Hydrodynamics at non-zero net baryon density and critical fluctuation dynamics

Lipei Du

Department of Physics, The Ohio State University, USA

at HENPIC online seminar

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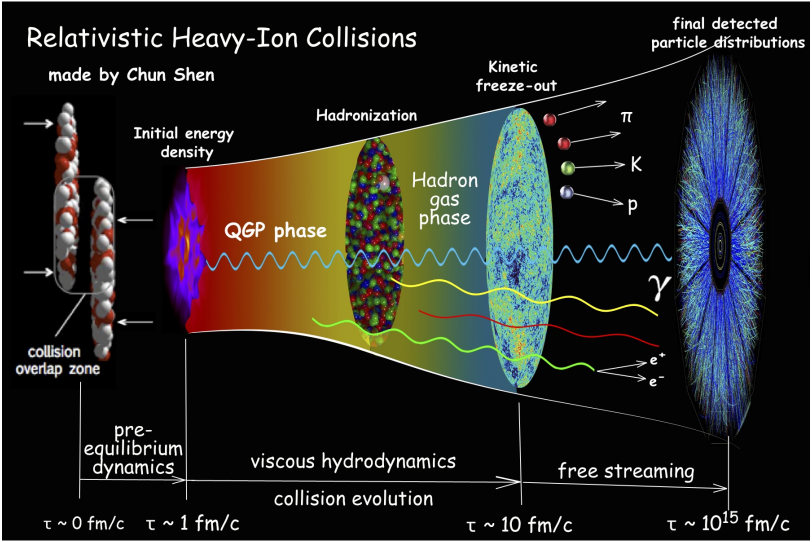
Outline

- Motivation
- Part I: Dissipative hydrodynamics at non-zero net baryon density
  - Dynamical initialization
  - Baryon evolution
  - BESHYDRO
- Part II: Fluctuation dynamics near the QCD critical point
  - The main effects controlling the dynamics of critical slow modes
  - Non-equilibrium contributions from the slow modes to bulk matter properties
- Conclusions

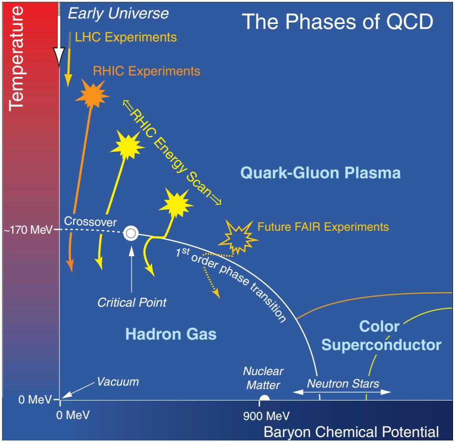
Motivation

Motivation: illustration of heavy-ion collisions



credit: Chun Shen

Motivation: exploring the QCD phase diagram with heavy-ion collisions



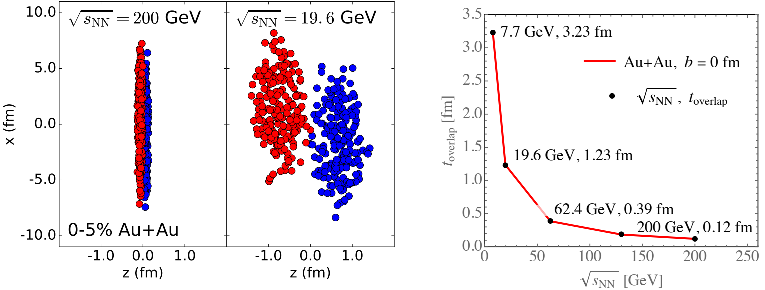
2007 NSAC Long Range Plan

Theoretical modeling of heavy-ion collisions at low energies faces:

- complicated interpenetration dynamics;
- dynamics of conserved charges, large baryon doping; (Part I)
- singularity associated with the QCD critical point in the thermal properties of the medium;
- large fluctuations and strong correlations near the critical point; their dynamics; (Part II)
- …

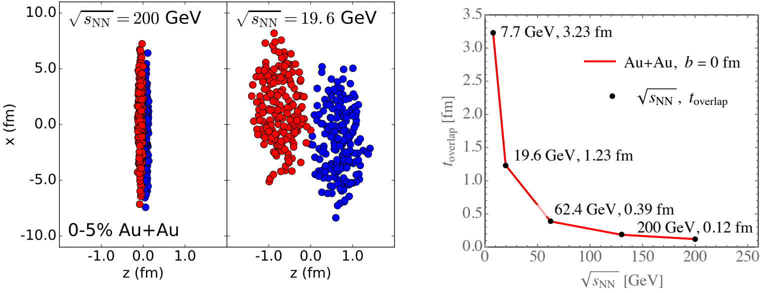
Hydrodynamics at non-zero net baryon density

Initial conditions: construction in (3+1)D



Collision geometry and the maximum overlap time of Au-Au collision [C. Shen and B. Schenke, 1710.00881; L. Du, D. Everett and U. Heinz, in preparation].

- At low collision energies (e.g. Beam Energy Scan energies), the space-time history of nuclear interpenetration is complicated;



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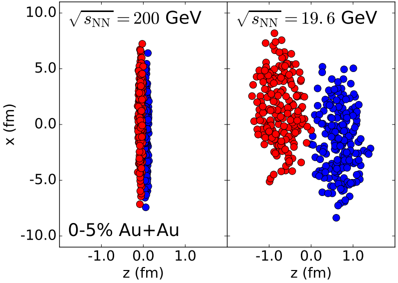

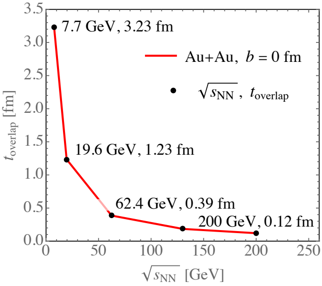

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- At low collision energies (e.g. Beam Energy Scan energies), the space-time history of nuclear interpenetration is complicated;
- Hybrid descriptions of heavy-ion collision involving hydrodynamics require (3+1)D initial conditions in these cases;
- Challenges: large theoretical uncertainties [G. Denicol et al, 1804.10557], e.g. amount of baryon stopping [A. Bialas et al, 1608.07041; L. Du et al, 1807.04721; C. Shen and B. Schenke, 1710.00881]; interplay between (3+1)D pre-hydrodynamics and hydrodynamics [C. Shen and B. Schenke, 1710.00881; Y. Akamatsu et al, 1805.09024; Y. Kanakubo et al, 1806.10329;]

Baryon evolution: recent developments

The conservation laws for energy, momentum and the baryon charge are

$$
\begin{aligned}
d_\mu T^{\mu\nu} &= 0\,, \quad \text{with} \quad T^{\mu\nu} = e u^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}\,, \\
d_\mu N^{\mu} &= 0\,, \quad \text{with} \quad N^{\mu} = n u^\mu + n^\mu\,.
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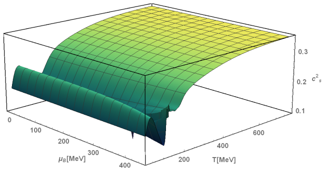
Required ingredients:

- Equation of State
  - At non-zero charge density: [A. Monnai et al, 1902.05095; J. Noronha-Hostler et al, 1902.06723]
  - With a critical point: [P. Parotto et al, 1805.05249]



Speed of sound from a Lattice-QCD-based Equation of State with a critical point [P. Parotto et al, 1805.05249]

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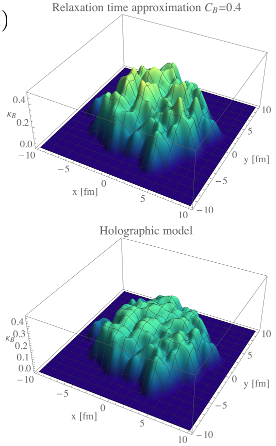
- Transport coefficients
  - Shear and bulk viscosity $(\eta/s)(\mu, T)$, $(\zeta/s)(\mu, T)$: e.g. [J. Noronha-Hostler et al, 0811.1571; G. Denicol, 1512.01538]
  - Baryon diffusion coefficient $\kappa_n(\mu, T)$: e.g. [G. Denicol et al, 1804.10557; R. Rougemont, 1507.06972; O. Soloveva et al, 1911.08547; Fotakis et al, 1912.09103]



Baryon diffusion coefficients from kinetic theory and holographic theory [L. Du et al, 1807.04721]

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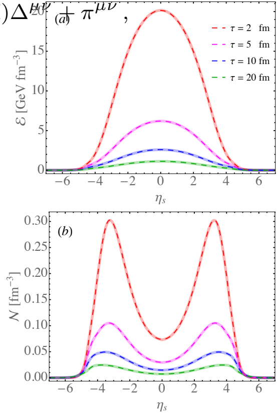
- (3+1)D hydrodynamic simulation
  - MUSIC [G. Denicol et al, 1804.10557]
  - BESHYDRO [L. Du and U. Heinz, 1906.11181]
  - CLVisc [X. Wu, QM2019]; Frankfurt [J. Fotakis, QM2019];



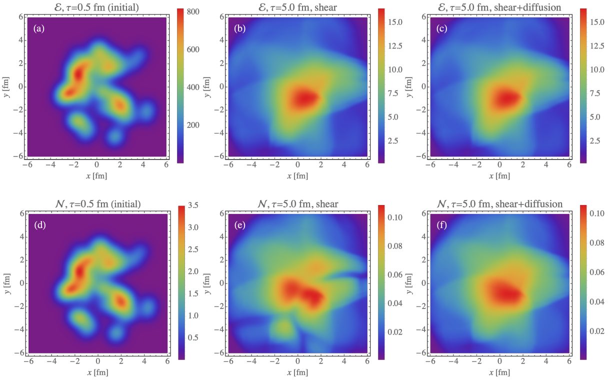
(1+1)D comparison ($r = 0$) between MUSIC and BESHYDRO [L. Du and U. Heinz, 1906.11181], both using DNMR theory [G. Denicol et al, 1004.5013; G. Denicol et al, 1202.4551]

Baryon diffusion: smoothen baryon gradients

- The relaxation equation for baryon diffusion:

$$u^\nu \partial_\nu n^\mu = -\frac{1}{\tau_n}\left[n^\mu - \kappa_n \nabla^\mu \left(\frac{\mu}{T}\right)\right] + \dots ,$$

where the baryon diffusion coefficient ($\kappa_n \propto C_B$) controls the response of diffusion current to the driving force.



Energy and baryon evolution in transverse plane [L. Du and U. Heinz, 1906.11181]

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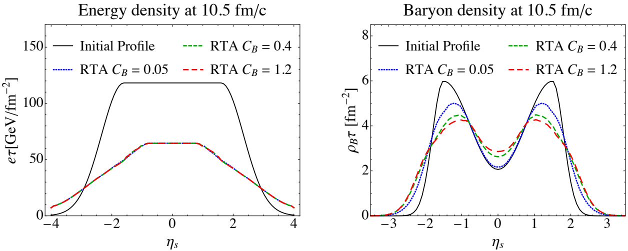

Energy and baryon evolution in longitudinal direction in (1+1)D evolution (adapted from [L. Du et al, 1807.04721])

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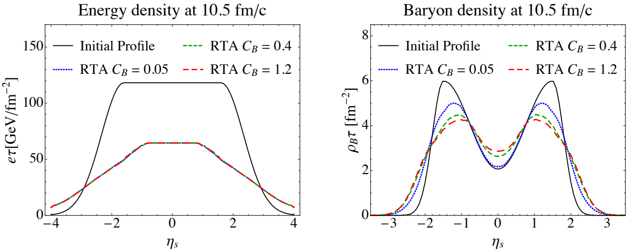
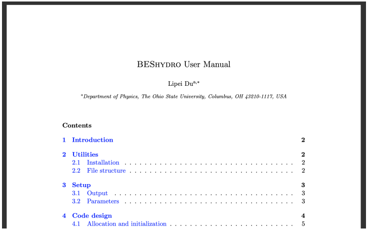


Energy and baryon evolution in longitudinal direction in (1+1)D evolution (adapted from [L. Du et al, 1807.04721])

- Baryon diffusion leaves no pronounced signatures in the evolution of the energy density but smoothes out gradients in baryon density [L. Du and U. Heinz, 1906.11181].
- An active topic: see also e.g. [A. Monnai, 1204.4713; C. Shen et al, 1704.04109; G. Moritz et al, 1711.08680; G. Denicol et al, 1804.10557; M. Li and C. Shen, 1809.04034; X. Wu, QM2019; Fotakis et al, 1912.09103].

BESHYDRO: internal workings

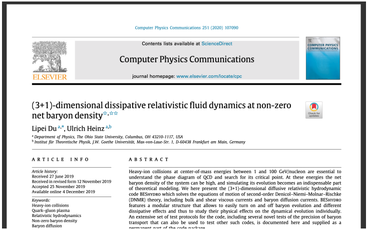


BESHYDRO User Manual

Lipei Du[a,*]

[a] Department of Physics, The Ohio State University, Columbus, OH 43210-1117, USA

Contents



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(3+1)-dimensional dissipative relativistic fluid dynamics at non-zero net baryon density☆,☆☆

Lipei Du[a,*], Ulrich Heinz[a,b]

[a] Department of Physics, The Ohio State University, Columbus, OH 43210-1117, USA
[b] Institut für Theoretische Physik, J.W. Goethe Universität, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany

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ABSTRACT

Heavy-ion collisions at center-of-mass energies between 1 and 100 GeV/nucleon are essential to understand the phase diagram of QCD and search for its critical point. At these energies the net baryon density of the system can be high, and simulating its evolution becomes an indispensable part of theoretical modeling. We here present the (3+1)-dimensional diffusive relativistic hydrodynamic code BESHYDRO which solves the equations of motion of second-order Denicol–Niemi–Molnar–Rischke (DNMR) theory, including bulk and shear viscous currents and baryon diffusion currents. BESHYDRO features a modular structure that allows to easily turn on and off baryon evolution and different dissipative effects and thus to study their physical effects on the dynamical evolution individually. An extensive set of test protocols for the code, including several novel tests of the precision of baryon transport that can also be used to test other such codes, is documented here and supplied as a

- Physics, internal workings and validation schemes are well documented:
  - BESHYDRO paper [L. Du and U. Heinz, 1906.11181]
  - User manual ▸ on GitHub
  - Stay tuned for more validation schemes, which can be used for other codes ▸ on GitHub
- With documents and validation schemes, BESHYDRO can be a good tool for knowing internal workings of hydrodynamic code, and it is especially suitable for doing exploratory studies.

Fluctuation dynamics near the QCD critical point

Critical fluctuations: necessity of off-equilibrium dynamics

- A critical point features large fluctuations and correlations [Gitterman, *Rev. Mod. Phys.*, 1978]; dependence on the correlation length ξ stronger in higher-order cumulants, e.g. $K_3 \propto \xi^{4.5}$, $K_4 \propto \xi^7$; non-monotonic dependence of the cumulants on beam energies — proposed as telltale signature of QCD critical point [M. Stephanov, 0809.3450 and 1104.1627];

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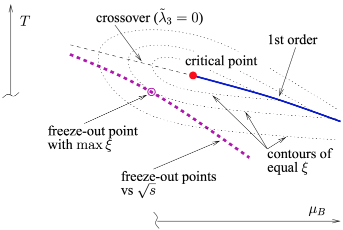

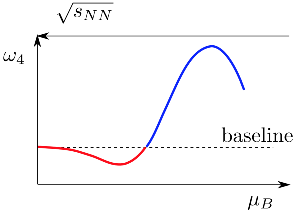

Illustration of the critical regime and normalized quartic cumulant of proton multiplicity as a function of μ (equivalently, collision energy $\sqrt{s_{NN}}$) [M. Stephanov, 0809.3450 and 1104.1627; A. Bzdak et al, 1906.00936]

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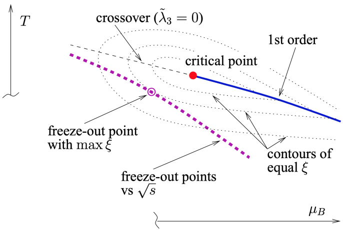

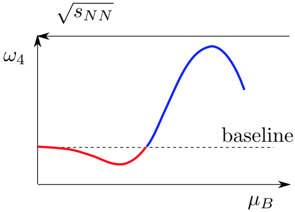

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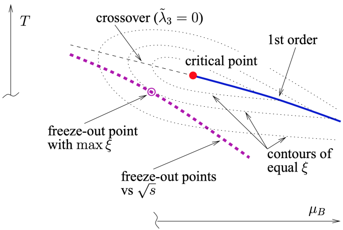

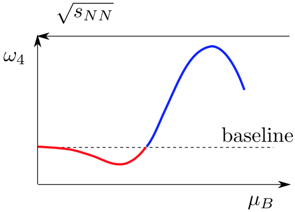

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- Hydro+ framework: conventional hydrodynamics coupled to slowly evolving critical modes. The slowest mode in the system created in heavy-ion collisions is, $\delta(s/n)_p$ [M. Stephanov and Y. Yin, 1712.10305].

Critical fluctuations: equilibrium value

- Introduce a phase-space density for the slow degrees of freedom, $\phi_Q(\boldsymbol{x})$, via the Wigner transform of the two-point correlation of $\delta(s/n)_p$ [M. Stephanov and Y. Yin, 1712.10305; see also X. An et al, 1902.09517 and 1912.13456]:

$$\phi_Q(\boldsymbol{x}) \sim \int_{\Delta \boldsymbol{x}} \left\langle \delta \frac{s}{n} \left(\boldsymbol{x} + \frac{\Delta \boldsymbol{x}}{2} \right) \delta \frac{s}{n} \left(\boldsymbol{x} - \frac{\Delta \boldsymbol{x}}{2} \right) \right\rangle e^{i Q \cdot \Delta \boldsymbol{x}} .$$

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- With the scale separation $\ell \gg \xi$, in the hydrodynamic limit $Q \to 0$, i.e. $Q \ll \xi^{-1}$ ($Q = |\boldsymbol{Q}|$), the equilibrium value $\bar{\phi}_0$ is [M. Stephanov and Y. Yin, 1712.10305; Y. Akamatsu et al, 1811.05081]

$$\bar{\phi}_0 = V \left\langle \left(\delta \frac{s}{n}\right)^2 \right\rangle = \frac{c_p}{n^2} ,$$

  where $c_p = nT(\partial(s/n)/\partial T)_p$ is the heat capacity.

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- In the critical regime ($Q \gg \xi^{-1}$), the Q- and ξ-dependent equilibrium value is given as [M. Stephanov and Y. Yin, 1712.10305; Y. Akamatsu et al, 1811.05081; X. An et al, 1912.13456]

$$\bar{\phi}_Q = \bar{\phi}_0 f_2(Q\xi) = \left[\left(\frac{c_p}{n^2} \right) \left(\frac{\xi}{\xi_0} \right)^2 \right] \frac{1}{1 + (Q\xi)^2} ,$$

  where spatial isotropy for the $\boldsymbol{Q}$-dependence is assumed (see also [K. Rajagopal et al, 1908.08539]).

Critical fluctuations: off-equilibrium dynamics

- The equations of motion of slow modes are of relaxation form [M. Stephanov and Y. Yin, 1712.10305; Y. Akamatsu et al, 1811.05081; X. An et al, 1912.13456]

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where u^μ the flow velocity and $u^\mu \partial_\mu$ the time-derivative in local rest frame.

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- Near the critical point, the Q-dependent relaxation rate Γ_Q is [Y. Akamatsu et al, 1811.05081; X. An et al, 1912.13456]

$$\Gamma_Q = \Gamma_\xi f_\Gamma(Q\xi) = \left[2\left(\frac{\lambda_T}{c_p \xi^2}\right)\left(\frac{\xi_0}{\xi}\right)^2\right]\left[(Q\xi)^2(1+(Q\xi)^2)\right],$$

with λ_T being the heat conductivity (note: "model B" in use, $\Gamma_\xi \propto \xi^{-4}$, *cf.* [K. Rajagopal et al, 1908.08539]).

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- Dynamics ϕ_Q is resulted from competition between expansion of the system and the relaxation of ϕ_Q. Introduce the "critical Knudsen number" to characterize this competition quantitatively:

$$\mathrm{Kn}(Q) \equiv \theta/\Gamma_Q \,.$$

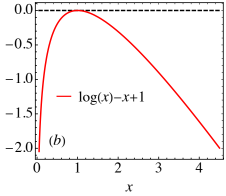
Slow modes with large critical Knudsen numbers will lag behind, not being able to follow the hydrodynamic evolution of the equilibrium value $\bar{\phi}_Q(x)$ [L. Du, U. Heinz, K. Rajagopal, and Y. Yin, 2004.02719].

Critical fluctuations: back-reaction to the fluid

- The slow modes are additional, non-thermal degrees of freedom, which contribute to the entropy density, $s_{(+)}(e,n,\phi) \equiv s_{\text{eq}}(e,n) + \Delta s$, with [M. Stephanov and Y. Yin, 1712.10305] (denoting $x \equiv \phi_Q/\bar{\phi}_Q$)

$$\Delta s(e,n,\phi) = \int dQ \frac{Q^2}{(2\pi)^2}\left[\log\frac{\phi_Q}{\bar{\phi}_Q} - \frac{\phi_Q}{\bar{\phi}_Q} + 1\right].$$

Note: Q^2 in the phase space factor.



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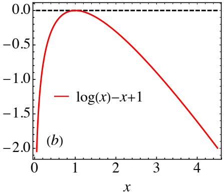

- Thus they modify the inverse temperature and chemical potential as follows:

$$\beta_{(+)} = \left(\frac{\partial s_{(+)}}{\partial e}\right)_{n\phi} \equiv \frac{1}{T} + \Delta\beta\,, \quad \alpha_{(+)} = -\left(\frac{\partial s_{(+)}}{\partial n}\right)_{e\phi} \equiv \frac{\mu}{T} + \Delta\alpha\,.$$

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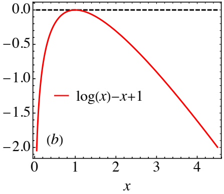

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- The pressure of the system at partial-equilibrium is

$$p_{(+)} = (s_{(+)} - \beta_{(+)}e + \alpha_{(+)}n)/\beta_{(+)} \equiv p + \Delta p\,,$$

where Δp is the additional contribution from the slow modes.

- Slow modes have back-reaction to the fluid.

Setup: parameterization and background fluid

- Parameterization: $c_p = s^2/(\alpha n)$ [Y. Akamatsu et al, 1811.05081] and $\lambda_T \propto T^2$ and $\xi(T)$ [K. Rajagopal et al, 1908.08539];

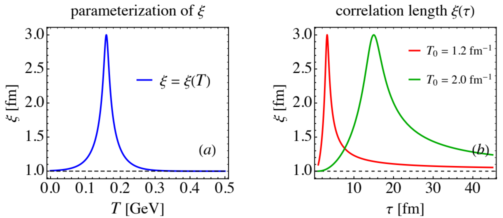

Illustration of the parameterization $\xi(T)$, and its evolution in a Bjorken expanding fluid

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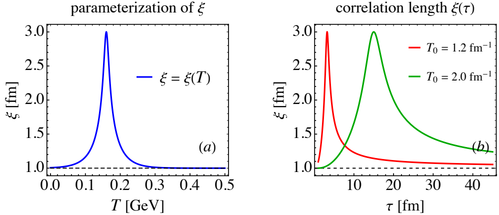
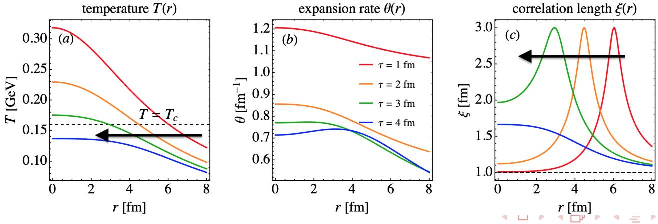
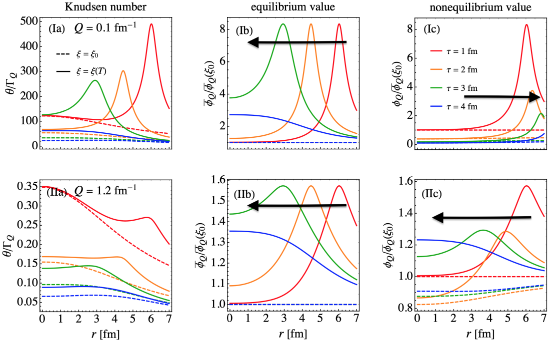


Illustration of the parameterization $\xi(T)$, and its evolution in a Bjorken expanding fluid

- Ideal Gubser flow [S. Gubser, 1006.0006] with no back-reaction; temperature profile given as $T(\tau, r) = C/(\tau \cosh^{2/3} \rho(\tau, r))$ (other quantities from conformal EoS, e.g., $e \propto T^4$ and $n \propto T^3$).



Off-equilibrium dynamics: dependence on Q and ξ



Dashed lines: $\xi = \xi_0$; solid lines: $\xi = \xi(T)$ [L. Du, U. Heinz, K. Rajagopal, and Y. Yin, 2004.02719].

- Equations of motion

$$(u^\tau \partial_\tau + u^r \partial_r)\phi_Q = -\Gamma_Q\,(\phi_Q - \bar{\phi}_Q)\,,$$

- Note: equilibrium value $\bar{\phi}_Q$ is evolving and a "moving target" for ϕ_Q

- Two effects at play: (1) the initial peak in the fluctuations carried outward by advection (outward through $u^r\partial_r$-term); (2) out-of-equilibrium fluctuations relax but more slowly due to critical slowing down (inward through Γ_Q-term);
- Small Q modes ($Q < \xi_{\max}^{-1}$): relaxation is invisible because of large Kn(Q). Large Q modes ($Q \gtrsim \xi_0^{-1}$): relax more quickly to equilibrium and the initial peak dissipates more rapidly; advection is invisible.

Off-equilibrium dynamics: effects on entropy density

- Test different aspects of the dynamics

$$(u^\tau \partial_\tau + u^r \partial_r)\phi_Q = -\Gamma_Q (\phi_Q - \bar{\phi}_Q)\,,$$

where the equilibrium value $\bar{\phi}_Q$ and relaxation rate Γ_Q in the critical regime given as

$$\bar{\phi}_Q = \left[\left(\frac{c_p}{n^2}\right)\left(\frac{\xi}{\xi_0}\right)^2\right] f_2(Q\xi)\,,$$

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- Four dynamical models:
  - (I) full dynamics of the slow modes,
  - (II) with the advection off ($u^r = 0$),
  - (III) with constant $\xi = \xi_0 = 1$ fm,
  - (IV) with (T, μ)-independent c_p/n^2, λ_T/c_p (*cf.* [K. Rajagopal et al, 1908.08539]).

Note: background fluid unchanged.

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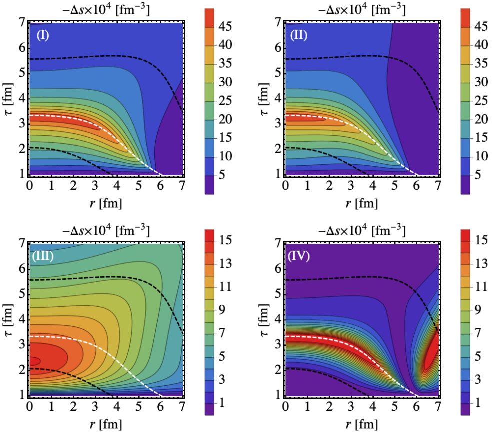
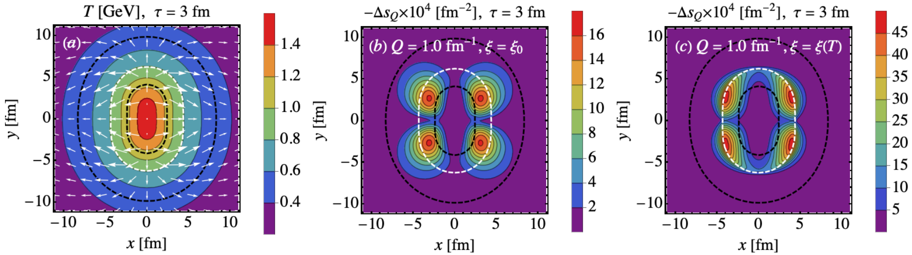
- Large difference in the correction between (I,II) and (III,IV). The correction is small ($\sim \mathcal{O}(10^{-4})$) compared to $s_{\rm eq}$.



Evolution of correction to entropy density
[L. Du, U. Heinz, K. Rajagopal, and Y. Yin, 2004.02719].

Off-equilibrium dynamics: effects on eccentricities



(a) temperature and flow profile, (b) with $\xi = \xi_0$, (c) with $\xi = \xi(T)$ [L. Du, U. Heinz, K. Rajagopal, and Y. Yin, 2004.02719].

- Using Gubser profile at early time regime ($\tau \ll 1/q$), with deformation [Y. Hatta et al, 1405.1984 and 1505.04226];

$$T_{\rm iso} \approx \frac{C}{\tau^{1/3}} \frac{(2q)^{2/3}}{(1+q^2r^2)^{2/3}}\,,\; T(\tau, r, \phi) = T_{\rm iso}(1 - \epsilon_n \mathcal{A}_n \delta)\,,\; \mathcal{A}_n \equiv \left(\frac{2qr}{1+(qr)^2}\right)^n \cos(n\phi)\,.$$

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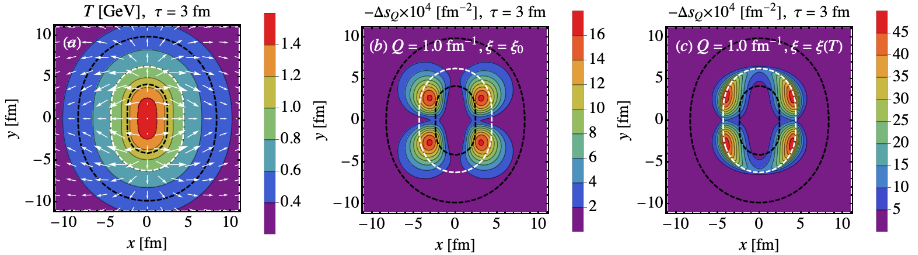

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- Large correction to entropy happens at large advection and is enhanced by large correlation length;
- The ellipticity is slightly increased by the correction from slow modes (of relative order $\lesssim 10^{-4}$).

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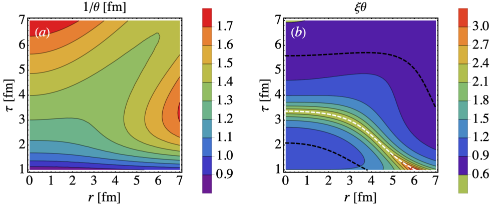

(a) estimation of hydrodynamic homogeneity length ℓ (b) estimation of $\xi/\ell \sim \xi\theta$
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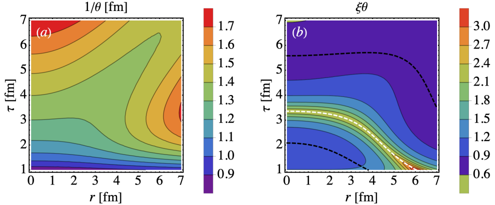

(a) estimation of hydrodynamic homogeneity length ℓ (b) estimation of $\xi/\ell \sim \xi\theta$
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- One sees that the framework gets challenged mostly in an arrow region around T_c but elsewhere it works well, even at very early times where the homogeneity length ℓ is short.
- Besides, critical dynamics for off-/far-away-from-equilibrium systems?

Conclusions

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- Part I: Dissipative hydrodynamics at non-zero net baryon density
  - Hydrodynamics at low collision energies requires (3+1)D dynamical initialization;
  - Baryon diffusion can have effects on observables and should be included in hydrodynamic simulations for Beam Energy Scan studies;
  - BESHYDRO is designed for this purpose and well documented.
- Part II: Fluctuation dynamics near the QCD critical point
  - Different aspects of the off-equilibrium dynamics controlling the evolution of critical fluctuations are analyzed in details;
  - It provides useful guidance for more realistic studies on observables affected by critical fluctuations;
  - Back-reaction off-equilibrium effects of critical fluctuations are small on the bulk properties of the fluid;

Thank you very much!