Fluid dynamics of non-fluids

严力

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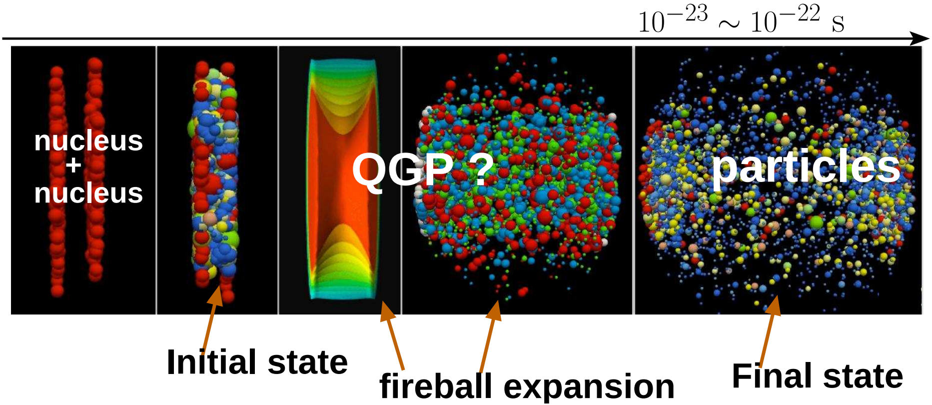
Sep. 2 2020, IOPP Forum at CCNU

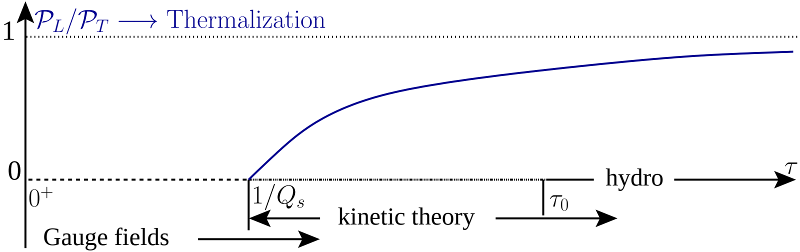
High-energy nucleus-nucleus collisions (ideal picture)



- Hydro modeling: I.C. + hydro + EoS + freeze-out + ...
- Ideally, local equilibrium is approached in initial state $\Rightarrow$ Thermalization.

$$\text{hence:} \quad T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \underbrace{(\eta,\zeta)(\text{Kn})}_{\text{1st order hydro}} + \underbrace{(\lambda,\tau_\pi,\tau_\Pi)(\text{Kn}^2)}_{\text{2nd order hydro}} + \cancel{\mathcal{O}(\text{Kn}^3)}$$

Initial state thermalization (ideal picture)



- Close to local thermal equilibrium: $\mathcal{P}_L = \mathcal{P}_T +$ viscous corrections.
- Strong coupling: based on AdS/CFT, τ_0 could be small

  Yaffe, Chesler, Shuryak, ...
- Weak coupling (QCD): $\alpha_s \ll 1$, τ_0 is generally large
  - Parametrical estimate (bottom-up): $\tau_0 \gtrsim 1.5\alpha_s^{-13/5} Q_s^{-1}$

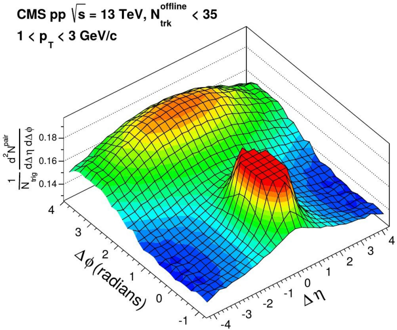
    Baier, Mueller, Schiff, Son, ...
  - More systematic analysis: solving Boltzmann equation

    EKT(Kurkela, Moore), BAMPS (Xu, Greiner, ...)

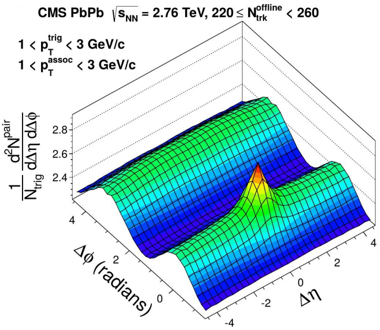
Long-range multi-particle correlations

- Example: Two-particle correlation function: $\langle N_a N_b \rangle (\Delta\phi_p, \Delta\eta)$

without long-range correlations



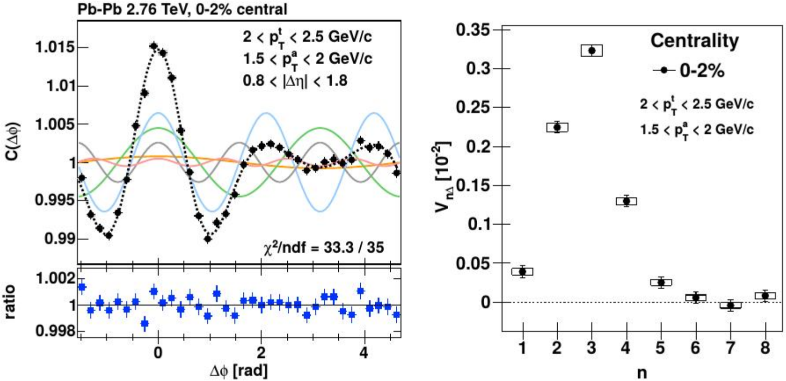
with long-range correlation



Near-side ($\Delta\phi = 0$) correlation with $\Delta\eta \Rightarrow$ system collective expansion.

Anisotropic flow V_n

ALICE Collaboration, 2012

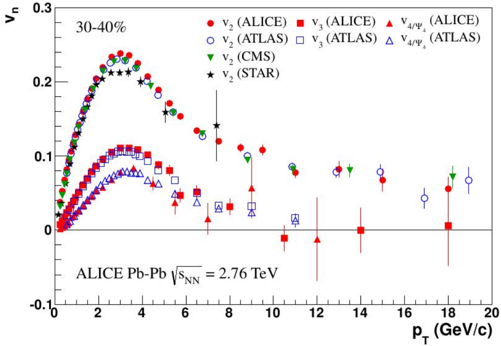


Implies the single-particle spectrum Fourier decomposition:

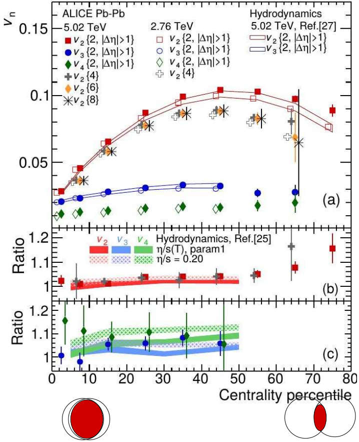
$$\frac{dN}{p_T dp_T d\phi_p dy} = \frac{dN}{2\pi p_T dp_T dy}\left[1 + \sum_{n=1}^{\infty} \underbrace{v_n(\mathrm{y}, p_T) e^{in\Psi_n}}_{V_n} e^{-in\phi_p} + c.c.\right]$$

Anisotropic flow : $V_n = v_n e^{in\Psi_n}$

Asymmetry regarding azimuthal angle ϕ_p



ALICE collaboration, 2013

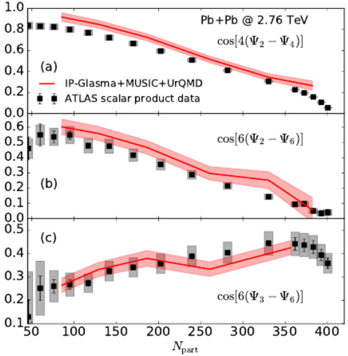


ATLAS Collaboration, 2016

- v_2 elliptic flow; v_3 triangular flow; v_4 quadrangular flow, etc.

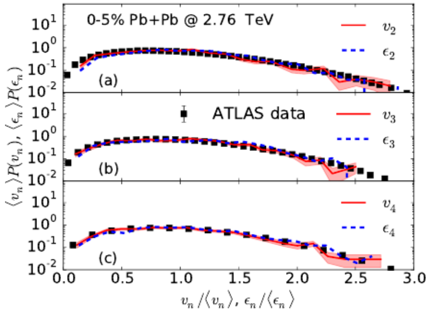
Flow observables and hydrodynamics: AA

Event-plane correlation



[S. McDonal et al., 1609.02658]

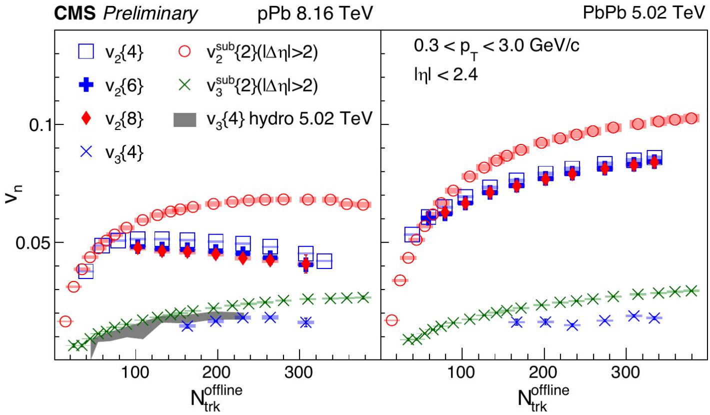
$P(v_n)$



Hydro successfully captures flow observables, (those from multi particle correlations, $v_n\{m\}$, event-plane correlations, symmetric cumulant, etc.. As a good approximation, in these results, $(n = 2, 3)$,

$$V_n \propto \mathcal{E}_n : \quad \text{reflecting the nature of fluid response to small perturbations}$$

Small colliding systems



CMS Collaboration, 1904.11519

- Collective expansion dominates evolution ⇔ Compatible with hydro
- Small system means shorter time scale ⇔ Unlikely to thermalize!

Hydro without local thermal equilibrium

Hydro is unreasonably successful.

(Relativistic) Fluid dynamics

- Relativistic hydrodynamics:

$$\text{conservation of energy and momentum} \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

  where $T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \sum_{n=1}^{\infty} \alpha_n \text{Kn}^{\text{n}}$: hydro gradient expansion.

- Kn is the dimensionless Knudsen number,

$$\text{Kn} \sim \frac{\text{mean free path}}{\text{system size}} \Rightarrow \text{how far away a system is from equilibrium}$$

- For fluid dynamics to apply, the system must be **close** to equilibrium

$$\text{Kn} \ll 1 \Rightarrow \text{hydro gradient expansion can be truncated}$$

  e.g., Space shuttle entering atmosphere at 20km experiences $\text{Kn} \sim 10^{-8}$.

- However, hydro gradient expansion is asymptotic!

Example: solve BRSSS hydro in Bjorken expansion

BRSSS hydro is a conformal 2nd order viscous hydrodynamics

$$
\begin{aligned}
T^{\mu\nu} =& e u^\mu u^\nu - \mathcal{P}\Delta^{\mu\nu} + \pi^{\mu\nu} \\
\pi^{\mu\nu} =& \eta\langle\nabla^\mu u^\nu\rangle - \tau_\pi\left[\langle D\pi^{\mu\nu}\rangle + \frac{4}{3}\pi^{\mu\nu}\nabla\cdot u\right] - \frac{\lambda_1}{\eta^2}\langle\pi^\mu{}_\alpha\pi^{\alpha\nu}\rangle + O(\mathrm{Kn}^3)
\end{aligned}
$$

Baier, Romatschke, Son, Stephanov and Starinets

These transport coefficients can be parameterized as

$$
\eta = C_\eta s \qquad \tau_\pi = \frac{C_\tau C_\eta}{T} \qquad \lambda_1 = C_\lambda \frac{s}{T}
$$

From kinetic theory of massless particles,

$$
C_\eta = \frac{1}{4\pi} \qquad C_\tau = 5 \qquad C_\lambda = \frac{5}{7}C_\eta C_\tau
$$

Note that the above parameterization is NOT unique.

Bjorken expansion: a good approximation for the very early-stage evolution,

$$(t, z) \longrightarrow \begin{cases} \tau = \sqrt{t^2 - z^2} \\ \xi = \operatorname{arctanh}\left(\frac{z}{t}\right) \end{cases}$$

BRSSS hydro EoM becomes coupled ODE,

$$\begin{aligned} \partial_\tau e &= -\frac{4}{3}\frac{e}{\tau} - \frac{\pi}{\tau} \\ \pi &= -\frac{4}{3}\frac{\eta}{\tau} - \tau_\pi \left(\partial_\tau \pi + \frac{4}{3}\frac{\pi}{\tau} \right) + \frac{\lambda_1}{2\eta^2}\pi^2 \end{aligned}$$

where $\pi = \pi^\xi_\xi$. Noting that tensor structures are simplified,

$$\frac{1}{\tau} \sim \nabla \cdot u \sim \nabla^\xi u_\xi$$

Knudsen number is

$$\mathrm{Kn}^{-1} = w \equiv \frac{\tau}{\tau_\pi}$$

Consider the function,

$$g(w) \equiv \frac{\mathrm{d}\ln e}{\mathrm{d}\ln \tau} = -1 - \frac{\mathcal{P}_L}{e} \longrightarrow \begin{cases} -1 & \text{anisotropic} \\ \\ -4/3 & \text{isotropic/thermallized} \end{cases}$$

which obeys a nonlinear ODE,

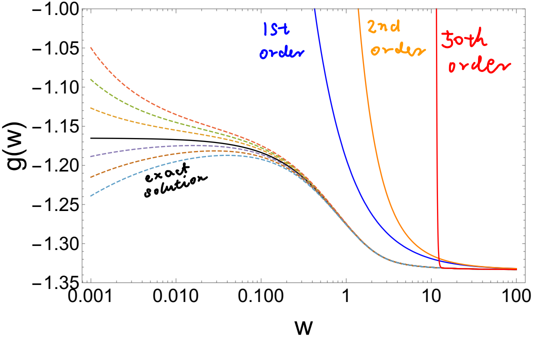
$$w\frac{\mathrm{d}g}{\mathrm{d}w}\left(1+\frac{g}{4}\right)+\left(g+\frac{4}{3}\right)^2\left[1+\frac{3w}{8}\frac{C_\lambda}{C_\eta}\right]+w\left(g+\frac{4}{3}\right)-\frac{16}{9}\frac{C_\eta}{C_\tau}=0 \qquad (\star)$$

We shall solve eq.($\star$),

1. Numerically.
2. Semi-analytically using form of hydro gradient expansion,

$$g(w) = g^{\text{hydro}}(w) \equiv \sum_{n=0}^{\infty} f_n^{(0)} w^{-n} = -\frac{4}{3} - \frac{16}{9}\frac{C_\eta}{C_\tau}\frac{1}{w} + \ldots$$

Numerical solution: attractor



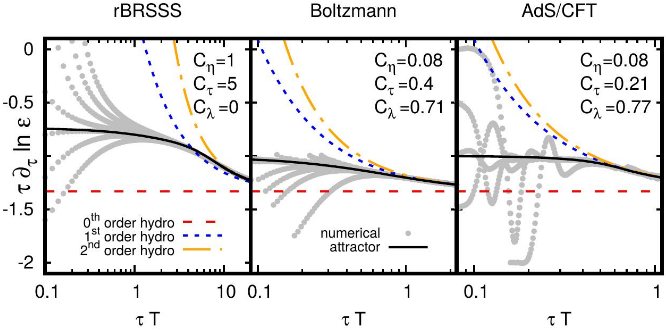
- Simple summation of hydro gradients would NOT help.
- The solution becomes insensitive to I.C. for $w \sim 1 \Rightarrow$ attractor.

  [M.Heller, M. Spalinski, R. Janik, G. Basar, G. Dunne, P. Romatschke, G. Denicol, M. Strickland, U. Heinz, ...]

In the mathematical field of dynamical systems, an attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system. System values that get close enough to the attractor values remain close even if slightly disturbed. – Wikipedia

Attractor from different dynamics

[P. Romatschke, PRL 120 (012301) 2018]

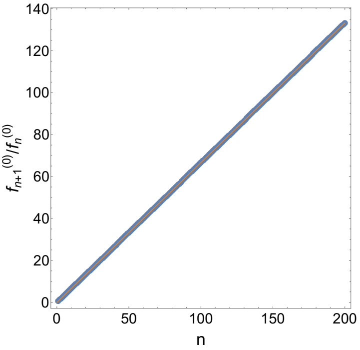


One may understand the attractor as a results of

- Existence of free-streaming fixed point, such that $\delta g \sim w^{-\chi}$.
  [J.P. Blaizot, LY]
- The slowest 'mode' that evolves adiabatically.
  [J. Brewer, Y. Yin, LY, 1910.00021]
- Borel resum of hydro gradients, trans-series solution – this talk.

Hydro gradient expansion is asymptotic!

Plug the hydro gradient expansion into eq. ($\star$), one solves $f_n^{(0)}$ iteratively,



- $f_{n+1}^{(0)}/f_n^{(0)} \sim S^{-1}(n+\beta) + O(1/n)$, with $S = 3/2$ and $\beta = -2C_\lambda/C_\tau$
  [Basar and Dunne, PRD 92, 125011]

$$\to \text{for large } n: \quad f_n^{0)} \sim \frac{\Gamma(n+\beta)}{S^{n+\beta}} \sim n!$$

- Hydro gradient expansion is asymptotic. (zero radius of convergence)

Borel resum of asymptotic series

For asymptotic series, Borel resummation technique can be applied:

- Borel transform of the hydro gradient expansion $g^{\text{hydro}}(w)$,

$$g^{\text{hydro}}(w) = \sum_{n=0}^{\infty} f_n^{(0)} w^{-n} \quad \rightarrow \quad \mathcal{B}[g](z) \equiv \sum_{n=0}^{\infty} \frac{f_n^{(0)}}{n!} z^n$$

  note that $\mathcal{B}[g](z)$ is convergent.
- Borel resum (Laplace transform),

$$\tilde{g}^{\text{hydro}}(w) \equiv w \int_0^{\infty} dz e^{-zw} \mathcal{B}[g](z) \quad \rightarrow \quad w \int_C dz e^{-zw} \mathcal{B}[g](z)$$

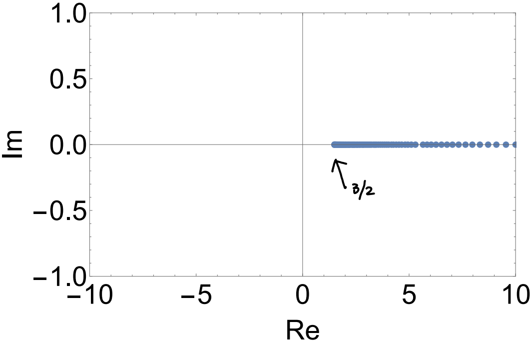
- Borel resum $\tilde{g}^{\text{hydro}}(w)$ can be the actual representation of the asymptotic series, if $g^{\text{hydro}}(w)$ is Borel summable, i.e., there is no singularity on R^+.

Hydro gradient expansion is not Borel summable

- In fact, one finds for the hydro gradient expansion,

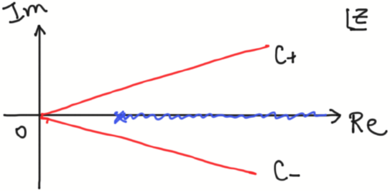
analytically: $\mathcal{B}[g](z) \sim \dfrac{1}{(S-z)^{\beta}}$

numerically: (also in practice) using a Padé approximation of $\mathcal{B}[g](z)$,



brunch cut exists on R^{+}, analytic continuation is required.

Imaginary ambiguity in Borel resum



A complex ambiguity arises with respect to the integration contour,

$$\int_{C_+} - \int_{C_-} \sim i e^{-Sw} w^{\beta}$$

accordingly, Borel resum of the hydro gradient expansion gives

$$\tilde{g}^{\text{hydro}}(w) = \text{Re}\left(\tilde{g}^{\text{hydro}}(w)\right) + \left(\propto i e^{-Sw} w^{\beta}\right)$$

Note that $g(w) = -1 - \mathcal{P}_L/e$ must be real!

Trans-series solution and resurgence

[Ecalle 1980s]

- One needs to consider instead solution of a trans-series form,

$$g(w) = \sum_{m=0} (\sigma e^{-Sw} w^{\beta})^m g^{(m)}(w) = \sum_{m=0} (\sigma e^{-Sw} w^{\beta})^m \sum_{n=0} f_n^{(m)} w^{-n}$$

* $g^{(m)}(w) = \sum_{n=0} f_n^{(m)} w^{-n}$ is asymptotic series for m=0,1, ... Especially,

$$g^{(0)}(w) = g^{\text{hydro}}(w)$$

each $g^{(m)}(w)$ should be Borel resummed (with analytic continuation).

* σ is a complex constant, to be determined via the relation of resurgence.

$$\sigma = \underbrace{\mathrm{Re}\sigma}_{\text{dep. on I.C.}} + i\mathrm{Im}\sigma$$

very commonly, it has be determined numerically.

Trans-series solution and resurgence

- Resurgence theory: $g(w)$ is real, hence in [Ecalle 1980s]

$$
\begin{aligned}
g(w) =& \mathrm{Re}\tilde{g}^{(0)}(w) + i\mathrm{Im}\tilde{g}^{(0)}(w) + \sigma e^{-Sw} w^{\beta} \left[\mathrm{Re}\tilde{g}^{(1)}(w) + i\mathrm{Im}\tilde{g}^{(1)}(w)\right] \\
&+ (\sigma e^{-Sw} w^{\beta})^2 \left[\mathrm{Re}\tilde{g}^{(2)}(w) + i\mathrm{Im}\tilde{g}^{(2)}(w)\right] + \ldots
\end{aligned}
$$

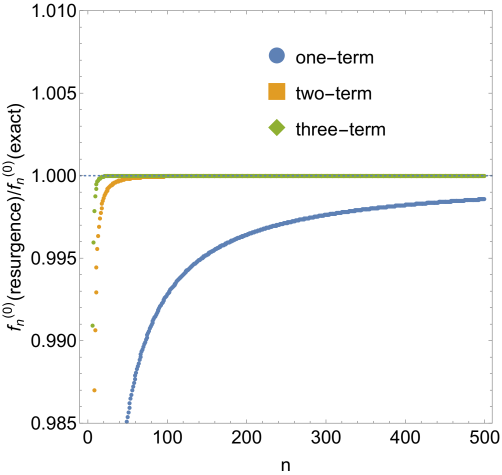
All imaginary parts cancel $\Leftrightarrow$ Reality condition.

$$
\rightarrow \begin{cases}
\mathrm{Im}\tilde{g}^{(0)}(w) \sim S_1 \mathrm{Re}\tilde{g}^{(1)}(w) + \ldots \\[2em]
f_n^{(0)} \sim S_1 \frac{\Gamma(n+\beta)}{2\pi i S^{n+\beta}} \left(f_0^{(1)} + \frac{S_1}{n+\beta-1} f_1^{(1)} + \ldots \right) \\[2em]
f_n^{(1)} \sim \ldots
\end{cases}
$$

where $S_1 = -2i\mathrm{Im}\sigma$ is the first Stoke constant.

- Resurgence: link between ‘hydro’ and ‘non-hydro’ physics.

Test of resurgence

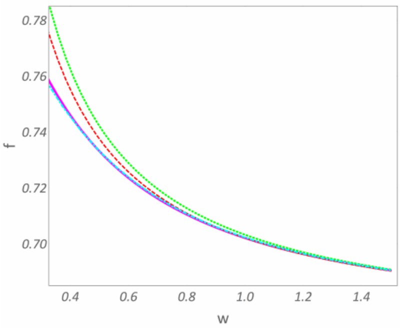


[see also, Basar and Dunne, PRD 92 (125011)]

Trans-series solution and attractor

Resurgence theory: attractor emerges in the resummed trans-series with respect to one specified I.C.

$$g^{\mathrm{att}}(w) = \tilde{g}^{(0)}(w) + (\sigma e^{-Sw} w^{\beta})\tilde{g}^{(1)}(w) + \dots$$



[M Heller and M Spanlinski, 115 (072501)]

Not easy to prove, one needs :

1. Find the specified I.C. in σ. 2. Resum properly w.r.t. resurgence.

Analytical solution for a time-dependent relaxation time

Instead of BRSSS, we consider the $\mathcal{L}$-moment equations,

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau}(a_0\mathcal{L}_0 + c_0\mathcal{L}_1)\,,$$
$$\frac{\partial \mathcal{L}_1}{\partial \tau} = -\frac{1}{\tau}(a_1\mathcal{L}_1 + b_1\mathcal{L}_0) - \frac{\mathcal{L}_1}{\tau_R}\,,$$

which is equivalent to BRSSS (or MIS) hydro. We parameterize τ_R as,

[J.P. Blaizot and LY, 2006.08815]

$$\tau_R \sim \tau^{1-\Delta}$$

so that the inverse Knudsen number is related to

$$w = \frac{\tau}{\tau_\pi} \sim \tau^{\Delta} \to \begin{cases} \Delta > 0 & \text{towards thermalization/hydro} \\ \Delta = 0 & \text{balance between expansion and collision} \\ & \text{[G. Denicol and J. Noronha, PRL 124 (152301) 2020]} \\ \Delta < 0 & \text{towards de-coupling} \\ & \text{[Chattopadhyay and Heinz, PLB 801, 135158, 2020]} \end{cases}$$

For $\Delta > 0$, analytical solution of $g(w)$ exists

$$g(w) = g_+ - w + aw \frac{\frac{1}{b} M\left(1+a, 1+b, \frac{w}{\Delta}\right) - AU\left(1+a, 1+b, \frac{w}{\Delta}\right)}{M\left(a, b, \frac{w}{\Delta}\right) + AU\left(a, b, \frac{w}{\Delta}\right)},$$

- M, U are confluent hypergeometric functions of the first and second kinds.
- g_+, a and b are constant parameters.
- A is constant of integration, to be determined by I.C. For attractor solution, $A = 0$,

$$g^{\text{att}}(w) = g_+ - w + w \frac{aM(1+a, 1+b, w/\Delta)}{bM(a, b, w/\Delta)}.$$

$\Delta < 0$ can be similarly obtained, while for $\Delta = 0$ solution is trivial.

Asymptotics of confluent hypergeometric functions

E.g., for large $|z|$, (similarly for $U(a,b,z)$)

$$M(a,b,z) \sim \frac{e^z z^{a-b}}{\Gamma(a)} \mathcal{F}(1-a, b-a, z) + \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b-a)} \mathcal{F}(a, a-b+1)$$

- $\mathcal{F}$ is an asymptotic series, and Borel summable $\Rightarrow \tilde{\mathcal{F}}$

$$\mathcal{B}[\mathcal{F}](z) = {}_2F_1(a,b,1,z)$$

The integral (7) can be "looked up" in standard references, but with caution: it is a special case of an integral that is given correctly [Eq. (30), p. 78] in Buchholz[18] and correctly once in Erdélyi *et al.*,[19] but that is given incorrectly twice in Erdélyi *et al.*,[19] twice in Gradshteyn and Ryzhik,[20] and twice in Roberts and Kaufman.[21] The general integral is

$$\int_0^\infty dt\, e^{-zt} t^{a-1}\, {}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}m-b, \tfrac{1}{2}-\tfrac{1}{2}m-b; a; -t) = z^{-a-b} e^{z/2} \Gamma(a) W_{b,m/2}(z) \,. \quad (8)$$

[H. Silverstone, S. Nakai and J. Harris, PRA 32, 1341, 1985]

Asymptotics of confluent hypergeometric functions

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- The factor $\zeta(w) = e^{-w/\Delta}\left(\frac{w}{\Delta}\right)^{b-2a+1}$, hence

$$g(w) \to g_+ - w + \frac{w\mathcal{F}(-a,b-a,w) - a\sigma\frac{\zeta(w)}{w}\mathcal{F}(1+a,1+a-b,-w)}{\mathcal{F}(1-a,b-a,w) + \sigma\frac{\zeta(w)}{w}\mathcal{F}(a,1+a-b,-w)}$$

where $\sigma = \frac{A\Gamma(a)}{\Gamma(b)} + e^{i\pi a}\frac{\Gamma(a)}{\Gamma(b-a)}$ can be identified easily.

Trans-series from the asymptotic form

- Expand w.r.t. $\zeta(w) = e^{-w/\Delta}\left(\frac{w}{\Delta}\right)^{b-2a+1}$ from

$$g(w) \to g_+ - w + \frac{w\mathcal{F}(-a, b-a, w) - a\sigma\frac{\zeta(w)}{w}\mathcal{F}(1+a, 1+a-b, -w)}{\mathcal{F}(1-a, b-a, w) + \sigma\frac{\zeta(w)}{w}\mathcal{F}(a, 1+a-b, -w)}$$

trans-series is recovered, $g(w) = g^{(0)}(w) + (\sigma\zeta(w))g^{(1)}(w) + \ldots$. E.g.,

$$g^{\text{hydro}}(w) \equiv g^{(0)}(w) = g_+ - w\left(1 - \frac{\mathcal{F}(-a, b-a, w/\Delta)}{\mathcal{F}(1-a, b-a, w/\Delta)}\right),$$

and

$$g^{(1)}(w) = -\frac{\mathcal{F}(-a, b-a, w/\Delta)}{\mathcal{F}(1-a, b-a, w/\Delta)}\left(\frac{\alpha\mathcal{F}(1+a, 1+a-b, -w/\Delta)}{w\mathcal{F}(-a, b-a, w/\Delta)} + \frac{\mathcal{F}(a, 1+a-b, -w/\Delta)}{\mathcal{F}(1-a, b-a, w/\Delta)}\right)$$

Analytical Borel resum of hydro gradients expansion

- The leading term gives hydro gradient expansion

$$g^{(0)}(w) = g_+ - w\left(1 - \frac{\mathcal{F}(-a, b-a, w)}{\mathcal{F}(1-a, b-a, w)}\right) = -\frac{4}{3} + \sum_{n=1} f_n^{(0)} w^{-n}$$

- Borel resum of hydro gradients (*analytically!*)

$$\tilde{g}^{(0)}(w) = g_+ - w\left(1 - \frac{\tilde{\mathcal{F}}(-a, b-a, w)}{\tilde{\mathcal{F}}(1-a, b-a, w)}\right)$$

Analytical Borel resum of the trans-series solution

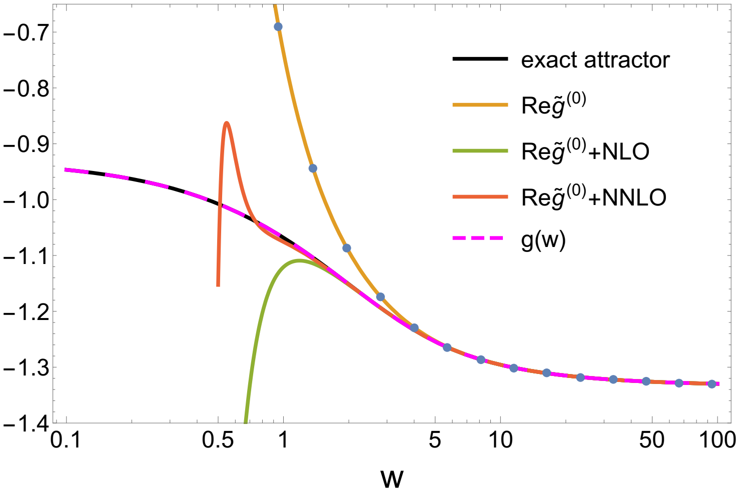
- Order by order, following resurgence relations,

$$g(w) = \mathrm{Re}\tilde{g}^{(0)}(w) + \sigma_{\mathcal{R}}\mathrm{Re}\tilde{g}^{(1)}(w) + (\sigma_{\mathcal{R}}^2 - \frac{S_1^2}{4})\mathrm{Re}\tilde{g}^{(2)}(w) + \cdots$$

- To infinite orders,

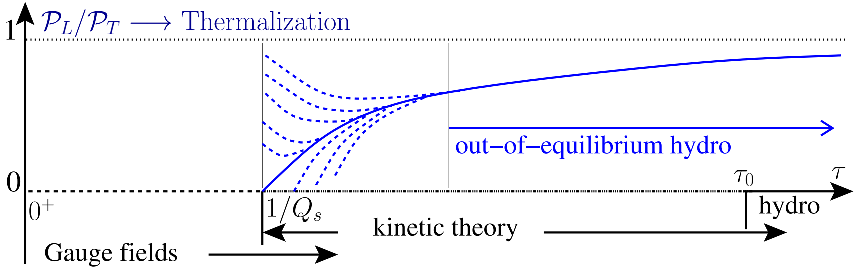
$$g(w) \to g_+ - w + \frac{w\tilde{\mathcal{F}}(-a, b-a, w) - a\sigma\frac{\zeta(w)}{w}\tilde{\mathcal{F}}(1+a, 1+a-b, -w)}{\tilde{\mathcal{F}}(1-a, b-a, w) + \sigma\frac{\zeta(w)}{w}\tilde{\mathcal{F}}(a, 1+a-b, -w)}$$

- By taking $A = 0$ in $\sigma \Rightarrow$ the exact attractor solution.



- attractor = Borel resum of the trans-series solution (to infinite orders).
- attractor contains transient structure $\propto e^{-Sw}w^{\beta}$.

Summary



- Attractor solution is the key to extend hydro to out of equilibrium.
- Attractor can be absorbed into redefinition of η/s in 2nd viscous hydro. [Romatschke, Blaitot, LY, Martinez]
- Hydro gradient expansion can be extened to trans-series solution.
- Out-of-equlibrium physics is implied in hydro $\Leftrightarrow$ resurgence theory.

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