Perturbation Problem and Adaptive Method

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arxiv: 1911.08211 [quant-ph]

November 23, 2019

Perturbation Problem

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- The most interesting problem in quantum field theory is QCD at the low-energy limit. Two main problems related to the low-energy regime are 1:Color Confinement; 2: Critical Points.
- This theory only can be studied from the standard perturbation method in the high-energy regime. Therefore, we suffer from the strong coupling problem in the above problems.

Zero-Dimensional ϕ^4 Model

The partition function is

$$Z(g) \equiv \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}}\, e^{-\frac{\phi^2}{2}-g\frac{\phi^4}{24}} = \sqrt{\frac{3}{2\pi g}}\, e^{\frac{3}{4g}} K_{\frac{1}{4}}\left(\frac{3}{4g}\right), \tag{1}$$

where $K_\alpha(x)$ is the modified Bell function.

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Decomposition

We demonstrate the unperturbed order of the adaptive perturbation method from the Hamiltonian

$$H_1 = \frac{p^2}{2} + \frac{\lambda_1}{6}x^4 + \frac{\lambda_2}{120}x^6, \tag{4}$$

where p and x are the momentum and position operators, and λ_1 and λ_2 are coupling constants. The p and x satisfy the usual commutation relation

$$[p, x] = -i. \tag{5}$$

Now we introduce the $A_\gamma^\dagger$ and A_γ as that:

$$x = \frac{1}{\sqrt{2\gamma}}(A_\gamma^\dagger + A_\gamma), \qquad p = i\sqrt{\frac{\gamma}{2}}(A_\gamma^\dagger - A_\gamma). \tag{6}$$

The commutation relation between A_γ and $A_\gamma^\dagger$ is

$$[A_\gamma, A_\gamma^\dagger] = 1. \tag{7}$$

The operators acting on a quantum state gives that:

$$N_\gamma|n_\gamma\rangle = n_\gamma|n_\gamma\rangle; \qquad A_\gamma|0_\gamma\rangle = 0, \tag{8}$$

where

$$N_\gamma \equiv A_\gamma^\dagger A_\gamma. \tag{9}$$

We first decompose the Hamiltonian to a solvable part and a perturbation part. The solvable part contains the diagonal elements of the Fock space from the interacting term. In other words, the diagonal elements in the solvable part of the Hamiltonian $H_0(\gamma)$ can be written in terms of the N_γ, which is

$$\frac{\gamma}{4}(2N_\gamma + 1) + \frac{\lambda_1}{4\gamma^2}\left(N_\gamma^2 + N_\gamma + \frac{1}{2}\right) + \frac{\lambda_2}{4\gamma^3}\left(\frac{1}{12}N_\gamma^3 + \frac{29}{240}N_\gamma^2 + \frac{1}{6}N_\gamma + \frac{1}{16}\right). \qquad (10)$$

The expectation value of the energy is:

$$\begin{aligned} & E_n(\gamma) \\ \equiv\ & \langle n_\gamma | H_1(\gamma) | n_\gamma \rangle \\ =\ & \langle n_\gamma | H_0(\gamma) | n_\gamma \rangle \\ =\ & \frac{\gamma}{4}(2n_\gamma + 1) + \frac{\lambda_1}{4\gamma^2}\left(n_\gamma^2 + n_\gamma + \frac{1}{2}\right) \\ & + \frac{\lambda_2}{4\gamma^3}\left(\frac{1}{12}n_\gamma^3 + \frac{29}{240}n_\gamma^2 + \frac{1}{6}n_\gamma + \frac{1}{16}\right). \end{aligned} \tag{11}$$

We still have one undetermined variable γ. To fix this variable, we choose the minimized expectation value of the energy to determine. The minimized expectation value of the energy occurs when $\gamma > 0$ satisfies

$$\gamma^4 - 2\lambda_1 \frac{n_\gamma^2 + n_\gamma + \frac{1}{2}}{2n_\gamma + 1}\gamma - 3\lambda_2 \frac{\frac{1}{12}n_\gamma^3 + \frac{29}{240}n_\gamma^2 + \frac{1}{6}n_\gamma + \frac{1}{16}}{2n_\gamma + 1} = 0. \quad (12)$$

Then we choose the minimized expectation value of the energy as the unperturbed spectrum.

When $\lambda_2 = 0$, the minimized energy is

$$E_n(\gamma)_{\min} = \frac{3}{8}\lambda_1^{\frac{1}{3}}(2n_\gamma + 1)^{\frac{2}{3}}\left(2n_\gamma^2 + 2n_\gamma + 1\right)^{\frac{1}{3}}. \quad (13)$$

n	$E_n(\gamma)_{\min}$	**Numerical Solution**
0	1.117	1.074
1	4.047	3.941
2	7.993	7.963
3	12.724	12.764
4	18.109	18.203
5	24.067	24.189
6	30.54	30.657
7	37.486	37.555

Table: The comparison between the $E_n(\gamma)_{\min}$ and the numerical solutions for the $\lambda_1 = 16$ and $\lambda_2 = 256$.

n	$E_n(\gamma)_{\min}$	**Numerical Solution**
0	0.343	0.326
1	1.258	1.218
2	2.512	2.504
3	4.039	4.072
4	5.795	5.87
5	7.753	7.869
6	9.892	10.048
7	12.197	12.391

Table: The comparison between the $E_n(\gamma)_{\min}$ and the numerical solutions for the $\lambda_1 = 0.25$ and $\lambda_2 = 4$.

Reference

- M. Weinstein, "Adaptive perturbation theory. I. Quantum mechanics," hep-th/0510159.

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- The eigenenergy calculated by the time-independent perturbation is

$$E_n = E_n^{(0)} + \lambda\langle n^{(0)}|V|n^{(0)}\rangle + \lambda^2 \sum_{k\neq n} \frac{|\langle k^{(0)}|V|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} + \cdots, \quad (14)$$

where E_n^0 is the n-th unperturbed eigenenergy, $|n^{(0)}\rangle$ is the n-th unperturbed eigenstate, λ is the coupling constant.

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where E_n^0 is the n-th unperturbed eigenenergy, $|n^{(0)}\rangle$ is the n-th unperturbed eigenstate, λ is the coupling constant.
- When we use the adaptive time-independent perturbation, the V is defined by $H_1 \equiv H_0 + \lambda_1 V/(24\gamma^2)$, and the $E_n^{(0)}$ is defined by the $E_n(\gamma)_{\min}$, and $\lambda \equiv \lambda_1/(24\gamma^2)$. Then we can show that each term is at the same order of the coupling constant $\lambda_1^{1/3}$.

- This is not a bad result and gives the consistency to the spectrum because we can do the transformations, $x \to x/\lambda_1^{1/6}$ and $p \to \lambda_1^{1/6} p$, to show that the H_1 or its spectrum must be proportional to $\lambda_1^{1/3}$.

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- In the final, we also find that $|\langle k^{(0)}|V|n^{(0)}\rangle|^2$ contributes n_γ^2 and $E_n^{(0)} - E_k^{(0)}$ also contribute so when a quantum number is large enough. Hence no divergence comes from a summation of the quantum numbers.

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- Indeed, it is also due to using the $E_n(\gamma)_{\mathrm{min}}$ because it includes information about the coupling constant. When we include the mass term in the standard time-independent perturbation, the unperturbed part is the harmonic oscillator. The unperturbed eigenenergy is proportional to a quantum number. Hence the divergence must appear when the quantum number becomes large.

- When we go to the higher-order of the time-independent perturbation, we can find more multiplications of the $E_n^{(0)} - E_k^{(0)}$. Even for the unperturbed ground-state, the calculation of the higher-order term will be suppressed by the multiplications of the $E_n^{(0)} - E_k^{(0)}$.

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- We also discuss why the adaptive perturbation method is better than before from the explicit Hamiltonian H_1 with the $\lambda_2 = 0$.
- One interesting application of quantum mechanics is to observe whether the spectrum can have a universal rule when the phase transition occurs.
- Now we only focus on checking the bosonic quantum mechanics, but the perturbation problem and theoretical formulation should be similar in bosonic quantum mechanics and quantum field theory.