Transasymptotics, dynamical systems and far from equilibrium hydrodynamics

Mauricio Martinez Guerrero

129th HENPIC Seminar
December 2020

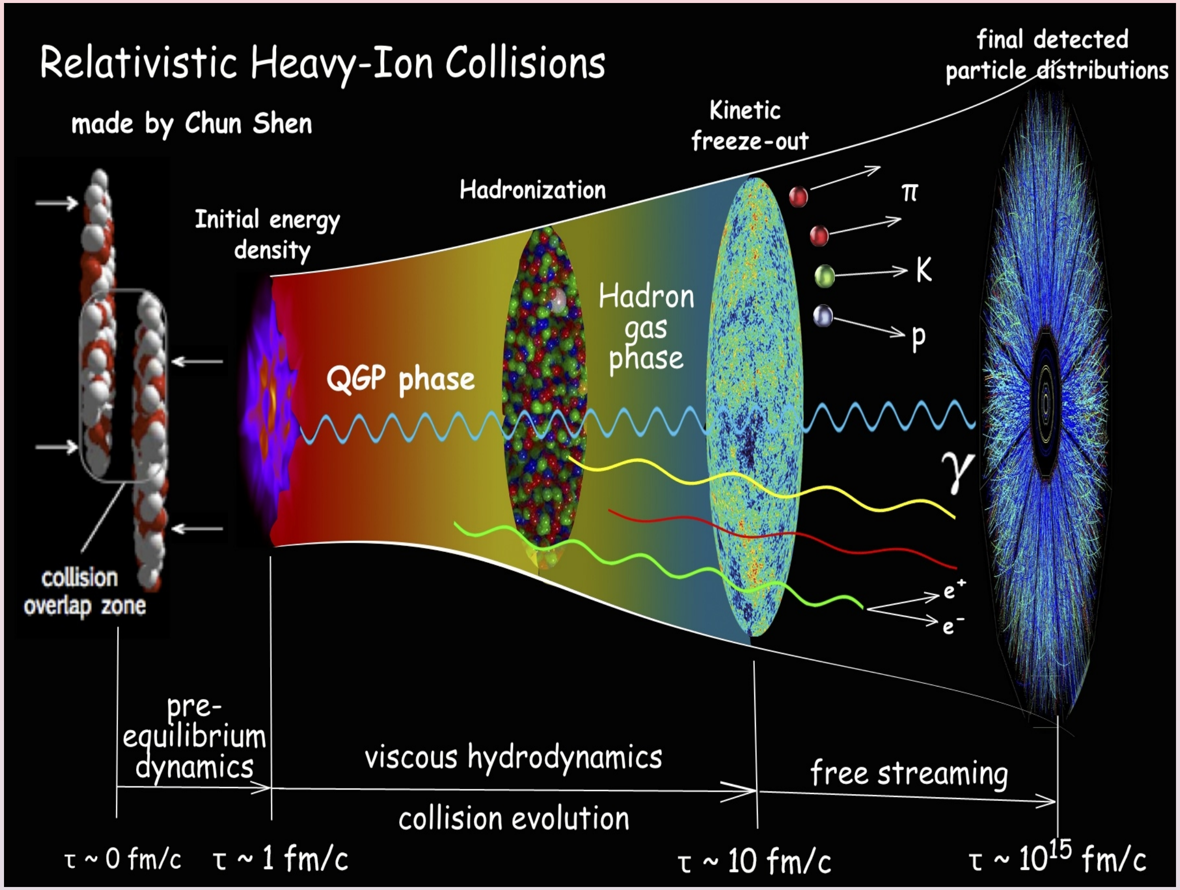
arXiv: 2011.08235

BEST COLLABORATION

NC STATE UNIVERSITY

Exploring nucleus at short distances

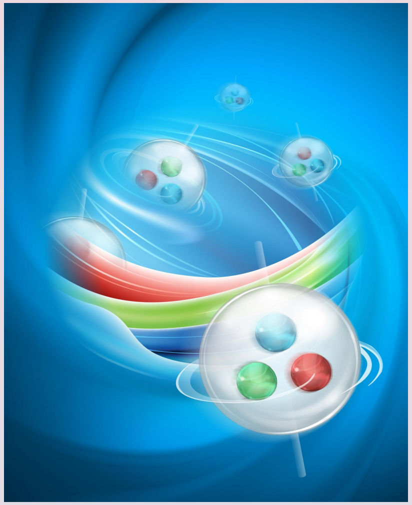
- Lorentz contracted nuclei collide as **squeezed pancakes**
- Pre-equilibrium dynamics leads to **emergent hydrodynamis**
- **Hydrodynamical flow** of the quark-gluon plasma
- Condensation into **confined hadrons** and **freeze-out**
- **Late-time** hadron cascade



Hydrodynamics:
one theory to rule them all

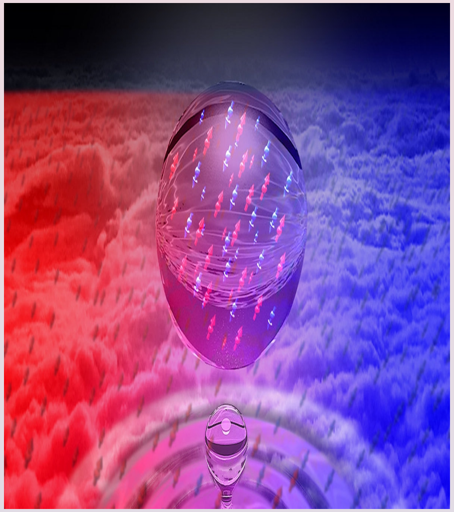
New discoveries
Fluids at extreme conditions

$T \sim 10^{12}$ K



QGP

$T \sim 10^{-7}$ K



Cold atoms

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**
- Near to equilibrium the energy-momentum tensor is expanded in gradients

$$T^{\mu\nu} = \sum_{k=0}^{\infty} T^{\mu\nu}_{(k)}$$

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**
- Near to equilibrium the energy-momentum tensor is expanded in gradients

$$T^{\mu\nu} = \sum_{k=0}^{\infty} T^{\mu\nu}_{(k)}$$

$$T_0^{\mu\nu} = (\epsilon + p(\epsilon))\, u^\mu u^\nu + p(\epsilon) g^{\mu\nu}$$

→ **0^{th} order:** Ideal fluid

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**
- Near to equilibrium the energy-momentum tensor is expanded in gradients

$$T^{\mu\nu} = \sum_{k=0}^{\infty} T^{\mu\nu}_{(k)}$$

$$T^{\mu\nu}_0 = (\epsilon + p(\epsilon))\, u^\mu u^\nu + p(\epsilon) g^{\mu\nu}$$

→ **0th order:** Ideal fluid

$$T^{\mu\nu}_1 = -\eta\, \sigma^{\mu\nu} - \zeta \partial_\mu u^\mu$$

→ **1st order:** Navier-Stokes

$$T^{\mu\nu}_2 = -\tau_\pi \left[\langle D\pi^{\mu\nu} \rangle + \frac{4}{3}\pi^{\mu\nu} \nabla \cdot u \right] + \cdots$$

→ **2nd order:** Israel-Stewart,etc

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**
- Near to equilibrium the energy-momentum tensor is expanded in gradients

$$T^{\mu\nu} = \sum_{k=0}^{\infty} T^{\mu\nu}_{(k)}$$

- Evolution of $T^{\mu\nu}$ is universal and determined by **conservation laws:**

$$\partial_\mu T^{\mu\nu} = 0$$

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**
- Near to equilibrium the energy-momentum tensor is expanded in gradients

$$T^{\mu\nu} = \sum_{k=0}^{\infty} T^{\mu\nu}_{(k)}$$

- Evolution of $T^{\mu\nu}$ is universal and determined by **conservation laws:**

$$\partial_\mu T^{\mu\nu} = 0$$

- Truncation in gradients leads to Euler, Navier-Stokes, Israel-Stewart, ...

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**
- Near to equilibrium the energy-momentum tensor is expanded in gradients

$$T^{\mu\nu} = \sum_{k=0}^{\infty} T^{\mu\nu}_{(k)}$$

- Evolution of $T^{\mu\nu}$ is universal and determined by **conservation laws:**

$$\partial_\mu T^{\mu\nu} = 0$$

- Truncation in gradients leads to Euler, Navier-Stokes, Israel-Stewart, ...
- Hydrodynamics works across **phase transitions**

Hydrodynamics: one theory to rule them all

- Effective field theory of **long-wavelength modes**
- Near to equilibrium the energy-momentum tensor is expanded in gradients

$$T^{\mu\nu} = \sum_{k=0}^{\infty} T^{\mu\nu}_{(k)}$$

- Evolution of $T^{\mu\nu}$ is universal and determined by **conservation laws:**

$$\partial_\mu T^{\mu\nu} = 0$$

- Truncation in gradients leads to Euler, Navier-Stokes, Israel-Stewart, ...
- Hydrodynamics works across **phase transitions**
- **Microscopic details** encoded in **transport parameters** and EOS

c_s

0$^{\text{th}}$ order:
Speed of sound

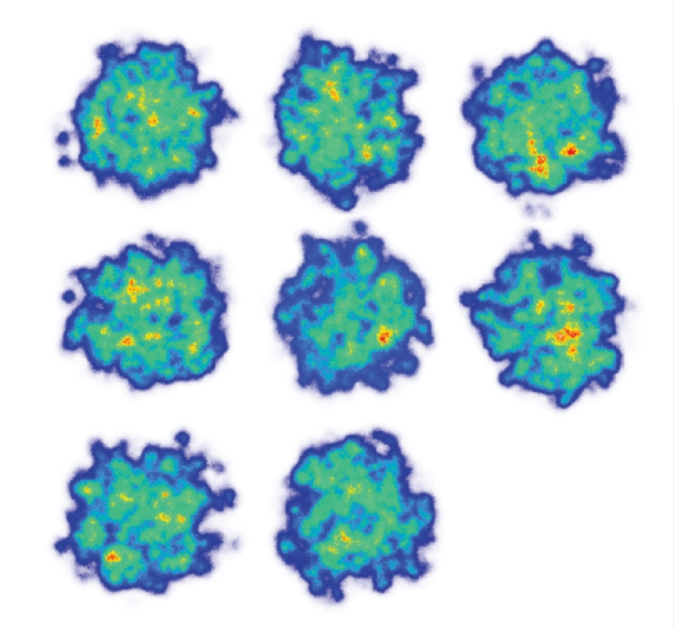
$\eta \, , \, \zeta$

1$^{\text{st}}$ order:
Shear and bulk viscosities

$\tau_\pi \, , \, \lambda_1 \, ,$
$\lambda_2 \, , \, \lambda_3 \, , \, \kappa \, ...$

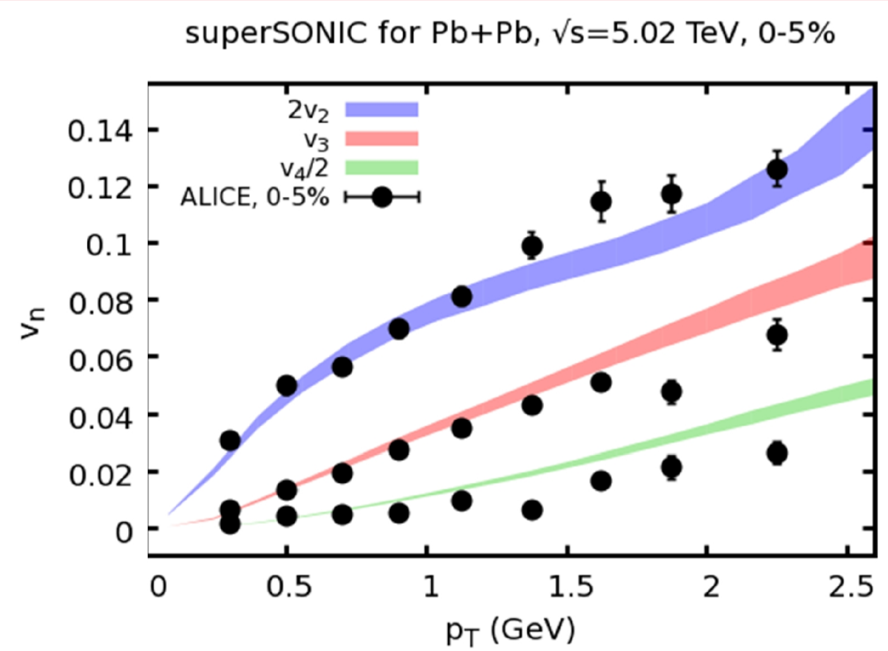
2$^{\text{nd}}$ order:
Relaxation coefficients

Fluidity in Heavy Ions



$$\partial_\mu T^{\mu\nu} = 0$$

Weller & Romatschke (2017)



$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n v_n \cos(n\phi)\right)$$

$n = 2$ $n = 3$ $n = 4$ $n = 5$

- Hydrodynamics is a **deterministic initial-value problem** and the QGP flows with **nearly zero viscosity**
- v_n is sensitive to the **initial geometry** of the collision

Paradox

Small gradient expansion $\equiv$ expansion in Knudsen number

Microscopic scale

$$l \sim \tau_\pi$$

Macroscopic scale

$$L^{-1} \sim \partial_i v^i$$

Paradox

Small gradient expansion $\equiv$ expansion in Knudsen number

Microscopic scale

$$l \sim \tau_\pi$$

Macroscopic scale

$$L^{-1} \sim \partial_i v^i$$

Knudsen number

$$Kn \equiv \frac{l}{L}$$

Paradox

Small gradient expansion $\equiv$ expansion in Knudsen number

Microscopic scale

$$l \sim \tau_\pi$$

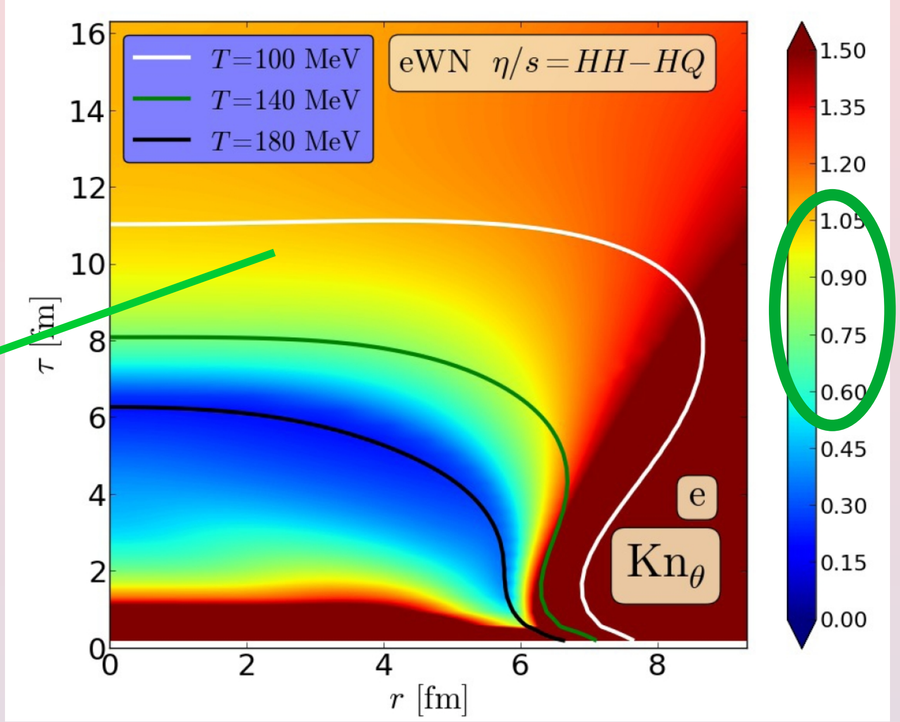
Macroscopic scale

$$L^{-1} \sim \partial_i v^i$$

Knudsen number

$$Kn \equiv \frac{l}{L}$$

Kn is not small !!



Denicol & Niemi (2014)

Paradox

Small gradient expansion $\equiv$ expansion in Knudsen number

Microscopic scale

$$l \sim \tau_\pi$$

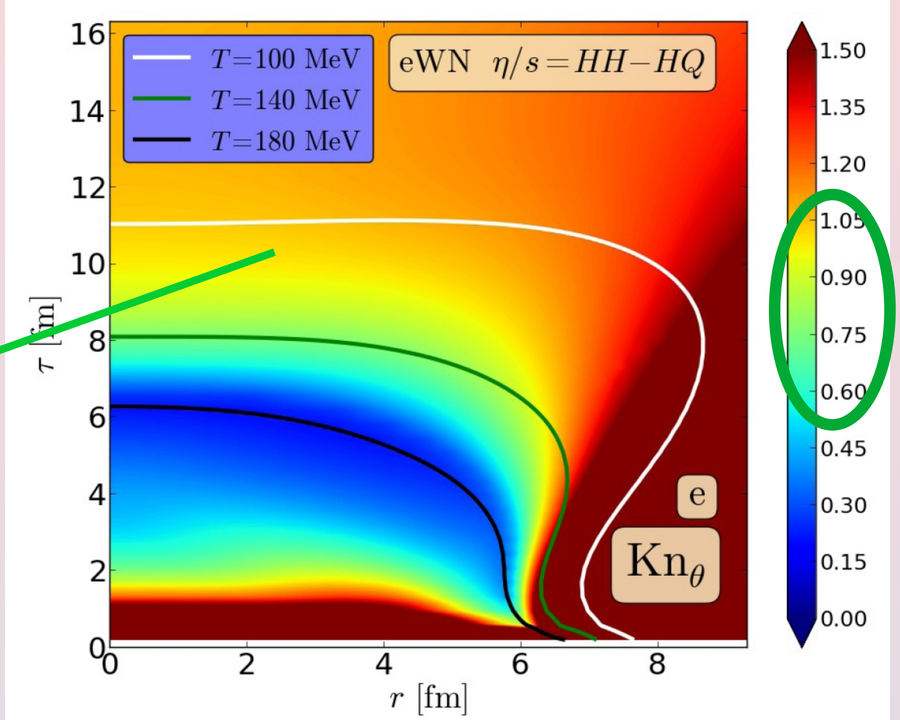
Macroscopic scale

$$L^{-1} \sim \partial_i v^i$$

Knudsen number

$$Kn \equiv \frac{l}{L}$$

Kn is not small !!

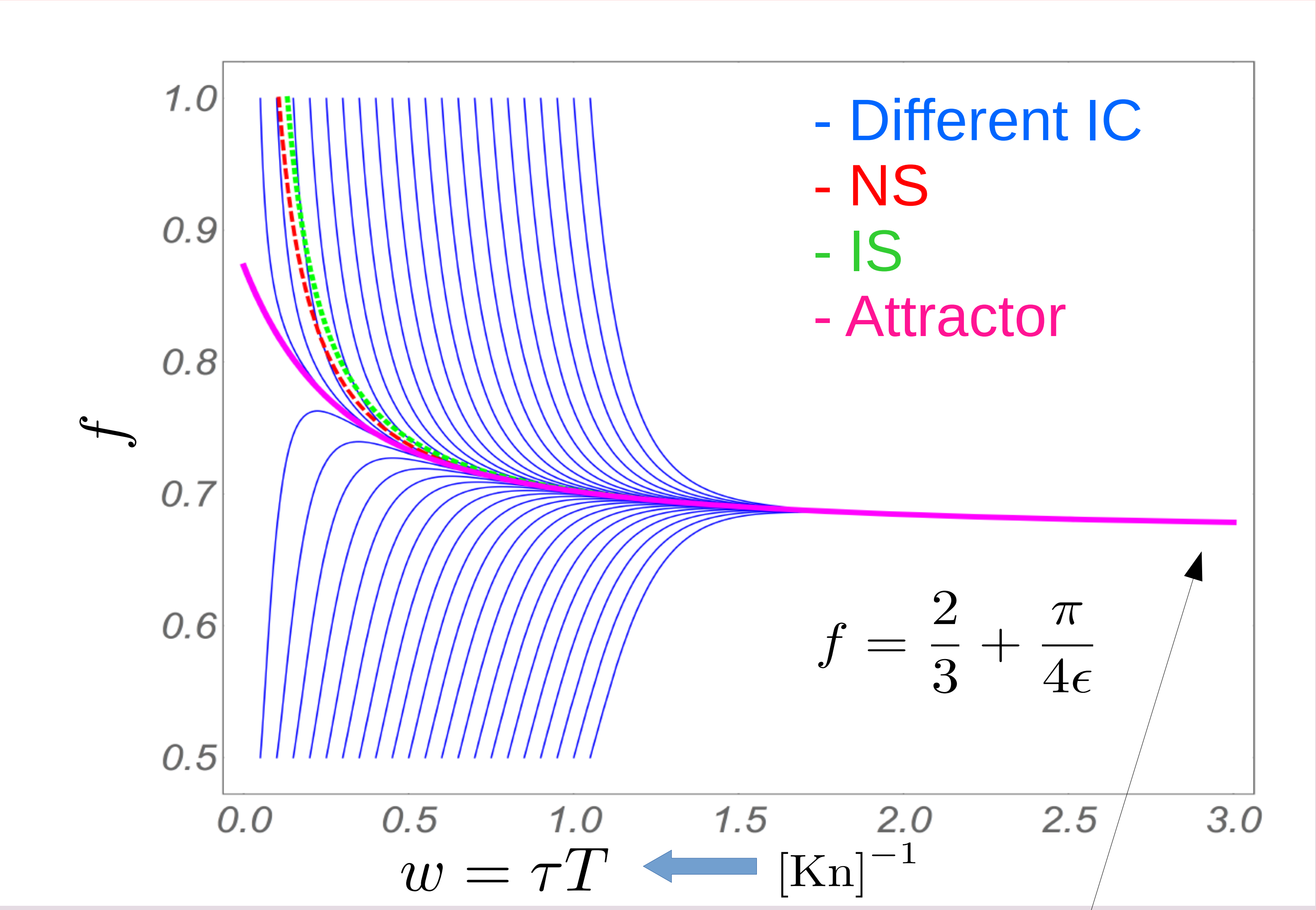


Denicol & Niemi (2014)

However, hydrodynamics works. Why?

New developments in far-from-equilibrium hydrodynamics

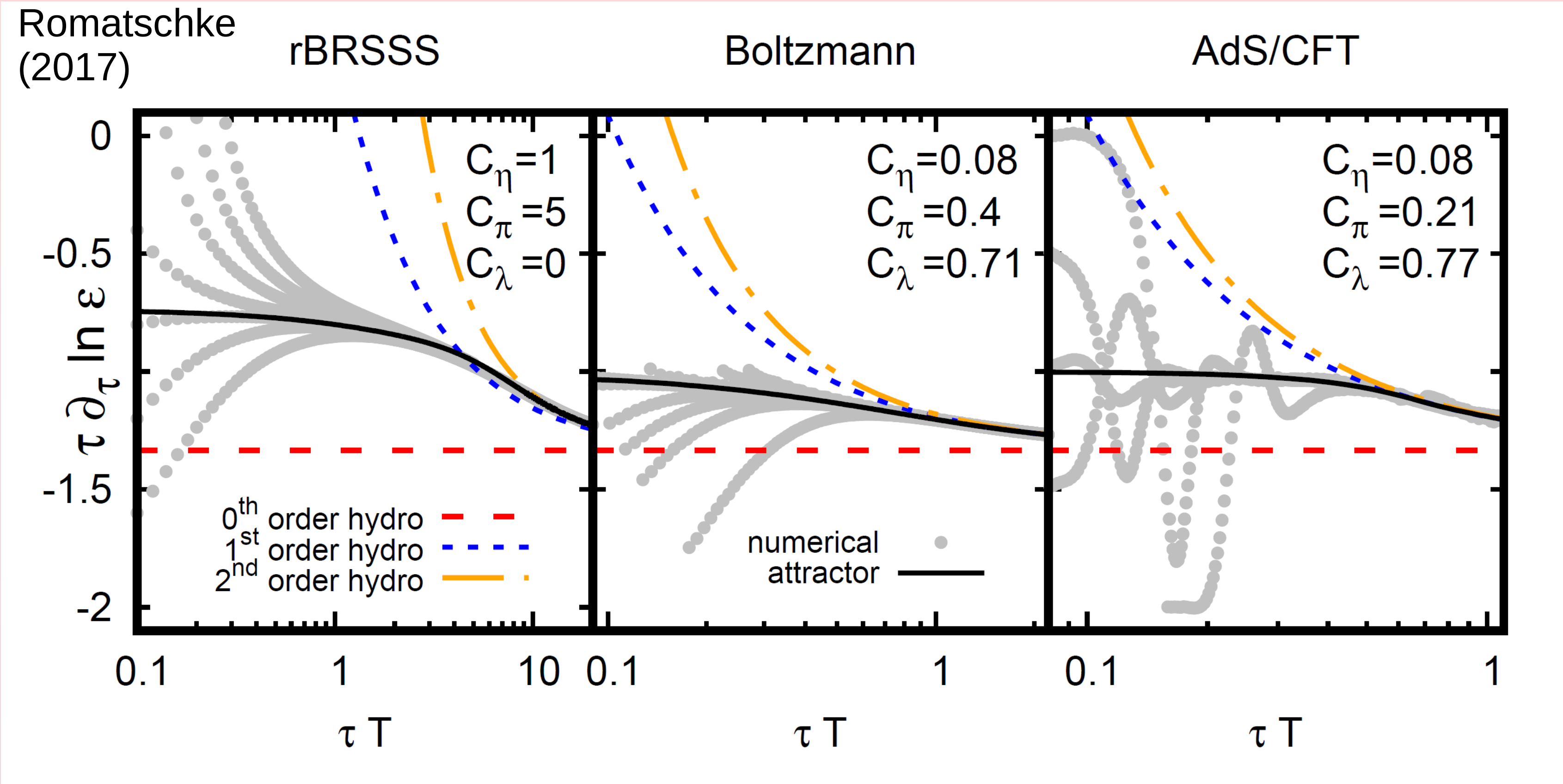
Attracting behavior in hydrodynamics



Same late time behavior independent of the IC!!!

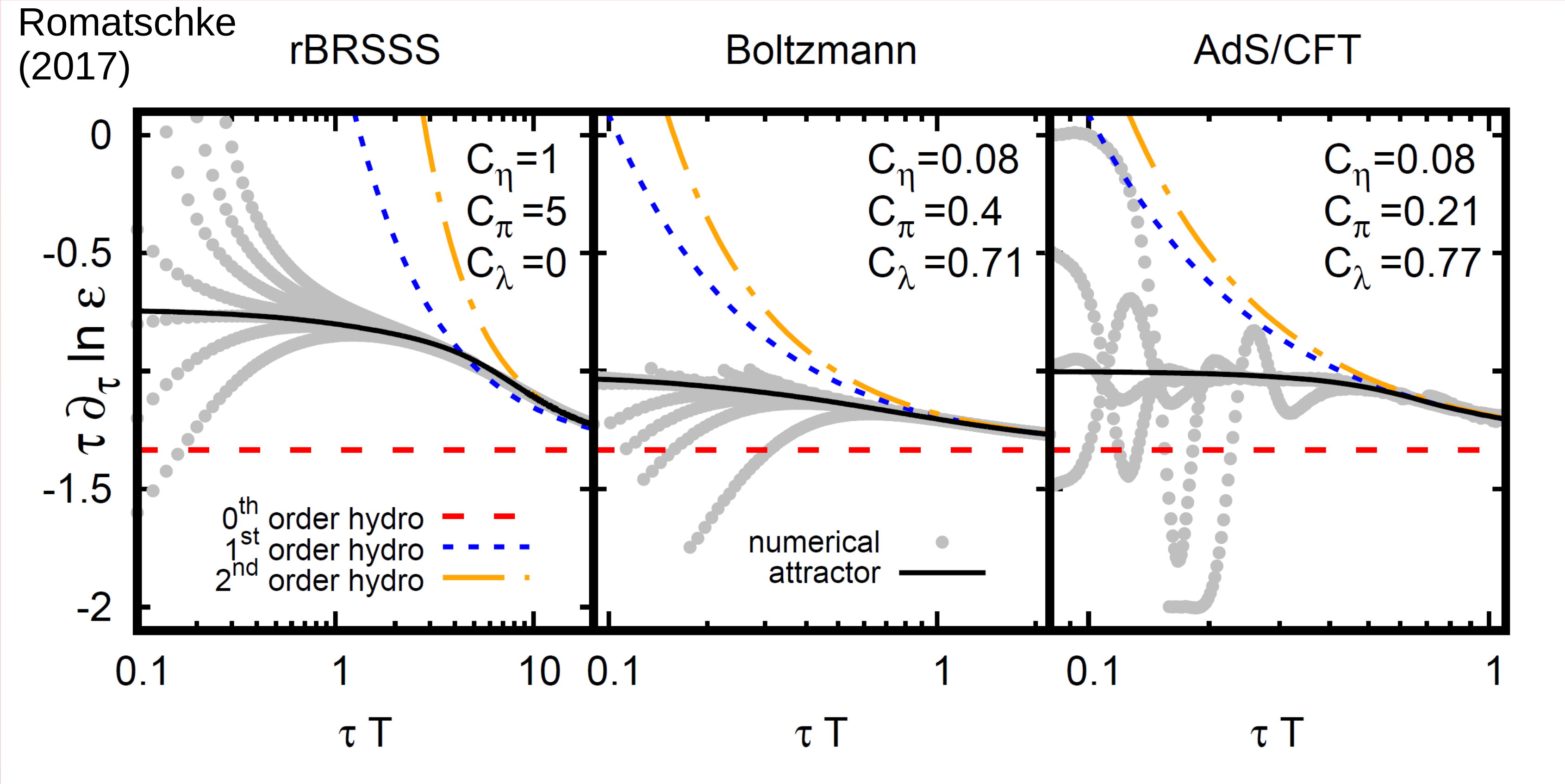
Heller and Spalinski (2015)

Message to take



- *arbitrarily far-from-equilibrium initial conditions used to solve hydro equations merge towards a unique line (attractor).*
- *Independent of the coupling regime.*
- *Attractors can be determined from very few terms of the gradient expansion*
- *At the time when hydrodynamical gradient expansion merges to the attractor, the system is far-from-equilibrium, i.e. large pressure anisotropies are present in the system* $P_L \neq P_T$

Message to take



Existence of a new theory for far-from-equilibrium fluids

- *What are their properties?*

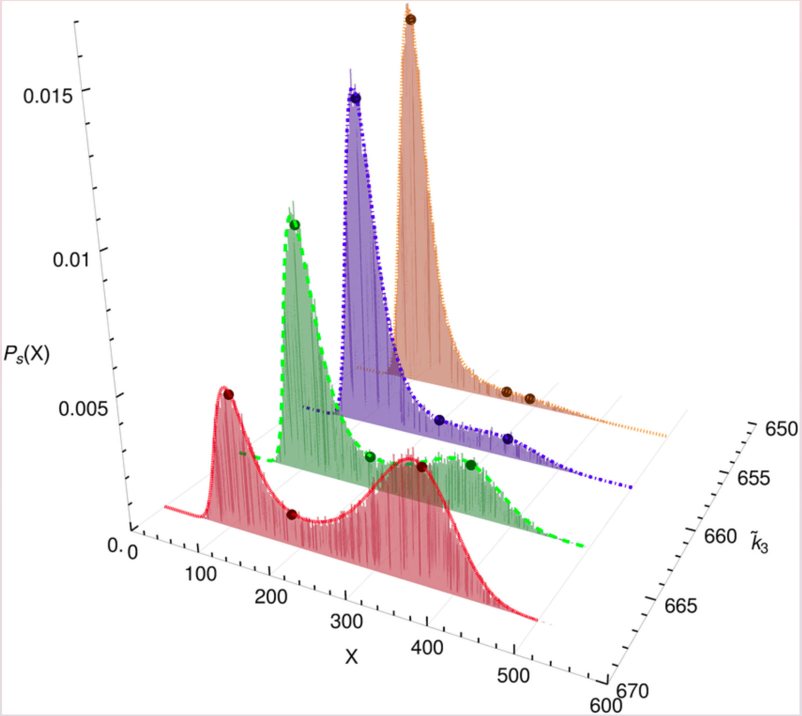
In this talk:

Far-from-equilibrium: Fokker Planck Equation

↓ **?**

Hydrodynamics

Fokker Planck Equation



Bjorken expansion and kinetics

Bjorken model

$$T^{\mu\nu} = \sum_{k=0}^{\infty} a_k^{\mu\nu} \left(Kn\right)^k$$

$$Kn = \left(\tau T(\tau)\right)^{-1}$$

Landau & Lifschitz, Physical kinetics
For gluons: A. Mueller (1999)

Bjorken expansion and kinetics

Bjorken model

$$T^{\mu\nu} = \sum_{k=0}^{\infty} a_k^{\mu\nu} \, (Kn)^k$$

$$Kn = \left(\tau \, T(\tau)\right)^{-1}$$

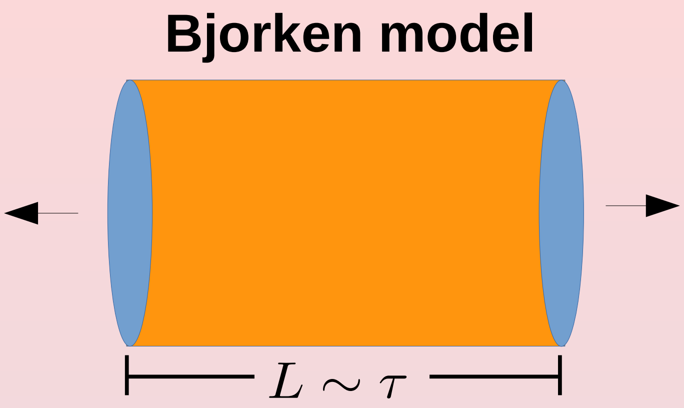
$$\partial_\tau f(\tau, p_T, p_z) = \mathcal{C}_{diff.}[f]$$

Free expansion

Particle imbalance

Landau & Lifschitz, Physical kinetics
For gluons: A. Mueller (1999)

Bjorken expansion and kinetics



$$T^{\mu\nu} = \sum_{k=0}^{\infty} a_k^{\mu\nu} (Kn)^k$$

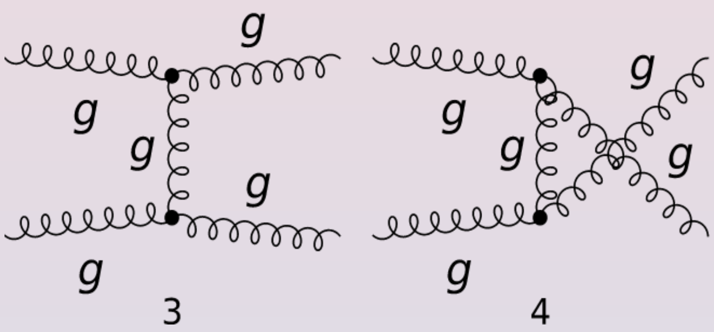
$$Kn = (\tau T(\tau))^{-1}$$

$$\partial_\tau f(\tau, p_T, p_z) = \mathcal{C}_{diff.}[f]$$

Free expansion

Particle imbalance

$$\mathcal{C}_{diff.}[f] =$$



Landau & Lifschitz, Physical kinetics
For gluons: A. Mueller (1999)

Moments expansion

$$\partial_\tau f(\tau, p_T, p_z) = \mathcal{C}_{diff.}[f]$$

By expanding the distribution function in orthogonal polynomials

$$f(x, \mathbf{p}) = f_{eq.}\left(E_\mathbf{p}/T(\tau)\right) \sum_{l=0}^{\infty} c_l(\tau)\, \mathcal{P}_{2l}(\cos\theta_\mathbf{p})$$

Physical observables:

$$T^{\mu\nu} = \int_\mathbf{p} p^\mu p^\nu f(x^\mu, \mathbf{p}) \equiv \text{ diag.}\left(\epsilon, P_T, P_T, P_L\right)$$

$$\epsilon \sim T^4 \qquad P_T = \epsilon\left(\frac{1}{3} - \frac{c_1}{15}\right) \qquad P_L = \epsilon\left(\frac{1}{3} + \frac{2}{15}c_1\right)$$

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Moments expansion

$$\partial_\tau f(\tau, p_T, p_z) = \mathcal{C}_{diff.}[f]$$

By expanding the distribution function in orthogonal polynomials

$$f(x, \mathbf{p}) = f_{eq.}\left(E_{\mathbf{p}}/T(\tau)\right) \sum_{l=0}^{\infty} c_l(\tau)\, \mathcal{P}_{2l}(\cos\theta_{\mathbf{p}})$$

The problem of solving the FP Eqn is mapped into solving a nonlinear set of ODEs for the Legendre moments

$$\frac{d\mathbf{c}}{dw} = F(\mathbf{c}, w)$$

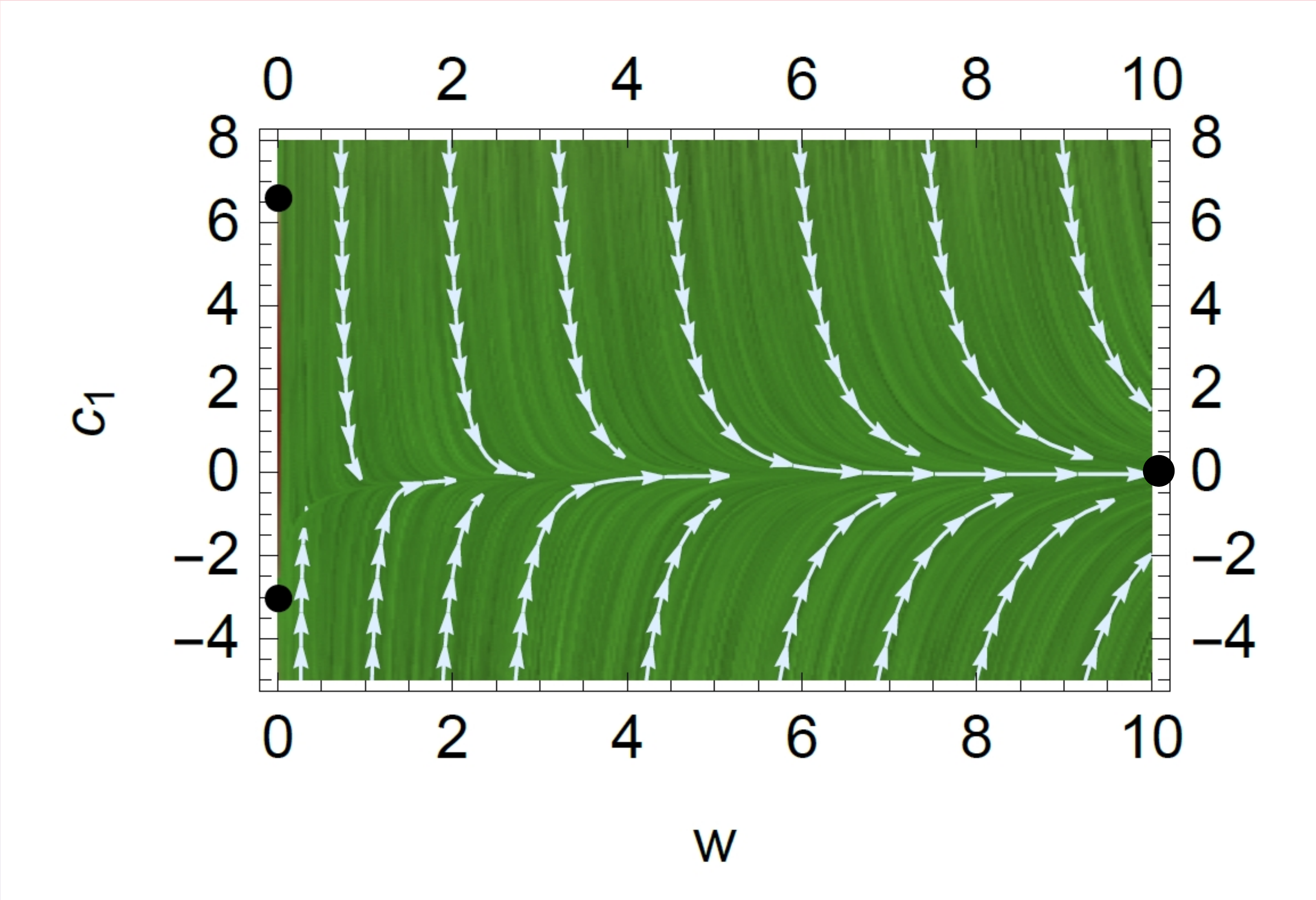
$$w = T\tau \sim Kn^{-1}$$

Non-autonomous dynamical system

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Flow lines in phase space

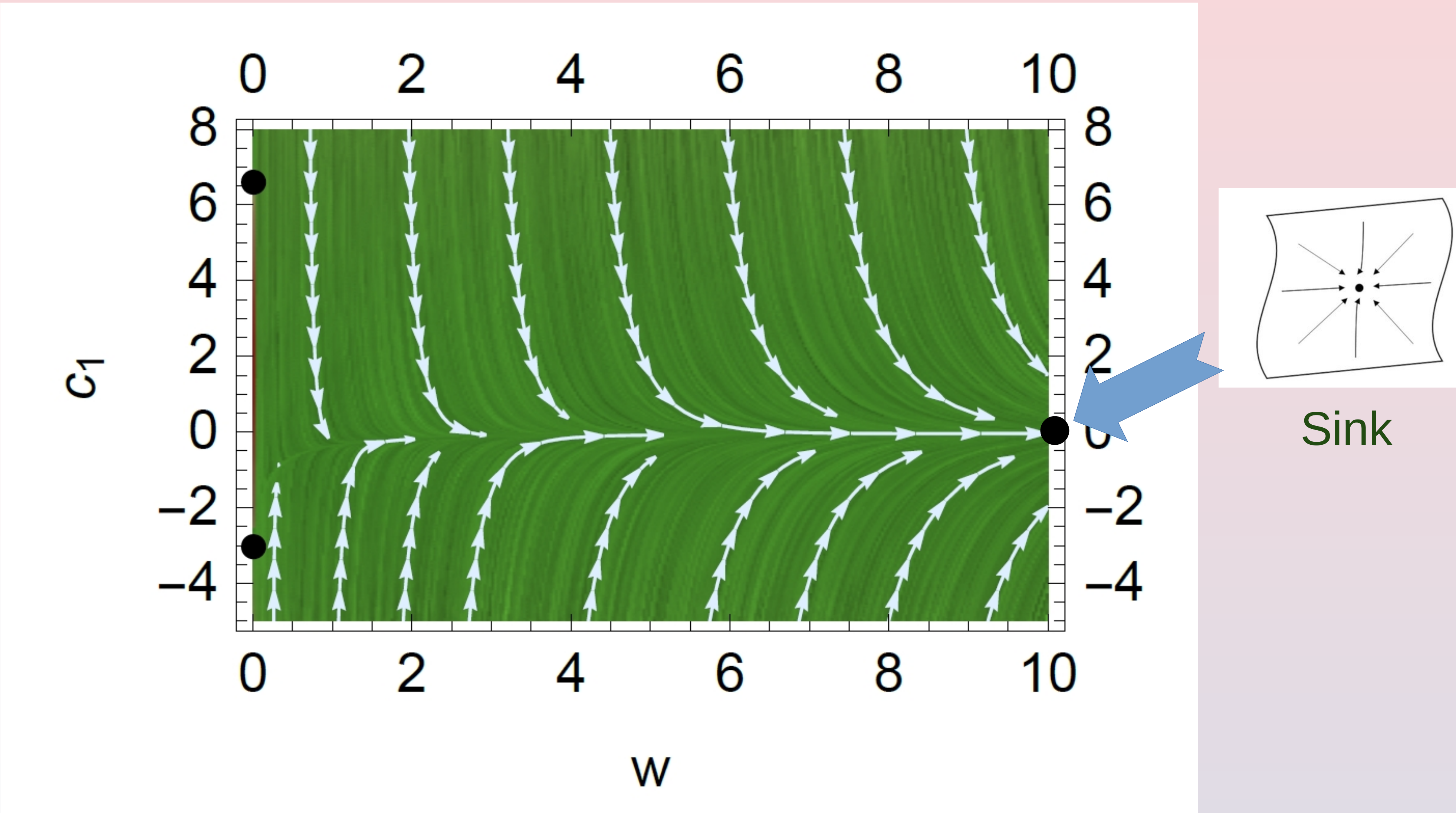
$$\frac{d\mathbf{c}}{dw} = F(\mathbf{c}, w)$$



Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Flow lines in phase space

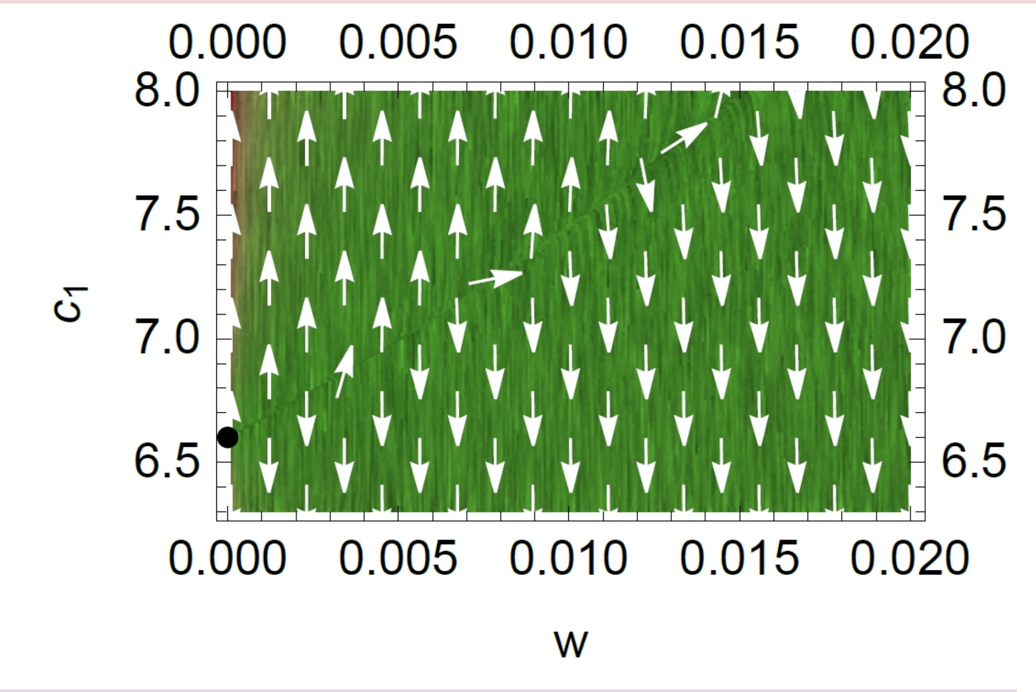
$$\frac{d\mathbf{c}}{dw} = F(\mathbf{c}, w)$$

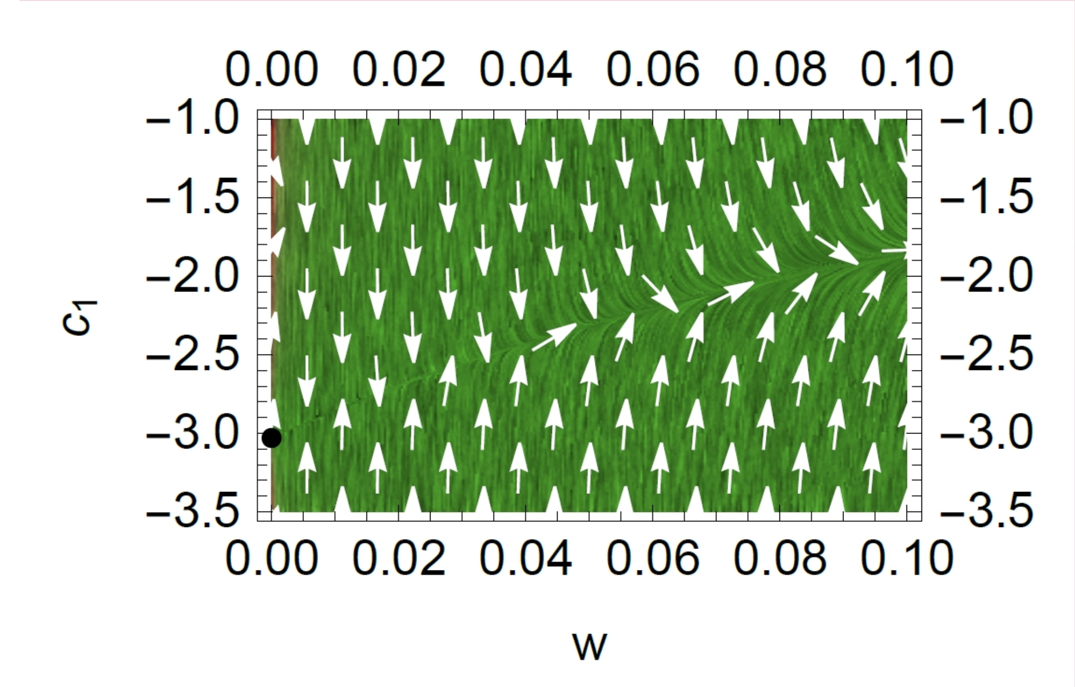


Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Flow lines in phase space

$$\frac{d\,c_1}{dw} = F(c_1, w)$$



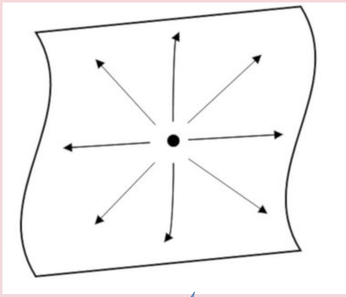


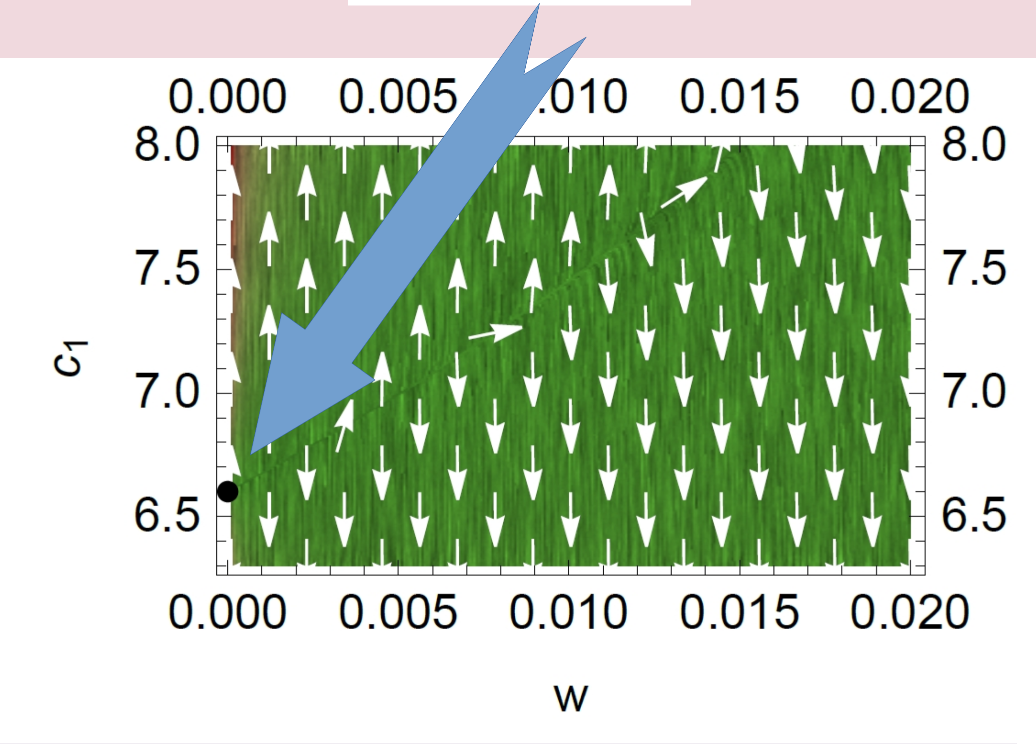
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

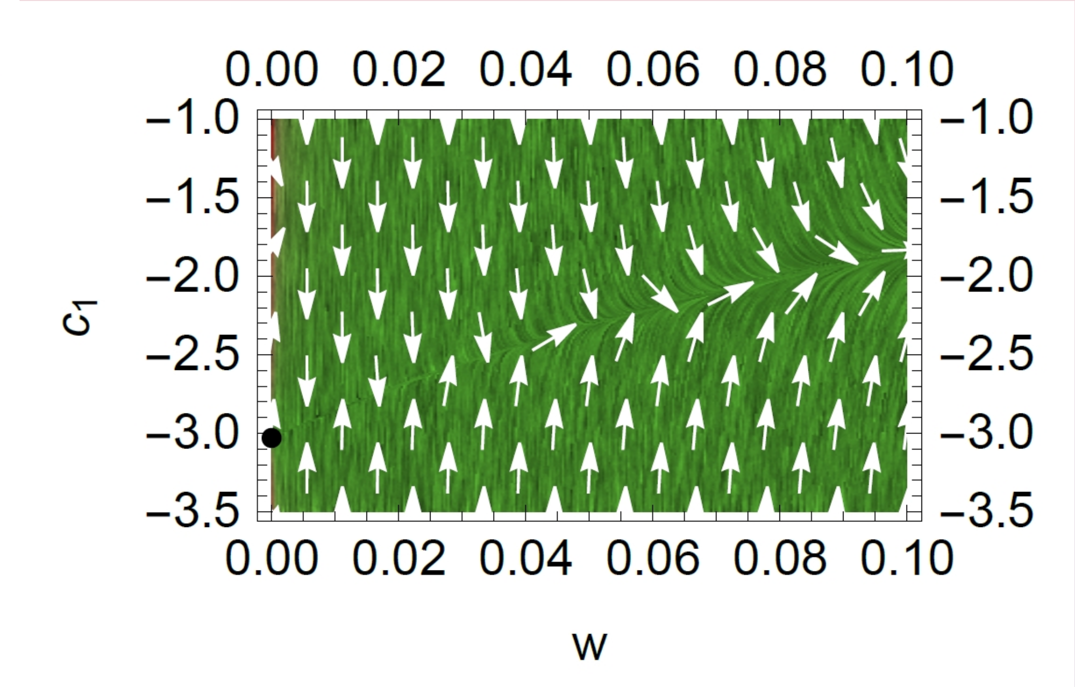
Flow lines in phase space

$$\frac{d\,c_1}{dw} = F(c_1, w)$$

Source





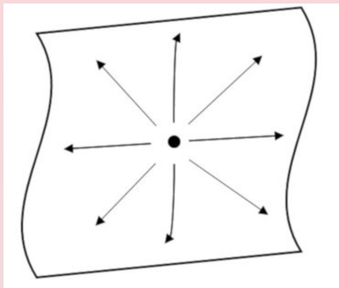


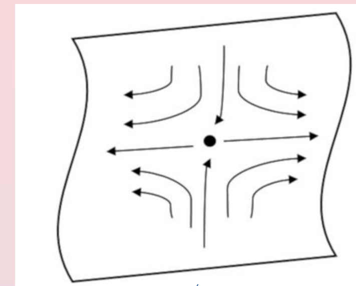
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Flow lines in phase space

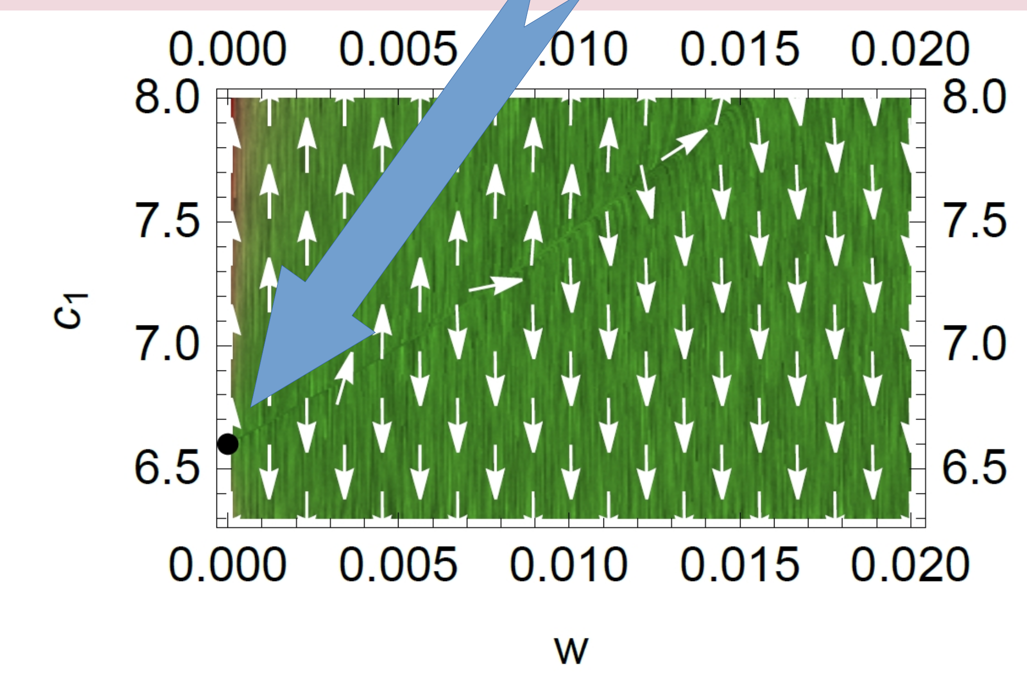
$$\frac{d\,c_1}{dw} = F(c_1, w)$$

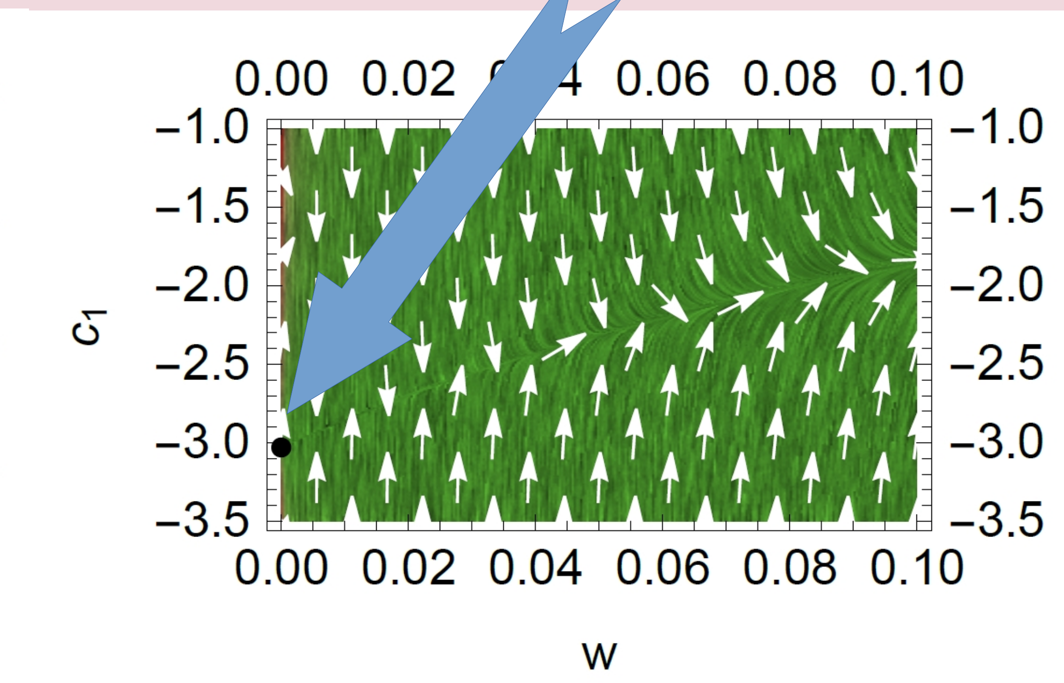
Source





Saddle





Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

UV and IR regimes

$$\frac{d\mathbf{c}}{dw} = F(\mathbf{c}, w)$$

IR: w >> 1

- Near equilibrium
  Linear response theory

UV: w << 1

- Extremely far from equilibrium
- Behavior of solutions depends on fixed point

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

UV and IR regimes

$$\frac{d\mathbf{c}}{dw} = F(\mathbf{c}, w)$$

IR: w >> 1

- Near equilibrium
  Linear response theory

Today

UV: w << 1

- Extremely far from equilibrium
- Behavior of solutions depends on fixed point

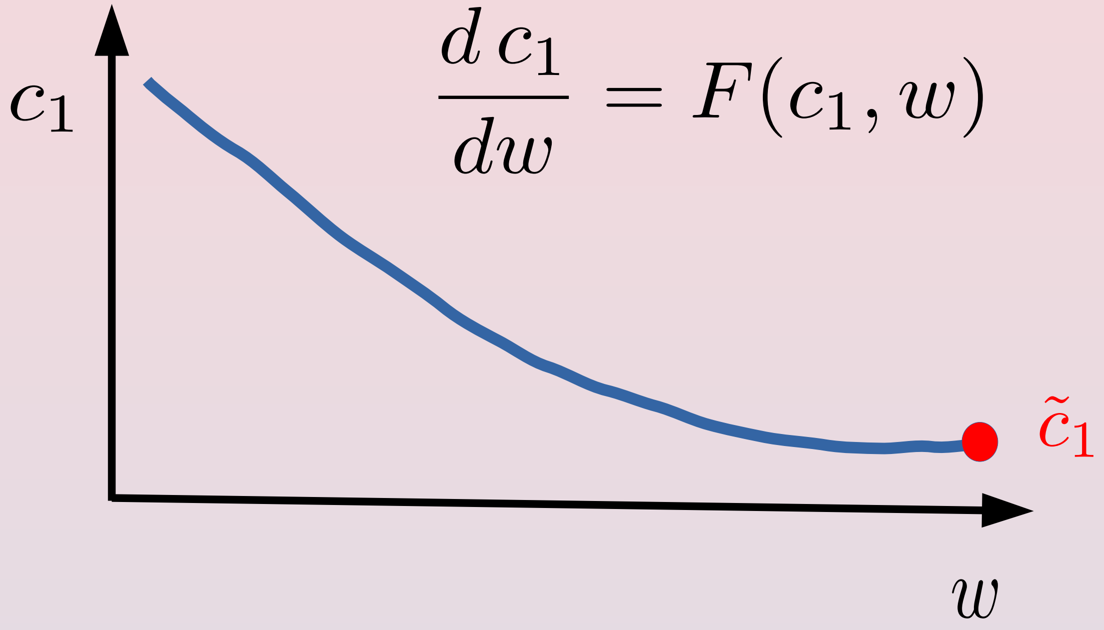
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Transasymptotics and resurgence

Basic idea:

Reconstruct the solution of ODE by knowing its asymptotic behavior

$$\frac{d\,c_1}{dw} = F(c_1, w)$$

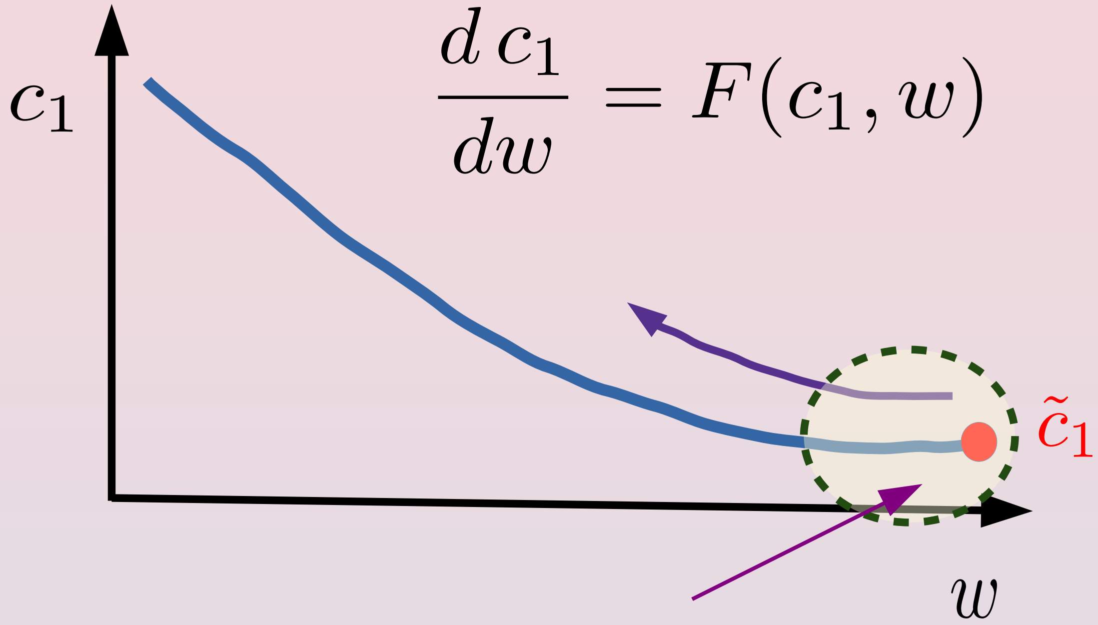


Transasymptotics and resurgence

Basic idea:

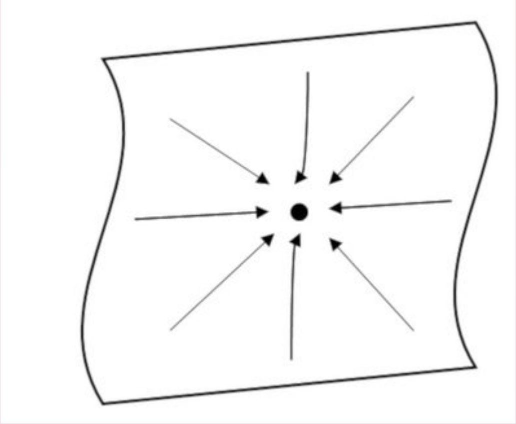
Reconstruct the solution of ODE by knowing its asymptotic behavior

$$\frac{d\,c_1}{dw} = F(c_1, w)$$



In some cases it is possible provided the knowledge of the fluctuations around the fixed points of the ODE

Transseries solutions in the IR regime

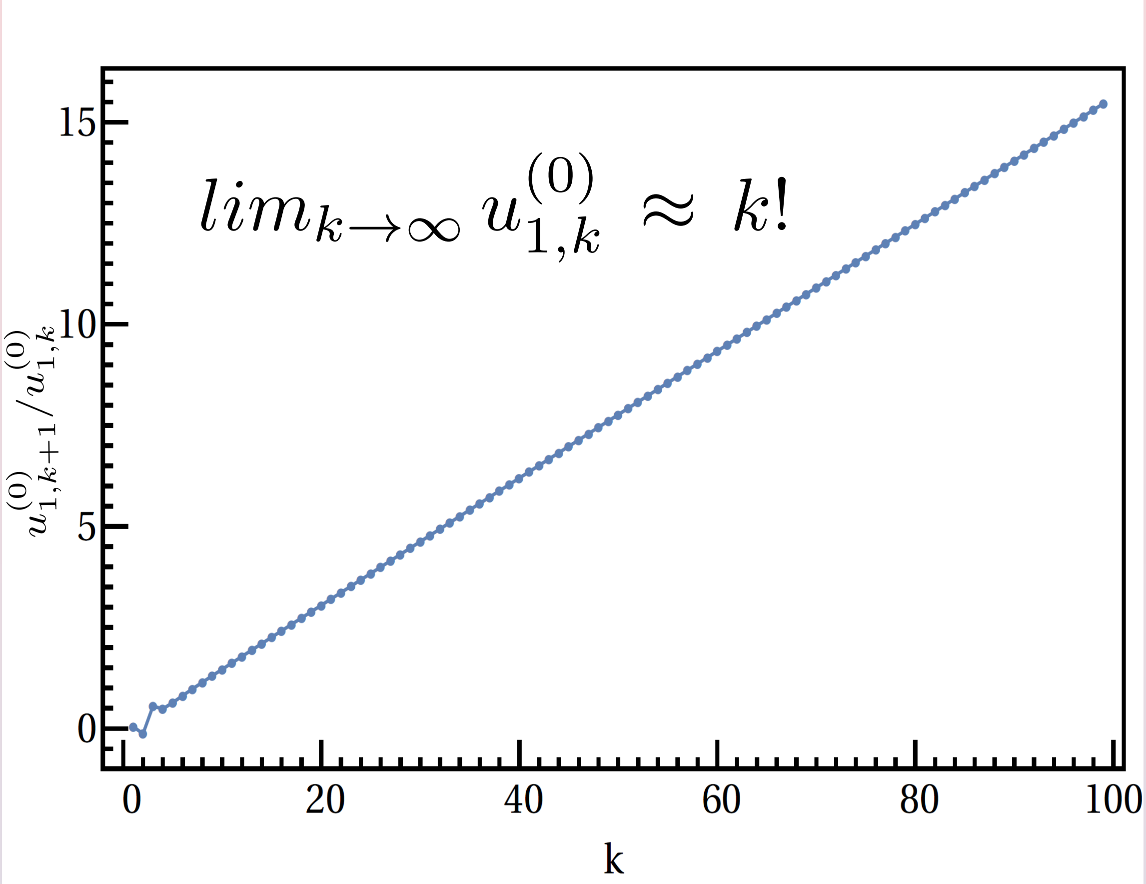


IR perturbative expansion

$$\frac{d\,c_1}{dw} = F(c_1, w)$$

Asymptotic solution looks like

$$c_1 = \sum_{k=1} u^{(0)}_{1,k}\, w^{-k}$$



Perturbative asymptotic expansion is divergent!!!!

Borel resummation is one way to sort out this type of situations.

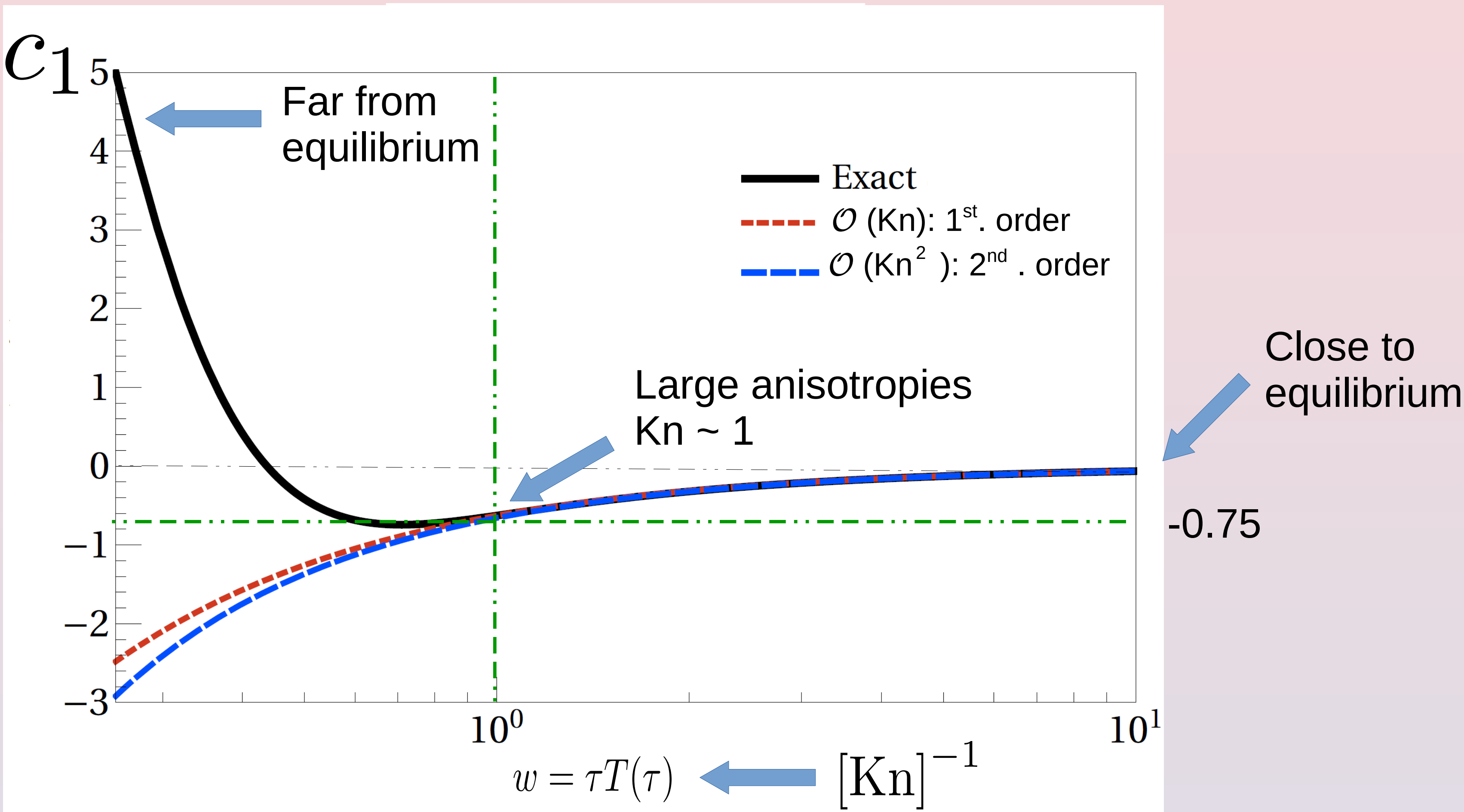
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

IR perturbative expansion

$$\frac{d\,c_1}{dw} = F(c_1, w)$$

Asymptotic solution looks like

$$c_1 = \sum_{k=1} u^{(0)}_{1,k}\, w^{-k}$$



Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Fluctuation around IR

Linearize around the the perturbative expansion series

$$\frac{d\delta c_1}{dw} = \frac{\partial F_1}{\partial c_1}\bigg|_{c_1=\bar{c}_1} \delta c_1$$

$$\delta c_1(w) = \sigma_1 \, e^{-S_1 w} \, w^{-b_1}$$

Lyapunov exponent (S_1)   Anomalous dimension (b_1)

Continue doing this procedure to all perturbative orders

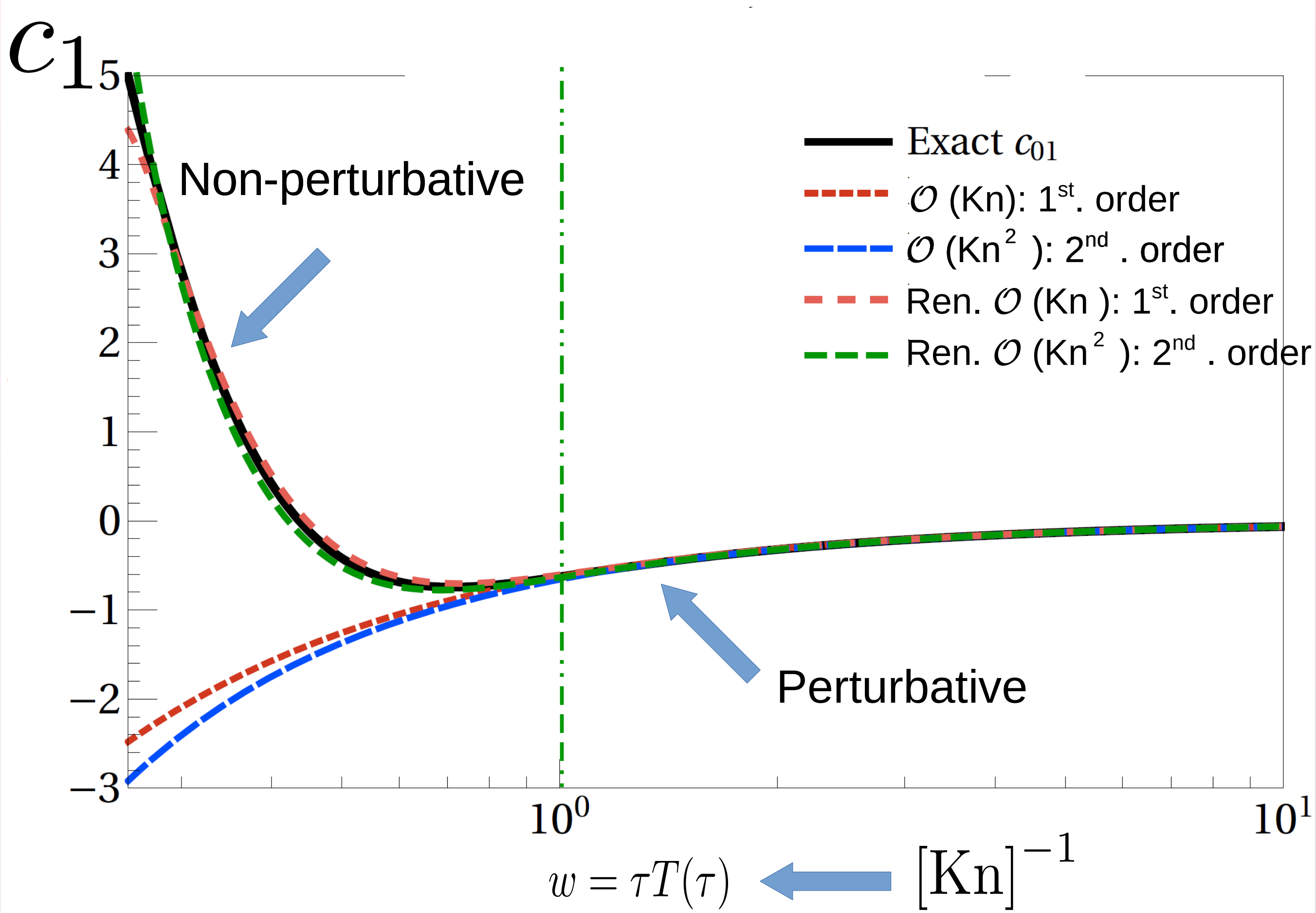
Martinez et. al.   1805.07881, 1901.08632, 1911.06406, 2011.08235

IR transseries solutions

Asymptotic expansion → Transseries solutions
Costin (1998)

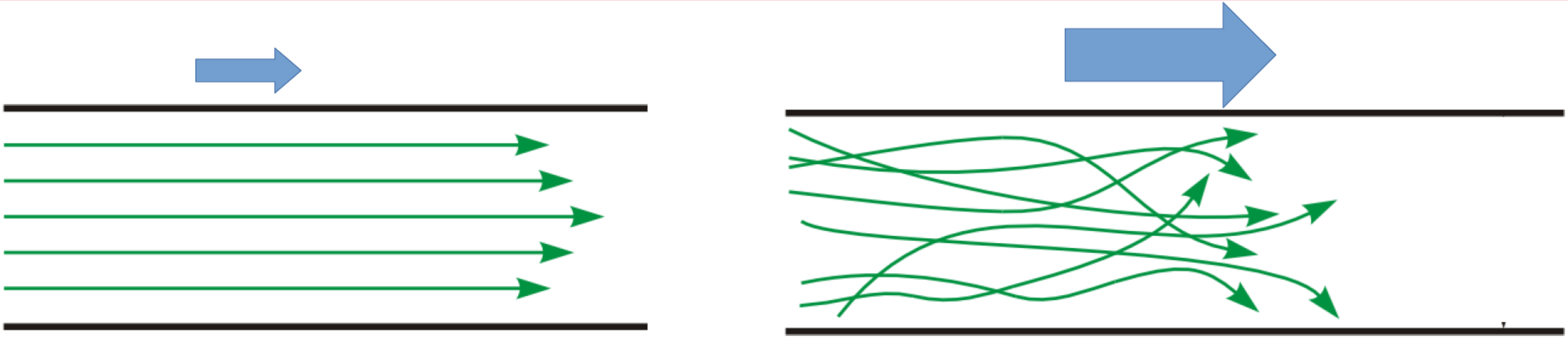
$$c_1 = \sum_{k=1} u_{1,k}^{(0)} w^{-k} \quad \rightarrow \quad c_1 = \sum_{k=1}^{\infty} \left[u_{1,k}^{(0)} + \sum_{l=1}^{\infty} u_{1,k}^{(l)} \left(\sigma\, e^{-S/\mathrm{Kn}} w^{\beta} \right)^{l} \right] w^{-k}$$

'Instanton'

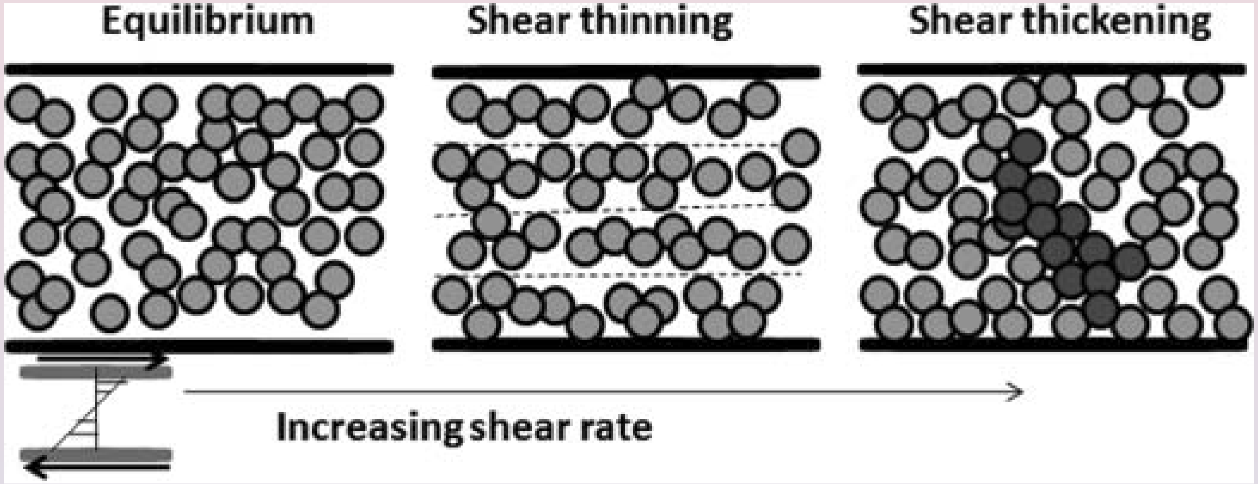


Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Non-newtonian fluids and rheology

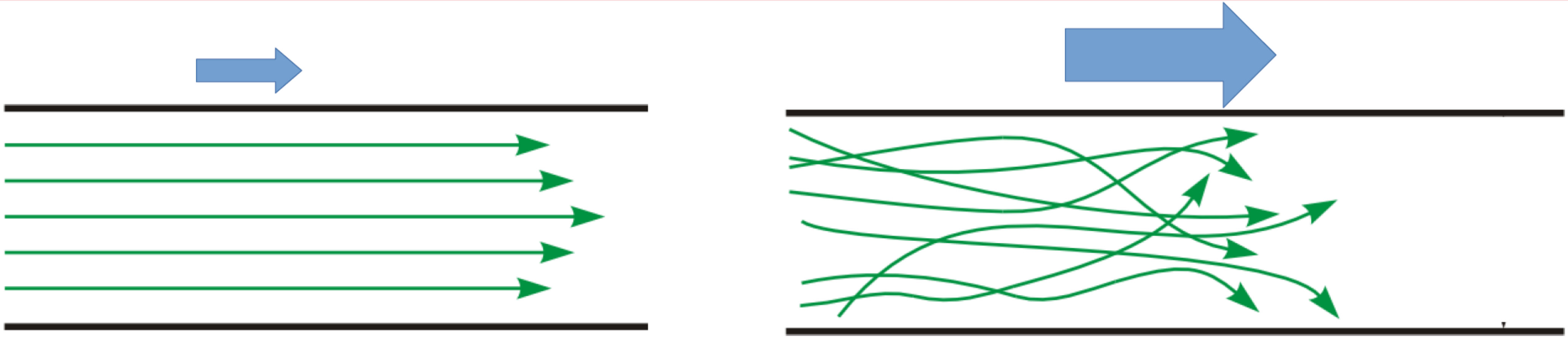


$$\pi_{xy} = -\eta\, \partial_x v_y \qquad \pi_{xy} = -\eta(\partial_x v_y)\, \partial_x v_y$$



Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Non-newtonian fluids and rheology



$$\pi_{xy} = -\eta\,\partial_x v_y$$

$$\pi_{xy} = -\eta(\partial_x v_y)\,\partial_x v_y$$

Shear viscosity

- Becomes a **function** of the **gradient of the flow velocity**
- can **increase** or **decrease depending** on the **size** of the **gradient of the flow velocity**

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Transasymptotic matching

$$c_1(w) = \left[u_{1,0}^{(1)}\,\sigma_1\,\zeta_1(w) + u_{1,0}^{(2)}\,[\sigma_1\,\zeta_1(w)]^2 + \cdots\right]$$
$$+ \frac{1}{w}\left[u_{1,1}^{(0)} + u_{1,1}^{(1)}\,\sigma_1\,\zeta_1(w) + u_{1,1}^{(2)}\,[\sigma_1\,\zeta_1(w)]^2 + \cdots\right]$$
$$+ \frac{1}{w^2}\left[u_{1,2}^{(0)} + u_{1,2}^{(1)}\,\sigma_1\,\zeta_1(w) + u_{1,2}^{(2)}\,[\sigma_1\,\zeta_1(w)]^2 + \cdots\right]$$

$$\zeta_1 = e^{-S_1 w} w^{b_1}$$

Perturbative IR data

Non-Perturbative Resummation of fluctuations around the IR perturbative expansion

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Transasymptotic matching

$$c_1 \equiv \sum_{k=0}^{\infty} G_{1,k}(\sigma_1 \zeta_1(w))\, w^{-k}$$

$$\zeta_1 = e^{-S_1 w} w^{b_1}$$

$$G_{1,k}(\sigma_1 \zeta_1(w)) = \sum_{n=0}^{\infty} u_{1,k}^{(n)} \left[\sigma_1\, \zeta_1(w)\right]^n$$

Transasymptotic matching

$$c_1 \equiv \sum_{k=0}^{\infty} G_{1,k}(\sigma_1 \zeta_1(w))\, w^{-k}$$

$$\zeta_1 = e^{-S_1 w} w^{b_1}$$

$$G_{1,k}(\sigma_1 \zeta_1(w)) = \sum_{n=0}^{\infty} u_{1,k}^{(n)} \left[\sigma_1\, \zeta_1(w)\right]^n$$

Each function $G_{1,k}$ satisfies:

$$\lim_{w \to \infty} G_{1,k} = u_{1,k}^{(0)}$$

→ Asymptotic value of the transport coefficient

Transasymptotic matching

$$c_1 \equiv \sum_{k=0}^{\infty} G_{1,k}(\sigma_1 \zeta_1(w))\, w^{-k}$$

$$\zeta_1 = e^{-S_1 w} w^{b_1}$$

$$G_{1,k}(\sigma_1 \zeta_1(w)) = \sum_{n=0}^{\infty} u_{1,k}^{(n)} \left[\sigma_1\, \zeta_1(w)\right]^n$$

Each function $G_{1,k}$ satisfies:

$$\lim_{w \to \infty} G_{1,k} = u_{1,k}^{(0)}$$

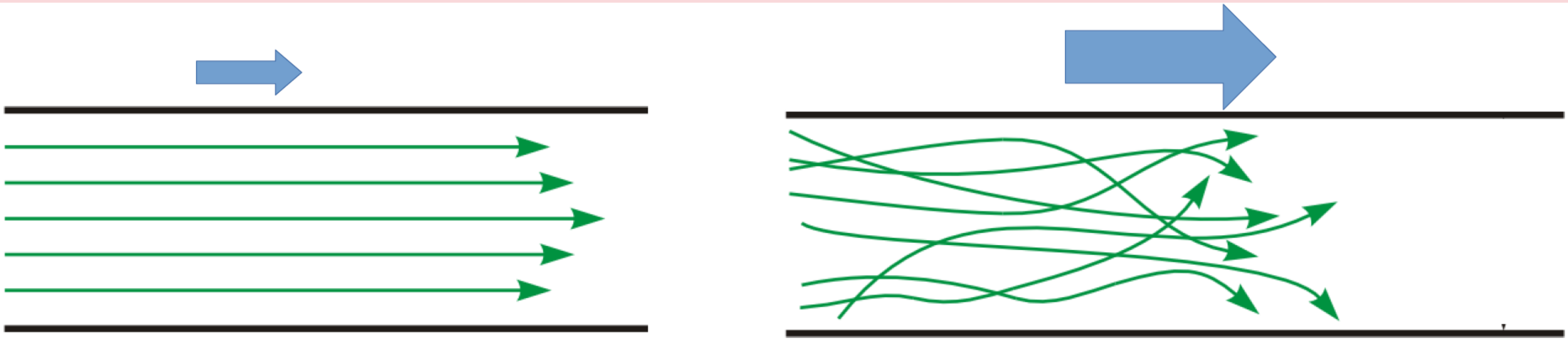
→ Asymptotic value of the transport coefficient

E.g. $$\frac{\eta}{s} = -\frac{3}{40} \lim_{w \to \infty} G_{1,1}(\sigma_1 \zeta(w))$$

⇒ $$\frac{\eta}{s}(w) = -\frac{3}{40} G_{1,k}(\sigma_1 \zeta(w))$$

Non-equilibrium transport coefficient!!!

Non-newtonian fluids and rheology



$$\pi_{xy} = -\eta\, \partial_x v_y \qquad \pi_{xy} = -\eta(\partial_x v_y)\, \partial_x v_y$$

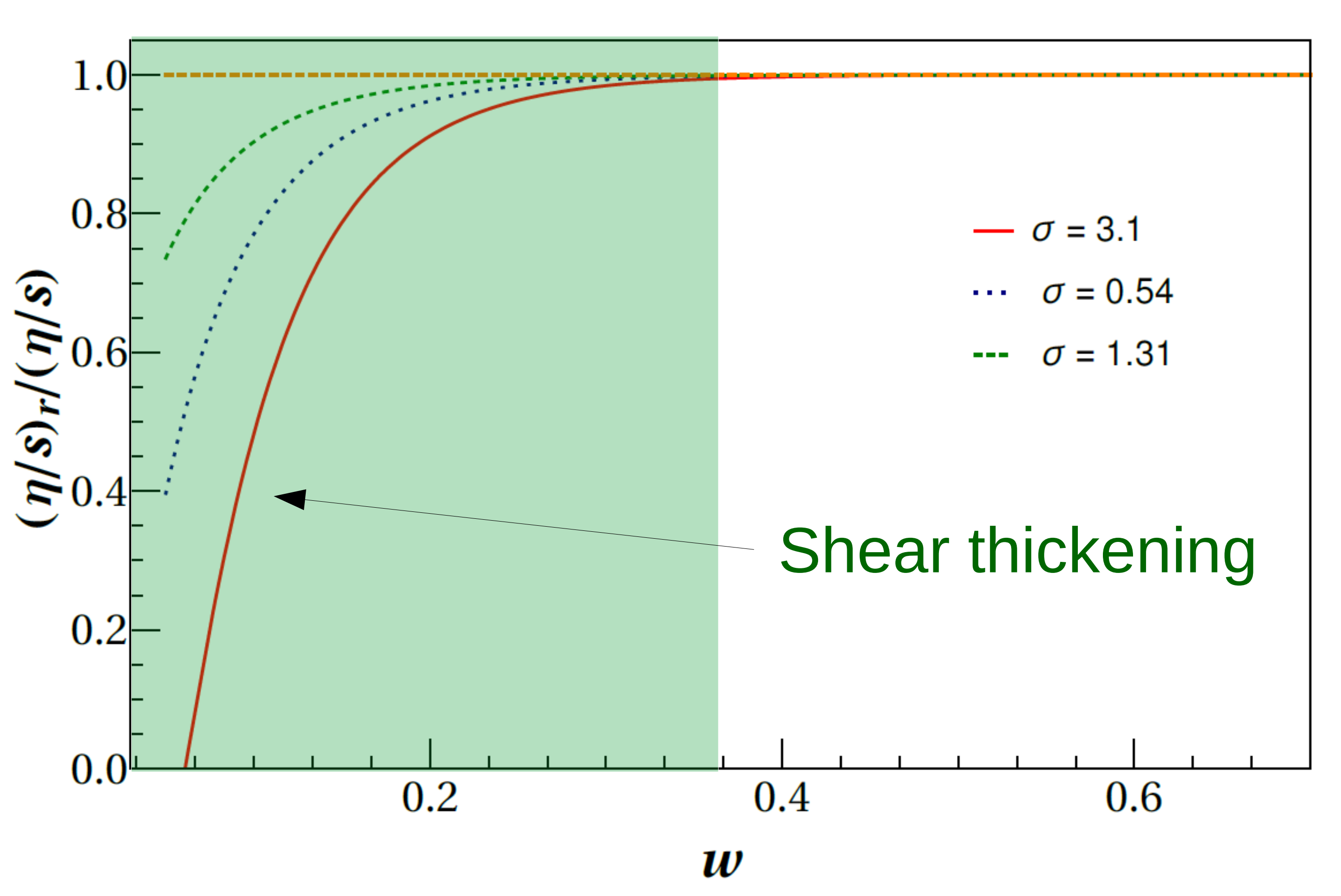
$$\frac{\eta}{s}(w) = -\frac{3}{40} G_{1,1}(\sigma_1 \zeta(w)) \qquad \zeta_1 = e^{-S_1 w} w^{b_1}$$

Thus, transseries solutions resummes non-perturbative contributions when the dissipative corrections are large.
As a result, **each transport coefficient is renormalized**

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Transient rheological behavior

$$\frac{\eta}{s}(w) = -\frac{3}{40} G_{1,1}(\sigma_1 \zeta(w))$$

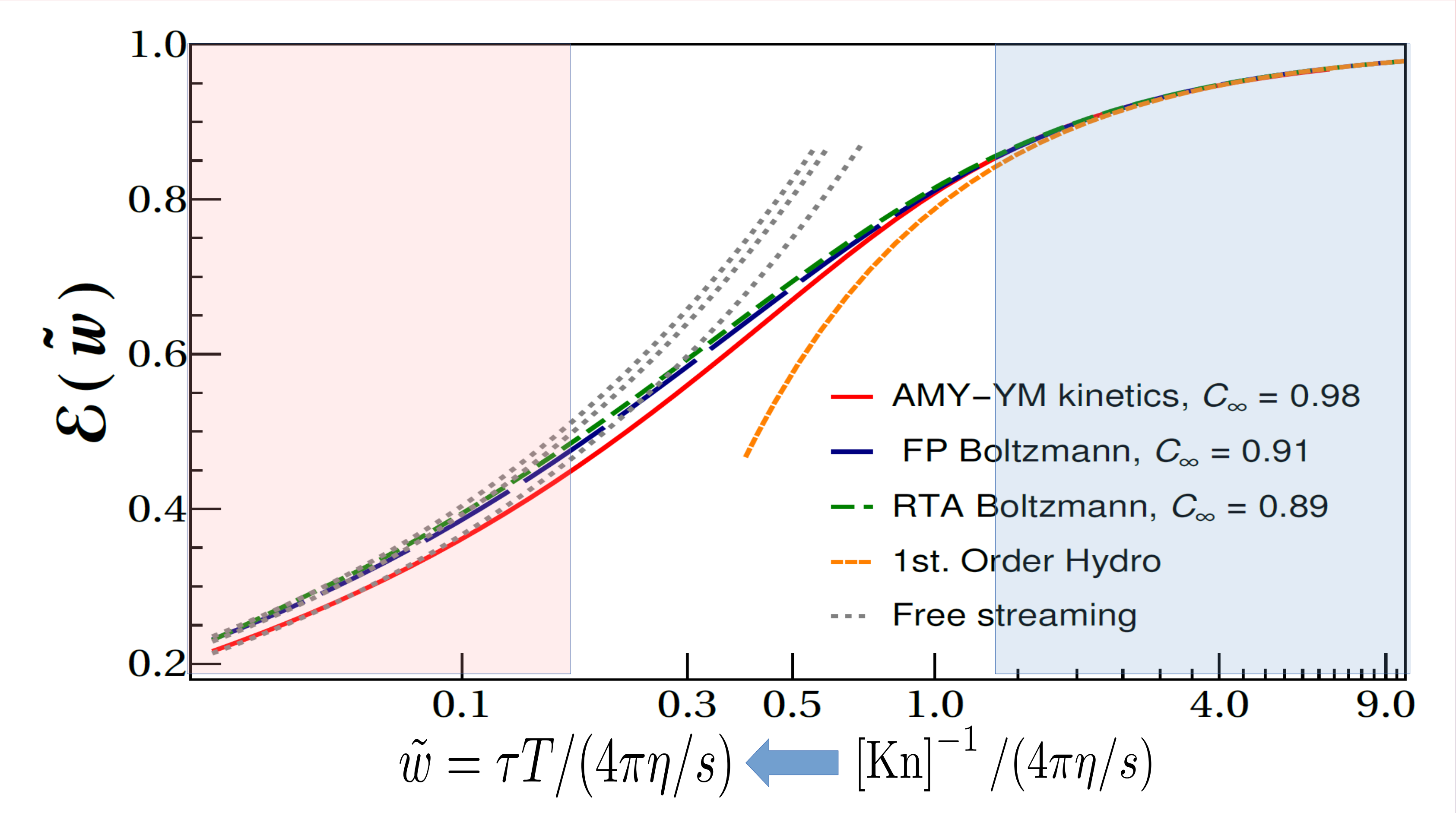


Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Universal properties

Universal features of attractors

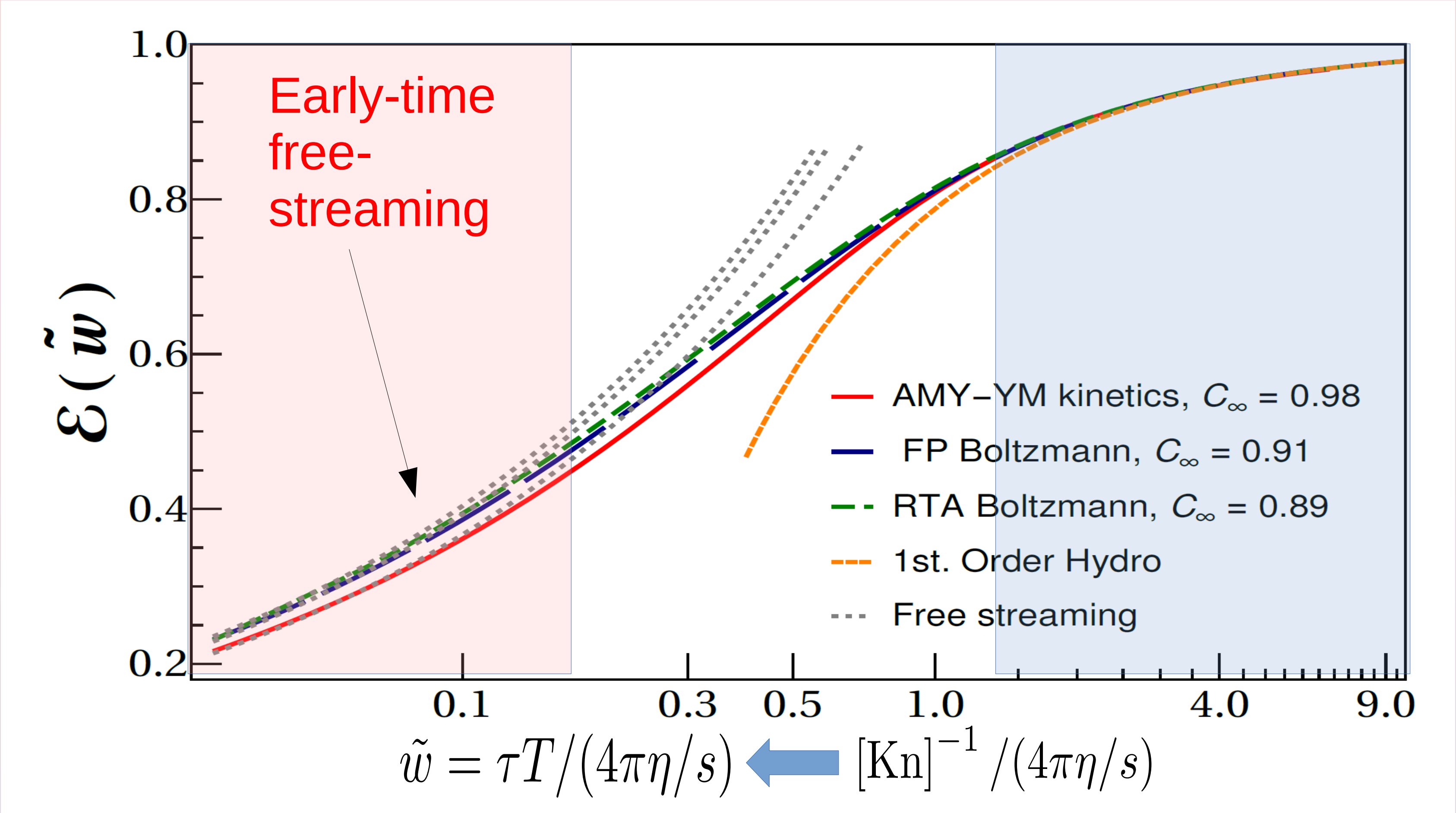
$$\mathcal{E} = \frac{\tau^{4/3}\,\epsilon(\tau)}{\left(\tau^{4/3}\epsilon\right)_{hydro}}$$



Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Universal features of attractors

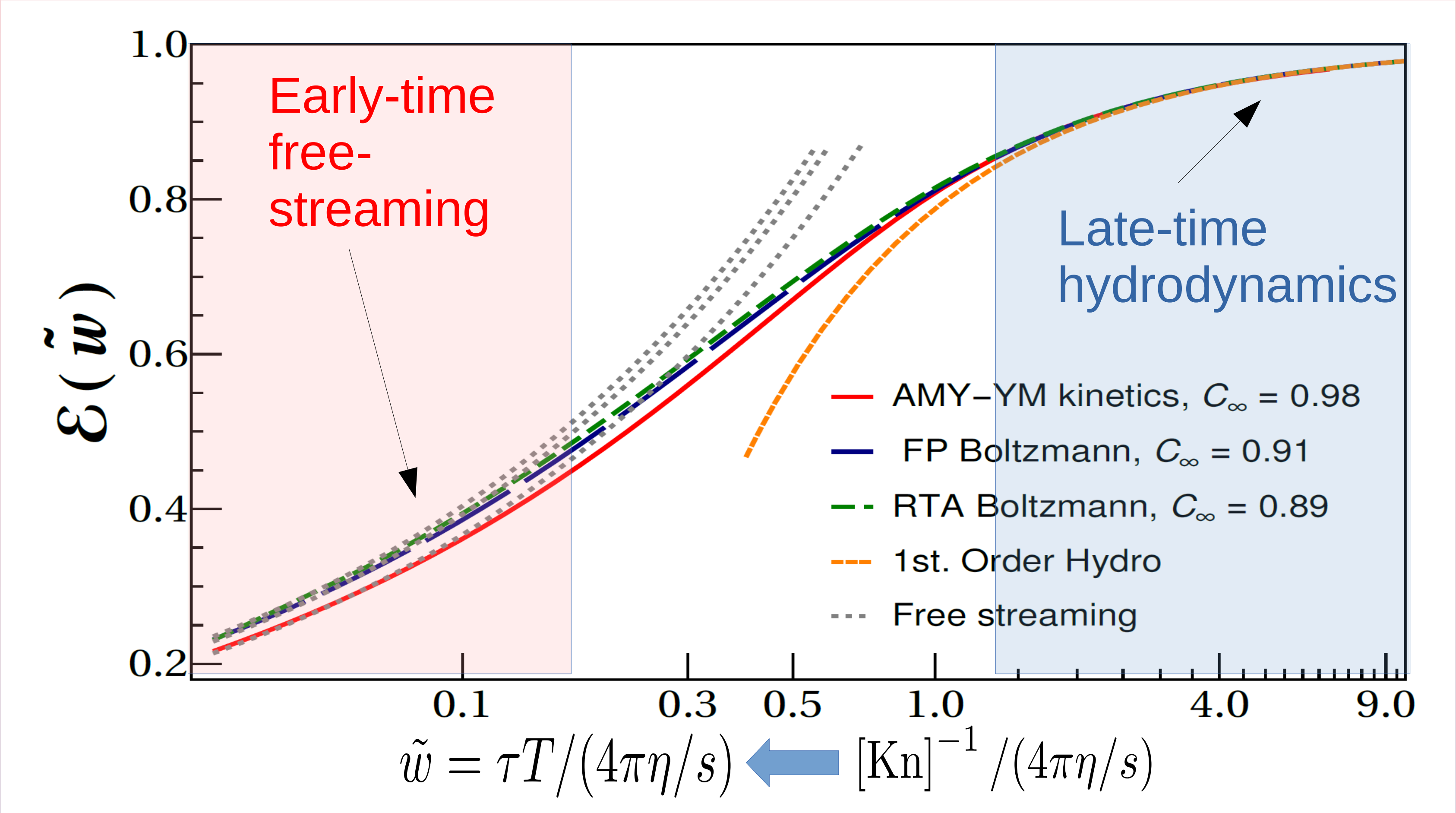
$$\mathcal{E} = \frac{\tau^{4/3}\,\epsilon(\tau)}{\left(\tau^{4/3}\epsilon\right)_{hydro}}$$



Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Universal features of attractors

$$\mathcal{E} = \frac{\tau^{4/3}\,\epsilon(\tau)}{\left(\tau^{4/3}\epsilon\right)_{hydro}}$$



Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Conclusions

1. Hydrodynamics can be formulated even if the system is far-from-equilibrium

2. Transient rheological behavior is intimately related with the formulation of a new theory of far-from-equilibrium hydrodynamics.

3. Transport coefficients get renormalized effectively after resumming non-perturbative instanton-like contributions

4. Early and late time behavior of different kinetic models are determined by free streaming and viscous hydrodynamics at early and late times respectively.

Excellent group of collaborators

A. Behtash

C. N. Camacho

S. Kamata

H. Shi

T. Schaefer

V. Skokov

Outlook

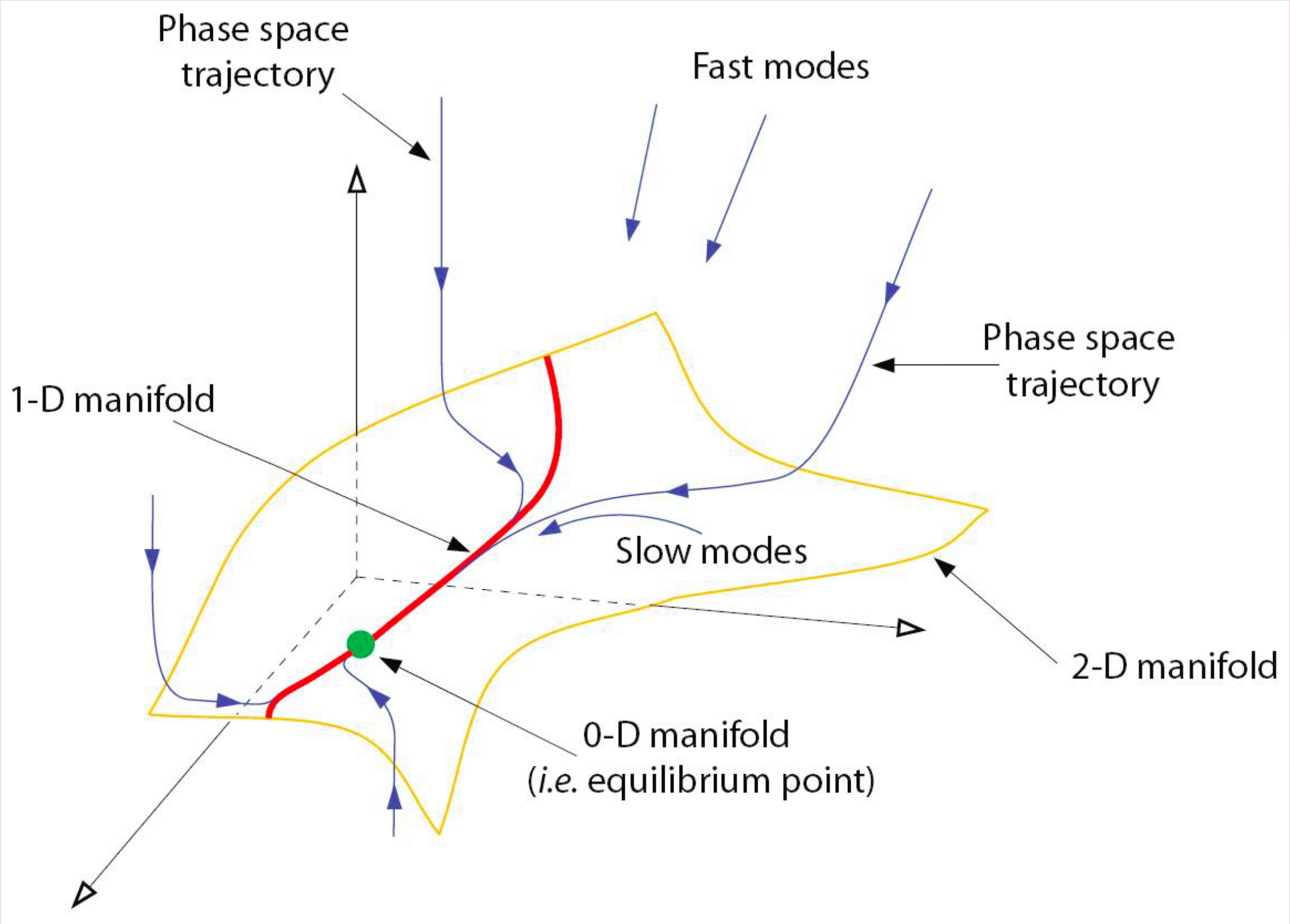
- Resurgence analysis of other relevant systems
- Relevance of attractors and connection to experiments

  - Giacalone et. al. PRL 123 (2019) 262301
  - Martinez et. al. 2012.02184

- Challenges:
  1. How to generalize to arbitrarily expanding geometries?
  2. Phase transitions?
  3. Effective action (Lyapunov functionals)
  For Gubser flow: Behtash. et. al. PRD 97 044041 (2018)

Backup slides

Slow invariant manifold picture



Transseries solutions to ODEs

If you have a non-linear differential equation of the form

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} - \frac{1}{x}\hat{B}\mathbf{y} + \mathbf{g}(x, \mathbf{y})$$

Then

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0;|\mathbf{k}|>0} C_1^{k_1}\cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\mathbf{m}} \tilde{\mathbf{y}}_{\mathbf{k}}$$

$$\tilde{\mathbf{y}}_{\mathbf{k}} = x^{-\mathbf{k}(\beta+\mathbf{m})} \sum_{l=0}^{\infty} \mathbf{a}_{\mathbf{k};l} x^{-l}$$

O. Coustin

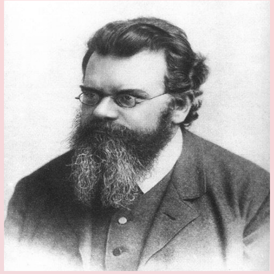
1. Non-resonance condition: Λ *does not have null eigenvalues*
2. *Regularity when* $x \to \infty$

Duke Math. J. vol 93, No 2, 1998

Moments method

Grad's moments method

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = -\mathcal{C}[f]$$



$$f(x^\mu, \mathbf{p}) = f_0 \left(1 + \sum_{l=0}^{\infty} \sum_{n=0}^{N_l} \mathcal{H}^l_{\mathbf{p},n} \, \rho_n^{\mu_1 \mu_2 \cdots \mu_l} \, p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \right)$$

Background distribution

Polynomials of energy

Irreducible moments

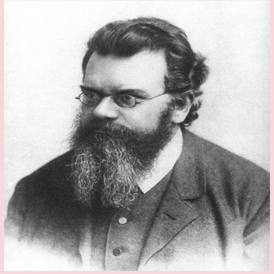
Irreducible tensors

Grad (1949), Israel-Stewart (1976), DNMR (2010)

Moments method

Grad's moments method

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = -\mathcal{C}[f]$$



$$f(x^\mu, \mathbf{p}) = f_0 \left(1 + \sum_{l=0}^{\infty} \sum_{n=0}^{N_l} \mathcal{H}^l_{\mathbf{p},n} \rho_n^{\mu_1 \mu_2 \cdots \mu_l} p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \right)$$

Relaxation to the asymptotic state of the distribution function is determined by analyzing the non-linear evolution equation of the moments

$$\frac{d\rho_r^{\mu_1 \mu_2 \cdots \mu_l}}{dt} \sim \frac{d}{dt} \left[\int_{\mathbf{p}} E^r_{\mathbf{p}} p^{\langle \mu_1} \cdots p^{\mu_l \rangle} \delta f \right]$$

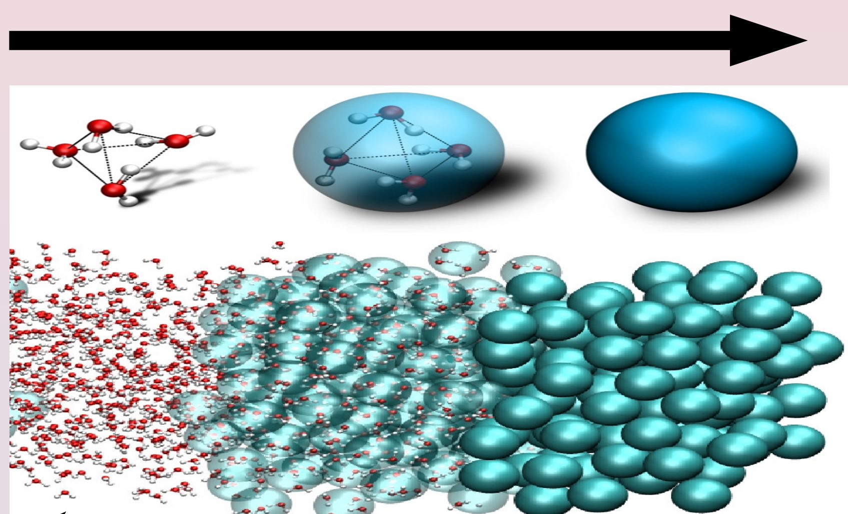
Grad (1949), Israel-Stewart (1976), DNMR (2010)

Hydro as an coarse-grained approach

How many moments do we need?

$$f(x^\mu, \mathbf{p}) = f_0 \left(1 + \sum_{l=0}^{\infty} \sum_{n=0}^{N_l} \mathcal{H}^{l}_{\mathbf{p},n} \, \rho_n^{\mu_1 \mu_2 \cdots \mu_l} p_{\langle \mu_1} \cdots p_{\mu_l \rangle} \right)$$

- **Coarse-grained procedure reduces # of degrees of freedom**
- **The slowest degrees of freedom determine hydrodynamics**
- **However, kinetic theory is highly non-linear…..**



Microscopic:
10^{23} particles

Mesoscopic:
$10^7 - 10^9$ particles

Continuum:
$T, \mu, \mu_i, \epsilon, n, p, \ldots.$

Non-autonomous dynamical system

$$\frac{d\mathbf{c}}{dw} = F(\mathbf{c}, w)$$

- Any solution, aka flow, depends on its initial value, initial and final values of w

$$\mathbf{c} \equiv \mathbf{c}(\mathbf{c}_0, w, w_0)$$

- Since future and past are not the same one requires to consider the following limits

$$\lim_{w \to \infty, w_0 fixed} \mathbf{c}(\mathbf{c}_0, w, w_0) \qquad \lim_{w_0 \to 0, w fixed} \mathbf{c}(\mathbf{c}_0, w, w_0)$$

Forward Attractor

Pullback Attractor

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Dynamical system as a RG flow

Let's rewrite the ODEs in a precise manner

$$\frac{dc_1}{d\log w} = \beta_1(c_1, w)$$

Any observable $\mathfrak{O} = \mathfrak{O}(G_{1,k}(\sigma_1\zeta_1))$

$$\frac{d\mathfrak{O}(G_{1,k}(\sigma_1\zeta_1))}{d\log w} = -\sum_{k=0}^{\infty}\left[(b_1 + S_1 w)\,\hat{\zeta}_1 G_{1,k}(\sigma_1\zeta_1)\right]\frac{\partial\mathfrak{O}}{\partial G_{1,k}}$$

RG flow equation for shear viscosity over entropy ratio is simply obtained by using

$$\frac{\eta}{s}(w) = -\frac{3}{40}G_{1,k}(\sigma_1\zeta(w))$$

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Non-autonomous dynamical system

$$\frac{d\mathbf{c}}{dw} = F(\mathbf{c}, w)$$

- The evolution parameter w appears explicitly in the RHS. This is a non-autonomous dynamical system.
- When w does not appear explicitly the system is an autonomous one
- For autonomous systems the fixed points are simply dc/dw =0.
- For non-autonomous dynamical systems the invariance under translations in the w parameter is broken
- For non-autonomous dynamical systems one requires to consider limits in the past and in the future.
- These limits are not commutative.

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Chapman-Enskog expansion

$$\frac{dc_1}{dw} = F_1(w, c_1)$$

$$c_1 = \sum_{k=0}^{\infty} u_{1,k}^{(0)}\, w^{-k}$$

From linear response theory

$$c_1 = -\frac{40}{3}\frac{1}{w}\frac{\eta}{s} - \frac{80}{9}\frac{1}{w^2}\frac{T(\eta\tau_\pi - \lambda_1)}{s} \ldots$$

Transport coefficients

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

Fokker-Planck equation

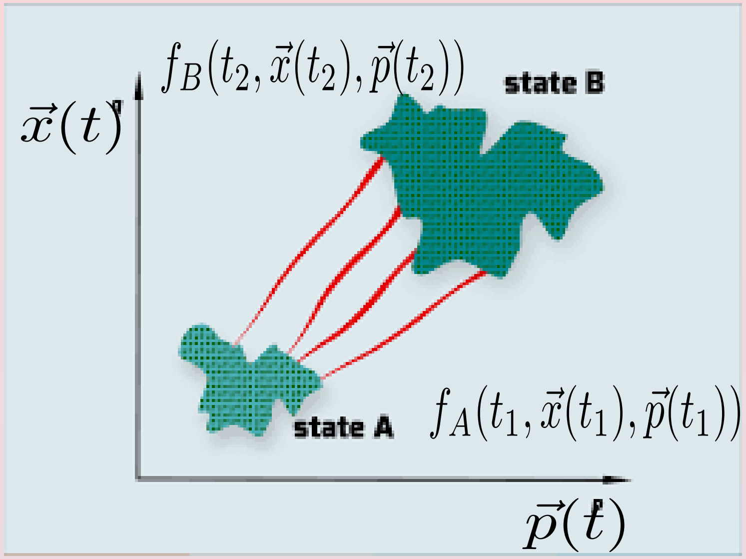
Microscopic dynamics is encoded in the distribution function f(t,**x**,**p**)

$$\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} = \mathcal{C}[f]$$

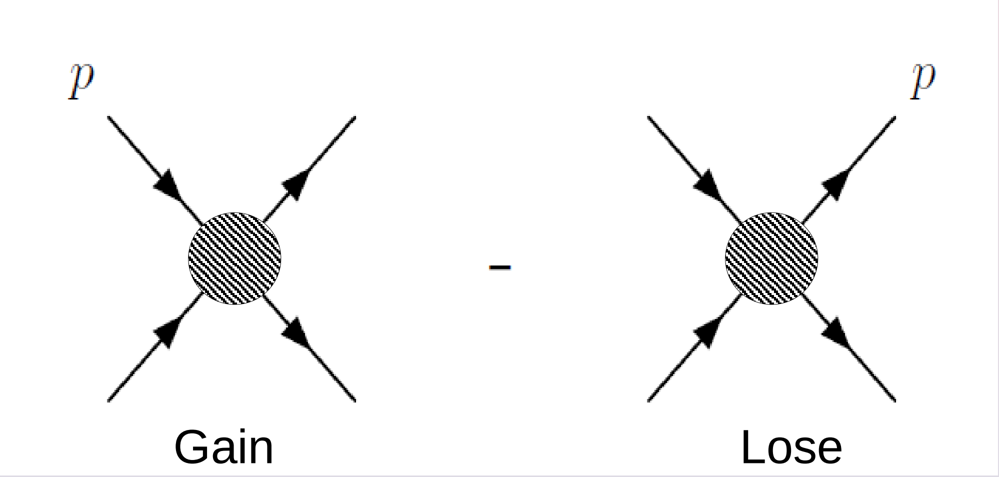
Free expansion

External Force

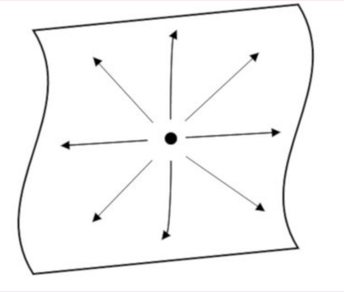
Particle imbalance

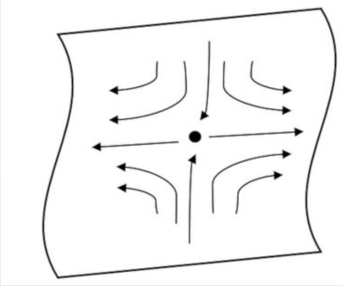


$$\mathcal{C}[f] =$$



UV regime





UV expansion

By expanding w ⇨ 0 around the UV fixed points

$$\frac{d\,c_1}{dw} = F(c_1, w)$$

Perturbative solutions $$c_1 = \sum_{k=1} v_{1,k}^{(0)}\, w^k$$

Linearized perturbations $$\delta c_1 = \frac{\mu_1^{\pm}}{w^{\alpha_1^{\pm}}}$$

Power law behavior

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

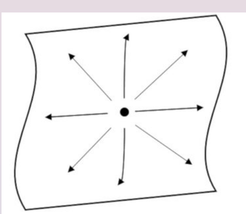
UV expansion

By expanding w ⇨ 0 around the UV fixed points

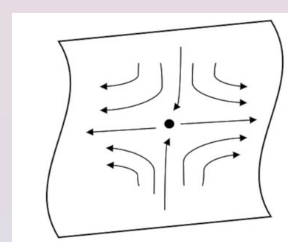
$$\frac{d\,c_1}{dw} = F(c_1, w)$$

Perturbative solutions $$c_1 = \sum_{k=1} v_{1,k}^{(0)}\, w^k$$

Linearized perturbations $$\delta c_1 = \frac{\mu_1^{\pm}}{w^{\alpha_1^{\pm}}}$$



α_1^{+} : Fast decay



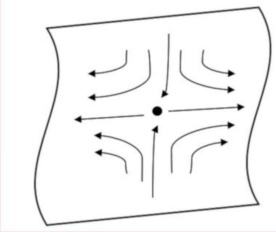
α_1^{-} : Growth

Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

UV expansion around saddle fixed point

Consider the expansion around saddle point

$$c_1 = \sum_{k=1} v_{1,k}^{(0)} \, w^k$$



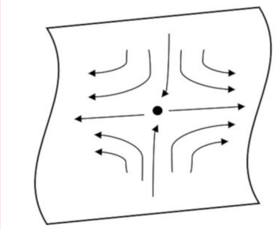
Power law series:

- Divergent.
- Fluctuations grow so one cannot perform any resummation scheme around saddle fix point.
- However, radius of convergence is extended by analytical continuation!!

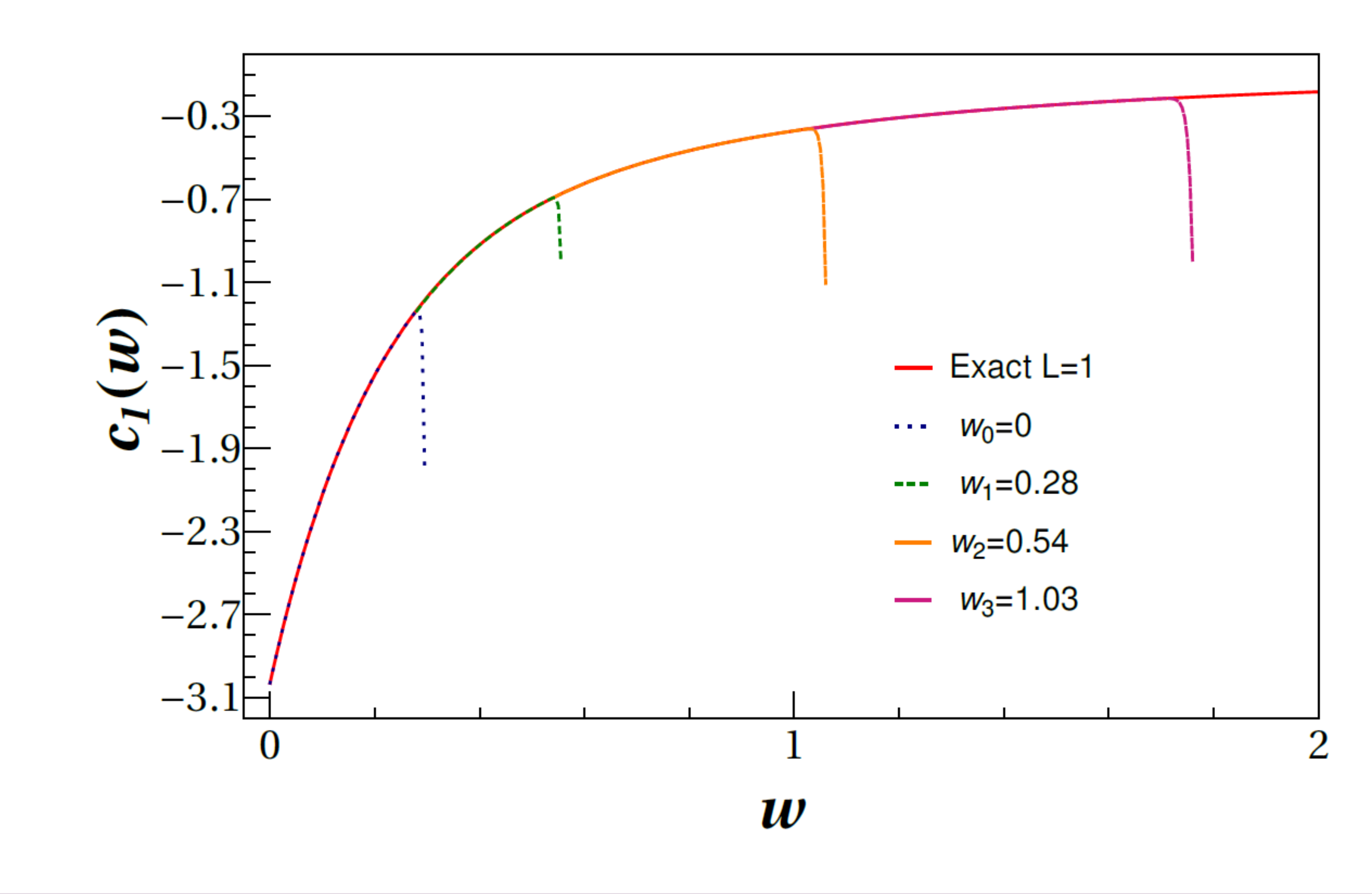
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

UV expansion around saddle fixed point

Analitical continuation extends the finite radius of convergence



$$c_1 = \sum_{k=1} v_{1,k}^{(0)}\, w^k$$

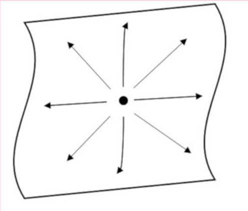


Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

UV expansion around source fixed point

Consider the expansion around source point

$$c_1 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} v_{1,k}^{(n)} \, \varphi_k^{(n)}(w)$$



$$\varphi^{(n)}(w) = \left[\frac{\mu_1^+}{w^{\alpha_1^+}} \right]^n w^k$$

UV transseries

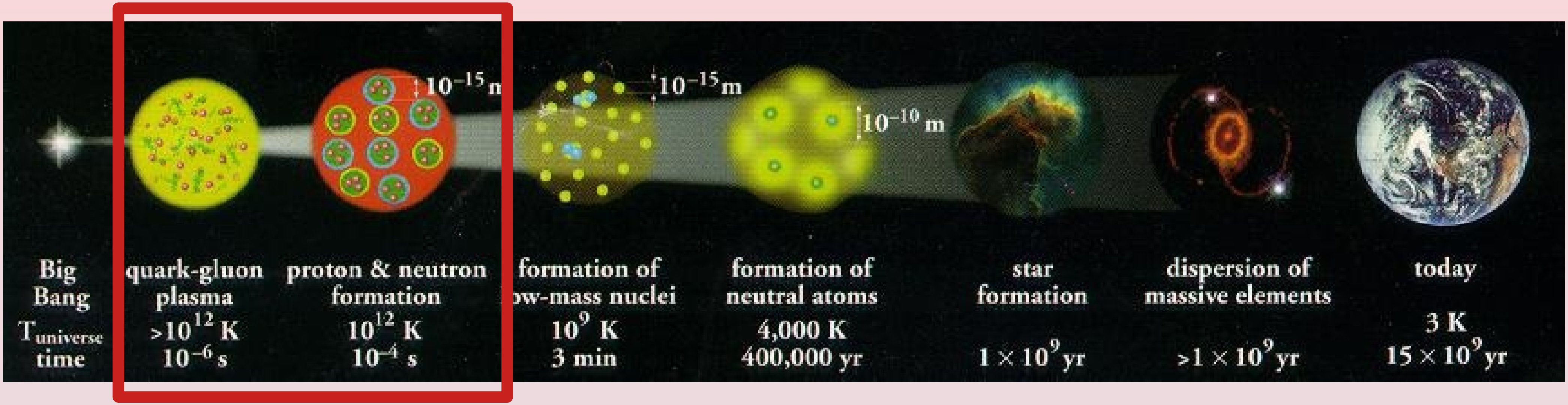
- Finite radius of convergence
- Fluctuations are not suppressed

Power law decay
No Instanton-like contributions

Perturbative

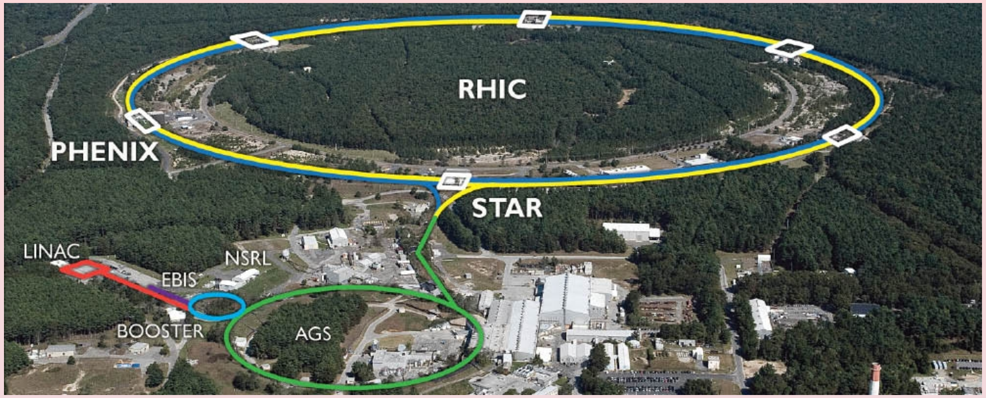
Martinez et. al. 1805.07881, 1901.08632, 1911.06406, 2011.08235

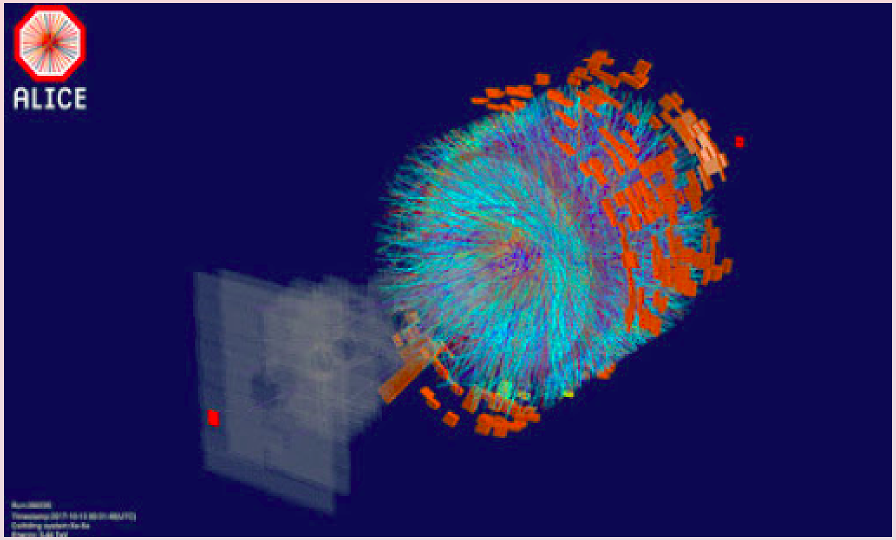
Early universe and the quark-gluon plasma



At extremely high temperatures the universe was filled with **quarks and gluons** which 'condensate' into **hadronic bound states**

Little Big Bangs





Large Hadron Collider (LHC) and the **Relativistic Heavy Collider (RHIC)** create **‘Little Big Bangs’**

- A deconfined plasma of **Quarks and Gluons**
- What about using heavy ions to understand neutron star mergers?

E. R. Most et. al. PRL 122 (2019) 061101