

# Flavor Invariants and CP Violation in the Leptonic Sector with Majorana Neutrinos

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2021/8/17



Talk based on the following works:

**Phys.Lett.B 800 (2020) 135085, Phys.Rev.D 103 (2021) 3, 035017, arXiv:2107.06274,**  
**arXiv:2107.11928**

In collaboration with **Yilin Wang** and Prof. **Shun Zhou**

# Outline

- ① Motivation
- ② Invariant Theory
- ③ Basic Invariants in Seesaw Model
- ④ Flavor Invariants and CP Violation in Leptonic Sector
- ⑤ Summary

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1 Motivation

2 Invariant Theory

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5 Summary

# Flavor Transformation

The effective Lagrangian in the leptonic sector of the SM extended with Majorana neutrinos reads

$$\mathcal{L}_{\text{lepton}} = -\overline{l}_L M_l l_R - \frac{1}{2} \overline{\nu}_L M_\nu \nu_L^C + \frac{g}{\sqrt{2}} \overline{l}_L \gamma^\mu \nu_L W_\mu^- + \text{h.c.} . \quad (1)$$

with  $\nu_L^C \equiv \mathcal{C} \overline{\nu}_L^T$  and  $\mathcal{C} \equiv i\gamma^2\gamma^0$ .

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with  $\nu_L^C \equiv \mathcal{C} \bar{\nu}_L^T$  and  $\mathcal{C} \equiv i\gamma^2\gamma^0$ . The Lagrangian in Eq. (1) is *unchanged* under the unitary transformation in the flavor space

$$l_L \rightarrow l'_L = U_L l_L , \quad \nu_L \rightarrow \nu'_L = U_L \nu_L , \quad l_R \rightarrow l'_R = U_R l_R , \quad (2)$$

if the lepton mass matrices transform as

$$M_l \rightarrow M'_l = U_L M_l U_R^\dagger , \quad M_\nu \rightarrow M'_\nu = U_L M_\nu U_L^T , \quad (3)$$

where  $U_L$  and  $U_R$  are arbitrary  $3 \times 3$  unitary matrices.

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- ☞ Values of lepton mass matrices  $M_l$  and  $M_\nu$  *do* change under the flavor transformation.
- ☞ Physical observables such as the lepton masses, should *not* depend on the chosen basis (**flavor invariants**).
- ☞ Defining

$$H_l \equiv M_l M_l^\dagger, \quad H_\nu \equiv M_\nu M_\nu^\dagger, \quad G_{l\nu} \equiv M_\nu H_l^* M_\nu^\dagger,$$
$$G_{l\nu}^{(n)} \equiv M_\nu (H_l^*)^n M_\nu^\dagger \text{ (} n \geq 2 \text{ integers) ,}$$

it is straightforward to construct a set of flavor invariants

$$\mathcal{I}_{rstu\cdots}^{abcd\cdots} \equiv \text{Tr} \left\{ H_l^a H_\nu^b G_{l\nu}^c \left[ G_{l\nu}^{(n)} \right]^d H_l^r H_\nu^s G_{l\nu}^t \left[ G_{l\nu}^{(n')} \right]^u \cdots \right\}, \quad (4)$$

with the power indices  $\{a, b, c, d, r, s, t, u\}$  all non-negative integers.

## An Example: Jarlskog Invariant

A well-known example is the Jarlskog invariant,\* which is unchanged under the flavor transformation

$$\mathcal{I} \equiv \text{Det} ([H_l, H_\nu]) \rightarrow \mathcal{I}' = \text{Det} \left( \left[ U_L H_l U_L^\dagger, U_L H_\nu U_L^\dagger \right] \right) = \mathcal{I}.$$

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Substituting into  $\mathcal{I}$  the standard parametrization of PMNS matrix

$$V_{\text{PMNS}} = \begin{pmatrix} c_{13} c_{12} & c_{13} s_{12} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{13} s_{23} e^{i\delta} & +c_{12} c_{23} - s_{12} s_{13} s_{23} e^{i\delta} & c_{13} s_{23} \\ +s_{12} s_{23} - c_{12} s_{13} c_{23} e^{i\delta} & -c_{12} s_{23} - s_{12} s_{13} c_{23} e^{i\delta} & c_{13} c_{23} \end{pmatrix} \cdot \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

one obtains the Jarlskog invariant in terms of physical observables

$$\mathcal{I} = 2i\Delta_{12}\Delta_{23}\Delta_{31}\Delta_{e\mu}\Delta_{\mu\tau}\Delta_{\tau e}s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2\sin\delta,$$

with  $\Delta_{ij} \equiv m_i^2 - m_j^2$  ( $i, j = 1, 2, 3$ ) and  $\Delta_{\alpha\beta} \equiv m_\alpha^2 - m_\beta^2$  ( $\alpha, \beta = e, \mu, \tau$ ).

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$$I' = P(I_1, I_2, \dots, I_m) . \quad (5)$$

Furthermore, how to determine the number  $m$  and how to explicitly construct all the basic invariants?

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- b) How are the flavor invariants related to physical observables (e.g., lepton masses, flavor mixing angles, CP-violating phases), and given basic invariants, how to extract physical observables from them?

The above two questions can be answered using the mathematical tools in the **invariant theory**, which is a branch of the computational algebraic geometry.

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# Invariant Theory

Given a theory with  $n$  parameters  $\vec{x} = (x_1, \dots, x_n)$  and a symmetry group  $G$ , we are interested in those quantities that are invariant under the group action, i.e.,

$$I(\vec{x}) = I(R(g)\vec{x}) , \quad \forall g \in G , \tag{6}$$

where  $R$  is a representation of  $G$  and  $I(\vec{x})$  is a polynomial function of  $\vec{x}$ .

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Since all the invariants are closed under the addition and multiplication, they form a **ring**. The ring is finitely generated if all invariants in the ring can be expressed as the polynomials of a finite number of basic invariants.

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Since all the invariants are closed under the addition and multiplication, they form a **ring**. The ring is finitely generated if all invariants in the ring can be expressed as the polynomials of a finite number of basic invariants.

## Theorem

*If  $G$  is a reductive group, then the invariant ring is finitely generated.<sup>abc</sup>*

<sup>a</sup>Bernd Sturmfels. *Algorithms in invariant theory*. Springer Science & Business Media, 2008.

<sup>b</sup>Harm Derksen and Gregor Kemper. *Computational invariant theory*. Springer, 2015.

<sup>c</sup>All the finite groups and semi-simple Lie groups are reductive.

# Hilbert Series I

- In order to find the number and degrees of all the basic invariants, one can use the tool called Hilbert series (HS), which serves as the generating function of the invariants

$$\mathcal{H}(q_1, \dots, q_n) \equiv \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} c_{k_1 \dots k_n} q_1^{k_1} \dots q_n^{k_n}, \quad (7)$$

where  $q_i$  (for  $i = 1, 2, \dots, n$ ) label the degree of the  $i$ -th building block and satisfy  $|q_i| < 1$ , while  $c_{k_1 \dots k_n}$  (with  $c_{0 \dots 0} \equiv 1$ ) denote the number of linearly-independent invariants when the  $n$  building blocks are at the degree of  $(k_1, \dots, k_n)$ , respectively.

## Hilbert Series II

- Given the HS, one can calculate its plethystic logarithm (PL), which counts the number and degrees of the basic invariants

$$\text{PL} [\mathcal{H}(q_1, \dots, q_n)] \equiv \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \ln \left[ \mathcal{H}(q_1^k, \dots, q_n^k) \right], \quad (8)$$

where  $\mu(k)$  is the Möbius function. The key observation of PL is<sup>†</sup>

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## Observation

The leading positive terms of PL correspond to the basic invariants while the leading negative terms of PL correspond to the polynomial relations among these basic invariants (syzygies).

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# Hilbert Series III

- Therefore, once obtaining the HS, one can conveniently read off the number and degrees of basic invariants from its PL. However, computing HS from the definition is usually very difficult, sometimes even impossible, since the number of linear-independent invariants grows very quick with the degree.

- ☞ Therefore, once obtaining the HS, one can conveniently read off the number and degrees of basic invariants from its PL. However, computing HS from the definition is usually very difficult, sometimes even impossible, since the number of linear-independent invariants grows very quick with the degree.
- ☞ A systematic method to calculate the HS is to use the **Molien-Weyl formula**, which reduces the calculation of HS to several complex integrals and can be accomplished via the residue theorem.

# Hilbert Series IV

## Theorem (Molien-Weyle formula)

Given a symmetric group  $G$  with rank  $r_0$  and  $n$  independent building blocks, the Hilbert series is given by<sup>a</sup><sup>b</sup>

$$\mathcal{H}(q_1, \dots, q_n) = \int [d\mu]_G \text{PE}(z_1, \dots, z_{r_0}; q_1, \dots, q_n) , \quad (9)$$

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<sup>a</sup>T. Molien. "Über die Invarianten der linearen Substitutionsgruppe". In: *Sitzungber. König. Preuss. Akad. Wiss. (J. Berl. Ber.)*. 52 (1897), pp. 1152–1156.

<sup>b</sup>Hermann Weyl. "Zur Darstellungstheorie und Invariantenabzählung der projektiven, der Komplex-und der Drehungsgruppe". In: *Acta Mathematica* 48.3-4 (1926), pp. 255–278.

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where  $[d\mu]_G$  denotes the Haar measure of the symmetry group  $G$  and the integrand is the plethystic exponential (PE) defined as

$$\text{PE}(z_1, \dots, z_{r_0}; q_1, \dots, q_n) \equiv \exp \left[ \sum_{k=1}^{\infty} \sum_{i=1}^n \frac{\chi_{R_i}(z_1^k, \dots, z_{r_0}^k) q_i^k}{k} \right] , \quad (10)$$

with  $z_i$  (for  $i = 1, 2, \dots, r_0$ ) being the coordinates on the maximum torus of  $G$  and  $\chi_{R_i}$  being the character function of  $G$  in the representation  $R_i$ .

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# Hilbert Series in Effective Theory

- If the RH Majorana neutrinos are much heavier than electroweak scale, we can integrate out them to obtain the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\overline{l}_L M_l l_R - \frac{1}{2} \overline{\nu}_L M_\nu \nu_L^C + \frac{g}{\sqrt{2}} \overline{l}_L \gamma^\mu \nu_L W_\mu^- + \text{h.c.} . \quad (11)$$

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- The representations of the building blocks under U(3) transformation

$$H_l \equiv M_l M_l^\dagger : \mathbf{3} \otimes \mathbf{3}^* , \quad M_\nu : (\mathbf{3} \otimes \mathbf{3})_s , \quad M_\nu^\dagger : (\mathbf{3}^* \otimes \mathbf{3}^*)_s . \quad (12)$$

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- The character functions of the building blocks

$$\begin{aligned} \chi_l(z_1, z_2, z_3) &= (z_1 + z_2 + z_3)(z_1^{-1} + z_2^{-1} + z_3^{-1}) , \\ \chi_\nu(z_1, z_2, z_3) &= z_1^2 + z_2^2 + z_3^2 + z_1 z_2 + z_1 z_3 + z_2 z_3 + z_1^{-2} + z_2^{-2} \\ &\quad + z_3^{-2} + z_1^{-1} z_2^{-1} + z_1^{-1} z_3^{-1} + z_2^{-1} z_3^{-1} . \end{aligned} \quad (13)$$

# Hilbert Series in Effective Theory

From the character functions, it is straightforward to calculate the PE

$$\begin{aligned} & \text{PE}(z_1, z_2, z_3; q_l^2, q_\nu) \\ &= \left[ (1 - q_l^2)^3 (1 - q_l^2 z_2 z_1^{-1}) (1 - q_l^2 z_1 z_2^{-1}) (1 - q_l^2 z_3 z_1^{-1}) \right. \\ &\quad \times (1 - q_l^2 z_1 z_3^{-1}) (1 - q_l^2 z_2 z_3^{-1}) (1 - q_l^2 z_3 z_2^{-1}) (1 - q_\nu z_1^2) \\ &\quad \times (1 - q_\nu z_2^2) (1 - q_\nu z_3^2) (1 - q_\nu z_1 z_2) (1 - q_\nu z_1 z_3) (1 - q_\nu z_2 z_3) \\ &\quad \times (1 - q_\nu z_1^{-2}) (1 - q_\nu z_2^{-2}) (1 - q_\nu z_3^{-2}) (1 - q_\nu z_1^{-1} z_2^{-1}) \\ &\quad \left. \times (1 - q_\nu z_1^{-1} z_3^{-1}) (1 - q_\nu z_2^{-1} z_3^{-1}) \right]^{-1}. \end{aligned} \tag{14}$$

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The Haar measure of  $\text{U}(3)$  group reads

$$\begin{aligned} \int [d\mu]_{\text{U}(3)} &= \frac{1}{6(2\pi i)^3} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} \oint_{|z_3|=1} \frac{dz_3}{z_3} \\ &\quad \times \left[ -\frac{(z_2 - z_1)^2 (z_3 - z_1)^2 (z_3 - z_2)^2}{z_1^2 z_2^2 z_3^2} \right]. \end{aligned} \quad (15)$$

# Hilbert Series in Effective Theory

- Finally, substituting the Haar measure and PE into Molien-Weyl formula and performing complex integrals, we obtain the HS<sup>‡</sup>

$$\mathcal{H}(q_l, q_\nu) = \int [d\mu]_{U(3)} \text{PE}(z_1, z_2, z_3; q_l^2, q_\nu) = \frac{\mathcal{N}(q_l, q_\nu)}{\mathcal{D}(q_l, q_\nu)}, \quad (16)$$

where

$$\begin{aligned} \mathcal{N}(q_l, q_\nu) = & -q_l^{24}q_\nu^{18} - 2q_l^{20}q_\nu^{14} - 2q_l^{20}q_\nu^{12} - q_l^{20}q_\nu^{10} - 2q_l^{18}q_\nu^{14} \\ & - 3q_l^{18}q_\nu^{12} - q_l^{18}q_\nu^{10} - 3q_l^{16}q_\nu^{14} - 3q_l^{16}q_\nu^{12} - 3q_l^{16}q_\nu^{10} \\ & - q_l^{16}q_\nu^8 - q_l^{16}q_\nu^6 - q_l^{14}q_\nu^{14} - q_l^{14}q_\nu^{12} - q_l^{14}q_\nu^{10} - 2q_l^{14}q_\nu^8 \\ & - q_l^{14}q_\nu^6 - q_l^{12}q_\nu^{14} + q_l^{12}q_\nu^4 + q_l^{10}q_\nu^{12} + 2q_l^{10}q_\nu^{10} + q_l^{10}q_\nu^8 \\ & + q_l^{10}q_\nu^6 + q_l^{10}q_\nu^4 + q_l^8q_\nu^{12} + q_l^8q_\nu^{10} + 3q_l^8q_\nu^8 + 3q_l^8q_\nu^6 \\ & + 3q_l^8q_\nu^4 + q_l^6q_\nu^8 + 3q_l^6q_\nu^6 + 2q_l^6q_\nu^4 + q_l^4q_\nu^8 + 2q_l^4q_\nu^6 \\ & + 2q_l^4q_\nu^4 + 1, \end{aligned}$$

<sup>‡</sup>Elizabeth Ellen Jenkins and Aneesh V. Manohar. "Algebraic Structure of Lepton and Quark Flavor Invariants and CP Violation". In: *JHEP* 10 (2009), p. 094.

# Hilbert Series in Effective Theory

$$\begin{aligned}\mathcal{D}(q_l, q_\nu) = & (1 - q_l^2)(1 - q_l^4)(1 - q_l^6)(1 - q_\nu^2)(1 - q_\nu^4)(1 - q_\nu^6) \\ & \times (1 - q_l^2 q_\nu^2)(1 - q_l^4 q_\nu^2)^2 (1 - q_l^2 q_\nu^4)(1 - q_l^6 q_\nu^2) \\ & \times (1 - q_l^4 q_\nu^4)(1 - q_l^8 q_\nu^2).\end{aligned}$$

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§Yilin Wang, Bingrong Yu, and Shun Zhou. “Flavor Invariants and Renormalization-group Equations in the Leptonic Sector with Massive Majorana Neutrinos”. In: (July 2021). arXiv: 2107.06274 [hep-ph].

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From the HS, one can calculate its PL by definition<sup>§</sup>

$$\begin{aligned}\text{PL}[\mathcal{H}(q_l, q_\nu)] = & q_l^2 + q_\nu^2 + q_l^4 + q_l^2 q_\nu^2 + q_\nu^4 + q_l^6 + 2q_l^4 q_\nu^2 + q_l^2 q_\nu^4 \\ & + q_\nu^6 + q_l^6 q_\nu^2 + 3q_l^4 q_\nu^4 + q_l^8 q_\nu^2 + 2q_l^6 q_\nu^4 + 2q_l^4 q_\nu^6 \\ & + 3q_l^8 q_\nu^4 + 3q_l^6 q_\nu^6 + q_l^4 q_\nu^8 + q_l^{10} q_\nu^4 + 3q_l^8 q_\nu^6 + q_l^6 q_\nu^8 \\ & + q_l^{12} q_\nu^4 + q_l^{10} q_\nu^6 - \mathcal{O}([q_l q_\nu]^{18}).\end{aligned}\tag{17}$$

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$$\begin{aligned}\mathcal{D}(q_l, q_\nu) = & (1 - q_l^2)(1 - q_l^4)(1 - q_l^6)(1 - q_\nu^2)(1 - q_\nu^4)(1 - q_\nu^6) \\ & \times (1 - q_l^2 q_\nu^2)(1 - q_l^4 q_\nu^2)^2 (1 - q_l^2 q_\nu^4)(1 - q_l^6 q_\nu^2) \\ & \times (1 - q_l^4 q_\nu^4)(1 - q_l^8 q_\nu^2).\end{aligned}$$

☞ From the HS, one can calculate its PL by definition<sup>§</sup>

$$\begin{aligned}\text{PL}[\mathcal{H}(q_l, q_\nu)] = & q_l^2 + q_\nu^2 + q_l^4 + q_l^2 q_\nu^2 + q_\nu^4 + q_l^6 + 2q_l^4 q_\nu^2 + q_l^2 q_\nu^4 \\ & + q_\nu^6 + q_l^6 q_\nu^2 + 3q_l^4 q_\nu^4 + q_l^8 q_\nu^2 + 2q_l^6 q_\nu^4 + 2q_l^4 q_\nu^6 \\ & + 3q_l^8 q_\nu^4 + 3q_l^6 q_\nu^6 + q_l^4 q_\nu^8 + q_l^{10} q_\nu^4 + 3q_l^8 q_\nu^6 + q_l^6 q_\nu^8 \\ & + q_l^{12} q_\nu^4 + q_l^{10} q_\nu^6 - \mathcal{O}([q_l q_\nu]^{18}) .\end{aligned}\tag{17}$$

☞ From the leading positive terms of PL, one can conveniently read off the number and degrees of the basic invariants.

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<sup>§</sup>Yilin Wang, Bingrong Yu, and Shun Zhou. "Flavor Invariants and Renormalization-group Equations in the Leptonic Sector with Massive Majorana Neutrinos". In: (July 2021). arXiv: 2107.06274 [hep-ph].

# Basic Invariants in Effective Theory

$$H_l \equiv M_l M_l^\dagger, \quad H_\nu \equiv M_\nu M_\nu^\dagger, \quad G_{l\nu} \equiv M_\nu H_l^* M_\nu^\dagger, \quad G_{l\nu}^{(2)} \equiv M_\nu (H_l^*)^2 M_\nu^\dagger$$

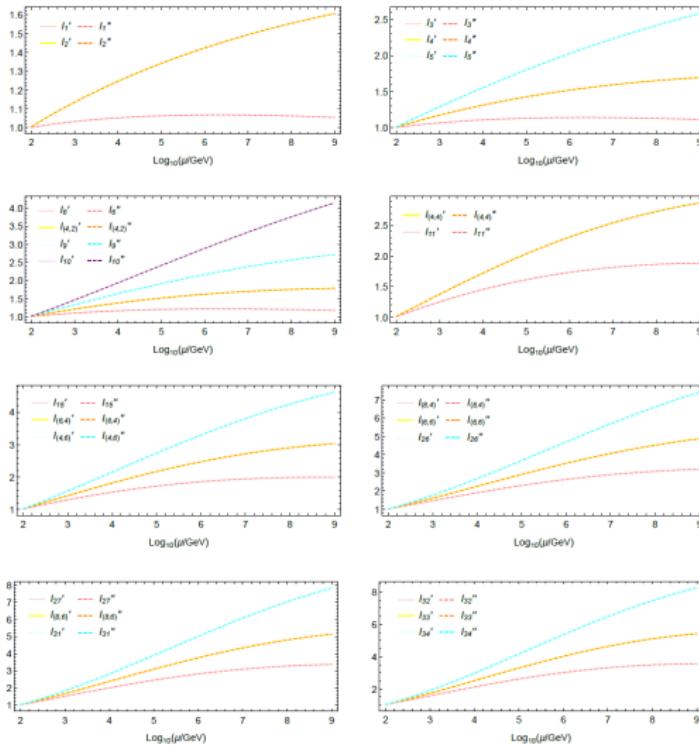
| flavor invariants                                    | $(q_l, q_\nu)$ | $q_l + q_\nu$ | CP parity |
|--|----------------|---------------|-----------|
| $I_1 \equiv \text{Tr}(H_l)$                          | (2, 0)         | 2             | +         |
| $I_2 \equiv \text{Tr}(H_\nu)$                        | (0, 2)         | 2             | +         |
| $I_3 \equiv \text{Tr}(H_l^2)$                        | (4, 0)         | 4             | +         |
| $I_4 \equiv \text{Tr}(H_l H_\nu)$                    | (2, 2)         | 4             | +         |
| $I_5 \equiv \text{Tr}(H_\nu^2)$                      | (0, 4)         | 4             | +         |
| $I_6 \equiv \text{Tr}(H_l^3)$                        | (6, 0)         | 6             | +         |
| $I_7 \equiv \text{Tr}(H_l^2 H_\nu)$                  | (4, 2)         | 6             | +         |
| $I_8 \equiv \text{Tr}(H_l G_{l\nu})$                 | (4, 2)         | 6             | +         |
| $I_9 \equiv \text{Tr}(H_l H_\nu^2)$                  | (2, 4)         | 6             | +         |
| $I_{10} \equiv \text{Tr}(H_\nu^3)$                   | (0, 6)         | 6             | +         |
| $I_{11} \equiv \text{Tr}(H_l^2 G_{l\nu})$            | (6, 2)         | 8             | +         |
| $I_{12} \equiv \text{Tr}(\{H_l, H_\nu\} G_{l\nu})$   | (4, 4)         | 8             | +         |
| $I_{13} \equiv \text{Tr}([H_l, H_\nu] G_{l\nu})$     | (4, 4)         | 8             | -         |
| $I_{14} \equiv \text{Tr}(H_l^2 H_\nu^2)$             | (4, 4)         | 8             | +         |
| $I_{15} \equiv \text{Tr}(H_l^2 G_{l\nu}^{(2)})$      | (8, 2)         | 10            | +         |
| $I_{16} \equiv \text{Tr}(\{H_l^2, H_\nu\} G_{l\nu})$ | (6, 4)         | 10            | +         |
| $I_{17} \equiv \text{Tr}([H_l^2, H_\nu] G_{l\nu})$   | (6, 4)         | 10            | -         |

| flavor invariants   | $(q_l, q_\nu)$ | $q_l + q_\nu$ | CP parity |
|---|----------------|---------------|-----------|
| $I_{18} \equiv \text{Tr}(\{H_l, H_\nu^2\} G_{l\nu})$  | (4, 6)         | 10            | +         |
| $I_{19} \equiv \text{Tr}([H_l, H_\nu^2] G_{l\nu})$  | (4, 6)         | 10            | -         |
| $I_{20} \equiv \text{Tr}(\{H_l^2, H_\nu\} G_{l\nu}^{(2)})$  | (8, 4)         | 12            | +         |
| $I_{21} \equiv \text{Tr}([H_l^2, H_\nu] G_{l\nu}^{(2)})$  | (8, 4)         | 12            | -         |
| $I_{22} \equiv \text{Tr}(H_l^2 H_\nu H_l G_{l\nu}) - \text{Tr}(H_l^2 G_{l\nu} H_l H_\nu)$                   | (8, 4)         | 12            | -         |
| $I_{23} \equiv \text{Tr}(\{H_l, H_\nu^2\} G_{l\nu}^{(2)})$  | (6, 6)         | 12            | +         |
| $I_{24} \equiv \text{Tr}([H_l, H_\nu^2] G_{l\nu}^{(2)})$  | (6, 6)         | 12            | -         |
| $I_{25} \equiv \text{Tr}(H_l^2 H_\nu^2 H_l H_\nu) - \text{Tr}(H_l^2 H_\nu H_l H_\nu^2)$                     | (6, 6)         | 12            | -         |
| $I_{26} \equiv \text{Tr}(H_l H_\nu^2 G_{l\nu} H_\nu) - \text{Tr}(H_l H_\nu G_{l\nu} H_\nu^2)$               | (4, 8)         | 12            | -         |
| $I_{27} \equiv \text{Tr}(H_l^2 H_\nu H_l G_{l\nu}^{(2)}) - \text{Tr}(H_l^2 G_{l\nu}^{(2)} H_l H_\nu)$       | (10, 4)        | 14            | -         |
| $I_{28} \equiv \text{Tr}(\{H_l^2, H_\nu\} G_{l\nu}^2)$  | (8, 6)         | 14            | +         |
| $I_{29} \equiv \text{Tr}([H_l^2, H_\nu] G_{l\nu}^2)$  | (8, 6)         | 14            | -         |
| $I_{30} \equiv \text{Tr}(H_l^2 H_\nu^2 H_l G_{l\nu}) - \text{Tr}(H_l^2 G_{l\nu} H_l H_\nu^2)$               | (8, 6)         | 14            | -         |
| $I_{31} \equiv \text{Tr}(H_l^2 H_\nu^2 G_{l\nu} H_\nu) - \text{Tr}(H_l^2 H_\nu G_{l\nu} H_\nu^2)$           | (6, 8)         | 14            | -         |
| $I_{32} \equiv \text{Tr}(H_l^2 G_{l\nu} H_l G_{l\nu}^{(2)}) - \text{Tr}(H_l^2 G_{l\nu}^{(2)} H_l G_{l\nu})$ | (12, 4)        | 16            | -         |
| $I_{33} \equiv \text{Tr}(H_l^2 H_\nu H_l G_{l\nu}^2) - \text{Tr}(H_l^2 G_{l\nu}^2 H_l H_\nu)$               | (10, 6)        | 16            | -         |
| $I_{34} \equiv \text{Tr}(H_l^2 H_\nu^2 G_{l\nu}^2) - \text{Tr}(H_l^2 G_{l\nu}^2 H_\nu^2)$                   | (8, 8)         | 16            | -         |

34 basic invariants, 19 CP-even,  
15 CP-odd

Y.Wang, B.Yu and S.Zhou, arXiv:2107.06274

# RGEs of Flavor Invariants



The renormalization-group evolutions of the 34 basic invariants in effective theory from  $10^2 \text{ GeV}$  to  $10^9 \text{ GeV}$

Y.Wang, B.Yu and S.Zhou,  
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# Extracting Physical Observables I

From the basic invariants, one can extract the physical observables, i.e., the lepton masses, flavor mixing angles and CP-violating phases.

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- For the hierarchical charged-lepton masses  $m_\tau \gg m_\mu \gg m_e$ ,

$$m_e^2 = \frac{I_1^3 - 3I_1I_3 + 2I_6}{3(I_1^2 - I_3)} , \quad m_\mu^2 = \frac{I_1^2 - I_3}{2I_6^{1/3}} , \quad m_\tau^2 = I_6^{1/3} ;$$

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- For the normal-ordering neutrino masses  $m_3 \gg m_2 > m_1$ ,

$$\begin{aligned} m_{1,2}^2 &= \frac{1}{2} \left[ \frac{I_2^2 - I_5}{2I_{10}^{1/3}} \mp \sqrt{\left( \frac{I_2^2 - I_5}{2I_{10}^{1/3}} \right)^2 - 4 \left( \frac{I_2^3 - 3I_2I_5 + 2I_{10}}{6I_{10}^{1/3}} \right)} \right], \\ m_3^2 &= I_{10}^{1/3}; \end{aligned}$$

# Extracting Physical Observables II

☞ For the inverted-ordering neutrino masses  $m_2 > m_1 \gg m_3$ ,

$$m_3^2 = \frac{I_2^3 - 3I_2I_5 + 2I_{10}}{3(I_2^2 - I_5)},$$

$$m_{1,2}^2 = \frac{1}{2} \left[ I_2 - m_3^2 \mp \sqrt{2I_5 - I_2^2 + 2I_2m_3^2} \right];$$

# Extracting Physical Observables II

- For the inverted-ordering neutrino masses  $m_2 > m_1 \gg m_3$ ,

$$m_3^2 = \frac{I_2^3 - 3I_2I_5 + 2I_{10}}{3(I_2^2 - I_5)},$$

$$m_{1,2}^2 = \frac{1}{2} \left[ I_2 - m_3^2 \mp \sqrt{2I_5 - I_2^2 + 2I_2m_3^2} \right];$$

- For the flavor mixing angles and CP-violating phases, though a little bit tedious and tricky, one can still extract all of them from the basic invariants.

Y.Wang, **B.Yu** and S.Zhou, arXiv:2107.06274

# Hilbert Series in MSM

- Then we consider a UV-complete model: the minimal seesaw model (MSM),<sup>¶</sup> which extends the SM by adding two RH neutrinos

$$\mathcal{L}_{\text{MSM}} = \mathcal{L}_{\text{SM}} - \overline{\ell_L} Y_\nu \tilde{H} \nu_R - \frac{1}{2} \overline{\nu_R^C} M_R \nu_R + \text{h.c.} . \quad (18)$$

<sup>¶</sup>P. H. Frampton, S. L. Glashow, and T. Yanagida. "Cosmological sign of neutrino CP violation". In: *Phys. Lett. B* 548 (2002), pp. 119–121.

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- The building blocks to construct flavor invariants are  $H_l \equiv M_l M_l^\dagger$ , Dirac neutrino mass matrix  $M_D \equiv Y_\nu v / \sqrt{2}$  and Majorana neutrino mass matrix  $M_R$ . Their representations under the  $U(3)_L$  and  $U(2)_R$  flavor transformation turn out to be

$$H_l : \mathbf{3} \otimes \mathbf{3}^* , \quad M_D : \mathbf{3} \otimes \mathbf{2}^* , \quad M_D^\dagger : \mathbf{2} \otimes \mathbf{3}^* , \\ M_R : (\mathbf{2}^* \otimes \mathbf{2}^*)_S , \quad M_R^\dagger : (\mathbf{2} \otimes \mathbf{2})_S . \quad (19)$$

<sup>¶</sup>P. H. Frampton, S. L. Glashow, and T. Yanagida. "Cosmological sign of neutrino CP violation". In: *Phys. Lett. B* 548 (2002), pp. 119–121.

# Hilbert Series in MSM

Using the Molien-Weyl formula, we calculate the HS and PL in MSM, which tell us the number and degrees of all basic invariants

$$\mathcal{H}(q_l, q_D, q_R) = \int [d\mu]_{U(3) \otimes U(2)} \text{PE}(z_1, z_2, z_3, z_4, z_5; q_l, q_D, q_R) = \frac{\mathcal{N}(q_l, q_D, q_R)}{\mathcal{D}(q_l, q_D, q_R)},$$

where

$$\begin{aligned} \mathcal{N}(q_l, q_D, q_R) = & -q_D^{12} q_R^{12} q_l^{22} - q_D^{18} q_R^{8} q_l^{22} + q_D^{20} q_R^{12} q_l^{20} - 2q_D^{18} q_R^{10} q_l^{20} - q_D^{16} q_R^{10} q_l^{20} - q_D^{18} q_R^{8} q_l^{20} + q_D^{16} q_R^{8} q_l^{20} \\ & + q_D^{14} q_R^{8} q_l^{20} + q_D^{16} q_R^{6} q_l^{20} - q_D^{18} q_R^{12} q_l^{18} - 2q_D^{18} q_R^{10} q_l^{18} + q_D^{14} q_R^{10} q_l^{18} - 2q_D^{18} q_R^{8} q_l^{18} + q_D^{16} q_R^{8} q_l^{18} \\ & + q_D^{14} q_R^{8} q_l^{18} - q_D^{12} q_R^{8} q_l^{18} + 2q_D^{16} q_R^{6} q_l^{18} + q_D^{14} q_R^{6} q_l^{18} - 2q_D^{18} q_R^{10} q_l^{16} + q_D^{14} q_R^{10} q_l^{16} - q_D^{18} q_R^{8} q_l^{16} \\ & + 3q_D^{16} q_R^{8} q_l^{16} + 3q_D^{14} q_R^{8} q_l^{16} - q_D^{12} q_R^{8} q_l^{16} + 2q_D^{16} q_R^{6} q_l^{16} - q_D^{14} q_R^{6} q_l^{16} - 2q_D^{12} q_R^{6} q_l^{16} - q_D^{14} q_R^{4} q_l^{16} \\ & + q_D^{10} q_R^{4} q_l^{16} + q_D^{14} q_R^{10} q_l^{14} - q_D^{18} q_R^{8} q_l^{14} + q_D^{16} q_R^{8} q_l^{14} + q_D^{14} q_R^{8} q_l^{14} - 2q_D^{12} q_R^{8} q_l^{14} + 2q_D^{16} q_R^{6} q_l^{14} \\ & + q_D^{14} q_R^{6} q_l^{14} + 2q_D^{10} q_R^{6} q_l^{14} - q_D^{14} q_R^{4} q_l^{14} + q_D^{10} q_R^{4} q_l^{14} - q_D^{8} q_R^{4} q_l^{14} - q_D^{16} q_R^{10} q_l^{12} + q_D^{14} q_R^{10} q_l^{12} \\ & + q_D^{12} q_R^{10} q_l^{12} + q_D^{16} q_R^{8} q_l^{12} + 3q_D^{14} q_R^{8} q_l^{12} + q_D^{16} q_R^{6} q_l^{12} - q_D^{14} q_R^{6} q_l^{12} - 3q_D^{12} q_R^{6} q_l^{12} - q_D^{8} q_R^{6} q_l^{12} \\ & - q_D^{14} q_R^{4} q_l^{12} + 2q_D^{10} q_R^{4} q_l^{12} - q_D^{8} q_R^{4} q_l^{12} - q_D^{8} q_R^{2} q_l^{12} + q_D^{14} q_R^{10} q_l^{10} + q_D^{14} q_R^{8} q_l^{10} - 2q_D^{12} q_R^{8} q_l^{10} \\ & + q_D^{8} q_R^{8} q_l^{10} + q_D^{14} q_R^{6} q_l^{10} + 3q_D^{10} q_R^{6} q_l^{10} + q_D^{8} q_R^{6} q_l^{10} - q_D^{6} q_R^{6} q_l^{10} - 3q_D^{8} q_R^{4} q_l^{10} - q_D^{6} q_R^{4} q_l^{10} \\ & - q_D^{10} q_R^{2} q_l^{10} - q_D^{8} q_R^{2} q_l^{10} + q_D^{6} q_R^{2} q_l^{10} + q_D^{14} q_R^{8} q_l^{8} - q_D^{12} q_R^{8} q_l^{8} + q_D^{8} q_R^{8} q_l^{8} - 2q_D^{12} q_R^{6} q_l^{8} \\ & - q_D^{8} q_R^{6} q_l^{8} - 2q_D^{6} q_R^{6} q_l^{8} + 2q_D^{10} q_R^{4} q_l^{8} - q_D^{8} q_R^{4} q_l^{8} - q_D^{6} q_R^{4} q_l^{8} + q_D^{4} q_R^{4} q_l^{8} - q_D^{8} q_R^{2} q_l^{8} \\ & - q_D^{12} q_R^{8} q_l^{6} + q_D^{8} q_R^{8} q_l^{6} + 2q_D^{10} q_R^{6} q_l^{6} - q_D^{8} q_R^{6} q_l^{6} - 2q_D^{6} q_R^{6} q_l^{6} + q_D^{10} q_R^{4} q_l^{6} - 3q_D^{8} q_R^{4} q_l^{6} \\ & - 3q_D^{6} q_R^{4} q_l^{6} + q_D^{4} q_R^{4} q_l^{6} - q_D^{8} q_R^{2} q_l^{6} + 2q_D^{4} q_R^{2} q_l^{6} - q_D^{8} q_R^{4} q_l^{4} - 2q_D^{6} q_R^{4} q_l^{4} + q_D^{4} q_R^{4} q_l^{4} \\ & + q_D^{10} q_R^{4} q_l^{4} - q_D^{8} q_R^{4} q_l^{4} - q_D^{6} q_R^{4} q_l^{4} + 2q_D^{6} q_R^{4} q_l^{4} - q_D^{8} q_R^{2} q_l^{4} + 2q_D^{4} q_R^{2} q_l^{4} - q_D^{6} q_R^{4} q_l^{2} \\ & - q_D^{8} q_R^{4} q_l^{2} - q_D^{6} q_R^{4} q_l^{2} + q_D^{4} q_R^{4} q_l^{2} - q_D^{2} q_R^{2} q_l^{2} + 2q_D^{4} q_R^{2} q_l^{2} + 2q_D^{4} q_R^{2} q_l^{2} + q_D^{4} q_R^{4} q_l^{1}, \\ \mathcal{D}(q_l, q_D, q_R) = & (1 - q_l^2) (1 - q_D^2) (1 - q_R^2) (1 - q_l^4) (1 - q_D^4) (1 - q_R^4) (1 - q_l^2 q_D^2)^2 (1 - q_D^2 q_R^2) \\ & \times (1 - q_l^6) (1 - q_l^4 q_D^2) (1 - q_l^2 q_D^4) (1 - q_l^2 q_D^2 q_R^2) (1 - q_D^4 q_R^2) (1 - q_l^4 q_D^2 q_R^2) \\ & \times (1 - q_l^4 q_D^4 q_R^2) (1 - q_l^6 q_D^2 q_R^2). \end{aligned}$$

$$\begin{aligned} \text{PL}[\mathcal{H}(q_l, q_D, q_R)] = & (q_l^2 + q_D^2 + q_R^2) + (q_l^4 + q_l^2 q_D^2 + q_l^2 + q_D^2 q_R^2 + q_R^4) + (q_l^6 + q_l^4 q_D^2 + q_l^2 q_R^4) \\ & + q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4) + (q_l^4 q_D^2 + q_l^4 q_R^2 q_D^2 + 2q_l^2 q_D^2 q_R^2 + 2q_l^2 q_R^4 q_D^2) + (3q_l^2 q_D^2 q_R^2) \\ & + (q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4 q_D^2) + (q_l^4 q_D^2 + q_l^4 q_R^2 q_D^2 + 2q_l^2 q_D^2 q_R^2 + 2q_l^2 q_R^4 q_D^2) + (q_l^6 q_D^2 q_R^2) \\ & + 2q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4 q_D^2) + (q_l^2 q_D^2 + 2q_l^2 q_R^2 q_D^2 + 2q_l^2 q_D^2 q_R^2 + 2q_l^2 q_R^4 q_D^2) + (q_l^6 q_D^2 q_R^2) \\ & - 5q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4 q_D^2 - 2q_l^2 q_R^2 q_D^2 - 2q_l^2 q_D^2 q_R^2 - q_l^2 q_R^4 q_D^2) + (2q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4 q_D^2 - q_l^2 q_D^2 q_R^2) \\ & - 5q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4 q_D^2 - 2q_l^2 q_R^2 q_D^2 - 2q_l^2 q_D^2 q_R^2 - q_l^2 q_R^4 q_D^2) + (q_l^{10} q_D^2 q_R^2 - q_l^6 q_R^10 q_D^2) \\ & - 5q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4 q_D^2 - 2q_l^2 q_R^2 q_D^2 - 2q_l^2 q_D^2 q_R^2 - q_l^2 q_R^4 q_D^2) + (q_l^{10} q_D^2 q_R^2 - q_l^6 q_R^10 q_D^2) \\ & - q_l^2 q_D^2 q_R^2 + q_l^2 q_R^4 q_D^2 - 8q_l^2 q_R^2 q_D^2 - q_l^2 q_D^2 q_R^2 - 6q_l^2 q_R^4 q_D^2 - 2q_l^2 q_D^2 q_R^2 - q_l^2 q_R^4 q_D^2) \\ & - \mathcal{O}((q_l q_D q_R)^{20}), \end{aligned}$$

**B.Yu and S.Zhou, arXiv:2107.11928**

# Basic Invariants in MSM

| flavor invariants  | degree | CP | flavor invariants  | degree | CP |
|--|--------|----|--|--------|----|
| $I_{200} \equiv \text{Tr}(H_i)$  | 2      | +  | $I_{442}^{(2)} \equiv \text{Tr}(\tilde{H}_{\text{D}} G_{\text{ID}}^{(2)})$                         | 10     | +  |
| $I_{020} \equiv \text{Tr}(H_{\text{D}})$   | 2      | +  | $I_{442}^{(3)} \equiv \text{Tr}([\tilde{H}_{\text{D}}, H_{\text{R}}] G_{\text{ID}}^{(2)})$         | 10     | -  |
| $I_{002} \equiv \text{Tr}(H_{\text{R}})$   | 2      | +  | $I_{262} \equiv \text{Tr}([\tilde{H}_{\text{D}}, G_{\text{ID}}] G_{\text{DR}})$                    | 10     | -  |
| $I_{400} \equiv \text{Tr}(H_i^2)$  | 4      | +  | $I_{244} \equiv \text{Tr}([H_{\text{R}}, G_{\text{ID}}] G_{\text{DR}})$                            | 10     | -  |
| $I_{220} \equiv \text{Tr}(H_i H_{\text{D}})$   | 4      | +  | $I_{660} \equiv \text{Tr}([\tilde{H}_{\text{D}}, G_{\text{ID}}] G_{\text{ID}}^{(2)})$              | 12     | -  |
| $I_{040} \equiv \text{Tr}(H_{\text{D}}^2)$   | 4      | +  | $I_{642}^{(1)} \equiv \text{Tr}([H_{\text{R}}, G_{\text{ID}}] G_{\text{ID}}^{(2)})$                | 12     | -  |
| $I_{022} \equiv \text{Tr}(\tilde{H}_{\text{D}} H_{\text{R}})$                        | 4      | +  | $I_{642}^{(2)} \equiv \text{Tr}(G_{\text{ID}} G_{\text{IDR}}^{(2)})$                               | 12     | +  |
| $I_{004} \equiv \text{Tr}(H_{\text{R}}^2)$   | 4      | +  | $I_{462}^{(1)} \equiv \text{Tr}([\tilde{H}_{\text{D}}, G_{\text{ID}}] G_{\text{IDR}})$             | 12     | -  |
| $I_{600} \equiv \text{Tr}(H_i^3)$  | 6      | +  | $I_{462}^{(2)} \equiv \text{Tr}([\tilde{H}_{\text{D}}, G_{\text{DR}}] G_{\text{ID}}^{(2)})$        | 12     | -  |
| $I_{420} \equiv \text{Tr}(H_i^2 H_{\text{D}})$                                       | 6      | +  | $I_{444}^{(1)} \equiv \text{Tr}([H_{\text{R}}, G_{\text{ID}}] G_{\text{IDR}})$                     | 12     | -  |
| $I_{240} \equiv \text{Tr}(H_i H_{\text{D}}^2)$                                       | 6      | +  | $I_{444}^{(2)} \equiv \text{Tr}([H_{\text{R}}, G_{\text{DR}}] G_{\text{ID}}^{(2)})$                | 12     | -  |
| $I_{222} \equiv \text{Tr}(H_{\text{R}} G_{\text{ID}})$                               | 6      | +  | $I_{842} \equiv \text{Tr}(G_{\text{ID}}^{(2)} G_{\text{IDR}}^{(2)})$                               | 14     | +  |
| $I_{042} \equiv \text{Tr}(\tilde{H}_{\text{D}} G_{\text{DR}})$                       | 6      | +  | $I_{662}^{(1)} \equiv \text{Tr}([\tilde{H}_{\text{D}}, G_{\text{ID}}] G_{\text{IDR}}^{(2)})$       | 14     | -  |
| $I_{440} \equiv \text{Tr}(H_i^2 H_{\text{D}}^2)$                                     | 8      | +  | $I_{662}^{(2)} \equiv \text{Tr}([\tilde{H}_{\text{D}}, G_{\text{ID}}] G_{\text{ID}}^{(2)})$        | 14     | -  |
| $I_{422} \equiv \text{Tr}(H_{\text{R}} G_{\text{ID}}^{(2)})$                         | 8      | +  | $I_{644} \equiv \text{Tr}([H_{\text{R}}, G_{\text{ID}}^{(2)}] G_{\text{IDR}})$                     | 14     | -  |
| $I_{242}^{(1)} \equiv \text{Tr}(G_{\text{ID}} G_{\text{IDR}})$                       | 8      | +  | $I_{862}^{(1)} \equiv \text{Tr}([\tilde{H}_{\text{D}}, G_{\text{ID}}^{(2)}] G_{\text{IDR}}^{(2)})$ | 16     | -  |
| $I_{242}^{(2)} \equiv \text{Tr}([H_{\text{R}}, \tilde{H}_{\text{D}}] G_{\text{ID}})$ | 8      | -  | $I_{862}^{(2)} \equiv \text{Tr}([G_{\text{ID}}, G_{\text{ID}}^{(2)}] G_{\text{IDR}})$              | 16     | -  |
| $I_{044} \equiv \text{Tr}([H_{\text{R}}, \tilde{H}_{\text{D}}] G_{\text{DR}})$       | 8      | -  | $I_{844} \equiv \text{Tr}([H_{\text{R}}, G_{\text{ID}}^{(2)}] G_{\text{IDR}}^{(2)})$               | 16     | -  |
| $I_{442}^{(1)} \equiv \text{Tr}(G_{\text{ID}} G_{\text{IDR}})$                       | 10     | +  | $I_{10,6,2} \equiv \text{Tr}([G_{\text{ID}}, G_{\text{ID}}^{(2)}] G_{\text{IDR}}^{(2)})$           | 18     | -  |

38 basic invariants, 20 CP-even, 18 CP-odd

$$\begin{aligned}
 H_l &\equiv M_l M_l^\dagger, \quad H_R \equiv M_R^\dagger M_R \\
 H_D &\equiv M_D M_D^\dagger, \quad \tilde{H}_D \equiv M_D^\dagger M_D \\
 G_{lD} &\equiv M_D^\dagger H_l M_D, \quad G_{lD}^{(2)} \equiv M_D^\dagger H_l^2 M_D \\
 G_{DR} &\equiv M_R^\dagger \tilde{H}_D^* M_R, \quad G_{lDR} \equiv M_R^\dagger G_{lD}^* M_R \\
 G_{lDR}^{(2)} &\equiv M_R^\dagger (G_{lD}^{(2)})^* M_R
 \end{aligned}$$

**Matching conditions:** The flavor invariants in effective theory can be expressed as the rational functions of those in full MSM.

B.Yu and S.Zhou, arXiv:2107.11928

# Outline

- 1 Motivation
- 2 Invariant Theory
- 3 Basic Invariants in Seesaw Model
- 4 Flavor Invariants and CP Violation in Leptonic Sector
- 5 Summary

# Flavor Invariants and CP Violation

- ☞ CP violation is the key issue in elementary particle physics and an indispensable ingredient to dynamically generate cosmological matter-antimatter asymmetry.\*

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†C. Jarlskog. "Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Violation". In: *Phys. Rev. Lett.* 55 (1985), p. 1039. DOI: [10.1103/PhysRevLett.55.1039](https://doi.org/10.1103/PhysRevLett.55.1039).

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§G. C. Branco, L. Lavoura, and M. N. Rebelo. "Majorana Neutrinos and CP Violation in the Leptonic Sector". In: *Phys. Lett. B* 180 (1986), pp. 264–268.

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$$\mathcal{I} = \text{Det} \left[ M_u M_u^\dagger, M_d M_d^\dagger \right] \propto \sin \delta_{\text{CP}} . \quad (20)$$

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- Flavor invariants are very useful to construct the **sufficient and necessary** conditions for CP conservation in the quark and leptonic sector in a basis-independent way.<sup>‡§</sup>

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# Conditions for CP Conservation

TABLE I. Summary of the number of independent  $CP$  phases and the weak-basis invariants chosen to guarantee  $CP$  conservation in the low-energy effective theory. Notice that the choice of weak-basis invariants is by no means unique.

| Low-energy effective theory                  | Number of $CP$ phases | Weak-basis invariants  |
|--|-----------------------|--|
| No degeneracy<br>$(m_1 \neq m_2 \neq m_3)$   | 3                     | $\mathcal{I}_1 \equiv \text{Tr}\{[H_u, H_d]^3\}$<br>$\mathcal{I}_2 \equiv \text{Im}\{\text{Tr}[H_l H_u G_h]\}$<br>$\mathcal{I}_4 \equiv \text{Im}\{\text{Tr}[H_l H_u^2 G_h]\}$ |
| Partial degeneracy<br>$(m_1 = m_2 \neq m_3)$ | 2                     | $\mathcal{I}_2 \equiv \text{Im}\{\text{Tr}[H_l H_u G_h]\}$<br>$\mathcal{I}_3 \equiv \text{Tr}\{[G_{lu}, H_l]^3\}$  |
| Full degeneracy<br>$(m_1 = m_2 = m_3)$       | 1                     | $\mathcal{I}_3 \equiv \text{Tr}\{[G_{lu}, H_l]^3\}$  |
| No degeneracy<br>with $m_1 = 0$              | 2                     | $\mathcal{I}_1 \equiv \text{Tr}\{[H_u, H_d]^3\}$<br>$\mathcal{I}_2 \equiv \text{Im}\{\text{Tr}[H_l H_u G_h]\}$   |

TABLE II. Summary of the number of independent  $CP$  phases and the weak-basis invariants chosen to guarantee  $CP$  conservation in the canonical seesaw model. Notice that the choice of weak-basis invariants is by no means unique.

| Canonical seesaw model                       | Number of $CP$ phases | Weak-basis invariants  |
|--|-----------------------|--|
| No degeneracy<br>$(M_1 \neq M_2 \neq M_3)$   | 6                     | $\tilde{\mathcal{I}}_1 \equiv \text{Im}\{\text{Tr}[H_D H_R G_{DR}]\}$<br>$\tilde{\mathcal{I}}_2 \equiv \text{Im}\{\text{Tr}[H_D H_R^2 G_{DR}]\}$<br>$\tilde{\mathcal{I}}_3 \equiv \text{Tr}\{[H_R, H_D]^3\}$<br>$\tilde{\mathcal{I}}_4 \equiv \text{Im}\{\text{Tr}[H_1 H_R^2 G_1]\}$<br>$\tilde{\mathcal{I}}_5 \equiv \text{Im}\{\text{Tr}[H_1 H_R^2 G_1]\}$<br>$\tilde{\mathcal{I}}_6 \equiv \text{Im}\{\text{Tr}[H_1 H_R G_1 H_R]\}$ |
| Partial degeneracy<br>$(M_1 = M_2 \neq M_3)$ | 5                     | $\tilde{\mathcal{I}}_1 \equiv \text{Im}\{\text{Tr}[H_0 H_R G_{DR}]\}$<br>$\tilde{\mathcal{I}}_7 \equiv \text{Tr}\{[G_{DR}, H_D]^3\}$<br>$\tilde{\mathcal{I}}_8 \equiv \text{Tr}\{[G_1, H_1]^3\}$<br>$\tilde{\mathcal{I}}_9 \equiv \text{Tr}\{[G_2, H_2]^3\}$<br>$\tilde{\mathcal{I}}_{10} \equiv \text{Tr}\{[G_3, H_3]^3\}$  |
| Full degeneracy<br>$(M_1 = M_2 = M_3)$       | 4                     | $\tilde{\mathcal{I}}_7 \equiv \text{Tr}\{[G_{DR}, H_D]^3\}$<br>$\tilde{\mathcal{I}}_8 \equiv \text{Tr}\{[G_1, H_1]^3\}$<br>$\tilde{\mathcal{I}}_9 \equiv \text{Tr}\{[G_2, H_2]^3\}$<br>$\tilde{\mathcal{I}}_{10} \equiv \text{Tr}\{[G_3, H_3]^3\}$   |
| Minimal seesaw model<br>$(M_1 \neq M_2)$     | 3                     | $\tilde{\mathcal{I}}_1 \equiv \text{Im}\{\text{Tr}[H_D H_R G_{DR}]\}$<br>$\tilde{\mathcal{I}}_4 \equiv \text{Im}\{\text{Tr}[H_1 H_R G_1]\}$<br>$\tilde{\mathcal{I}}_1$ in Eq. (3.32)   |
| Minimal seesaw model<br>$(M_1 = M_2)$        | 2                     | $\tilde{\mathcal{I}}_1$ in Eq. (3.32)<br>$\tilde{\mathcal{I}}_2$ in Eq. (3.33)   |

B.Yu and S.Zhou, Phys.Lett.B 800 (2020) 135085

B.Yu and S.Zhou, Phys.Rev.D 103 (2021) 3, 035017

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- ☞ Flavor invariants, which are basis-independent and contain only physical degrees of freedom, have proved to be extremely useful in studying the flavor structures of fermions as well as the CP violation in the quark and leptonic sector.
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# Summary

- ☞ Flavor invariants, which are basis-independent and contain only physical degrees of freedom, have proved to be extremely useful in studying the flavor structures of fermions as well as the CP violation in the quark and leptonic sector.
- ☞ Flavor invariants provide a novel way to establish the relationship between the UV-complete theories and their low-energy effective counterparts.
- ☞ Invariant theory supplies a systematic approach to studying the algebraic structure of the invariant ring and constructing all the basic invariants. The applications of invariant theory to flavor physics need more dedicated studies.

**Thank you!**