ν_τ

- Antifermions are filled into $\bar{\mathbf{4}} \otimes \mathbf{6}$ similarly.

Fermions



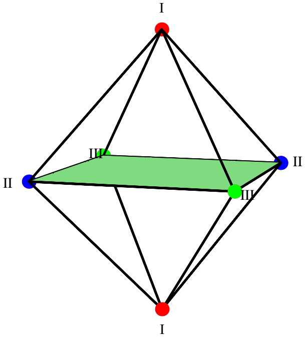


Fig 6: Weight diagram of SU(4) fundamental representation **4**.

Fig 7: Weight diagram of SU(4) fundamental representation **6**.

Conclusion

- This theory unify 64 “entities” into a single “entity”, square root metric.
- The interactions between those fields might be determined by these two Lagrangians

$$\mathrm{tr}\nabla l = 0 \quad \Leftrightarrow \quad \begin{cases} \mathcal{L} = \bar{u}_i \gamma^a (i\partial_\mu u_i - u_j A_\mu^\alpha \mathcal{T}_{ji}^\alpha)\theta_a^\mu + \bar{u}_i \phi u_i, \\ \phi = \frac{i}{2}\gamma^b \Gamma^a{}_{b\mu}\theta_a^\mu, \end{cases}$$

$$\mathrm{tr}\nabla^2 l^2 = 0 \quad \Leftrightarrow \quad \mathcal{L}_g = R\bar{u}_i\gamma^0 u_i - 2F_{abij}\bar{u}_j\gamma^a\gamma^0\gamma^b u_i.$$

New physics on this UFT

- Weak bosons: $W^{\pm}$ and Z are composite particles.
- Higgs: Originated from gravity fields.
- Neutrinos: Right handed neutrinos should be existed.
- Fiona(芳): Fiona(芳) almost not interacts with electro-magnetism, weak and strong, but has important gravity effect for our universe.
- Space-time: Signature of space-time might be $(-,-,-,-)$.

Thanks!