# Quantum Field Theory: A Introductory Primer

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# 1

# Synopsis

This book is an introduction to the mathematical formalism of *quantum fields*. The various topics discussed in this book are focused on the elaboration of the *quantum mathematics* that is encoded in the theory of quantum fields. Quantum mathematics refers to the system of mathematical concepts that arise in quantum systems – some of the leading concepts being that of quantum fields, vacuum expectation values, Hamiltonians, state spaces, operators, Lagrangians and Feynman path integrals.

The phenomena of phase transition arises in equilibrium statistical mechanics and is completely based on classical physics. Nonetheless, Wilson (1983) solved the problem of phase transition using the mathematics of quantum field theory. A major area of application of quantum mathematics is the field of pure and applied mathematics. Donaldson obtained ground breaking results in the classification of four dimensional manifolds by studying the classical solutions of Yang-Mills gauge theory in four dimensions. Witten (1989) applied the techniques of quantum field theory to derive the Jones polynomial, which classifies knots and links in three space dimension. More recently, superstring theory has led to a plethora of results in pure mathematics using quantum mathematics.

Quantum mathematics has been applied to finance in Baaquie (2004) and Baaquie (2010). The classical random behavior of financial instruments is interpreted in terms of the symbols of quantum mathematics; the framework for connecting the observed market behavior of these instruments with their mathematical description turns out to be quite different from that of quantum mechanics. Quantum mathematics has also been applied to the analysis of problems in the social sciences by Haven and Khrennikov (2013).

Needless to say, neither critical phenomena, nor mathematics nor mathematical finance nor the social sciences are quantum systems. The interpretations of quantum mathematics that are valid for quantum systems are not

directly applicable to these problems; instead one has to reason from first principles to interpret the structures of quantum mathematics. In particular, the results obtained in mathematics and finance using quantum mathematics entail a radically different interpretation of its symbols. Similar to the approach taken in Baaquie (2014) in analyzing path integrals, the focus of this book is on the underlying mathematics of quantum field theory – and which could form the basis of applications of quantum mathematics to disciplines that go beyond theoretical physics. All the topics and subjects in the various chapters have been specifically chosen to illustrate the structure of quantum mathematics.

The prime objective of this book is to study quantum mathematics by examining different mathematical aspect of quantum fields. The derivations in this book are not tied down to the application of quantum fields to physics – as this would require many concepts that are not necessary for understanding the mathematical formalism of quantum fields.

## 1.1 What is a quantum field?

Quantum fields arose from the synthesis of quantum mechanics and special relativity. To maintain causality, special relativity - when combined with quantum indeterminacy – requires the existence of anti-particles. The existence of anti-particles makes non-relativistic quantum mechanics, which always has a fixed number of particles, inconsistent; particle-antiparticle annihilation and creation from the vacuum needs a state space with an indefinite number of particles.

The requirement for indefinite number of particles finds its realization in a state space based on infinitely many degree of freedom. Recall a degree of freedom is defined to be an independent variable for each instant of time. For example, a non-relativistic quantum mechanical particle in three dimensions has three degrees of freedom.

A quantum field theory can be self-interacting, as is the case for the quantized Goldstone Lagrangian, or is nonlinear due to the coupling of two fields, as is the case of scalar quantum electrodynamics, which consists of the photon field coupled to the complex scalar field. Quantum field theory is appropriate for describing a nonlinear relativistic quantum system as well as other random system with infinitely coupled degrees of freedom such as classical phase transitions.

*In summary, a quantum field describes a system, either quantum or classical, with infinitely many coupled degrees of freedom.*

The distinction between quantum mechanics and quantum field theory

is that quantum mechanics is a system with a finite number of degrees of freedom while quantum fields consist of infinitely many coupled degrees of freedom.

## 1.2 Focus of the book

There are many voluminous and encyclopedic books on quantum field theory, such as Weinberg (2010) that runs for over 1,500 pages and are meant for professionals and researchers – being inaccessible to non-specialists and beginners. Some books on quantum fields are geared towards specific applications, such as the book by Peskin and Schroeder (1995), which is written for applications in high energy theory and phenomenology, or the book by Zinn-Justin (1993) that seeks to explain critical phenomenon.

Many of the standard books on quantum field theory are written primarily for a readership that is drawn from theoretical physics. The proposed book eschews this approach and provides a quick, short and direct introduction to quantum fields to a wide audience who neither has the patience nor any use for the voluminous books. The book avoids advanced topics such as Yang-Mills gauge fields or supersymmetric quantum fields. Instead, the book gives a quick and direct route to the leading ideas of quantum fields, from free fields to the concept of renormalization and the renormalization group.

The book is divided into the following three parts.

- Part One is the Introduction to the two underlying themes of the book, which are the quantum principle and classical field theory. The mathematical formalism of these two fields contain the seeds for all the mathematics that is developed all the way to superstring theory.
- Part Two focusses on *Linear Quantum Fields* and is a necessary preparation for the study of nonlinear quantum fields. Linear fields are important in their own right, and this is demonstrated in the detailed analysis of both the Maxwell and Dirac fields. The importance of two-dimensional free fields is exemplified by the exact solution of the Dirac field coupled to the gauge field as well as an introductory discussion of bosonic superstring theory.
- Part Three discusses *Nonlinear Quantum fields*. The nonlinear properties of quantum fields are, in general, mathematically formidable as well as being fairly intractable. The nonlinearity also yields novel and unexpected results. To address the nonlinear aspect of quantum fields two different approaches have been taken. The nonlinear scalar quantum fields is studied to discuss the key idea of renormalization and of the renormalization

group. The effective action is evaluated for scalar electrodynamics and its spontaneous symmetry breaking is shown to be renormalization group invariant.

# PART ONE
# INTRODUCTION

Part One is an introduction to two underlying concepts that are the starting point of quantum fields, which are

- The quantum principle
- Classical field theory

The theoretical framework of quantum mechanics and classical field theory provides the mathematical tools for studying both quantum fields and superstrings.

### Chapter 2: The quantum principle

The quantum principle is discussed in Chapter 2 and states that all physical systems are constituted by degrees of freedom that are fundamentally indeterminate. The indeterminacy is realized in two different manners [Baaquie (2013)]

- In the Hamiltonian formulation all physical observables are Hermitian operators acting on a state space built on the underlying degree of freedom. Physically observable quantities are the expectation value of the operators obtained by either using the state space approach of Schrödinger or the operator algebra of Heisenberg.
- In the path integral formulation, the degrees of freedom are integration variables and hence having no fixed value and being intrinsically indeterminate. Physical observables are defined by the matrix elements of operators representing physical quantities, which in turn can be evaluated using the path integral.

Chapter 2 discusses both the path integral and state space/operator formulations of the quantum principle. The mathematics of quantum mechanics is generalized in defining quantum fields and further generalized in the formulation of superstings. But what remains valid for quantum fields and superstrings is that both these entities are based on degrees of freedom that are quantum mechanical in nature, being indeterminate, and that the operator algebra and path integral mathematics of quantum mechanics continues to be the mathematical backbone of these objects.

### Chapter 3: Classical field theory

Chapter 3 introduces the concept of the classical field. The Standard Model of high energy physics as well as the geometrical theory of gravitation holds 'stuff' that Nature is made out of is a variety of fields. The key feature

of a field – in fact, its most important and defining property – is that the field carries both energy and momentum at *every point* of space (and time). This is the reason that a field is considered to be a *physical entity* – as physical as a classical particle – with the difference that the field's energy and momentum can flow from one part to another, unlike a particle for which its energy and momentum is at the point that it occupies (of course, this point can move). Just as is the case for particles, the time evolution of a field exactly conserves the field's energy and momentum.

As the field evolves in time, the energy and momentum of the field can be re-distributed from one point of space to another by the variation in the field's strength. The term 'a propagating field' is a short hand for describing the re-distribution of the field's energy and momentum at the different points of space; this re-distribution can take place in many ways, with the most commonly studied case being the wave-like oscillations of the field's strength at different points of space. And conversely a propagating electromagnetic field impinging on a charged particle can transfer energy to it and cause the particle to accelerate. The total energy and momentum of the field coupled to charged particles, taken together, is exactly conserved.

A propagating electromagnetic field can, in principle, spread out over infinite distances.

In general, a classical field is a *determinate entity*, which is completely specified by assigning a numerical value to the field (with appropriate dimensions) at every point in spacetime. The values specifying the field at every point can change as it evolves in time. A particle of classical physics occupies a single point whereas a scalar and vector field are spread over space. The numerical value of a field can be a single number, as in the case a scalar field, or it can consist of several numbers, as in the case of a spinor or a vector field. The gravitational field is a tensor field requiring 10 real numbers at every spacetime point.

The description of a classical field using Lagrangian mechanics is discussed, including the symmetries of the Lagrangian and the conserved quantities tha arise from such symmetries. A few classical fields, which include the effective field theory that describes superconductivity as well as the theory that describes the Higgs mechanism, are discussed to illustrate some of the classical field's key features.

# 2

# Quantum Mechanics

The starting point of both quantum fields and superstrings is from the principles of quantum mechanics. We briefly review the foundations of quantum mechanics.

## 2.1 Fundamental principles

The three fundamental ingredients of quantum mechanics are the following:

- The degree of freedom; this can be discrete or continuous and can be one or infinitely(many). The degree of freedom will be denoted by $\phi$, and all its values form a space $\mathcal{F}$
- The state space, which in quantum mechanics is a Hilbert space, but can be larger in systems that don't conserve probability; it is denoted by $\mathcal{V}$ and an element of $\mathcal{V}$ is $|\Psi\rangle \in \mathcal{V}$. $\mathcal{V} : \mathcal{F} \to \Phi$
- Operators $\widehat{O}$ that act on $\mathcal{V}$ and map it to itself $\widehat{O} : \mathcal{V} \to \mathcal{V}$. The space of operators is denoted by $\mathcal{Q}$.

In summary, quantum mechanics consists of the mathematical triple $\{\mathcal{F}, \mathcal{V}, \mathcal{Q}\}$. Physical observations are represented by Hermitian operators

$$\widehat{O}_i^\dagger = \widehat{O}_i, \ i = 1, 2, ...N$$

In general, $[\widehat{O}_i, \widehat{O}_j] \neq 0$. The physically observed value of a physical quantity $\mathcal{Q}$, such as position, energy..., is given by $\langle\Psi|\mathcal{Q}|\Psi\rangle$, where $|\Psi\rangle$ represents the quantum state of the physical entity.

Consider the eigenfunctions and eigenvalues of a Hermitian operator given by

$$\widehat{O}|\psi_n\rangle = \lambda_n|\psi_n\rangle; \ \langle\psi_m|\psi_n\rangle = \delta_{m-n} \tag{2.1.1}$$

All Hermitian operators have the following spectral decomposition in

terms of their eigenvalues

$$\widehat{O} = \sum_n \lambda_n |\psi_n\rangle\langle\psi_n| \tag{2.1.2}$$

Define projection operators $\Pi_n = |\psi_n\rangle\langle\psi_n|$, such that

$$\Pi_n^2 = \Pi_n \text{ and } \widehat{O} = \sum_n \lambda_n \Pi_n \tag{2.1.3}$$

The completeness equation on $\mathcal{V}$ yields

$$\mathbb{I} = \sum_n \Pi_n;\ \mathbb{I}^2 = \mathbb{I} \tag{2.1.4}$$

Every state vectors has the decomposition

$$|\chi\rangle = \mathcal{I}|\chi\rangle = \sum_n c_n |\psi_n\rangle, \tag{2.1.5}$$

where

$$c_n = \langle\psi_n|\chi\rangle \tag{2.1.6}$$

Since

$$\langle\chi|\chi\rangle = 1$$

it follows that

$$1 = \sum_n |c_n|^2 \Rightarrow |c_n|^2 \in [0,1] \tag{2.1.7}$$

## 2.2 Theory of Measurement

To measure a physical quantity represented by operators $\mathcal{Q}$, a device has to be made that is designed to experimentally realize all the projection operators $\Pi_n$. The probability $P_n$ of a detector representing $\Pi_n$ detects the quantum states $|\psi\rangle$ is given $E_\chi[\Pi_n] = \langle\chi|\Pi_n|\chi\rangle = P_n$.

Note quantum theory of measurement requires that only one of the detectors, represented by $\Pi_n$ detects the quantum states. This also the so called collapse of the wave function. Define

$$P_n = |c_n|^2 \in [0,1] \tag{2.2.1}$$

Tracking the expectation value of $\mathcal{I}$ yields

$$1 = E_\chi[\mathcal{I}] = \sum_n P_n \tag{2.2.2}$$

Eqs. 2.2.1 and 2.2.2 show that quantum mechanics is a theory probability

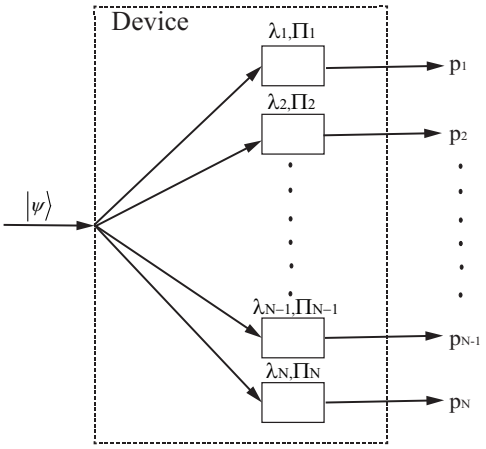


Figure 2.1 Projection operators observing the quantum state function $\psi$.

that is a synthesis of classical theory of probability with the concept of an underlying linear vector space $\mathcal{V}$ and Hermitian operators based on it.

Note the fundamental paradox of quantum mechanics, namely that the foundation of the quantum entity, namely the degree of freedom, can never, in principle be observed by any experiment. Furthermore, two orthogonal projection operators $\Pi_n, \Pi_m$ can never simultaneously observe the state function $\psi$. A measurement results in the state function collapsing to *either* the state $|\psi_n\rangle = \Pi_n|\psi\rangle$ or $|\psi_m\rangle = \Pi_m|\psi\rangle$; the state vector $|\psi\rangle$ and is *never* simultaneously observed by both the projection operators. If any experiment, two orthogonal projection operators simultaneously observe the state function $\psi$, then that would spell the end of the current (Copenhagen) interpretation of quantum mechanics.

## 2.3 The Schrodinger and Heisenberg Formulation

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}|\psi_t\rangle = H|\psi_t\rangle \tag{2.3.1}$$

The coordinate eigenstate $|\phi\rangle$ and operator $\widehat{\phi}$ is defined by

$$\widehat{\phi}|\phi\rangle = \phi|\phi\rangle \tag{2.3.2}$$

and yield the coordinate representation

$$-\frac{\hbar}{i}\langle\phi|\frac{\partial}{\partial t}|\psi_t\rangle = \langle\phi|H|\psi_t\rangle$$

and hence

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi_t(\phi) = \widehat{H}(\phi, \frac{\partial}{\partial \phi})\psi_t(\phi) \tag{2.3.3}$$

Note the Hamiltonian operator acts on the dual basis state, and this is the rule for all differential operators. From Eq. 2.3.1

$$|\psi_t\rangle = e^{-\frac{it}{\hbar}H}|\psi_0\rangle \tag{2.3.4}$$

The expectation value of a Schrodinger operator $\widehat{O}$ is given by ($\hbar = 1$)

$$E_\psi[\widehat{O}(t)] = \langle\psi_t|\widehat{O}|\psi_t\rangle \tag{2.3.5}$$

$$= \langle\psi|e^{itH}\widehat{O}e^{-itH}|\psi\rangle \tag{2.3.6}$$

$$= \langle\psi|\widehat{O}(t)|\psi\rangle = tr(\widehat{O}(t)\rho) \tag{2.3.7}$$

where

$$\rho = |\psi\rangle\langle\psi|$$

is the density operator.

The Heisenberg operator is defined by

$$\widehat{O}(t) = e^{itH}\widehat{O}e^{-itH}$$

$$\Rightarrow i\frac{\partial\widehat{O}(t)}{\partial t} = [\widehat{O}(t), \widehat{H}] : \text{ Heisenberg operator equation}$$

Heisenberg's formulation is more suitable for measurement theory. A device represents physical projection operators $\Pi_n$; the quantum state is $\rho$. The result of repeated measurements yields $P_n = tr(\Pi_n\rho)$.

## 2.4 Feynman Path Integral

The Dirac-Feynman formulation of path integrals is derived from the Hamiltonian operator. Although every Hamiltonian yields a path integral, it is not the case that every path integral can in turn be expressed in terms of a Hamiltonian. This aspect of path integrals come to the fore in studying path integrals for curved manifolds, but will be not addressed in this book.

Consider the continuation of time to Euclidean time given by $x_t = x_\tau,\ t = -i\tau$ Hence

$$\mathcal{L}_E = \frac{m}{2}(\frac{dx}{dt})^2 - V(x) = -\frac{m}{2}(\frac{dx}{d\tau})^2 - V(x)$$

The canonical momenta changes sign but gives the same Hamiltonian

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x) \ \ ; \ \ H_E = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

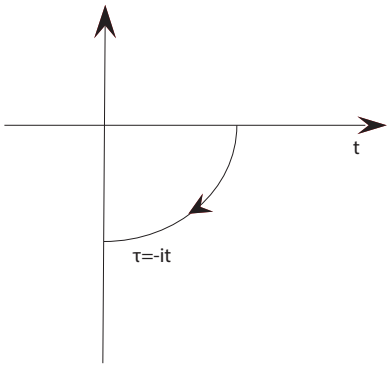


Figure 2.2 Euclidean continuation of real (Minkowski) time.

The Dirac-Feynman Formula is given by

$$\begin{aligned} \langle x'|e^{-i\epsilon H}|x\rangle &= \mathcal{N}(\epsilon)e^{i\epsilon\mathcal{L}(x,x',\epsilon)} \quad : \quad \text{Minkowski} \\ \langle x'|e^{-\epsilon H_E}|x\rangle &= \mathcal{N}(\epsilon)e^{\epsilon\mathcal{L}_E(x,x',\epsilon)} \quad : \quad \text{Euclidean} \end{aligned} \tag{2.4.1}$$

where $\mathcal{N}(\epsilon)$ is a normalization term. The subscript $E$ will be dropped unless necessary.

The Euclidean path integral has the following derivation.

$$\langle x'|e^{-\epsilon H_E}|x\rangle \simeq \langle x'|e^{-\epsilon\frac{p^2}{2m}}|x\rangle e^{-\epsilon V(x)} \tag{2.4.2}$$

Note

$$\int \frac{dp}{2\pi}|p\rangle\langle p| = \mathcal{I} \tag{2.4.3}$$

Hence

$$\begin{aligned} \langle x'|e^{-\epsilon\frac{p^2}{2m}}|x\rangle &= \int \frac{dp}{2\pi}\langle x'|e^{-\epsilon\frac{P^2}{2m}}|p\rangle\langle p|x\rangle = \int \frac{dp}{2\pi}e^{-\epsilon\frac{p^2}{2m}}e^{-ip(x-x')} \\ &= \sqrt{\frac{m}{2\pi\epsilon}}e^{-\frac{m}{2\epsilon}(x-x')^2} \end{aligned}$$

Hence

$$\epsilon\mathcal{L} = -\frac{m}{2\epsilon}(x_{t+\epsilon}-x_t)^2 - \epsilon V(x_t) \quad \Rightarrow \quad \mathcal{L} = -\frac{m}{2}(\frac{dx}{d\tau})^2 - V(x)$$

Using the completeness equation

$$\int dx|x\rangle\langle x| = \mathcal{I} \tag{2.4.4}$$

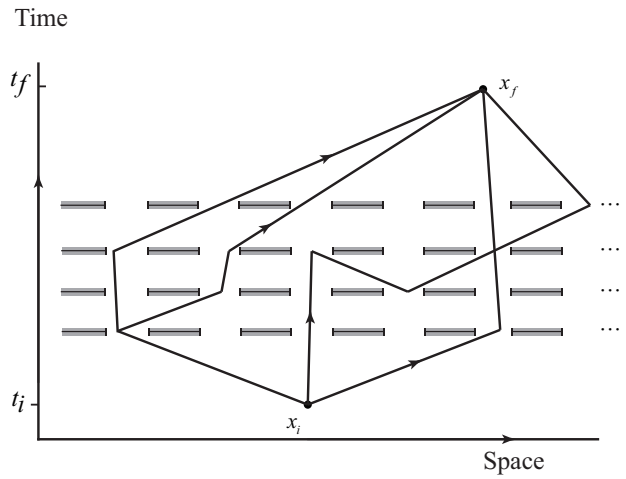


Figure 2.3 The paths between initial and final state.

yields for the path integral, for $\epsilon = \tau/N$, the following

$$K(x', x; \tau) = \langle x'|e^{-\tau H}|x\rangle \tag{2.4.5}$$
$$= \int \prod_{n=1}^{N-1} dx_n \langle x'|e^{-\epsilon H}|x_{N-1}\rangle\langle x_{N-1}|...\langle x_{n+1}|e^{-\epsilon H}|x_n\rangle...\langle x_1|e^{-\epsilon H}|x\rangle$$

Let $x_N = x'$; $x_0 = x$, then

$$K(x', x; \tau) = \Big\{ \int \prod_{n=1}^{N-1} dx_n \Big\} \Big\{ \prod_{n=0}^{N-1} \langle x_{n+1}|e^{-\epsilon H}|x_n\rangle \Big\} \tag{2.4.6}$$
$$= \int Dx e^{\sum_{n=0}^{N-1} \mathcal{L}(x_{n+1}, x_n)} \tag{2.4.7}$$

The lattice action is

$$\mathcal{S}[x] = -\frac{\epsilon m}{2} \sum_{n=0}^{N-1} (\frac{x_{n+1} - x_n}{\epsilon})^2 - \epsilon \sum_{n=0}^{N-1} V(x_n) \tag{2.4.8}$$

$$\int Dx = \prod_{n=1}^{N-1} \int (\frac{m}{2\pi\epsilon})^{\frac{1}{2}} dx_n \tag{2.4.9}$$

In the $\epsilon \to 0$ limit

$$\mathcal{S}[x] = \int_0^\tau \mathcal{L}(t)dt \;\; ; \;\; \mathcal{L} = -\frac{m}{2}(\frac{dx}{d\tau})^2 - V(x) \tag{2.4.10}$$

$$\int Dx = \mathcal{N}\prod_{t=0}^{\tau}\int_{-\infty}^{+\infty} dx(t) \tag{2.4.11}$$

The evolution kernel $\langle x_f|e^{-\tau H}|x_i\rangle$ is given by the Euclidean Feynman Path Integral

$$K(x_i, x_f; \tau) = \langle x_f|e^{-\tau H}|x_i\rangle = \int_{B.C.} Dxe^S \;\; : \text{Euclidean} \tag{2.4.12}$$
$$\text{Boundary Condition} : x(0) = x_i \;,\; x(\tau) = x_f$$

The Minkowski time path integral, which in fact is the original version written down by Feynman, can be obtained by analytically continuing Euclidean to Minkowski time. Using subscript $M$ for Minkowski time, we have $\tau = it_M$; hence, Eqs. 2.4.9 and 2.4.10 yield

$$\mathcal{S}_M[x] = \int_{t_i}^{t_f} dt_M \mathcal{L}_M(t) = -iS[x]; \;\; \mathcal{L}_M = \frac{m}{2}(\frac{dx}{dt_M})^2 - V(x)$$

$$\int Dx_M = \lim_{N\to\infty}\prod_{n=1}^{N-1}\int (\frac{m}{2\pi i\epsilon})^{\frac{1}{2}} dx_n \;\; ; \;\; t_f - t_i = N\epsilon$$

The Minkowski time path integral, from Eq. 2.4.12, is given by

$$K_M(x_i, x_f; \tau) = \langle x_f|e^{-i(t_f-t_i)H}|x_i\rangle = \int_{B.C.} Dx_M e^{iS_M} \;\; : \text{Minkowski}$$
$$\text{Boundary Condition} : \; x(t_i) = x_i \;,\; x(t_f) = x_f$$

## 2.5 Hamiltonian and path integral

For a continuous degree of freedom the Lagrangian consists a kinetic term that is usually the same for a wide class of systems; one needs to choose an appropriate potential $V(x)$ to fully describe the system. For the sake of rigor, consider the Euclidean Lagrangian and action given by

$$\mathcal{L} = -\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x) \;\; ; \;\; S = \int_{t_i}^{t_f} dt\mathcal{L}$$

The evolution kernel $\langle x_f|e^{-\tau H}|x_i\rangle$ is given by the superposition of all the indeterminate (indistinguishable) paths and is equal to sum of $e^S$ over all possible paths as obtained in Eq. 2.4.12.

Path integral quantization is more general than starting from the Schrödinger equation and the Hamiltonian operator for following reasons.

- The Schrödinger approach is based on the properties of state space *in addition* to the Hamiltonian driving the Schrödinger equation.
- The spacetime symmetries of the quantum system are explicit in the Lagrangian based path integral approach whereas in the Schrödinger approach these are implicit and need to be extracted using the properties of the Hamiltonian and state space. In particular, one has to derive the symmetry operators that commute with the Hamiltonian.
- Path integral quantization yields a transparent formulation of constrained systems, as for example discussed in Baaquie (2014). In the Schrödinger formulation, one needs *both* the Hamiltonian and commutation relations, which for a constrained system are far from obvious and require a fair amount of derivations.

These considerations come to the forefront for complicated systems like non-Abelian gauge fields that are discussed in Part V on lattice gauge theory, where the starting point is the Lagrangian; path integral quantization turns out to be more efficient than the Schrödinger approach.

## 2.6 Hamiltonian from Lagrangian

Recall in Section 2.4, the Lagrangian was derived from the Hamiltonian using the Dirac-Feynman formula. In this Section, it is shown how to derive the Hamiltonian $H$ if the Lagrangian is known; one can use the procedure of classical mechanics for the derivation, but instead a quantum mechanical deviation is given in this Section.

A Lagrangian that is more general than the one discussed in Section 2.4, and arises in the study of option theory in finance Baaquie (2004), is chosen to illustrate some new features. Option theory is based on classical random processes that are similar to the diffusion equation and hence the time parameter $t$ in the path integral appears as 'Euclidean time' $t$, which for option theory is in fact calendar time.

Let the degree of freedom be the real variable $\phi$. Consider the following Lagrangian and action

$$\begin{aligned}
\mathcal{L}(t) &= -\frac{1}{2}\left[me^{-2\nu\phi}\Big\{\frac{d\phi}{dt}+\alpha(\phi,t)\Big\}^2+V(\phi)\right] \\
S &= \int_0^\tau dt\mathcal{L}(t) = -\frac{1}{2}\int_0^\tau dt\left[me^{-2\nu\phi}\Big\{\frac{d\phi}{dt}+\alpha(\phi,t)\Big\}^2+V(\phi)\right] \quad (2.6.1)
\end{aligned}$$

For greater generality, a $\phi$ dependent mass equal to $me^{-2\nu\phi}$ and a drift term $\alpha(\phi, t)$ have been included in $\mathcal{L}$.

The path integral is given by the following generalization of Eq. 2.4.12

$$K(\phi_i, \phi_f; \tau) = \int D\phi e^{-\nu\phi} e^S \tag{2.6.2}$$
$$\int D\phi e^{-\nu\phi} \equiv \prod_{t=0}^{\tau} \int_{-\infty}^{+\infty} d\phi(t) e^{-\nu\phi(t)}$$
$$\text{Boundary Conditions } \phi(\tau) = \phi_f \ ; \ \phi(t=0) = \phi_i$$

Note the path integral integration measure $\int D\phi$, has a factor of $e^{-\nu\phi}$ needed to obtain a well-defined Hamiltonian.

Recall from the discussion of the evolution kernel in Section 2.4, the path integral is related to the Hamiltonian $H$ by Eq. 2.4.12, namely

$$K(\phi_i, \phi_f; T) = \int D\phi e^{-\nu\phi} e^S = \langle \phi_f | e^{-\tau H} | \phi_i \rangle \tag{2.6.3}$$

One needs to extract the Hamiltonian $H$ from the path integral on the left hand side of Eq. 2.6.3.

The Hamiltonian propagates the system through infinitesimal time; the time index $t$ is discretized into a lattice with spacing $\epsilon$, where $t = n\epsilon$ with $N = T/\epsilon$ and $\phi(x) \to \phi_n$. The path integral reduces to a finite $(N-1)$-fold multiple integral, analogous to what was obtained in Eq. 2.4.6. Discretizing the time derivative $d\phi/dt \to (\phi_{n+1} - \phi_n)/\epsilon$ yields the following lattice action and Lagrangian

$$\langle \phi_N | e^{-\epsilon N H} | \phi_0 \rangle = \prod_{n=1}^{N-1} \int d\phi_n e^{-\nu\phi_n} e^{S(\epsilon)} \tag{2.6.4}$$
$$S(\epsilon) = \epsilon \sum_{n=0}^{N-1} L(n)$$
$$L(n) = -\frac{me^{-2\nu\phi_n}}{2\epsilon^2} \left[\phi_{n+1} - \phi_n + \epsilon\alpha_n\right]^2 - \frac{1}{2}[V(\phi_{n+1}) + V(\phi_n)]$$

As in Section 2.4, the completeness equation given in Eq. 2.4.4 yields

$$\int d\phi_n |\phi_n\rangle\langle\phi_n| = \mathbb{I}$$

and is used $N-1$ times to write out the expression for $e^{-\epsilon N H}$. The Hamil-

tonian is identified as follows

$$\langle\phi_{n+1}|e^{-\epsilon H}|\phi_n\ \rangle = \mathcal{N}(\epsilon)e^{-\nu\phi_n}e^{\epsilon L_n}$$
$$= \mathcal{N}(\epsilon)e^{-\nu\phi_n}\exp\{-\frac{me^{-\nu\phi}}{2\epsilon}\big[\phi_{n+1}-\phi_n+\epsilon\alpha_n\big]^2-\frac{\epsilon}{2}[V(\phi_{n+1})+V(\phi_n)]\}$$

Since the Hamiltonian depends on the value of $\phi$ at two different instants, to simplify notation let

$$\phi_{n+1}=\phi\ \ ;\ \ \phi_n=\phi'\ \ ;\ \ \alpha_n=\alpha$$

Ignoring terms that are of $O(\epsilon)$ in Eq. 2.6.4, the matrix elements of the Hamiltonian are given by

$$\langle\phi|e^{-\epsilon H}|\phi'\rangle = \mathcal{N}(\epsilon)e^{-\nu\phi}\exp\Big\{-\frac{me^{-2\nu\phi}}{2\epsilon}\big[\phi-\phi'+\epsilon\alpha\big]^2-\epsilon V(\phi)\Big\} \quad (2.6.5)$$

Note that unlike Eq. 2.4.12, for which the Hamiltonian is known and the Lagrangian was derived from it, in Eq. 2.6.5 one needs to *derive* the Hamiltonian *from* the known Lagrangian. This derivation is the quantum mechanical analog of the derivation of $H$ given by Hamiltonian mechanics in classical mechanics and discussed by Baaquie (2014).

The key feature of the Lagrangian that in general allows one to derive its Hamiltonian is that the Lagrangian contains *only first order* time derivatives; hence on discretization the Lagrangian involves only $\phi_n$ that are nearest neighbours in time, thus allowing it to be represented as the matrix element of $e^{-\epsilon H}$, as in Eq. 2.6.5.

In contrast, for Lagrangians that contain *second order* or higher order time derivatives, discussed by Baaquie (2014), the derivation of the Hamiltonian from the Lagrangian and path integral is non-trivial since the entire framework of coordinate and canonical momentum is no longer applicable. Instead, one has to employ the Dirac method required for quantizing constrained systems, and in particular, evaluate the Dirac brackets for the system in order to obtain the Hamiltonian and commutation relations.

In Eq. 2.6.5, the time derivatives appears in a quadratic form; hence one can use Gaussian integration to re-write Eq. 2.6.5 in the following manner[1]

$$\langle\phi|e^{-\epsilon H}|\phi'\rangle = e^{-\nu\phi}e^{-\epsilon V(\phi)}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\exp\Big\{-\frac{\epsilon}{2m}p^2+ip[\phi-\phi'+\epsilon\alpha]e^{-\nu\phi}\Big\}$$
$$= e^{-\epsilon V(\phi)}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\exp\{-\frac{\epsilon e^{2\nu\phi}}{2m}p^2+ip\big(\phi-\phi'+\epsilon\alpha\big)\} \quad (2.6.6)$$

[1] Henceforth $\mathcal{N}(\epsilon)$ is ignored since it is an irrelevant constant contributing to only the definition of the zero of energy.

where the pre-factor of $e^{-\nu\phi}$ has been canceled by re-scaling the integration variable $p \to pe^{\nu\phi}$.

The Hamiltonian $H = H(\phi, \partial/\partial\phi)$ is a differential operator and acts on the dual co-ordinate $\phi$, as is required for all differential operators, as mentioned earlier after Eq. 2.3.3. Hence, for the state function $|\psi\rangle$, which is an element of the state space, the Hamiltonian acts on the *dual basis state* $\langle\phi|$, and yields $\langle\phi|H|\psi\rangle = H(\phi, \partial/\partial\phi)\psi(\phi)$, similar to the result given in Eq. 2.3.3.

The Hamiltonian is hence given by the following representation[2]

$$\langle\phi|e^{-\epsilon H}|\phi'\rangle = e^{-\epsilon H(\phi,\partial/\partial\phi)}\langle\phi|\phi'\rangle = e^{-\epsilon H(\phi,\partial/\partial\phi)}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}e^{ip(\phi-\phi')} \quad (2.6.7)$$

since $\langle\phi|\phi'\rangle = \delta(\phi - \phi')$. Ignoring overall constants and using the property of the exponential function under differentiation, one can re-write Eq. 2.6.6 as

$$\langle\phi|e^{-\epsilon H}|\phi'\rangle = \exp\{\frac{1}{2m}\epsilon e^{2\nu\phi}\frac{\partial^2}{\partial\phi^2} + \epsilon\alpha\frac{\partial}{\partial\phi} - \epsilon V(\phi)\}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}e^{ip(\phi-\phi')} \quad (2.6.8)$$

Comparing Eq. 2.6.8 above with Eq. 2.6.7 yields the Hamiltonian

$$H = -\frac{1}{2m}e^{2\nu\phi}\frac{\partial^2}{\partial\phi^2} - \alpha(\phi)\frac{\partial}{\partial\phi} + V(\phi) \quad (2.6.9)$$

The Hamiltonian is quite general since both $V(\phi)$ and $\alpha(\phi)$ can be functions of the degree of freedom $\phi$. Note that the Hamiltonian $H$ is non-Hermitian – and is Hermitian only for $\nu = 0$ and a pure imaginary $\alpha$. The path integral has a non-trivial integration measure $\exp\{-\nu\phi\}$ that needs to specified in addition to the Hamiltonian.

## 2.7 Summary

The principles of quantum mechanics are realized by indeterminate degrees of freedom. An entity in quantum mechanics is described by degrees of freedom that, due to quantum indeterminacy, take simultaneously take all possible values.This is realized by the operator formalism and by the Feynman path integral.

Both the operator formalism and the Dirac-Feynman path integral formalism were briefly reviewed as these form the basis for the generalizations

[2] From Eq. **??**, the convention for scalar product is $\langle p|\phi_n\rangle = \exp(-ip\phi_n)$, and the sign of the exponential in Eq. 2.6.7 reflects this choice. The definition of $H$ requires it to act on the dual state vector $\langle\phi|$; if one choose to write the Hamiltonian as acting of the state vector $|\phi\rangle$, $H^\dagger$ would then have obtained instead. Since $H$ is not Hermitian, this would lead to an incorrect result.

that are required for the description of quantum fields. The path integral was derived starting from the Hamiltonian and in turn the Hamiltonian was obtained starting from path integral quantization.

The interplay of the path integral with the underlying state space and Hamiltonian is one of the foundations and unique feature of quantum mathematics that distinguishes the Feynman path integral from functional integration in general.

# 3

# Classical field theory

The two pillars of contemporary physics are Einstein's theory of gravitation and the Standard Model of particle physics. Both these pillars are based on the concept of the field, with gravity being a nonlinear classical field and the Standard Model being described by quantum fields. The classical field is a determinate function of spacetime, having a unique and determinate (fixed) value for every point of spacetime. In contrast, quantum fields are indeterminate and 'have' all possible values at every point of spacetime.

The field exists at all points of space with the energy and momentum of the field being *spread* all over space. The strength (numerical value) of a classical field at the different points of spacetime is a measure of the energy and momentum at that point.

Newtonian mechanics of a point particle, specified by position $x(t)$ is determined by $md^2x/dt^2 = F$. For conservative system $F = -\partial V/\partial x$ and we obtain

$$m\frac{d^2x_c}{dt^2} = -\frac{\partial V(x_c)}{\partial x}$$

Energy conservation follows from equation of motion

$$\frac{dE}{dt} = 0;\ E = \frac{1}{2}m\left(\frac{dx_c}{dt}\right)^2 + V(x_c) \tag{3.0.1}$$

The Hamiltonian $H$ is

$$H = T + V = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x)$$

Note the Hamiltonian $H$ is defined for trajectory $x(t)$ and not necessarily for the classical solution, as is the case for the conserved energy $E$.

A classical particle has a Lagrangian given by

$$\mathcal{L} = T - V = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V(x)$$

with action $\mathcal{S} = \int_{t_i}^{t_f} dt\mathcal{L}(t)$. The equation of motion given in Eq. 3.0.1 is obtained by demanding that the solution $x_c(t)$ minimise the action $\mathcal{S}$, with boundary conditions $x_c(t_i) = x_i;\ x_c(t_f) = x_f$.

Consider paths that are close to the classical path: $x' = x_c + \epsilon$ with boundary conditions

$$b.c:\ x'(t_i) = x_c(t_i);\ x'(t_f) = x_c(t_f)$$

Hence for $\dot{x} = dx/dt$

$$\mathcal{S}[x_c + \epsilon] = \int_{t_i}^{t_f} dt[\frac{m}{2}(\dot{x}_c + \dot{\epsilon})^2 - V(x_c + \epsilon)] \tag{3.0.2}$$

$$= \mathcal{S}[x_c] + \int_{t_i}^{t_f} dt[m\dot{x}_c\dot{\epsilon} - \epsilon V'(x_c)] + O(\epsilon^2) \tag{3.0.3}$$

$$= \mathcal{S}_c + \int_{t_i}^{t_f} dt[-m\ddot{x}_c - V'(x_c)]\epsilon + m\int_{t_i}^{t_f} dt\frac{d}{dt}(\epsilon\dot{x}_c) \tag{3.0.4}$$

The action is minimized by $x_c(t)$ is

$$m\ddot{x}_c = -V'(x_c) + \text{boundary term}$$

and the boundary term is zero, namely

$$\int_{t_i}^{t_f} dt\frac{d}{dt}(\epsilon\dot{x}_c) = 0$$

More generally, for canonical coordinate $q_i,\ i = 1, 2...N$

$$\mathcal{L} = \frac{1}{2}m\sum_i \dot{q}_i^2 - V(q_i) = \mathcal{L}(q_i, \dot{q}_i)$$

and

$$\mathcal{S} = \int \mathcal{L}dt$$

The variation of the coordinates, keeping the boundary conditions fixed,

yields

$$\delta\mathcal{S} = \int \delta\mathcal{L} dt \tag{3.0.5}$$

$$= \int \Big[\frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i\Big] dt \tag{3.0.6}$$

$$= \int \Big[\frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big]\delta q_i dt + \int \frac{d}{dt}\Big(\delta q_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big) dt = 0 \tag{3.0.7}$$

$$\frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = 0 : \text{ Euler-Lagrange Equation} \tag{3.0.8}$$

Boundary term must be zero, namely

$$\int \frac{d}{dt}\Big(\delta q_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\Big) dt = 0 \tag{3.0.9}$$

Note $\mathcal{L}$ is defined only up to a term $d\Lambda/dt$ since

$$\mathcal{L}' = \mathcal{L} + \frac{d\Lambda}{dt}$$

gives the same Euler-Lagrange Equation up to a boundary term.

Noteworthy 3.1: Relativistic notation

The metric $\eta_{\mu\nu}$ for Minkowski space is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} ; \; x^\mu = (x^0, x^1, x^2, x^3) \tag{3.0.10}$$

This yields, using the convention that repeated indices are summed over

$$x_\mu = \eta_{\mu\nu}x^\nu = (ct, -x, -y, -z) = (x_0, x_1, x_2, x_3)$$

A Lorentz invariant scalar product is defined by using metric $\eta_{\mu\nu}$ and yields

$$A^\mu B_\mu = \eta^{\mu\nu}A_\mu B_\nu = \eta_{\mu\nu}A^\mu B^\nu$$

Furthermore

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \;\; ; \;\; \partial^\mu\partial_\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

### 3.1 Classical field equation

The field $\phi$ is defined on space time $x,\ t$ and can be density, pressure, temperature fields as well as the electromagnetic and gravitation fields. All fields $\phi(t,x)$ carry energy and momentum at each space time point. The kinetic energy of the field is defined for finite volume $\mathcal{R}^3$ by

$$T = \frac{1}{2}\widetilde{m}\int_{\mathcal{R}^3} d^3x\Big(\frac{\partial\phi(t,x)}{\partial t}\Big)^2 \tag{3.1.1}$$

and its potential energy is

$$V = \int_{\mathcal{R}^3} d^3x\Big[\frac{1}{2}\widetilde{m}(\frac{\partial\phi}{\partial\vec{x}})^2 + V(\phi)\Big] \tag{3.1.2}$$

The Lagrangian density $\mathcal{L}(t,x)$ is given by

$$\mathcal{L} = \frac{1}{2}\widetilde{m}(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}\widetilde{m}(\frac{\partial\phi}{\partial\vec{x}})^2 - V(\phi) \tag{3.1.3}$$

and the action is

$$\mathcal{S} = \int_{t_i}^{t_f} dt\int_{\mathcal{R}^3} d^3x\mathcal{L}(t,x) \equiv \int_V \mathcal{L}(\phi,\partial_\mu\phi) \tag{3.1.4}$$

where $V = \mathcal{R}^3\bigotimes[t_f,t_i]$.

A general transformation for the scalar field has the form

$$\phi(x) \to \phi'(x) = \phi(x)$$

and leads to the variation

$$\delta\phi \equiv \phi'(x) - \phi(x)$$

with the constraint that the variation at the initial and final surface is zero; that is

$$\delta\phi\Big|_{t=t_i} = 0 = \delta\phi\Big|_{t=t_f}$$

The variation of the action is given by

$$\begin{aligned}\delta\mathcal{S} &= \int_V \Big[\frac{\partial\mathcal{L}}{\partial\phi(t,x)}\delta\phi(t,x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi\Big]\\ &= \int_V \Big[\frac{\partial\mathcal{L}}{\partial\phi(t,x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big]\delta\phi(t,x) + \int_V \Big[\partial_\mu(\delta\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi})\Big]\\ &= \delta\mathcal{S}_V + \delta\mathcal{S}_{\partial V}\end{aligned} \tag{3.1.5}$$

The constraint that the variation on the boundary be zero yields

$$\delta\mathcal{S}_{\partial V} = 0$$

and the field equation is given by $\delta\mathcal{S} = \delta\mathcal{S}_V = 0$; hence

$$\delta\mathcal{S} = \delta\mathcal{S}_V = 0 \quad \Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0 : \text{ Classical field equation} \quad (3.1.6)$$

Note the label $i$ in $q_i(t)$ of the canonical coordinate has become a continuous label $\vec{x}$. In this sense, at every point $\vec{x}$, the field has an independent canonical coordinate $\phi_{\vec{x}}(t)$. Writing 3.1.6 explicitly yields

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\frac{\partial\phi}{\partial t}} + \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{L}}{\partial\frac{\partial\phi}{\partial x^i}}\right) = 0 \quad (3.1.7)$$

### *3.1.1 Free scalar field*

The free scalar field $\phi(t,x)$ is a real valued function of $t, x$ and is an infinite dimensional generalization of the simple harmonic oscillator; its Lagrangian and is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2$$

In terms of the space and time coordinates

$$\mathcal{L} = \frac{1}{2}(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}(\frac{\partial\phi}{\partial\vec{x}})^2 - \frac{m^2}{2}\phi^2$$

The Lagrangian yields

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi; \ \frac{\partial\mathcal{L}}{\partial\frac{\partial\phi}{\partial t}} = \frac{\partial\phi}{\partial t}; \ \frac{\partial\mathcal{L}}{\partial\frac{\partial\phi}{\partial\vec{x}}} = -\frac{\partial\phi}{\partial\vec{x}}$$

In relativistic notation

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$$

and the Euler-Lagrange Equation is

$$-m^2\phi - \partial_\mu\partial^\mu\phi = 0 \quad \Rightarrow (\partial_\mu\partial^\mu + m^2)\phi = 0$$

Or equivalently

$$[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial\vec{x}^2} + m^2]\phi = 0 : \quad \text{Klein-Gordon Equation}$$

The Hamiltonian density is

$$H = T + V = \frac{\widetilde{m}}{2}\dot{\phi}^2 + \frac{\widetilde{m}}{2}(\vec{\nabla}\phi)^2 + \frac{\omega^2}{2}\phi^2$$

Hence the Klein-Gordon Equation is

$$[\partial^\mu \partial_\mu + \frac{m^2c^2}{\hbar^2}]\phi = 0 : \quad \text{Klein-Gordon Equation}$$

In all the subsequent discussions, the units are chosen so that $c = \hbar = 1$.

## 3.2 Symmetries; Noether Theorem

The symmetries of a system are those transformations of the field $\phi$ and of the coordinates $x = (t, \vec{x})$ that leave the field equations of the theory unchanged. The symmetry is realized by the transformation leaving the Lagrangian and hence the action $\mathcal{S}$ *invariant*.

Consider a general transformation that has the form

$$x \to \tilde{x} = \tilde{x}(x) \;\; ; \;\; \phi(x) \to \tilde{\phi}(\tilde{x}) \tag{3.2.1}$$

The transformation leaves the Lagrangian invariant (up to a divergence) and hence, for any arbitrary volume $V$, we have

$$S = \int d^4\tilde{x}\mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x}), \tilde{x}) = \int d^4x\mathcal{L}(\phi, \partial\phi, x)$$

The coordinate transformation yields

$$\tilde{x}^\mu = x^\mu + \delta x^\mu$$

The Jacobian $J$ is given by

$$\frac{\partial\tilde{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu(\delta x^\mu) \quad \Rightarrow \quad J = \det(\frac{\partial\tilde{x}^\mu}{\partial x^\nu}) = 1 + \partial_\mu(\delta x^\mu)$$

Expanding the action to lowest order in the coordinate yields

$$S = \int d^4\tilde{x}\mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x})) = \int d^4x J\{\mathcal{L}(\tilde{\phi}(x), \partial\tilde{\phi}(x)) + \partial_\mu\mathcal{L}\delta x^\mu\}$$

Using the expression for the Jacobian $J$ yields

$$\begin{aligned} S &= \int d^4x\{\mathcal{L}(\tilde{\phi}(x), \partial\tilde{\phi}(x)) + \partial_\mu\mathcal{L}\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu)\} \\ &= \int d^4x\{\mathcal{L}(\tilde{\phi}(x), \partial\tilde{\phi}(x)) + \partial_\mu(\mathcal{L}\delta x^\mu)\} \end{aligned} \tag{3.2.2}$$

Define the variation of the field at the same spacetime point $x$ by

$$\delta\phi(x) = \tilde{\phi}(x) - \phi(x) \tag{3.2.3}$$

Since the transformation is a symmetry, it leaves the action invariant and yields

$$\delta S = 0 = \int d^4x \Big[\mathcal{L}(\tilde{\phi}(x), \partial\tilde{\phi}(x)) + \partial_\mu(\mathcal{L}\delta x^\mu) - \mathcal{L}(\phi(x), \partial_\mu\phi(x))\Big] \tag{3.2.4}$$

Note the fact that $\delta S = 0$ in Eq. 3.2.4 is a *consequence* of the symmetry of the Lagrangian under the transformation in question. This is quite different from the derivation in Section 3.1 in deriving the field equations, where one *imposes* of the condition of $\delta S = 0$; this condition in turn constrains the classical field to obey the classical field equation.

Hence, for an infinitesimal transformation, using Eq. 3.1.5

$$\begin{aligned}\delta\mathcal{S} &= \int_V \Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi + \partial_\mu(\mathcal{L}\delta x^\mu)\Big] \\ &= \int_V \Big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big]\delta\phi(t,x) + \int_V \Big[\partial_\mu(\delta\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}) + \partial_\mu(\mathcal{L}\delta x^\mu)\Big] \\ &= \delta\mathcal{S}_V + \delta\mathcal{S}_{\partial V} = 0\end{aligned}$$

Hence, in general

$$0 = \delta\mathcal{S} = \delta\mathcal{S}_V + \delta\mathcal{S}_{\partial V}$$

The equation of motion makes $\delta\mathcal{S}_V = 0$ and the symmetry transformation must respect

$$\delta\mathcal{S}_{\partial V} = 0 \quad : \text{Symmetry} \tag{3.2.5}$$

Using Gauss's theorem, the boundary term can be re-written.[1] For $d\Sigma^\mu$ being the vector of the surface element

$$\begin{aligned}0 = \delta\mathcal{S}_{\partial V} &= \int_V d^4x\partial_\mu(\delta\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \mathcal{L}\delta x^\mu) = \int_{\partial V} d\Sigma_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \mathcal{L}\delta x^\mu\Big) \\ &= \int_{\partial V} d\Sigma_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}[\delta\phi + (\partial_\nu\phi)\delta x^\nu] - [\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}]\delta x^\nu\Big)\end{aligned} \tag{3.2.6}$$

Note that

$$\delta\phi + (\partial_\nu\phi)\delta x^\nu = \tilde{\phi}(\tilde{x}) - \phi(x) = \Delta\phi$$

and yields the final result

$$0 = \delta\mathcal{S}_{\partial V} = \int_{\partial V} d\Sigma_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - [\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}]\delta x^\nu\Big) \tag{3.2.7}$$

[1] Under a symmetry transformation, the equations of motion are left invariant as long as

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial_\mu\mathcal{J}^\mu \quad \Rightarrow \quad \delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu$$

The current $\mathcal{J}^\mu$ given above does not change the equations of motion because, using Gauss's theorem, it integrates to zero in the action. It, however, does contribute to the boundary term $\delta\mathcal{S}_{\partial V}$. We will ignore the extra current $\mathcal{J}^\mu$ as it is not required for subsequent discussions.

We analyze special cases of Eq. 3.2.6 to understand the significance of Noethers theorem.

### 3.3 Noether Theorem: Internal symmetries

For internal symmetries, $\tilde{x} = x$ and only the degrees of freedom are transformed. Hence

$$\delta x = 0 \;\; ; \;\; \delta\phi \neq 0$$

Hence, from Eq. 3.2.5

$$\delta\mathcal{S}_{\partial V} = \int_V \partial_\mu(\delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}) = 0$$

Since the volume $V$ is arbitrary, we have for every spacetime point, the following

$$\partial_\mu j^\mu = \partial_\mu j^\mu(t,x) = \delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}) = 0 : \text{ conserved invariant}$$

and we obtain the Noether current

$$j^\mu = \delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{3.3.1}$$

The conserved current $j^\mu$ given in Eq. 3.3.1 is a classical result. On quantizing a classical field, some of the classically conserved currents may no longer be conserved; in particular, a classical field have the symmetry of scale invariance and chiral invariance is broken by the quantized field.

If the field has many components, denoted by $\phi_a$, the conserved current is given by

$$j^\mu = \sum_a \delta\phi_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \tag{3.3.2}$$

The derivation of the Noether current is based on the field equation and is essentially equivalent to the field equation; in other words, the Noether currents can be obtained directly by studying the field equations. Studying the boundary term is in practice a more efficient and transparent procedure for identifying the conserved currents.

The Dirac field is defined in Chapter 5 in terms of multi-component anticommuting field variables $\bar{\psi}_a, \psi_a$. For Dirac field the Noether conserved current is given by

$$j^\mu = \sum_a \{\delta\bar{\psi}_a \frac{\partial\mathcal{L}_D}{\partial(\partial_\mu\bar{\psi}_a)} + \frac{\partial\mathcal{L}_D}{\partial(\partial_\mu\psi_a)}\delta\psi_a\} \tag{3.3.3}$$

where $\mathcal{L}_D$ is the Dirac Lagrangian. The ordering of the variations $\delta\bar{\psi}_a, \delta\psi_a$ reflects the anticommuting nature of the Dirac field.

The current $j^\mu$ yields the charge

$$Q = \int_{\mathcal{R}^3} d^3x j^0$$

which is conserved since

$$\frac{dQ}{dt} = \int_{\mathcal{R}^3} \partial_0 j^0 = \int_{\mathcal{R}^3} (-\partial_i j^i) = -\int_{\partial\mathcal{R}^3} j^i ds_i = 0$$

## 3.4 Noether Theorem: Energy-momentum stress tensor

Recall from Eq. 3.2.7

$$0 = \delta\mathcal{S}_{\partial V} = \int_{\partial V} d\Sigma_\mu \Big( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \Delta\phi - [\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\nu\phi - \delta^\mu_\nu \mathcal{L}] \delta x^\nu \Big)$$

Consider the infinitesimal coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu - \epsilon^\mu \quad \Rightarrow \quad \delta x^\nu = -\epsilon^\mu$$

Since the coordinate transformation is a Lorentz transformation, for a scalar field

$$\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(x) \quad \Rightarrow \quad \Delta\phi = 0$$

Hence Eq. 3.2.7 simplifies to

$$0 = \delta\mathcal{S}_{\partial V} = \int_{\partial V} d\Sigma_\mu [\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\nu\phi - \delta^\mu_\nu \mathcal{L}] \epsilon^\nu \Big)$$

Using Gauss's law yields

$$\int_{\partial V} d^4x \partial_\mu (\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \partial_\nu\phi - \delta^\mu_\nu \mathcal{L}) \epsilon^\nu = 0 \tag{3.4.1}$$

Define the energy-momentum stress tensor by

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \partial_\nu\phi - \delta^\mu_\nu \mathcal{L} \tag{3.4.2}$$

The conservation of energy-momentum conservation, from Eq. 3.4.1, is given by

$$\partial_\mu (T^\mu_\nu) = 0$$

The energy of the field is given by

$$E = \int d^3x T^{00} = \int_{\vec{x}} (\dot{\phi}^2 - \mathcal{L}) = H$$

The momentum of the field is given by

$$P^i = \int d^3x T^{0i} = \int d^3x (\dot{\phi}\partial^i\phi) \tag{3.4.3}$$

Note $T^{\mu\nu} = T^{\nu\mu}$ is symmetric and this leads to angular momentum conservation.

Note that the stress-tensor in Eq. 3.4.2 is only defined up to to a total divergence. An equivalent and modified stress-tensor that is conserved can be defined as follows

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda(K^{\lambda\mu\nu}) \quad ; \quad K^{\lambda\mu\nu} = -K^{\mu\lambda\nu} \tag{3.4.4}$$

In some cases, such as electrodynamics, the stress tensor has to be modified to make it symmetric, as required by angular momentum conservation.

### *3.4.1 Energy-momentum tensor: Klein-Gordon field*

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2$$

and yields

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = \partial^\mu\phi$$

The stress tensor is

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

### *3.4.2 Energy-momentum tensor: Electromagnetic field*

The Maxwell Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Note that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta}$$

Hence, from Eq. 3.4.2, the stress tensor ($J^\mu = 0$) is given by

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \delta^\mu_\nu\mathcal{L}$$

and hence

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}\delta^{\nu\mu}F^{\alpha\beta}F_{\alpha\beta} \neq T^{\nu\mu}$$

The stress-tensor needs to be symmetric for the conservation of angular momentum. Using the result from Eq. 3.4.4, the new stress tensor is defined by

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

with

$$K^{\lambda\mu\nu} = F^{\lambda\nu}A^\nu = -K^{\mu\lambda\nu}$$

The sought for symmetric stress tensor is given by

$$\tilde{T}^{\mu\nu} = F^{\mu\lambda}F^\nu_\lambda + \frac{1}{4}\delta^{\nu\mu}F^{\alpha\beta}F_{\alpha\beta} = \tilde{T}^{\nu\mu}$$

## 3.5 Spontaneous Symmetry Breaking

The Goldstone model consists of a *complex scalar field* $\phi(t,x)$ with a non-linear Lagrangian given by

$$\mathcal{L}_G = \partial_\mu\phi^*\partial^\mu\phi - \mu^2\phi^*\phi - \lambda[\phi^*\phi]^2$$

The parameter $\mu^2$ in the Lagrangian yields a well defined and convergent (quantum) field theory for both positive and negative values.

Representing the complex scalar field by the following two real scalar fields

$$\phi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)] \;\; ; \;\; \phi^*(x) = \frac{1}{\sqrt{2}}[\phi_1(x) - i\phi_2(x)]$$

yields the following

$$\mathcal{L}_G = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}\mu^2(\phi_1^2+\phi_2^2) - \frac{1}{4}\lambda[\phi_1^2+\phi_2^2]^2$$

Consider the global $U(1)$ symmetry transformation

$$\phi(x) \to \phi'(x) = e^{-i\alpha}\phi(x);\;\; \phi^*(x) \to \phi'^*(x) = e^{i\alpha}\phi^*(x)$$

The phase $\alpha$ is constant and hence the transformation is a global one.

The Lagrangian is invariant and hence

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x)$$

The complex field can be represented using polar coordinate as follows

$$\phi = \frac{1}{\sqrt{2}}re^{i\theta} \;\; \Rightarrow \;\; \phi_1 = \frac{1}{\sqrt{2}}r\cos(\theta) \;\; , \;\; \phi_2 = \frac{1}{\sqrt{2}}r\sin(\theta)$$

This yields

$$V(r) \equiv \mu^2\phi^*\phi + \lambda[\phi^*\phi]^2 = \frac{1}{2}\mu^2r^2 + \frac{1}{4}\lambda r^4$$

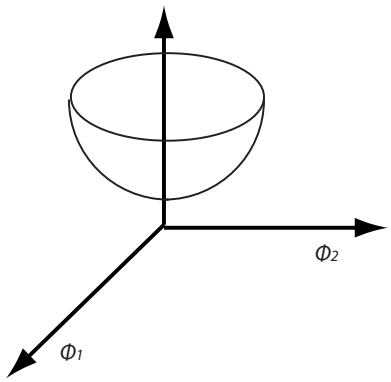


Figure 3.1 Potential for $\mu^2 > 0$ has a unique minimum.

and hence

$$\frac{\partial V}{\partial r} = r(\mu^2 + \lambda r^2) \; ; \; \frac{\partial^2 V}{\partial r^2} = \mu^2 + 3\lambda r^2$$

Consider a system undergoing a second order phase transition at temperature $T_c$; in the Landau-Ginzburg-Wilson phenomenological approach, $\mu^2$ has the following dependence on temperature $T$

$$\mu^2 \propto T - T_c$$

Hence

$$\begin{array}{l} \text{Non-condensed phase} \;\; : \;\; \mu^2 \; > 0 \;\; : \;\; T > T_c \\ \text{Condensed phase} \;\; : \;\; \mu^2 \; < 0 \;\; : \;\; T < T_c \end{array} \tag{3.5.1}$$

The non-condensed phase, for $\mu^2 > 0$, is given by

$$\frac{\partial V}{\partial r} = 0 \;\; \Rightarrow r_0 = 0 \;\; ; \;\; \frac{\partial^2 V}{\partial r^2} = \mu^2 > 0 \;\; \Rightarrow \;\; \text{Minima}$$

The potential for the non-condensed phase is shown in Figure 3.1. For the condensed phase $\mu^2 < 0$ and yields

$$\frac{\partial V}{\partial r} = 0 \;\; \Rightarrow r_0^2 = 0, -\frac{\mu^2}{\lambda}$$

Hence

$$\frac{\partial^2 V}{\partial r^2}\Big|_{r_0=0} = \mu^2 < 0 \;\; \Rightarrow \;\; \text{Maxima}$$

and

$$\frac{\partial^2 V}{\partial r^2}\Big|_{r_0^2=-\frac{\mu^2}{\lambda}} = -2\mu^2 > 0 \;\; \Rightarrow \;\; \text{Minima}$$

In other words, for $\mu^2 > 0$, the minimum of the potential is at $|\phi| = 0$. The

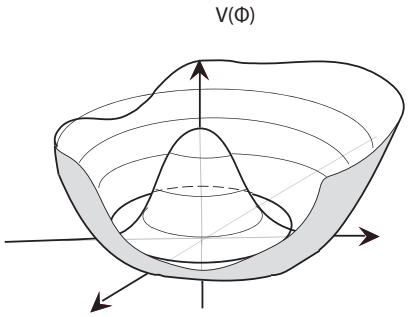


Figure 3.2 There are multiple minimas for $\mu^2 < 0$, all of which break global gauge symmetry.

potential $V$ for $\mu^2 < 0$, that is, less than zero is shown in Figure 3.2; for $\mu^2 < 0$, the minima of the field lie on the circle defined by

$$r_0^2 = -\frac{\mu^2}{\lambda}$$

Let $\upsilon^2 > 0$ be a constant; in the broken phase, to leading order, the value of the field is

$$|\phi| \simeq |\phi_0| = \frac{1}{\sqrt{2}}\upsilon \tag{3.5.2}$$

To study the theory around the nonzero minima given by $|\phi_0|$, consider the change of variables

$$\phi(x) = \frac{1}{\sqrt{2}}[\upsilon + \sigma(x) + i\pi(x)];\ \phi^*(x) = \frac{1}{\sqrt{2}}[\upsilon + \sigma(x) - i\pi(x)]$$

In terms of the new field variables the potential is given by

$$\begin{aligned}
V &= \frac{1}{2}\mu^2[(\upsilon+\sigma)^2 + \pi^2] + \frac{1}{4}\lambda[(\upsilon+\sigma)^2 + \pi^2]^2 \\
&= \frac{1}{2}\mu^2[\sigma^2 + \pi^2 + 2\upsilon\sigma + \upsilon^2] + \frac{1}{4}\lambda\Big[\left(\sigma^2+\pi^2\right)^2 \\
&\quad + 2(\upsilon^2\sigma^2 + \upsilon^2\pi^2 + 2\upsilon\sigma^3 + 2\upsilon\sigma\pi^2) + 4\upsilon^2\sigma^2 + 4\upsilon^3\sigma + \upsilon^4\Big] \\
&= \frac{1}{2}\mu^2[\sigma^2 + \pi^2 + 2\upsilon\sigma + \upsilon^2] + \frac{1}{2}\lambda\Big[\upsilon^2\sigma^2 + \upsilon^2\pi^2 + 2\upsilon^2\sigma^2 + 2\upsilon^3\sigma\Big] \\
&\quad + \frac{1}{4}\lambda\Big[\left(\sigma^2+\pi^2\right)^2 + 4\upsilon\sigma^3 + 4\upsilon\sigma\pi^2 + \upsilon^4\Big]
\end{aligned}$$

and which yields the final result that

$$V = (\mu^2 + \lambda\upsilon^2)[\upsilon\sigma + \frac{1}{2}\pi^2] + \frac{1}{2}(\mu^2 + 3\lambda\upsilon^2)\sigma^2 + \frac{1}{2}\mu^2\upsilon^2 + \frac{1}{4}\lambda\Big[\left(\sigma^2 + \pi^2\right)^2 + 4\upsilon\sigma^3 + 4\upsilon\sigma\pi^2 + \upsilon^4\Big]$$

Choosing $\upsilon$ to be at the minima of the potential yields

$$\upsilon^2 = r_0^2 = -\frac{\mu^2}{\lambda}$$

Since the field is being expanded about the minima of the potential, the choice for $\upsilon^2$ eliminates the first term in the potential – that has *both* the linear term in $\sigma$ as well as the quadratic $\pi^2$ term. The mass of the $\pi$ field is zero because the $\pi^2$ term is zero and this is due to the excitations of the field along the valley of the potential are massless; the massive field $\sigma$ results from excitations away from the valley.

The field $\pi$ is massless and is called the 'Goldstone boson' field. It is a result of the phase transition having long range correlation functions.

Replacing $\mu^2$ by $-\lambda\upsilon^2$ in the Lagrangian, we obtain

$$\begin{aligned}\mathcal{L}_G = {} & \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}(2\lambda\upsilon^2)\sigma^2 : \text{massive scalar} \\ & + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi : \text{massless scalar} \\ & - \lambda\upsilon\sigma(\sigma^2 + \pi^2) - \frac{\lambda}{4}[\sigma^2 + \pi^2]^2 : \text{interaction} \\ & + \frac{1}{4}\lambda\upsilon^4 + \frac{1}{2}\upsilon^2\mu^2 : \text{constant}\end{aligned} \tag{3.5.3}$$

Expanding the field about the nonzero minimum *breaks the global symmetry* of $\phi \to e^{i\alpha}\phi$ – that exists in the original Lagrangian – and the vacuum state of the corresponding quantum field theory is said to have *spontaneously broken* the global U(1) symmetry. The reason the symmetry is broken is because the integration variable are now $\sigma,\ \pi$. The transformation $\sigma \pm i\pi \to e^{\pm i\alpha}(\sigma \pm i\pi)$ no longer leaves the potential term $V$ invariant since $\upsilon^2 > 0$ is a physically observable quantity and it cannot depend on the gauge and hence cannot be changed under a gauge transformation.

Hence, for $\upsilon^2 > 0$, the Lagrangian $\mathcal{L}$ is no longer $U(1)$ invariant. The spontaneous symmetry breaking has given rise to the massless Goldstone boson field $\pi$.

## 3.6 The Landau-Ginzburg Lagrangian

Scalar quantum electrodynamics consists of a complex scalar field coupled to an Abelian gauge field. The Landau-Ginzburg provides a phenomenological description of the superconducting phase of ordinary conductors using scalar quantum electrodynamics. The complex scalar field $\phi$ is an *order parameter* that describes the phase transition. The development of the microscopic BCS theory showed that, in the condensed superconducting phase, the complex field's 'mass term' – given by $\phi^*\phi$ – represents the density of superconducting Cooper pairs of electrons.

Consider the local $U(1)$ gauge symmetry given by

$$\phi(x) \to \phi'(x) = e^{-i\alpha(x)}\phi(x);\ \ \phi^*(x) \to \phi'^*(x) = e^{i\alpha(x)}\phi^*(x)$$

and yields

$$\partial_\mu\phi(x) \to \partial_\mu\phi'(x) = e^{-i\alpha(x)}\partial_\mu\phi - i(\partial_\mu\alpha)\phi$$

Under a $U(1)$ local gauge symmetry, we have

$$\mathcal{L}_G \to \mathcal{L}'_G = \mathcal{L}_G + \partial_\mu\alpha\partial^\mu\alpha\phi^*\phi + i\partial_\mu\alpha\phi^*\partial^\mu\phi$$

$$-\mu^2\phi^*\phi - i\partial_\mu\alpha(\partial^\mu\phi^*)\phi - \mu^2\phi^*\phi \neq \mathcal{L}_G$$

Hence the Goldstone Lagrangian $\mathcal{L}_G$ does not have local gauge symmetry.

A gauge field is introduced to obtain exact local gauge symmetry. Consider the Landau-Gingzburg Lagrangian

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where the gauge covariant derivative and gauge field tensor are given by

$$D_\mu\phi \equiv \partial_\mu\phi(x) + ieA_\mu(x)\phi(x)\ \ ;\ \ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

Note that

$$|D_\mu\phi|^2 \sim |\partial_\mu\phi|^2 + \phi^* A\partial_\mu\phi + (A\phi)^2$$

Local gauge transformations are defined by

$$\begin{aligned}\phi(x) \to \phi'(x) &= e^{-ief(x)}\phi(x);\\ \phi^*(x) \to \phi'^*(x) &= e^{ief(x)}\phi^*(x);\\ A_\mu(x) \to A'_\mu(x) &= A_\mu(x) + \partial_\mu f(x)\end{aligned}$$

Hence

$$\begin{aligned} D_\mu\phi &\to [\partial_\mu + ie(A_\mu + \partial_\mu f)]\Big(e^{-ief}\phi\Big) \\ &= e^{-ief}\partial_\mu\phi + ieA_\mu e^{-ief}\phi - ie\partial_\mu f e^{-ief}\phi + ie\partial_\mu f e^{-ief}\phi \\ &= e^{-ief}D_\mu\phi \end{aligned}$$

and yields

$$(D_\mu\phi)^* D^\mu\phi \to (D_\mu\phi)^* D^\mu\phi$$

Furthermore, for the gauge field

$$\begin{aligned} F_{\mu\nu} &\to \partial_\mu(A_\nu + \partial_\nu f) - \partial_\nu(A_\mu + \partial_\mu f) \\ &= \partial_\mu A_\nu + \partial_\mu\partial_\nu f - \partial_\nu A_\mu - \partial_\nu\partial_\mu f = F_{\mu\nu} \end{aligned}$$

So we conclude that $\mathcal{L}$ has $U(1)$ local gauge symmetry.

The conserved Noether current for symmetric Lagrangian ($\upsilon = 0$) is given by the global gauge symmetry transformation and from Eq. 3.3.1

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\delta\phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\delta\phi_2 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A)}\delta A_\mu$$

Recall

$$\phi' = e^{-i\alpha}\phi;\ \phi'^* = e^{i\alpha}\phi^* \quad \Rightarrow \quad \delta\phi = -i\alpha\phi;\ \delta\phi^* = i\alpha\phi^*$$

and

$$\delta A_\mu = \partial_\mu\alpha = 0$$

Hence, in terms of the complex scalar fields

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A)}\delta A_\mu = -(D^{*\mu}\phi^*)i\alpha\phi + D^\mu\phi(i\alpha\phi^*)$$

The overall scale of the current is fixed by the units used, and for $\alpha = 1$,

$$j^\mu = i(\phi D^{*\mu}\phi^* - \phi^* D^\mu\phi) \tag{3.6.1}$$

More explicitly

$$j_\mu = i\Big(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi - 2ieA_\mu\phi^*\phi\Big) = i(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + 2e\phi^*\phi A_\mu$$

The condition that $\partial_\mu j^\mu = 0$ is a result of the Noether's theorem that $j^\mu$ is a conserved current.

The field equations are obtained by considering $\phi$ and $\phi^*$ to be independent fields; one can obtain the same result by writing the complex field in

terms of two real fields. The field equations, for $\lambda = 0$, are the following

$$\phi : 0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}) = -\mu^2 \phi^* - \partial_\mu (D^{\mu *} \phi^*)$$

and

$$\phi^* : 0 = \frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu (\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^*}) = -\mu^2 \phi - \partial_\mu (D^\mu \phi) \tag{3.6.2}$$

$$A_\mu : 0 = \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu (\frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu}) = -\partial_\nu F^{\mu\nu} + j^\mu \tag{3.6.3}$$

Note that the last equation above states that

$$\partial_\nu F^{\mu\nu} = j^\mu$$

Since $F^{\mu\nu}$ is antisymmetric, one has the identity $\partial_\mu \partial_\nu F^{\mu\nu} = 0$, and hence consistency requires that

$$\partial_\mu j^\mu = 0 \tag{3.6.4}$$

In other words, the gauge field $A_\mu$ can only be coupled to a conserved current, reflecting the conservation of electric charge. Eq. 3.6.4 shows that the Noether conserved current is in fact already contained by the field equation.

### *3.6.1 Meissner Effect*

The Meissner effect is the fact that a superconductor expels magnetic fields, upto a critical value of the field.

A superconductor is in thermodynamic equilibrium with *no time dependence*; hence all the time derivatives in Landau-Ginzburg Lagrangian are set to zero. All indices are taken to run over only space coordinates.

The conserved current, from Eq. 3.6.1, is given by

$$j_\mu = i(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi) + 2e\phi^* \phi A_\mu$$

For the symmetry breaking superconductor phase, since the Landau-Ginzburg has the same potential as the Goldstone Lagrangian, the leading order of the field is given from Eq. 3.5.2 by

$$|\phi_0| = \frac{1}{\sqrt{2}} \upsilon$$

and we obtain

$$\phi^* \phi \simeq |\phi_0|^2 = \frac{1}{2} \upsilon^2$$

Since the fluctuation of the field is small over space, we have that $\phi \partial_\mu \phi^* -$

$\phi^* \partial_\mu \phi$ is negligible. Hence, for a symmetry breaking superconductor phase we obtain, to the leading order value for the current given by

$$j_\mu \simeq e\upsilon^2 A_\mu : \text{London equation}$$

The field equation Eq. 3.6.3 now yields

$$\partial_\nu F_{\mu\nu} = \partial_\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) = e\upsilon^2 A_\mu \tag{3.6.5}$$

Note that

$$\partial_\mu \partial_\nu F_{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\mu A_\mu = 0.$$

This is a gauge condition on $A_\mu$.

Hence, from Eq. 3.6.5

$$\partial_\nu \partial^\nu A_\mu + e\upsilon^2 A_\mu = 0 : \text{massive Klein Gordon equation}$$

The gauge field has become massive since the phase transition of the scalar field leads to the breaking of gauge symmetry. We will see in next Section 3.7 this is the same as the Higgs mechanism.

Consider a superconductor occupying the half plane, with its surface at $x = 0$. The magnetic field is given by $\vec{B} = \vec{\nabla} \times \vec{A}$. Imposing a constant magnetic field of strength $\vec{B}_0$ in the z-direction, the London equation yields the magnetic field as given by

$$(-\partial_x^2 + e\upsilon^2)\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = e^{-x/g}\vec{B}_0 \quad ; \quad g = 1/(e\upsilon^2) : \quad \text{Meissner effect}$$

As shown in Figure 4.3, the magnetic field penetrates into the superconductor only to a depth of $g$, called the penetration depth; for a low temperature superconductor, $g \simeq 10^{-9}$m.

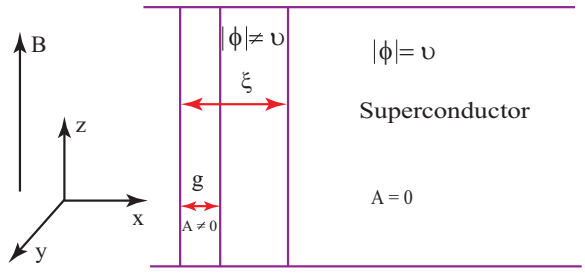


Figure 3.3 The magnetic field penetrates only upto a length of $g$ inside a superconductor. The condensed phase is perturbed upto a skin depth of $\xi$.

For the condensed phase, the mass of the scalar field, the analog of the $\sigma$

field given in Eq. 3.5.3, has a mass of $2\lambda v^2$; hence the field equation for the field $\phi$ in the condensed phase –the analog of Eq. 3.6.2 – yields

$$(+\partial_x^2 - 2\lambda v^2)|\phi| + O(\phi A_\mu) = 0$$

Using the boundary condition that for $\phi$ in the condensed phase, we have

$$\lim_{x\to\infty} |\phi| = \frac{1}{\sqrt{2}} v$$

and the field equation yields

$$|\phi(x)| = \frac{1}{\sqrt{2}} v(1 - e^{-x/\xi}) \;\; ; \;\; \xi = 1/(2\lambda v^2)$$

Hence, the condensed phase is broken near the boundary of the superconductor, and the length is given by $\xi = 1/(2\lambda v^2)$. The behavior of the gauge and scalar field for the condensed phase is shown in Figure 4.3.

In particle physics, for spontaneously broken gauge theories, the mass of the gauge field $A_\mu$ is equal to the inverse of the penetration depth $g$ and the mass of the Higgs boson is equal to the inverse of the correlation length $\xi$.

## 3.7 The Higgs Mechanism

The interplay between local gauge invariance and spontaneous breaking of symmetry is the basis of the Higgs mechanism. A direct way of seeing this interplay is to choose the so called **unitary gauge**. Consider polar coordinates for the complex scalar field given by

$$\phi(x) = \frac{1}{\sqrt{2}} r(x) e^{i\theta(x)}$$

Choose a specific a gauge transformation that is determined by the $\theta$ field and given by[2]

$$A_\mu \to A_\mu - \partial_\mu \theta$$

The gauge covariant derivative term then yields

$$D_\mu \phi \to \frac{1}{\sqrt{2}} \Big\{ \partial_\mu + ie(A_\mu(x) - \partial_\mu \theta(x)) \Big\} \big\{ r(x) e^{i\theta(x)} \big\}$$

[2] The unitary gauge is a singular limit of the so called $R_\xi$ gauge; it can be shown [Peskin and Schroder Eq 21.29] that in the unitary gauge, the gauge field $A_\mu$ is massless for the case of $\mu^2 > 0$. The $R_\xi$ gauge is defined by the gauge fixing term

$$\frac{1}{\sqrt{\xi}} \left[ \frac{\partial A_\mu}{\partial x_\mu} - \xi e v r \sin(\theta) \right]$$

and the unitary gauge is by the limit of $\xi \to \infty$.

$$= \frac{1}{\sqrt{2}}\Big\{\partial_\mu r(x) + ieA_\mu(x)r(x)\Big\}$$

In other words, the gauge transformation completely removes one degree of freedom and makes the complex field $\phi(x)$ into a real field $r(x)$. The gauge transformation produces no change in $F_{\mu\nu}$ and hence we obtain

$$\mathcal{L}(r, A_\mu) = \frac{1}{2}|\partial_\mu r(x) + ieA_\mu(x)r(x)|^2 - \frac{1}{2}\mu^2 r^2(x) - \frac{1}{4}\lambda r^4(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

For the spontaneously broken phase $\mu^2 = -\lambda\upsilon^2$, with $\upsilon \neq 0$. Let

$$r(x) = \upsilon + \sigma(x)$$

The scalar field $\sigma$ is the Higgs boson and yields

$$\mathcal{L}(r, A_\mu) = \frac{1}{2}[\partial_\mu\sigma(x)]^2 + \frac{1}{2}e^2A_\mu^2(x)(\upsilon + \sigma(x))^2 - \frac{1}{2}\mu^2(\upsilon + \sigma(x))^2$$

$$- \frac{1}{4}\lambda(\upsilon + \sigma(x))^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This yields the following Lagrangian, using the result obtained earlier for the Goldstone potential

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}(2\lambda\upsilon^2)\sigma^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2\upsilon^2A_\mu A^\mu - \lambda\upsilon\sigma^3$$
$$- \frac{1}{4}\lambda\sigma^4 + \frac{1}{2}e^2A_\mu A^\mu(2\upsilon\sigma + \sigma^2)$$

The massless Goldstone field $\pi$ in Eq. 3.5.4 has been removed using the gauge symmetry of the Lagrangian. Instead of the Goldstone field, one now has the gauge field acquiring a mass; the extra degree of freedom for the gauge field mass arises by the gauge field absorbing the Goldstone degree of freedom.

On quantizing the fields, symmetry breaking is the result of the properties of the quantum vacuum $|\Omega\rangle$ of the system. The broken phase is characterized by

$$\langle\Omega|A_\mu|\Omega\rangle = 0 \;\;;\;\; \langle\Omega|\phi^*\phi|\Omega\rangle = \upsilon$$

In the path integral formalism, this condition yields

$$E[A_\mu] = 0 \;\;;\;\; E[\phi^*\phi] = \upsilon$$

where $E[\mathcal{O}]$ is the expectation value of the field variables $\mathcal{O}$.

In summary, the Higgs transition and mechanism describes the following phenomenon.

- Before the phase transition, the system consists of a complex scalar + massless gauge boson. The number of the degrees of freedom before the phase transition is $4 = 2 + 2$.
- After the system undergoes a phase transition, the condensed phase consists of a real scalar and a massive gauge boson. After the phase transition, the number of the degrees of freedom is also $4 = 1 + 3$.

The process of a massless gauge field acquiring a mass through a phase transition is called the Higgs mechanism. The magic of gauge invariance and symmetry breaking combine to give a new result. In the absence of the gauge field, a phase transition leads to the appearance of the massless Goldstone boson, but when coupled to the gauge field, the Goldstone boson is completely removed from the theory and instead, gauge field becomes massive and the complex massive charged field becomes a massive real field with zero charge.

## 3.8 Lorentz transformations

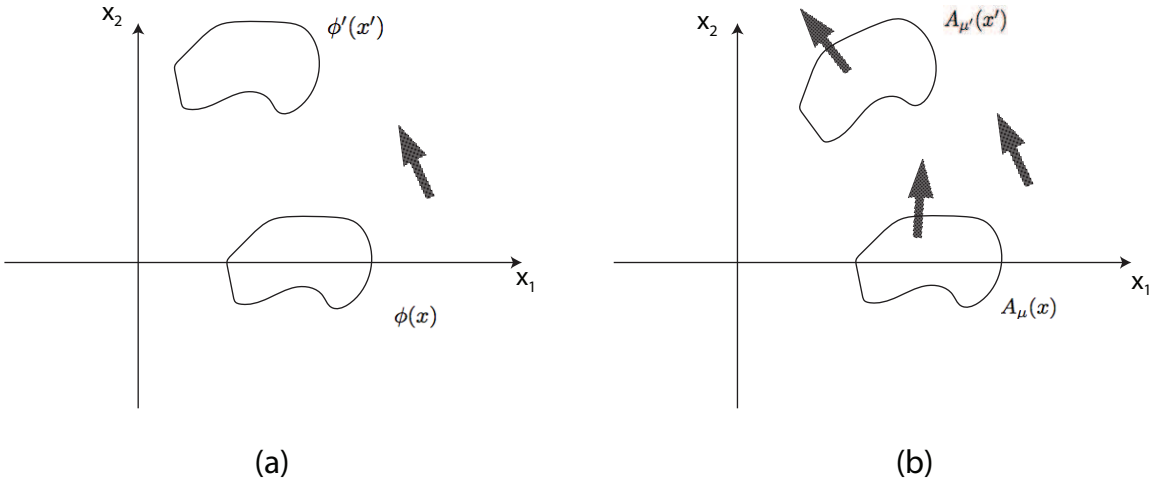


Figure 3.4 Lorentz tranformations for scalar and vector fields.

## 3.9 Summary

# PART TWO
# LINEAR QUANTUM FIELDS

Part Two introduces the concept of a quantum field starting with linear quantum fields. A quantum field is a mathematical system consisting of infinitely many coupled degrees of freedom. Furthermore, a quantum field is an indeterminate field, defined on a given manifold, that is equivalent to set of all possible configurations of the field.

The mathematics of quantum field theory, originating in the *synthesis* and *fusion* of the quantum principle with classical field theory, defines a mathematical system that is the main focus of this book. The term *quantum mathematics* is appropriate for the mathematics that emerges from this fusion since it is quite distinct from other branches of mathematics [Baaquie (2014)].

A free field is a special case of a quantum field for which the infinitely many degrees of freedom can be decoupled, leading to essentially a system consisting of a few independent degrees of freedom. Although this may sound like a retreat from the general case where all the degrees of freedom are coupled, it is logical to start from the case of free fields for a number of reasons.

- Free fields are linear since their field equations are linear. All free fields are described by Lagrangians that are quadratic functions of the underlying degrees of freedom. In the path integral formalism, all free fields are described by Gaussian path integration.
- Free fields have quadratic Hamiltonians and hence can be mapped to an underlying oscillator algebra, leading to great simplifications. In particular, one can introduce creation and destruction operators for obtaining all the excitations of the quantum field.
- Free quantum fields can be solved exactly, in the sense that all the expressions for quantities of interest can be obtained explicitly, using either the oscillator algebra or Gaussian integration.
- A fruitful approach to the study of weakly coupled nonlinear quantum fields is to start from the limit of a free quantum field when the couplings are equal to zero, and then obtain the properties of the nonlinear quantum field. This is the approach of using Feynman perturbation theory and Feynman diagrams, for which the fundamental ingredient is the underlying free field.
- The bulk of the discussion on nonlinear quantum fields in Part Three is based on Feynman perturbation expansions around the properties of the free scalar field.

The free field comes in many varieties, depending on the nature of the underlying degrees of freedom. The three most widely used and most useful free

quantum fields are the scalar, spinor and vector quantum fields. These fields have many specific features of great generality and hence need to studied one by one.

- The free scalar quantum field has many features of a quantum field and is an ideal theoretical laboratory for starting one's study of a system with infinitely many degrees of freedom. A scalar quantum field has one degree of freedom for each spacetime point.

  The Fourier transform is introduced to decouple the infinitely many degrees of freedom since this technique, appropriately generalized, applies to all free fields. The Heisenberg operator equations for the scalar quantum field are studied as are the various commutators.

  The continuation of Minkowski to Euclidean quantum fields is introduced and is of fundamental importance since almost all the derivations in this book are based on Euclidean quantum fields.
- The free spinor quantum field, of which the Dirac field is a leading exemplar, introduces the concept of a quantum field obeying fermion statistics. The Dirac field is based on fermionic degrees of freedom that obey canonical anticommutation equations.

  The quantization of the free Dirac field requires a multi-component spinor field, having four degrees of freedom at each spacetime point, and which is shown to have particles as well as antiparticles in its spectrum of states.

  The Casimir force is evaluated for the Dirac field to study the importance of boundary conditions for the Dirac field and the associated state space.
- The photon field is a free vector field with the local symmetry of *gauge invariance*. The symmetry of gauge invariance is so important that the photon field is also referred to as an Abelian gauge field. To quantize the photon field, one has to choose a gauge.

  Choosing a gauge is necessary for quantizing both for Abelian and non-Abelian Yang-Mills gauge fields. The mathematics required for choosing a gauge is studied in great detail, using both the path integral formalism – that leads to Fadeev-Popov quantization – and the Hamiltonian formalism that leads to the Coulomb gauge.

  The state space that results for both the path integral and Hamiltonian quantization are discussed. The BRST symmetry exhibited by the gauge-fixed action in the Fadeev-Popov scheme is utilized to define the state space and is shown to be equivalent to the Gupta-Bleurel quantization for a covariant gauge.

- Two dimensional quantum electrodynamics – also called the Schwinger model – consists of the massless electron quantum field coupled to the massless photon field in two spacetime dimensions. It is a simple model to start the analysis of interacting quantum fields. Although the theory is formally nonlinear, a careful analysis reveals that it can in fact be treated as two free fields, with the electron contributing a mass term to the photon field. Since the theory is linear, many exact results can be derived, from the breaking of chiral symmetry to the generation of mass for the photon.

  The most intriguing and far reaching result is that although one starts off by having a Lagrangian being composed of the electron and photon degrees of freedom, due to the nature of the interaction the spectrum of states exhibits the presence of only massive bosons. Hence, although one starts of with massless fermions and bosons, interaction results in the permanent confinement of the fermions, with the electrons only appearing in bound states and hence having boson statistics.

  The Schwinger model exhibits many features, such as the confinement of the fundamental fermions, that is expected to occur in quantum chromodynamics but is difficult to demonstrate due to the complexity of the four dimensional quantum fields.
- The pure bosonic string is analyzed and the conformal gauge is chosen. It is shown that the formalism of BRST quantization developed for the Abelian gauge field can be used to describe the gauge-fixed bosonic string in terms of a two dimensional free quantum field. The pure bosonic string demonstrates the continuing vitality and utility of free quantum fields.

# 4

# Scalar quantum field

## 4.1 Introduction

Quantum fields come in many varieties and defined on a great range of underlying manifolds. We start our study of quantum fields with a scalar field defined on flat Minkowski spacetime. The simplest but not unimportant case of a quantum field is a scalar field, which is described by a single real degree of freedom at every point of spacetime. To foreground the study of nonlinear scalar fields, we analyze a free scalar field, which has no self-interaction.

In this Chapter, we focus on the basic ingredients that constitute a quantum field, and study the relation of the underlying spacetime manifold with the behavior of the scalar quantum field.

A special case of a quantum field is when it is linear and all its degrees of freedom can consequently be decoupled. This, for example, is the case of the photon field, which by itself is linear and hence can be completely decoupled – using the momentum representation for the field – into an infinite product of single degrees of freedom.

A free quantum field is linear and all its modes can be exactly factorized. The main difference between a linear and nonlinear quantum field is that the free field does not require renormalization. The free quantum field is studied as a precursor to nonlinear quantum fields.

The utility of the creation and destruction operators in the study of quantum fields is illustrated by using the example of free fields. Furthermore, the free quantum field is analyzed in Minkowski spacetime to examine its relativistic structure – and in particular to verify that it fulfills the requirement of causality.

## 4.2 Free Two Dimensional Quantum Field

The simple harmonic oscillator, with coordinate degree of freedom $x$, has a Hamiltonian given by

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \frac{1}{2}m\omega^2 x^2$$

Consider a collection of infinitely many decoupled oscillators, , with coordinate degrees of freedom $x_n$ and frequency given by $\omega_n^2 = n^2\omega^2$ so that each oscillator is distinct. The Hamiltonian is given by

$$H = -\frac{1}{2m}\sum_{n=-\infty}^{\infty}\frac{\partial^2}{\partial x_n^2} - \frac{1}{2}m\omega^2\sum_{n=-\infty}^{\infty} n^2 x_n^2$$

$\{x_n : \ n \in \mathbb{Z}\}$ is a $1+1$ dimensional quantum field. Eigenstate are given

$$H|\ell_{-\infty}...\ell_k...\ell_\infty\rangle = E(\ell_{-\infty}...\ell_\infty)|n_{-\infty},...n_k,...,n_\infty\rangle$$

The state vector is the product of the separate oscillators and given by

$$|n_{-\infty}...n_k...n_\infty\rangle = \prod_{k=-\infty}^{k=+\infty}\otimes|n_k\rangle$$

The energy of the quantum state is given by

$$\begin{aligned} E &= \sum_{k=-\infty}^{\infty}\ell_k\omega(k) + \frac{1}{2}\sum_{k=-\infty}^{\infty}\omega(k) = \omega\sum_{k=-\infty}^{\infty}\ell_k|k| + \frac{1}{2}\omega\sum_{k=-\infty}^{\infty}|k| \\ &= \omega\sum_{k=-\infty}^{\infty}\ell_k|k| + E_0 \end{aligned}$$

where $E_0 =$ vacuum energy $= \infty$.

The coordinate and momentum eigenstates are given by

$$|x\rangle = \prod_{n=-\infty}^{\infty}|x_n\rangle \ \ ; \ \ |p\rangle = \prod_{n=-\infty}^{\infty}|p_n\rangle$$

and their scalar product is

$$\langle x|p\rangle = e^{i\sum_n x_n p_n} \ \ ; \ \ \int Dp = \prod_{n=-\infty}^{\infty}\int\frac{dp_n}{2\pi}$$

Similar to the derivation for the case of quantum mechanics given in Section

2.4, the evolution kernel is given by

$$\langle\tilde{x}|e^{-\epsilon H}|x\rangle = \int Dp\ \exp\{-\frac{\epsilon}{2m}\sum_n p_n^2 + i\epsilon\sum_n p_n(\tilde{x}_n - x_n) - \frac{\epsilon m\omega^2}{2}\sum_n n^2 x_n^2\}$$
$$= \mathcal{N}\exp\{-\frac{m^2}{2\epsilon}\sum_n(\tilde{x}_n - x_n)^2 - \frac{\epsilon m\omega^2}{2}\sum_n n^2 x_n^2\}$$

where $\mathcal{N}$ is a normalization. Taking the limit of $\epsilon \to 0$ yields the Euclidean Lagrangian

$$\mathcal{L} = -\frac{m}{2}\sum_n(\frac{\partial x_n}{\partial t})^2 - \frac{m\omega^2}{2}\sum_n n^2 x_n^2 \tag{4.2.1}$$

The $x_n = x_n(t)$ degrees of freedom define a periodic quantum field

$$x(t,\sigma) = \sum_n e^{in\sigma}x_n(t) = x(t,\sigma+2\pi)$$

Since $x_n^* = x_n$: real, this yields

$$x_n(t) = \int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}e^{-in\sigma}x(t,\sigma) \quad \Rightarrow \quad x^*(t,\sigma) = x(t,-\sigma)$$

The Lagrangian in Eq. 4.2.1 is given by

$$\mathcal{L} = -\frac{m}{2}\int d\sigma\frac{\partial x(t,\sigma)}{\partial t}\frac{\partial x(t,-\sigma)}{\partial t} - \frac{m\omega^2}{2}\int d\sigma\frac{\partial x(t,\sigma)}{\partial\sigma}\frac{\partial x(t,-\sigma)}{\partial\sigma}$$
$$= -\frac{m}{2}\int d\sigma\Big(\Big|\frac{\partial x(t,\sigma)}{\partial t}\Big|^2 + \frac{\omega^2}{2}\Big|\frac{\partial x(t,\sigma)}{\partial\sigma}\Big|^2\Big)$$

with action given by

$$S = \int dtd\sigma\mathcal{L}$$

The path integral is given by

$$Z = \langle\tilde{x}|e^{-\tau H}|x\rangle = \int Dxe^S \tag{4.2.2}$$

with the boundary $x(0,\sigma) = x(\sigma), x(\tau,\sigma) = \tilde{x}(\sigma)$.

Noteworthy 4.1: Functional derivative

For a continuous index $\sigma$ and function $f(\sigma)$, let a *functional* of $f$ be denoted by $G[f]$. The functional derivative is defined by

$$\frac{\delta G[f]}{\delta f(\sigma)} = \lim_{\epsilon\to 0}\frac{G[f(\sigma') + \epsilon\delta(\sigma'-\sigma)] - G[f(\sigma)]}{\epsilon}$$

The definition of functional derivative yields

$$\frac{\delta f(\sigma)}{\delta f(\sigma')} = \delta(\sigma' - \sigma)$$

Consider the function

$$G[f] = \exp\{\int d\sigma f^2(\sigma)\}$$

Then the definition of the functional derivative yields

$$\frac{\delta G[f]}{\delta f(\sigma)} = \frac{\exp\{\int d\sigma'[f(\sigma') + \epsilon\delta(\sigma' - \sigma)]^2 - G[f]}{\epsilon}$$

$$= \frac{\delta\{\int d\sigma' f^2(\sigma')\}}{\delta f(\sigma)} G[f] = 2f(\sigma)[G[f]$$

Taylor's expansion can be defined using functional differentiation. Consider functional $G$ of two functions $f(\sigma), h(\sigma)$ and let $\lambda$ be a real parameter. Then

$$G[f + \lambda h] = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int d\sigma_1 \cdots d\sigma_n h(\sigma_1) \cdots h(\sigma_n) \frac{\delta^n G[f]}{\delta f(\sigma_1) \cdots \delta f(\sigma_n)}$$

In more compact notation

$$G[f + \lambda h] = \exp\left\{\lambda \int d\sigma h(\sigma) \frac{\delta}{\delta f(\sigma)}\right\} G[f]$$

To derive the translation operator, let $\sigma \to \sigma + \epsilon$; then

$$f(\sigma + \epsilon) = f(\sigma) + \epsilon f'(\sigma) \quad ; \quad f'(\sigma) \equiv df(\sigma)/d\sigma$$

Hence, for $h(\sigma) = f'(\sigma)$

$$G[f(\sigma + \epsilon)] = \exp\left\{\epsilon \int d\sigma f'(\sigma) \frac{\delta}{\delta f(\sigma)}\right\} G[f] = e^{i\epsilon\mathcal{P}} G[f]$$

and we obtain the Hermetian momentum operator

$$\mathcal{P} = -i \int d\sigma f'(\sigma) \frac{\delta}{\delta f(\sigma)}$$

Consider a potential function $V(\phi(x))$ that depends only on the field variable $\phi(x)$; the functional derivative of $V(\phi(x))$ is given by

$$\frac{\delta V(\phi(x))}{\delta \phi(y)} = \frac{1}{\epsilon}\Big(V\big(\phi(x) + \epsilon\delta(x - y)\big) - V(\phi(x))\Big) = \frac{\partial V(\phi(x))}{\partial \phi(x)} \delta(x - y)$$

Or equivalently, using the chain rule

$$\frac{\delta V(\phi(x))}{\delta\phi(y)} = \frac{\partial V(\phi(x))}{\partial\phi(x)}\frac{\delta\phi(x)}{\delta\phi(y)} = \frac{\partial V(\phi(x))}{\partial\phi(x)}\delta(x-y)$$

---

Consider operator

$$\hat{O} = \frac{1}{2\pi}\sum_n e^{-in\sigma}\frac{\partial}{\partial x_n}$$

Then

$$\hat{O}x(\tilde{\sigma}) = \frac{1}{2\pi}\sum_{mn} e^{-in\sigma}e^{im\tilde{\sigma}}\frac{\partial x_m}{\partial x_n} = \frac{1}{2\pi}\sum_n e^{in(\tilde{\sigma}-\sigma)} = \delta(\tilde{\sigma}-\sigma)$$

Hence

$$\hat{O}(\sigma) = \frac{\delta}{\delta x(\sigma)}$$

is a functional derivative and

$$\frac{\partial}{\partial x_n} = \int_{-\pi}^{\pi} d\sigma e^{in\sigma}\frac{\delta}{\delta x(\sigma)}$$

The Hamiltonian is defined at an instant of time and hence the time index on the operators is suppressed. In terms of functional derivatives, the Hamiltonian can be written as

$$H = -\frac{1}{2m}\sum_n \int d\sigma d\sigma' e^{in(\sigma-\sigma')}\frac{\delta^2}{\delta x(\sigma)\delta x(\sigma')}$$

$$-\frac{1}{2}m\omega^2\sum_n n^2\int\frac{d\sigma}{2\pi}\frac{d\sigma'}{2\pi}e^{-in(\sigma+\sigma')}x(\sigma)x(\sigma')$$

and hence, for $x'(\sigma) = dx(\sigma)/d\sigma$, the Hamiltonian is

$$H = -\frac{1}{2m}2\pi\int d\sigma\frac{\delta^2}{\delta x(\sigma)\delta x(-\sigma)} - \frac{1}{2}\tilde{m}\omega^2\int\frac{d\sigma}{2\pi}x'(\sigma)x'(-\sigma)$$

$$= -\frac{1}{2\tilde{m}}\int d\sigma|\frac{\delta}{\delta x(\sigma)}|^2 - \frac{1}{2}\tilde{m}\omega^2\int d\sigma|x'(\sigma)|^2$$

with $\tilde{m} = m/(2\pi)$.

The Lagrangian can be obtained from the Hamiltonian using the Dirac-Feynman formula

$$\langle\tilde{x}|e^{-\epsilon H}|x\rangle = Ne^{\epsilon\mathcal{L}(\tilde{x},x;\epsilon)}$$

Consider the momentum basis states

$$\int Dp|p\rangle\langle p| = \prod_\sigma \int dp_\sigma \prod_\sigma |p_\sigma\rangle\langle p_\sigma|$$

where $\langle x|p\rangle = e^{i\int x(\sigma)p(\sigma)}$. Then

$$\begin{aligned}\int Dp\langle\tilde{x}|e^{-\epsilon H}|p\rangle\langle p|x\rangle &= \int Dp\ e^{-\frac{\epsilon}{2m}\int p(\sigma)p(-\sigma)}e^{-i\int p(\sigma)[\tilde{x}(\sigma)-x(\sigma)]}e^{-\epsilon V[x]} \\ &= \exp\{-\frac{m}{2\epsilon}\int_0^{2\pi} d\sigma[\left|\tilde{x}(\sigma)-x(\sigma)\right|^2 - \epsilon V(x)]\}\end{aligned}$$

Hence, as obtained earlier in Eq. 4.2.1 , the Lagrangian is given by

$$\mathcal{L} = -\frac{m}{2}\left|\frac{\partial x(t,\sigma)}{\partial t}\right|^2 - \frac{m\omega^2}{2}\left|\frac{\partial x(t,\sigma)}{\partial \sigma}\right|^2$$

and the path integral is

$$Z = \int DX \exp\{\int dt \int_0^{2\pi} d\sigma\mathcal{L}\} \tag{4.2.3}$$

In conclusion, $x(t,\sigma)$ is a *two dimensional free scalar quantum field* that consists, at each instant of time, of infinitely many independent degrees of freedom. To render the quantum field nonlinear, one needs to include nonlinear terms in the Lagrangian. An important example is to add a $|x(t,\sigma)|^4$ term and which yields

$$\mathcal{L} = -\frac{m}{2}\left|\frac{\partial x(t,\sigma)}{\partial t}\right|^2 - \frac{m\omega^2}{2}\left|\frac{\partial x(t,\sigma)}{\partial \sigma}\right|^2 - \frac{1}{4!}|x(t,\sigma)|^4$$

The nonlinear term *couples* the infinitely many degrees of freedom and leads to new features absent in quantum mechanics, and discussed in later Chapters on nonlinear quantum fields.

## 4.3 Scalar Field: Path integral formulation

Consider the free scalar field Lagrangian in four dimensional Minkowski spacetime

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

The canonical momentum $\pi$ is given by

$$\pi(t,\vec{x}) = \frac{\partial\mathcal{L}}{\partial(\frac{\partial\phi}{\partial t})} = \frac{\partial\phi(t,\vec{x})}{\partial t} \tag{4.3.1}$$

The Euclidean time Lagrangian, for $\tau = -it$, is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2$$

The path integral for Euclidean time is given by

$$Z = \int D\phi e^{\int_0^\tau \mathcal{L} d^3x d\tau}; \;\; \int D\phi = \prod_{t,\vec{x}} \int d\phi_{t,\vec{x}}$$

For the infinitesimal time, the Lagrangian yields

$$\begin{aligned}
\langle \phi_\epsilon | e^{-\epsilon H} | \phi_0 \rangle &= e^{\epsilon \mathcal{L}(x,x';\epsilon)} \\
&= \exp\{-\frac{\epsilon}{2}\int_{\vec{x}} [(\frac{\phi - \phi'}{\epsilon})^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2]\} \\
&= e^{-\frac{\epsilon}{2}\int_{\vec{x}} (\frac{\phi-\phi'}{\epsilon})^2} e^{-\frac{\epsilon}{2}\int_{\vec{x}}[(\vec{\nabla}\phi)^2 + m^2\phi^2]} \\
&= e^{-\epsilon V(\phi)} (\prod_{\vec{x}} \int \frac{dp_{\vec{x}}}{2\pi}) e^{-\frac{\epsilon}{2}\int_{\vec{x}} p_{\vec{x}}^2} e^{i\int_{\vec{x}} p_{\vec{x}}(\phi_{\vec{x}} - \phi'_{\vec{x}})}
\end{aligned}$$

To extract the Hamiltonian, note

$$\langle \phi | e^{-\epsilon H} | \phi' \rangle = e^{-\epsilon H(\phi, \partial\phi)} \langle \phi | \phi' \rangle$$

and the identity

$$\begin{aligned}
\int Dp e^{-\frac{\epsilon}{2}\int_{\vec{x}} p_{\vec{x}}^2} e^{i\int_{\vec{x}} p_{\vec{x}}(\phi_{\vec{x}} - \phi'_{\vec{x}})} &= e^{\frac{\epsilon}{2}\int_{\vec{x}} \frac{\delta^2}{\delta\phi_{\vec{x}}^2}} \int Dp e^{i\int_{\vec{x}} p_{\vec{x}}(\phi_{\vec{x}} - \phi'_{\vec{x}})} \\
&= \exp\{\frac{\epsilon}{2}\int_{\vec{x}} \frac{\delta^2}{\delta\phi_{\vec{x}}^2}\} \delta[\phi - \phi']
\end{aligned} \tag{4.3.2}$$

Hence, collecting all the terms yields the Hamiltonian – valid for both Euclidean and Minkowski time – is given by

$$\begin{aligned}
H &= -\frac{1}{2}\int d^3x \frac{\delta^2}{\delta\phi_{\vec{x}}^2} + \frac{1}{2}\int d^3x [(\vec{\nabla}\phi(\vec{x}))^2 + m^2\phi(\vec{x})^2] \\
&= \frac{1}{2}\int d^3x [\pi^2 + (\vec{\nabla}\phi)^2 + \phi^2]
\end{aligned} \tag{4.3.3}$$

where the canonical momentum is defined by

$$\pi = -i\frac{\delta}{\delta\phi(\vec{x})} \tag{4.3.4}$$

The definition of canonical momentum $\pi$ given in Eq. 4.3.1 is consistent with the one given above in Eq. 4.3.4 since, in the Hamiltonian representation, all time dependent operators are considered to be Heisenberg operators. The consistency of Eqs. 4.3.1 and 4.3.4 will become more clear in Section 4.4.

Eq. 4.3.4 yields the canonical commutation equations

$$[\phi(\vec{x}), \pi(\vec{x'})] = i\frac{\delta\phi(\vec{x})}{\delta\phi(\vec{x'})} = i\delta^3(\vec{x} - \vec{x'}), \tag{4.3.5}$$

and

$$[\phi(\vec{x}), \phi(\vec{x'})] = [\pi(\vec{x}), \pi(\vec{x'})] = 0$$

## 4.4 Operator field equations

To have some practice with the operator structure of the scalar quantum field and its momentum, a detailed derivation is given of the Heisenberg evolution equation. For Minkowski time, the Heisenberg operator obey

$$\phi(t, \vec{x}) = e^{itH}\phi(\vec{x})e^{-itH}$$

and hence

$$\frac{\partial\phi(t, \vec{x})}{\partial t} = -i[\phi(t, \vec{x}), H] \tag{4.4.1}$$

Similarly, the time evolution of the canonical momentum is given by

$$\frac{\partial\pi(t, \vec{x})}{\partial t} = -i[\pi(t, \vec{x}), H] \tag{4.4.2}$$

Recall that the Hamiltonian is given by

$$H = \frac{1}{2}\int d^3x[\pi_{\vec{x}}^2 + (\vec{\nabla}\phi_{\vec{x}})^2 + m\phi_{\vec{x}}^2]$$

with the operator realization of the canonical momentum that yields the canonical commutation equations as follows

$$\pi = -i\frac{\delta}{\delta\phi(\vec{x})}$$

Consider the Heisenberg operator equation

$$\frac{\partial\phi_{t,\vec{x}}}{\partial t} = -i[\phi_{t,\vec{x}}, H] = -ie^{itH}[\phi_{\vec{x}}, H]e^{-itH} \tag{4.4.3}$$

The only non-zero term in the commutator above is

$$[\phi_{\vec{x}}, H] = \frac{1}{2}\int_{\vec{x'}}[\pi_{\vec{x'}}^2, \phi_{\vec{x}}]$$

Note

$$[AB, C] = [A, C]B + A[B, C]$$

and this yields

$$[\phi_{\vec{x}}, H] = \frac{1}{2}\int_{\vec{x'}} \{\pi_{\vec{x'}}[\phi_{\vec{x}}, \pi_{\vec{x'}}] + [\phi_{\vec{x}}, \pi_{\vec{x'}}]\pi_{\vec{x'}}\} = i\frac{1}{2}\int \pi_{\vec{x'}}\delta^{(3)}(\vec{x} - \vec{x'}) = i\pi_{\vec{x}}$$

Hence

$$\frac{\partial \phi_{t,\vec{x}}}{\partial t} = -i^2 e^{itH}\pi_{\vec{x}}e^{-itH} = \pi_{t,\vec{x}}$$

Note $\partial\phi/\partial t = \pi$ is the result expected from the classical definition of the canonical momentum.

The Heisenberg operator equation for the canonical momentum is given by

$$\frac{\partial \pi_{t,\vec{x}}}{\partial t} = -ie^{itH}[\pi_{\vec{x}}, H]e^{-itH}$$

There are two non-zero terms in the commutator above. The first commutator is given by

$$\begin{aligned}\frac{1}{2}\int_{\vec{x'}}[\pi_{\vec{x'}}, (\vec{\nabla}\phi_{\vec{x}})^2] &= \frac{1}{2}\int_{\vec{x'}}\{\partial_i^x\phi_{\vec{x'}}\partial_i^x[\pi_{\vec{x'}}, \phi_{\vec{x}}] + \partial_i^x[\pi_{\vec{x'}}, \phi_{\vec{x}}]\partial_i^x\phi_{\vec{x}}\} \\ &= -i\int \partial_i^x\phi_{\vec{x}}\partial^x\delta(\vec{x} - \vec{x'}) = i\int \vec{\nabla}^2\phi_{\vec{x}}\delta(\vec{x} - \vec{x'}) \\ &= i\vec{\nabla}^2\phi_{\vec{x}} \qquad (4.4.4)\end{aligned}$$

The second commutator is given by

$$\frac{1}{2}\int_{\vec{x'}}[\pi_{\vec{x'}}, (\phi_{\vec{x}})^2] = -i\phi_{\vec{x}}$$

The two terms yield the final result

$$\frac{\partial \pi_{t,\vec{x}}}{\partial t} = i^2 e^{itH}[-\vec{\nabla}^2\phi_{\vec{x}} + m^2\phi_{\vec{x}}]e^{-itH} = -(-\vec{\nabla}^2 + m^2)\phi_{t,\vec{x}} = \frac{\partial^2\phi_{t,\vec{x}}}{\partial t^2}$$

Therefore, we obtain the Klein-Gordon operator equation

$$(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 - m^2)\phi_{t,\vec{x}} = 0$$

or

$$(\partial^\mu\partial_\mu + m^2)\phi = 0$$

The classical field equations, for the free scalar field, appear as the operator equations for the Heisenberg field operators. The analogy with the classical field equation breaks down for nonlinear quantum fields due to the requirement of rendering such field theories finite using the procedure of renormalization.

## 4.5 Fourier expansion

The Heisenberg operators $\hat{\phi}(t,x)$ and $\hat{\pi}(t,x)$ can be expanded using the plane wave basis. Consider the relativistic invariant expansion

$$\hat{\phi}(t,x) = \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p_\mu p^\mu - m^2) e^{-ip^\mu x_\mu} \phi(p^0, \vec{p})$$

where

$$p^\mu = (p^0, p^1, p^2, p^3) = (p^0, \vec{p});\ p_\mu = (p^0, -\vec{p});\ p_\mu p^\mu = (p^0)^2 - (\vec{p})^2 = p^2$$

The Heisenberg operator $\hat{\phi}(t,x)$ obeys the Klein-Gordon operator equation

$$(\partial^\mu \partial_\mu + m^2)\hat{\phi}(t,x) = \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)(-p^2 + m^2) e^{ip^\mu x_\mu} \phi(p^0, \vec{p}) = 0$$

Recall for the $\delta$-function obeys

$$\delta[f(x)] = \sum \frac{1}{f'(x_i)} \delta(x - x_i);\ f(x_i) = 0$$

Hence for $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$.

$$\delta((p^0)^2 - (\vec{p})^2 - m^2) = \delta\Big[(p^0 + E_{\vec{p}})(p^0 - E_{\vec{p}})\Big] = \frac{1}{2E_{\vec{p}}}\Big[\delta(p^0 + E_{\vec{p}}) + \delta(p^0 - E_{\vec{p}})\Big]$$

we have

$$\begin{aligned}
\phi(t,\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} dp^0 \frac{1}{2E_{\vec{p}}} [\delta(p^0 + E_{\vec{p}}) + \delta(p^0 - E_{\vec{p}})] e^{ip^\mu x_\mu} \phi(p^0, \vec{p}) \\
&= \int_{\vec{p}} \frac{1}{2E_{\vec{p}}} \{e^{-iE_p t + i\vec{p}\cdot\vec{x}} \phi(E_{\vec{p}}, \vec{p}) + e^{iE_p t + i\vec{p}\cdot\vec{x}} \phi(-E_{\vec{p}}, \vec{p})\} \\
&= \int_{\vec{p}} \frac{1}{2E_{\vec{p}}} \{e^{-iE_p t + i\vec{p}\cdot\vec{x}} \phi(E_{\vec{p}}, \vec{p}) + e^{iE_p t - i\vec{p}\cdot\vec{x}} \phi(-E_{\vec{p}}, -\vec{p})\}
\end{aligned}$$

The coordinate field operator is Hermitian since it is an observable operator; Hermiticity yields

$$\phi^\dagger(t,\vec{x}) = \phi(t,\vec{x}) \quad \Rightarrow \quad \phi^\dagger(E_p, \vec{p}) = \phi(-E_p, -\vec{p})$$

We hence obtain

$$\phi(t,\vec{x}) = \int_{\vec{p}} \frac{1}{2E_{\vec{p}}} \{e^{-iE_p t + i\vec{p}\cdot\vec{x}} \phi(E_{\vec{p}}, \vec{p}) + e^{iE_p t - i\vec{p}\cdot\vec{x}} \phi^\dagger(E_{\vec{p}}, \vec{p})\} \quad (4.5.1)$$

Furthermore

$$\pi(t,\vec{x}) = \frac{\partial\phi(t,\vec{x})}{\partial t} \tag{4.5.2}$$

$$= -i\int_{\vec{p}} \frac{E_{\vec{p}}}{2E_{\vec{p}}}\{e^{-iE_pt+i\vec{p}\cdot\vec{x}}\phi(E_{\vec{p}},\vec{p}) - e^{iE_pt-i\vec{p}\cdot\vec{x}}\phi^\dagger(E_{\vec{p}},\vec{p})\} \tag{4.5.3}$$

Define the destruction and creation operator for momentum $\vec{p}$ by

$$a_{\vec{p}} = \frac{1}{\sqrt{2E_{\vec{p}}}}\phi(E_{\vec{p}},\vec{p}) \quad ; \quad a^\dagger_{\vec{p}} = \frac{1}{\sqrt{2E_{\vec{p}}}}\phi^\dagger(E_{\vec{p}},\vec{p})$$

Hence we have the expansions

$$\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{-iE_{\vec{p}}t+i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{iE_{\vec{p}}t-i\vec{p}\cdot\vec{x}}) \tag{4.5.4}$$

$$\pi(t,\vec{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}(a_{\vec{p}}e^{-iE_{\vec{p}}t+i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{iE_{\vec{p}}t-i\vec{p}\cdot\vec{x}}) \tag{4.5.5}$$

The equal time canonical commutation equations given in Eq. 4.3.5 yields

$$[a_{\vec{p}}, a^\dagger_{\vec{p'}}] = (2\pi)^3\delta^3(\vec{p}-\vec{p'}),$$

and

$$[a_{\vec{p}}, a_{\vec{p'}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{p'}}] = 0,$$

## 4.6 Creation and destruction operators

The creation and destruction operators diagonalize the field Hamiltonian when it is expressed in terms of the creation and destruction operators. The initial values of the operators are the Schrodinger operators given in Eqs. 4.5.4 and 4.5.5 are the following

$$\phi_{\vec{x}} = \phi(\vec{x}) = \phi(0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}})$$

$$\pi_{\vec{x}} = \pi(\vec{x}) = \pi(0,\vec{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}})$$

and the Hamiltonian is given by

$$H = \frac{1}{2}\int d^3x[\pi^2_{\vec{x}} + (\vec{\nabla}\phi_{\vec{x}})^2 + m^2\phi^2_{\vec{x}}]$$

$$= \frac{1}{2}\int d^3x[\pi^2_{\vec{x}} - \phi_{\vec{x}}\vec{\nabla}^2\phi_{\vec{x}} + m^2\phi^2_{\vec{x}}] \tag{4.6.1}$$

The three terms in the Hamiltonian as given in Eq. 4.6.1 are expressed in terms of the creation and destruction operators.

$$\int_{\vec{x}} \pi^2(\vec{x}) = \frac{(-i)^2}{2} \int_{\vec{x}} \int_{\vec{p},\vec{p'}} \sqrt{E_{\vec{p}}}(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}})\sqrt{E_{\vec{p'}}}(a_{\vec{p'}} e^{i\vec{p'}\cdot\vec{x}} - a^\dagger_{\vec{p'}} e^{-i\vec{p'}\cdot\vec{x}})$$

$$= -\frac{1}{2}\int_{\vec{p},\vec{p'}} E_{\vec{p}}[\delta(\vec{p}+\vec{p'})a_{\vec{p}}a_{\vec{p'}} - \delta(\vec{p}-\vec{p'})a_{\vec{p}}a^\dagger_{\vec{p'}} - \delta(\vec{p}-\vec{p'})a^\dagger_{\vec{p}}a_{\vec{p'}} + \delta(\vec{p}+\vec{p'})a^\dagger_{\vec{p}}a^\dagger_{\vec{p'}}]$$

$$= \frac{1}{2}\int E_{\vec{p}}(a_{\vec{p}}a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}}) - \frac{1}{2}\int E_{\vec{p}}(a_{\vec{p}}a_{-\vec{p}} + a^\dagger_{\vec{p}}a^\dagger_{-\vec{p}})$$

and

$$\int_{\vec{x}} [m^2\phi^2(\vec{x}) - \phi_{\vec{x}}\vec{\nabla}^2\phi_{\vec{x}}]$$

$$= \frac{1}{2}\int_{\vec{x}}\int_{\vec{p},\vec{p'}} \frac{1}{\sqrt{E_{\vec{p}}E_{\vec{p'}}}}(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}})(m^2-\nabla^2)(a_{\vec{p'}} e^{i\vec{p'}\cdot\vec{x}} - a^\dagger_{\vec{p'}} e^{-i\vec{p'}\cdot\vec{x}})$$

$$= \frac{1}{2}\int \frac{(m^2+\vec{p'}^2)}{\sqrt{E_{\vec{p}}E_{\vec{p'}}}}[\delta(\vec{p}+\vec{p'})a_{\vec{p}}a_{\vec{p'}} - \delta(\vec{p}-\vec{p'})a_{\vec{p}}a^\dagger_{\vec{p'}} - \delta(\vec{p}-\vec{p'})a^\dagger_{\vec{p}}a_{\vec{p'}} + \delta(\vec{p}+\vec{p'})a^\dagger_{\vec{p}}a^\dagger_{\vec{p'}}]$$

$$= \frac{1}{2}\int_{\vec{p}} E_{\vec{p}}(a_{\vec{p}}a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}}) + \frac{1}{2}\int_{\vec{p}} E_{\vec{p}}(a_{\vec{p}}a_{-\vec{p}} + a^\dagger_{\vec{p}}a^\dagger_{-\vec{p}})$$

Adding the terms and taking note of a cancellation of two terms yields

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3}E(\vec{p})(a^\dagger_{\vec{p}}a_{\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}}) \\ &= \int \frac{d^3p}{(2\pi)^3}E(\vec{p})a^\dagger_{\vec{p}}a_{\vec{p}} + \delta^{(3)}(0)\int d^3pE(\vec{p}) \end{aligned} \tag{4.6.2}$$

since $[a_{\vec{p}}, a^\dagger_{\vec{p'}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p'})$.

The commutation of the creation and destruction operators with the Hamiltonian yields the following *spectrum generating algebra*

$$[a_{\vec{p}}, H] = \int_{\vec{p'}} E_{\vec{p'}}[a_{\vec{p}}, a^\dagger_{\vec{p'}}]a_{\vec{p'}} = E_{\vec{p}}a_{\vec{p}} \tag{4.6.3}$$

and

$$[a^\dagger_{\vec{p}}, H] = \int_{\vec{p'}} E_{\vec{p'}}[a_{\vec{p}}, a^\dagger_{\vec{p'}}]a^\dagger_{\vec{p'}} = -E_{\vec{p}}a^\dagger_{\vec{p}} \tag{4.6.4}$$

Time evolution of the destruction operator, from Eq. 4.6.3, is given by

$$a_{t,\vec{p}} = e^{itH}a_{\vec{p}}e^{-itH} \quad \Rightarrow \quad \frac{\partial a_{t,\vec{p}}}{\partial t} = i[H, a_{\vec{p}}] = -E_{\vec{p}}a_{\vec{p}}$$

and similarly for the creation operator. Hence

$$a_{t,\vec{p}} = e^{-iE_{\vec{p}}t} a_{\vec{p}} \;\; ; \;\; a^\dagger_{t,\vec{p}} = e^{iE_{\vec{p}}t} a^\dagger_{\vec{p}}$$

The time evolution of the creation and destruction operators yield the following time dependence for the field operator

$$\begin{aligned}\phi(t,\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (a_{t,\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{t,\vec{p}} e^{-i\vec{p}\cdot\vec{x}}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} (a_{\vec{p}} e^{-iE_{\vec{p}}t + i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{iE_{\vec{p}}t - i\vec{p}\cdot\vec{x}}) \qquad (4.6.5)\end{aligned}$$

Since

$$E_{\vec{p}}t - \vec{p}\cdot\vec{x} = p_\mu x^\mu = px, \;\; p^\mu = (p^0, \vec{p})$$

Eq. 4.6.5 yields the relativistic expansion as obtained in Eqs. 4.5.4.

## 4.7 Energy eigenstates

The scalar field is equivalent to a infinite set of decoupled simple harmonic oscillators. Hence, similar to the case of the harmonic oscillators, the state space for the scalar field is generated by acting on the vacuum state with the creation operators.

Recall the vacuum state $|0\rangle$ is defined by $a_{\vec{p}}|0\rangle = 0$ for all $\vec{p}$ and is an eigenstate of the Hamiltonian given by $H|0\rangle = E_0|0\rangle$. Furthermore, from Eqs. 4.6.2 and 4.6.4

$$H(a^\dagger_{\vec{p}}|0\rangle) = (a^\dagger_{\vec{p}} H + E_{\vec{p}} a^\dagger_{\vec{p}})|0\rangle = (E_{\vec{p}} + E_0) a^\dagger_{\vec{p}}|0\rangle$$

Hence $a^\dagger_{\vec{p}}$ acting on $|0\rangle$ creates an eigenstate $a^\dagger_{\vec{p}}|0\rangle$ with energy $E_{\vec{p}} + E_0$.

The vacuum state, for all $\vec{p}$, is given by

$$a_{\vec{p}}|0\rangle = 0 \quad \Rightarrow \langle 0|0\rangle = 1$$

The one-particle state is given by

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}} a^\dagger_{\vec{p}}|0\rangle = \phi^\dagger(E_{\vec{p}}, \vec{p})|0\rangle$$

with Lorentz invariant normalization

$$\langle \vec{p}|\vec{p'}\rangle = 2E_{\vec{p}}(2\pi)^3 \delta^3(\vec{p} - \vec{p'})$$

The multiple-particles states are given by

$$|\vec{p_1}, \vec{p_2} ... \vec{p_N}\rangle = \sqrt{2E_{\vec{p_1}} ... 2E_{\vec{p_N}}} a^\dagger_{\vec{p_1}} ... a^\dagger_{\vec{p_N}}|0\rangle$$

The most general energy eigenvector, upto a normalization $\mathcal{K}$, is

$$|\{n,\vec{p}\}\rangle = |n_1,\vec{p_1};n_2,\vec{p_2};...n_k,\vec{p_k};\rangle = \mathcal{K}\frac{(a^\dagger_{\vec{p_1}})^{n_1}}{\sqrt{n_1!}}\frac{(a^\dagger_{\vec{p_2}})^{n_2}}{\sqrt{n_2!}}...\frac{(a^\dagger_{\vec{p_k}})^{n_k}}{\sqrt{n_k!}}|0\rangle$$

and is an eigenstate with energy $E_n$ where

$$H|\{n,\vec{p}\}\rangle = E_n|\{n,\vec{p}\}\rangle \quad ; \quad E_n = \sum_{i=1}^{k} n_i E_{\vec{p_i}}$$

The state vector is the simultaneously the eigenstate of energy and momentum. The momentum operator of the field, from Eq. 3.4.3 is given by

$$\vec{P} = \int d^3x\varphi(\vec{x})\dot{\varphi}(\vec{x}) = \int d^3x\varphi(\vec{x})\pi(\vec{x})$$

Using the creation destruction representation of the field and its conjugate yields

$$\vec{P} = \int_{\vec{p}} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}}$$

and yields

$$\vec{P}|\{n,\vec{p}\}\rangle = \vec{p}|\{n,\vec{p}\}\rangle \quad ; \quad \vec{p} = \sum_{i=1}^{k} n_i\vec{p_i}$$

Since $[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p'}}] = 0$, multiple-particle states are symmetric under exchange of any two particles; this is a reflection of the fact that $\phi(t,\vec{x})$ is a bosonic scalar field.

## 4.8 Schrodinger Wave functional

The Schrodinger state functional $\Psi$ depends on the degree of freedom $\phi_{\vec{x}}$. The coordinate basis is given by

$$\bigotimes_{\vec{x}} |\phi_{\vec{x}}\rangle = |\phi\rangle$$

and the state functional $\Psi$ is given by

$$\Psi[\phi] = \langle\phi|\Psi\rangle$$

The ground state $|\Omega\rangle$ is defined by $a_{\vec{p}}|\Omega\rangle = 0$ for all $\vec{p}$. Recall the definition of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ is given by

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}})$$
$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}(-i)(a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}})$$

Hence

$$a_{\vec{p}} = \frac{1}{\sqrt{2E_{\vec{p}}}}\int_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}}[E_{\vec{p}}\phi(\vec{x}) + i\pi(\vec{x})] = \frac{1}{\sqrt{2E_{\vec{p}}}}\int_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}}[E_{\vec{p}}\phi(\vec{x}) + \frac{\delta}{\delta\phi(\vec{x})}]$$

The vacuum state is obtained by a procedure similar to the oscillator case; since the scalar field is free, consider the Gaussian ansatz for the vacuum state

$$\Omega = \mathcal{N}\exp(-\frac{1}{2}\int_{\vec{x},\vec{y}} \phi_{\vec{x}} f_{\vec{x}-\vec{y}}\phi_{\vec{y}})$$

The equation

$$a_{\vec{p}}\Omega = 0$$

yields

$$\int_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}}[E_{\vec{p}}\phi_{\vec{x}} - \int_{\vec{y}} f_{\vec{x}-\vec{y}}\phi_{\vec{x}}]\Omega = 0$$

Let $f_{\vec{x}} = \int_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} f_{\vec{k}}$, then

$$\int_{\vec{x},\vec{y},\vec{k}} e^{-i\vec{p}\cdot\vec{x}}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}f_{\vec{k}}\phi_{\vec{y}} = \int_{\vec{y}} e^{-i\vec{p}\cdot\vec{y}}f_{\vec{p}}\phi_{\vec{y}}$$

Hence

$$\int_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}}[E_{\vec{p}} - f_{\vec{p}}]\phi_{\vec{x}}\Omega = 0 \quad \Rightarrow \quad f_{\vec{p}} = E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

This yields the vacuum state function.

$$\Omega[\phi] = \mathcal{N}\exp\{-\int_{\vec{x},\vec{y}} f_{\vec{x}-\vec{y}}\phi_{\vec{x}}\phi_{\vec{y}}\} \quad ; \quad f_{\vec{x}-\vec{y}} = \int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\sqrt{\vec{p}^2 + m^2}$$

The normalization $\mathcal{N}$ is fixed by requiring $\langle\Omega|\Omega\rangle = 1$.

## 4.9 Functional formulation of quantum fields

Consider the Lagrangian for a real scalar field $\phi$ given by

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\varphi\partial_{\mu}\varphi - V(\varphi)$$

The probability amplitude for a transition from initial coordinate eigenstate $|\varphi\rangle$ to a final state $|\varphi'\rangle$ is given by

$$Z = \langle\varphi'|e^{-2iTH}|\varphi\rangle$$

where $H$ is the scalar field's Hamiltonian and

$$|\varphi\rangle = \Pi_{\vec{x}}|\varphi_{\vec{x}}\rangle \;\; ; \;\; |\varphi'\rangle = \Pi_{\vec{x}}|\varphi'_{\vec{x}}\rangle$$

The Feynman path integral yields the following representation

$$Z = \int D\varphi e^{iS} \;\; ; \;\; S = \int_{-T}^{+T} dt \int d^3\vec{x}\mathcal{L}(t,\vec{x})$$

The boundary conditions are

$$\varphi(-T,\vec{x}) = \varphi(\vec{x}) \;\; ; \;\; \varphi(T,\vec{x}) = \varphi'(\vec{x})$$

A compressed notation for the transition amplitude is given by

$$Z = \int_{\varphi}^{\varphi'} D\varphi e^{iS} \;\; ; \;\; \int D\varphi \equiv \prod_{t,\vec{x}} \int_{-\infty}^{+\infty} d\varphi(t,\vec{x})$$

Correlation functions are defined by a functional average overall possible field values and yields

$$E[\varphi(t_1,\vec{x_1})\cdots\varphi(t_N,\vec{x_N})] = \frac{1}{Z}\int D\varphi e^{iS}\varphi(t_1,\vec{x_1})\cdots\varphi(t_N,\vec{x_N})$$

Consider the two-point correlator

$$D(x_1,x_2) = E[\varphi(t_1,\vec{x_1})\varphi(t_2,\vec{x_2})]$$

Let $t_1 > t_2$; break up the path into three segments that go from $-T$ to $t_2$, from $t_2$ to $t_1$ and from $t_1$ to $T$, as shown in Figure 4.1

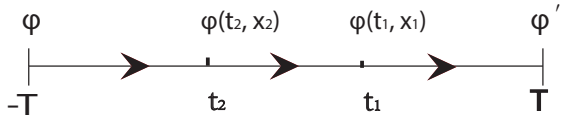


Figure 4.1 The three segments of time for the path integral.

The path integral can be written as follows

$$D(x_1,x_2) = \frac{1}{Z}\int_{\varphi}^{\varphi_2} D\varphi \int D\varphi_2 \varphi_2(t_2,\vec{x_2}) \int_{\varphi_2}^{\varphi_1} D\varphi \int D\varphi_1 \varphi_1(t_1,\vec{x_1}) \int_{\varphi_1}^{\varphi'} D\varphi e^{i\mathcal{S}}$$
$$= \frac{1}{Z}\int D\varphi_1 D\varphi_2 \langle\varphi'|e^{-i(T-t_1)H}|\varphi_1\rangle\varphi_1(t_1,\vec{x_1})$$
$$\langle\varphi_1|e^{-i(t_1-t_2)H}|\varphi_2\rangle\varphi_2(t_2,\vec{x_2})\langle\varphi_2|e^{-i(T+t_2)H}|\varphi\rangle \qquad (4.9.1)$$

where

$$\int D\varphi_i \equiv \prod_{\vec{x}} \int_{-\infty}^{+\infty} d\varphi(t_i,\vec{x}) \;\; ; \;\; i=1,2$$

The Schrodinger (coordinate) field operator is defined by

$$\hat{\varphi}(\vec{x}) = \int D\varphi |\varphi\rangle\varphi(\vec{x})\langle\varphi| \;\; ; \;\; |\varphi\rangle = \prod_{\vec{y}} |\varphi(\vec{y})\rangle \qquad (4.9.2)$$

Performing the functional integration over the field $\varphi$ and using the Schrodinger representation yields, from Eq. 4.9.1 the matrix element

$$D(x_1,x_2) = \frac{1}{Z}\langle\varphi'|e^{-iTH}e^{it_1H}\hat{\varphi}(\vec{x_1})e^{-i(t_1-t_2)H}\hat{\varphi}(\vec{x_2})e^{-it_2H}e^{-iTH}|\varphi\rangle$$

The definition of the Heisenberg operator yields

$$D(x_1,x_2) = \frac{1}{Z}\langle\varphi'|e^{-iTH}\hat{\varphi}_H(t_1,\vec{x_1})\hat{\varphi}_H(t_2,\vec{x_2})e^{-iTH}|\varphi\rangle$$

Tacking the limit of $T\to\infty$ yields

$$e^{-iTH}|\varphi\rangle \simeq e^{-iTE_0}|\Omega\rangle\langle\Omega|\varphi\rangle \;\; ; \;\; \langle\varphi'|e^{-iTH} \simeq e^{-iTE_0}\langle\varphi'|\Omega\rangle\langle\Omega|$$
$$\Rightarrow Z \simeq e^{-2iTE_0}\langle\varphi'|\Omega\rangle\langle\Omega|\varphi\rangle$$

Note in the limit of $T\to\infty$, the initial and final state vectors $|\varphi\rangle,\ \langle\varphi'|$ cancel out, as does the infinite vacuum energy $E_0$. Taking the limit of $T\to\infty$ yields

$$D(x_1,x_2) = \langle\Omega|\hat{\varphi}_H(t_1,\vec{x_1})\hat{\varphi}_H(t_2,\vec{x_2})|\Omega\rangle \qquad (4.9.3)$$

Note that time-ordering is built into the path integral. In general, for

$$D(x_1,x_2) = \langle\Omega|T(\hat{\varphi}_H(t_1,\vec{x_1})\hat{\varphi}_H(t_2,\vec{x_2}))|\Omega\rangle \qquad (4.9.4)$$

one needs to order the operators by hand before converting the matrix element into a functional integral.

## 4.10 Unequal time commutation equations

The spacetime properties are encoded in the light cone, as shown in Figure 4.2. The scalar field in Minkowski space is investigated to study interplay of a quantum field with the structure of spacetime, and in particular whether a quantum field is consistent with causality.

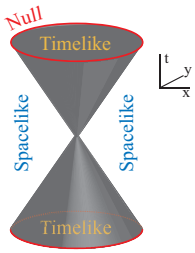


Figure 4.2 The light cone.

In particular, we expect that for two spacetime points with spacelike separation, the scalar quantum field at these two points should commute – as observations can be made at space like separation that are causally independent. We will, in fact, find that it is the *commutator* of the fields that vanish for two spacetime points, since all observations finally require the commutator of the field with the observable operator.

Define the notation

$$x = (x^0, \vec{x}) \quad ; \quad y = (y^0, \vec{y}) \quad ; \quad px = p_\mu x^\mu = p^0 x^0 - \vec{p} \cdot \vec{x}$$

Recall from Eq. 4.5.4

$$\phi(x) = \phi(t, \vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} (a_{\vec{p}} e^{-iE_{\vec{p}} t + i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{iE_{\vec{p}} t - i\vec{p}\cdot\vec{x}})$$

For unequal time $x^0 \neq y^0$, the commutator is given by[1]

$$\begin{aligned}[\phi(x), \phi(y)] = \int_{\vec{p},\vec{p'}} \frac{1}{2\sqrt{E_{\vec{p}} E_{\vec{p'}}}} \Big\{ & e^{-iE_{\vec{p}} x^0 + i\vec{p}\cdot\vec{x}} e^{iE_{\vec{p'}} y^0 - i\vec{p'}\cdot\vec{y}} [a_{\vec{p}}, a^\dagger_{\vec{p'}}] \\ & + e^{iE_{\vec{p}} x^0 - i\vec{p}\cdot\vec{x}} e^{-iE_{\vec{p'}} y^0 + i\vec{p'}\cdot\vec{y}} [a^\dagger_{\vec{p}}, a_{\vec{p'}}] \Big\} \\ = \mathbb{I} \int_{\vec{p}} \frac{1}{2E_{\vec{p}}} \Big\{ & e^{-ip(x-y)} - e^{ip(x-y)} \Big\} \end{aligned} \tag{4.10.1}$$

Note the operator on the right hand side of above equation is the unity

[1] Recall $px = E_p x_0 - \vec{p} \cdot \vec{x}$.

operator $\mathbb{I}$, and this fact simplifies our calculations. Using the fact that $\langle 0|\mathbb{I}|0\rangle = 1$ yields

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = \Delta(x-y) - \Delta(y-x) \tag{4.10.2}$$

For equal time $x^0 = y^0$ the commutator is zero since

$$[\phi(x^0, \vec{x}), \phi(x^0, \vec{y})] = \int_{\vec{p}} \frac{1}{2E_{\vec{p}}} \Big\{ e^{-i\vec{p}(\vec{x}-\vec{y})} - e^{i\vec{p}(\vec{x}-\vec{y})} \Big\} = 0 \tag{4.10.3}$$

The result is expected because the operators are at spacelike separation and hence are not causally connected.

The unequal time *operator product* is given by

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \int_{\vec{p},\vec{p'}} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{p'}}}} \langle 0|(e^{-ipx}a_{\vec{p}} + e^{ipx}a^{\dagger}_{\vec{p}})(e^{-ip'y}a_{\vec{p'}} + e^{ip'y}a^{\dagger}_{\vec{p'}})|0\rangle \\
&= \int_{\vec{p},\vec{p'}} \frac{e^{-ipx}e^{ip'y}}{2\sqrt{E_{\vec{p}}E_{\vec{p'}}}} \langle 0|a_{\vec{p}}a^{\dagger}_{\vec{p'}}|0\rangle = \int_{\vec{p}} \frac{e^{-ip(x-y)}}{2E_{\vec{p}}} \\
&= \Delta(x-y)
\end{aligned}$$

a) Time-like separation. Let the position be fixed at $\vec{x} = \vec{y}$ and let the two different instants of time be given by $x^0 = t, y^0 = t'$; $(x-y)^2 = (x^0-y^0)^2 > 0$.

$$\begin{aligned}
\Delta(x-y) &= \int \frac{d^3p}{(2\pi)^3} e^{-i(t-t')E_p} \frac{1}{2\sqrt{\vec{p}^2+m^2}} \\
&= \frac{4\pi}{(2\pi)^3} \int dp \frac{p^2}{2\sqrt{\vec{p}^2+m^2}} e^{-i(t-t')\sqrt{\vec{p}^2+m^2}} \\
&\sim e^{-im(t-t')} \qquad\qquad |t-t'| \to \infty
\end{aligned}$$

a) Space-like separation. Let time be equal $x^0 = y^0$ and the spatial positions be $\vec{x} - \vec{y} = \vec{r}$; $(x-y)^2 = -(\vec{x}-\vec{y})^2 < 0$.

$$\begin{aligned}
\Delta(x-y) &= \int \frac{d^3p}{(2\pi)^3} e^{ipr\cos\theta} \frac{1}{2E_p} \\
&= \frac{2\pi}{(2\pi)^3} \int dp \frac{p^2}{2E_p} \int_{-1}^{1} d\cos\theta e^{ipr\cos\theta} \\
&= \frac{-i}{2(2\pi)^2 r} \int_0^{\infty} p^2 dp \left(\frac{e^{ipr} - e^{-ipr}}{pE_p}\right) \\
&= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}} \\
&= \frac{m}{4\pi^2 r} K_1(mr) \sim \frac{m}{4\pi^2 r} \sqrt{\frac{\pi}{2mr}} e^{-mr} \quad (r \to \infty)
\end{aligned}$$

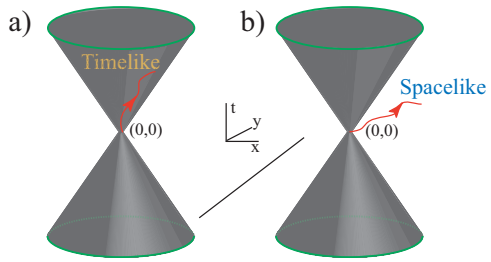


Figure 4.3 a) Timelike spacetime interval. b) Spacelike spacetime interval.

There is a non-zero amplitude for the operator product for spacelike separation. Does this violate causality?

We expect the commutator of physical observables to be zero for spacelike separation since these operators should be simultaneously diagonalizable. Hence, for Hermitian physical operators $\hat{O}(x)$, special relatively requires that for spacelike separation $[\hat{O}(x), \hat{O}(y)] = 0, (x-y)^2 < 0$ Recall

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = \Delta(x-y) - \Delta(y-x) \tag{4.10.4}$$

If $x, y$ have a spacelike separation, that is $(x-y)^2 < 0$, then we expect the coordinate operator of the field to be simultaneously observable, which in turn requires

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = 0 = \Delta(x-y) - \Delta(y-x) \quad : \quad x, y\text{: spacelike} \tag{4.10.5}$$

If $x, y$ are spacelike, a Lorentz transformation takes $(x-y) \to -(x-y)$. Hence to prove

$$[\phi(x), \phi(y)] = 0 \quad : \quad (x-y)^2 < 0$$

we have to show that $\Delta(x)$ is Lorentz invariant; it then follows that $\Delta(\Lambda x) = \Delta(x)$, where $\Lambda$ is Lorentz transformation.

Define the function

$$G(x-y) = i[\phi(x), \phi(y)] = i[\Delta(x-y) - \Delta(y-x)] = G^*(x-y)$$

From the definition of the $\Delta(x)$ function

$$\begin{aligned} G(x) &= i \int_{\vec{p}} \frac{1}{2E_{\vec{p}}} [e^{-ipx} - e^{ipx}]\Big|_{p^0 = E_{\vec{p}}} \\ &= i \int_{\vec{p}} \int dp_0 \frac{\epsilon(p_0)}{2E_{\vec{p}}} [\delta(p^0 - E_{\vec{p}}) + \delta(p^0 + E_{\vec{p}})] e^{-ip_0 x^0 + i\vec{p}\vec{x}} \\ &= i \int_p \epsilon(p_0) \delta(p^2 - m^2) e^{-ipx} \end{aligned} \tag{4.10.6}$$

where

$$\epsilon(p_0) = \begin{cases} +1 & p_0 > 0 \\ -1 & p_0 < 0 \end{cases} \tag{4.10.7}$$

A Lorentz transformation $\Lambda$ of four momentum $p_\mu$, given by $\Lambda p$, leaves the sign of $p_0$ and norm $p^2$ invariant. Hence, $G(\Lambda x) = G(x)$ is Lorentz invariant and satisfies the Klein Gordon equation

$$(\partial^\mu \partial_\mu + m^2)G(x-y) = 0] \tag{4.10.8}$$

The function yields

$$G(x-y) = \int_{\vec{p}} \frac{e^{i\vec{p}(\vec{x}-\vec{y})}}{E_{\vec{p}}} \sin[E_{\vec{p}}(x^0 - y^0)]$$

and hence

$$[\phi(x), \phi(y)]\Big|_{x^0=y^0} = -iG(x-y)\Big|_{x^0=y^0} = 0$$

By a Lorentz transformation, it follows that

$$[\phi(x), \phi(y)] = 0; \text{ for} (x-y)^2 < 0$$

Furthermore

$$\frac{\partial G(x-y)}{\partial y_0}\Big|_{x^0=y^0} = -\int_{\vec{p}} \frac{e^{i\vec{p}(\vec{x}-\vec{y})}}{E_{\vec{p}}} E_{\vec{p}} \cos[E_{\vec{p}}(x^0 - y^0)]$$

$$= -\int_{\vec{p}} e^{i\vec{p}(\vec{x}-\vec{y})} = -\delta^3(\vec{x}-\vec{y})$$

and which yields

$$[\phi(x), \pi(y)]\Big|_{x^0=y^0} = -i\frac{\partial G(x-y)}{\partial y_0}\Big|_{x^0=y^0} = i\delta^3(\vec{x}-\vec{y})$$

We obtain the expected equal time canonical commutation equation.

## 4.11 The Feynman Propagator

A fundamental feature of quantum fields is the Feynman propagator, which is the amplitude of an excitation propagating from one spacetime point to another. The Feynman propagator is defined by

$$D(x,y) = \begin{cases} \Delta(x-y) & x^0 > y^0 \\ \Delta(y-x) & x^0 < y^0 \end{cases} \tag{4.11.1}$$

and is written as

$$D(x-y) = \theta(x^0 - y^0)\Delta(x-y) + \theta(y^0 - x^0)\Delta(y-x)$$

The propagator is an even function

$$D(x-y) = D(y-x)$$

This definition takes into account relativistic causality as well as the fact that there is an underlying Hamiltonian propagating the field. The definition of $\Delta(x-y)$ yields

$$D(x-y) = \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$\equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

where $T$ stands for *time ordering* the product of operators.

Recall the operator product of the free field is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int_{\vec{p}} \frac{e^{-ip(x-y)}}{2E_{\vec{p}}} = \Delta(x-y)$$

and yields

$$D(x-y) = \int_{\vec{p}} \frac{1}{2E_{\vec{p}}}[\theta(x^0 - y^0)e^{-ip(x-y)} + \theta(y^0 - x^0)e^{ip(x-y)}]$$

or

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{4.11.2}$$

The expression given in Eq. 4.11.2 is derived in the following manner. The pole in the integration is at $p^0 = \pm\sqrt{E_{\vec{p}}^2(1 - \frac{i\epsilon}{E_{\vec{p}}^2})} \simeq \pm E_{\vec{p}}(1 - \frac{i\epsilon}{2E_{\vec{p}}^2})$.

- Consider $x^0 > y^0$. Then

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} e^{ip(\vec{x}-\vec{y})} \int \frac{dp_0}{2\pi} e^{-ip^0(x^0-y^0)} \frac{i}{(p^0)^2 - \vec{p}^2 - m^2 + i\epsilon}$$

  For $x^0 > y^0$ for convergence the contour is closed in the lower half plane and contour is traversed clockwise picking up the plane at $E_{\vec{p}} - i\epsilon$. Hence, the residue theorem yields

$$D(x-y) = \theta(x^0 - y^0)\frac{i}{2\pi} \int \frac{d^3p}{(2\pi)^3} e^{ip(\vec{x}-\vec{y})}(-2\pi i)\frac{-i}{2\pi}\frac{e^{-iE_p(x^0-y^0)}2\pi i}{2E_p}$$

$$= \theta(x^0 - y^0)\Delta(x-y)$$

- Similarly, for $x^0 < y^0$ the contour is closed anti-clockwise in the upper half plane picking up the plane at $-E_{\vec{p}} + i\epsilon$. Hence, the residue theorem yields

$$D(y-x) = \theta(y^0 - x^0)\frac{i}{2\pi}\int \frac{d^3p}{(2\pi)^3} e^{ip(\vec{x}-\vec{y})}(2\pi i)\frac{e^{iE_p(x^0-y^0)}}{-2E_p}$$

$$= \theta(y^0 - x^0)\Delta(y-x)$$

The Feynman propagator obeys the inhomogeneous Klein Gordon equation given by

$$\begin{aligned}(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2_{\vec{x}} + m^2)D(x-y) &= (\partial^\mu\partial_\mu + m^2)D(x-y)\\ &= i\int \frac{d^4p}{(2\pi)^4}\frac{-p^2+m^2}{p^2-m^2+i\epsilon}e^{-ip(x-y)}\\ &= -i\int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)} = -i\delta^4(x-y)\end{aligned}$$

The physical interpretation of the propagator is the following. The field is separated into the creation and annihilation components

$$\varphi(x) = \varphi^+(x) + \varphi^-(x)$$

where

$$\varphi^+(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} e^{-ipx} a_{\vec{p}}\,; \quad \varphi^-(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} e^{ipx} a^\dagger_{\vec{p}}$$

Note

$$\varphi^+|0\rangle = 0 = \langle 0|\varphi^- \quad \Rightarrow \quad \langle 0|(\varphi^- + \varphi^+)|0\rangle = 0 = \langle 0|\varphi|0\rangle$$

Hence we obtain

$$D(x-y) = \langle 0|T[\varphi(x)\varphi(y)]|0\rangle$$

$$= \theta(x^0-y^0)\langle 0|\varphi^+(x)\varphi^-(y)|0\rangle + \theta(y^0-x^0)\langle 0|\varphi^+(y)\varphi^-(x)|0\rangle$$

The interpretation of the components of the propagator is the following.

$\theta(x^0-y^0)\langle 0|\varphi^+(x)\varphi^-(y)|0\rangle$ : create particle at $y^0$ and destroying at $x^0$

$\theta(y^0-x^0)\langle 0|\varphi^+(y)\varphi^-(x)|0\rangle$ : create particle at $x^0$ and destroying at $y^0$

In the presence of a classical source $j(x)$, the Lagrangian for the scalar field is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

The field equation is given by

$$(\partial_\mu \partial^\mu + m^2)\phi^2 = j(x)$$

and yields the solution

$$\phi(x) = \phi_0(x) - i \int d^4y D(x-y) j(y)$$

Note $\phi_0(x)$ is the sourceless scalar field and $D(x-y)$ is the Feynman propagator. For a time-independent source $j(x)$, the Hamiltonian is given by

$$H = \frac{1}{2} \int d^3x [\pi_{\vec{x}}^2 + (\vec{\nabla}\phi_{\vec{x}})^2 + m^2 \phi_{\vec{x}}^2 + 2j(x)]$$

### *4.11.1 Retarded Greens function*

To examine the role of time-ordering, consider the following retarded propagator that is non-zero only in the future. Namely

$$D_R(x-y) = \theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle = \theta(x^0 - y^0)\{\Delta(x-y) - \Delta(y-x)\}$$

and

$$\Delta(x-y) = \int_{\vec{p}} \frac{e^{-ip(x-y)}}{2E_{\vec{p}}}\Big|_{p_0 = E_{\vec{p}}}$$

Let $\partial^\mu \equiv \partial/\partial x^\mu$, $\partial^\mu \partial_\mu \equiv \partial^2$ and $f = \theta(x^0 - y^0)$ and $g = \langle 0|[\phi(x), \phi(y)]|0\rangle$. Then

$$\partial^2(D_R) = (\partial^2 f)g + 2\partial^\mu f \partial_\mu g + f\partial^2 g = (\partial_0^2 f)g + 2\partial_0 f \partial_0 g + f\partial^2 g$$

Note, for $\partial h/\partial x^0 \equiv \dot{h}$, the identities

$$\frac{\partial}{\partial x^0}\theta(x^0 - y^0) = \delta(x^0 - y^0),\ \partial^2\theta(x^0 - y^0) = \dot{\delta}(x^0 - y^0)$$

and yields

$$\delta(x^0 - y^0)[\phi(x), \phi(y)] = 0 \ \Rightarrow\ \dot{\delta}(x^0 - y^0)[\phi(x), \phi(y)] = -\delta(x^0 - y^0)[\dot{\phi}(x), \phi(y)]$$

The above identities yield

$$(\partial^\mu \partial_\mu + m^2)D_R(x-y) = \big(\partial^2\theta(x^0 - y^0)\big)\langle 0|[\phi(x), \phi(y)]|0\rangle$$
$$+2\big(\frac{\partial}{\partial x^0}\theta(x^0 - y^0)\langle 0|[\dot{\phi}(x), \phi(y)]|0\rangle\big) + \theta(x^0 - y^0)\big(\partial^\mu \partial_\mu + m^2\big)\langle 0|[\phi(x), \phi(y)]|0\rangle$$

The free field operator $\phi(x)$ obeys the Klein Gordon equation that

$$\big(\partial^\mu \partial_\mu + m^2\big)[\phi(x), \phi(y)] = [\big(\partial^\mu \partial_\mu + m^2\big)\phi(x), \phi(y)] = 0$$

Hence, the retarded propagator obeys the Green's function equation since

$$(\partial^\mu \partial_\mu + m^2) D_R(x-y) = \delta(x^0 - y^0)[\dot{\phi}(x), \phi(y)] = -i\delta^{(4)}(x-y)$$

## 4.12 Complex scalar field

The complex scalar field $\varphi$ consists of a scalar particle together with its antiparticle. A complex scalar field carries charge and hence can couple to a gauge field. In contrast, a real scalar field has zero charge and cannot have any coupling to a gauge field. The particle and antiparticle interpretation is valid only if both the underlying real scalar fields have the same mass $m$; otherwise the interpretation fails.

In terms of the complex field $\varphi$, the Lagrangian is given by

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

where $m$ is the mass of the complex scalar field.

A complex scalar field consists of two real scalar fields $\phi_1, \phi_2$. The complex scalar field is defined by

$$\varphi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)];\ \varphi^\dagger(x) = \frac{1}{\sqrt{2}}[\phi_1(x) - i\phi_2(x)] \neq \varphi(x)$$

Using the real component field representation yields the Lagrangian

$$\mathcal{L} = \sum_{i=1}^{2} \frac{1}{2}(\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i^2)$$

Note that both the fields $\phi_1, \phi_2$ have the same mass $m$, and this is the basis of identifying the complex scalar field as consisting of a particle and antiparticle, since a necessary requirement for a particle-antiparticle pair is that their masses must be exactly equal.

The Hamiltonian is given by

$$H = \frac{1}{2} \sum_{i=1}^{2} [\pi_i^2 + (\vec{\nabla}\phi_i)^2 + m^2 \phi_i^2]$$

The complex field canonical momentum is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\dagger(x)} = \dot{\phi}(x) = \frac{1}{\sqrt{2}}(\pi_1(x) + i\pi_2(x)) \neq \pi(x)$$

and the Hermitian conjugate canonical momentum is given by

$$\pi^\dagger = \frac{1}{\sqrt{2}}(\pi_1(x) - i\pi_2(x))$$

In the complex field representation, the Hamiltonian is

$$H = \pi^\dagger \pi + \vec{\nabla}\varphi^\dagger \cdot \vec{\nabla}\varphi + m^2 \varphi^\dagger \varphi$$

The complex scalar field obeys the Klein-Gordon equation

$$[\partial_\mu \partial^\mu - m^2]\varphi = 0$$

Similar to the real scalar field, has a Fourier expansion given by

$$\varphi(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} (e^{i\vec{p}\cdot\vec{x}} A_{\vec{p}} + e^{-i\vec{p}\cdot\vec{x}} B^\dagger_{\vec{p}}) \tag{4.12.1}$$

$$\varphi^\dagger(\vec{x}) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} (e^{-i\vec{p}\cdot\vec{x}} A^\dagger_{\vec{p}} + e^{i\vec{p}\cdot\vec{x}} B_{\vec{p}}) \tag{4.12.2}$$

where

$$[A_{\vec{p}}, A^\dagger_{\vec{p'}}] = [B_{\vec{p}}, B^\dagger_{\vec{p'}}] = \delta^3(\vec{p} - \vec{p'})$$

The state space for the complex field $\varphi$ is

$$\mathcal{V}_\varphi = \mathcal{V} \otimes \mathcal{V}$$

where $\mathcal{V}$ is the state space of a single real scalar field.

In terms of the state space $\mathcal{V}_\varphi$, the creation and destruction operators have the following representation

$$a_{\vec{p_1}} = a_{\vec{p}} \otimes \mathbb{I} \;\; ; \;\; a_{\vec{p_2}} = \mathbb{I} \otimes a_{\vec{p}} \;\; ; \;\; a^\dagger_{\vec{p_1}} = a^\dagger_{\vec{p}} \otimes \mathbb{I} \;\; ; \;\; a^\dagger_{\vec{p_2}} = \mathbb{I} \otimes a^\dagger_{\vec{p}}$$

The destruction operators for the complex field are given by

$$A_{\vec{p}} = \frac{1}{\sqrt{2}}[a_{\vec{p}} \otimes \mathbb{I} + i\mathbb{I} \otimes a_{\vec{p}}] \;\; ; \;\; B_{\vec{p}} = \frac{1}{\sqrt{2}}[a_{\vec{p}} \otimes \mathbb{I} - i\mathbb{I} \otimes a_{\vec{p}}]$$

Instead of writing out the tensor products explicitly one instead uses the simplified notation given by

$$A_{\vec{p}} = \frac{1}{\sqrt{2}}[a_{\vec{p_1}} + i a_{\vec{p_2}}] \;\; ; \;\; B_{\vec{p}} = \frac{1}{\sqrt{2}}[a_{\vec{p_1}} - i a_{\vec{p_2}}]$$

The ground state $|\Omega\rangle$ is

$$|\Omega\rangle = |0\rangle \otimes |0\rangle$$

and

$$A_{\vec{p}}|\Omega\rangle = 0 = B_{\vec{p}}|\Omega\rangle \quad \Rightarrow \quad a_{\vec{p_1}}|\Omega\rangle = 0 = a_{\vec{p_2}}|\Omega\rangle$$

Hence

$$\langle\Omega|\varphi|\Omega\rangle = 0 = \langle\Omega|\varphi^\dagger|\Omega\rangle$$

The single real fields $\phi_1, \phi_2$ have the same propagator given by

$$\langle\Omega|T(\phi_1(x)\phi_1(y))|\Omega\rangle = \langle\Omega|T(\phi_2(x)\phi_2(y))|\Omega\rangle = D(x-y)$$

Consider

$$\langle\Omega|T[\varphi(x)\varphi(y)]|\Omega\rangle = \frac{1}{2}\langle\Omega|T[(\phi_1(x)+i\phi_2(x))(\phi_1(y)+i\phi_2(y))]|\Omega\rangle$$
$$\frac{1}{2}[\langle\Omega|T[\phi_1(x)\phi_1(y)]|\Omega\rangle - \langle\Omega|T[\phi_2(x)\phi_2(y)]|\Omega\rangle] = 0$$

The non-zero propagator is the one that conserves charge and is given by

$$\langle\Omega|T[\varphi(x)\varphi^\dagger(y)]|\Omega\rangle = \frac{1}{2}[\langle\Omega|T[\phi_1(x)\phi_1(y)+\phi_2(x)\phi_2(y)]|\Omega\rangle]$$
$$= D(x-y) = \langle\Omega|T[\varphi^\dagger(x)\varphi(y)]|\Omega\rangle \qquad (4.12.3)$$

The Hamiltonian, similar to the real scalar field, is given by

$$H = \int_{\vec{p}} E_{\vec{p}}(A^\dagger_{\vec{p}}A_{\vec{p}} + B^\dagger_{\vec{p}}B_{\vec{p}}) + E_0 \quad ; \quad E_{\vec{p}} = \sqrt{\vec{p}^2+m^2} \qquad (4.12.4)$$

The change operator is given by

$$Q = \int_{\vec{p}}(A^\dagger_{\vec{p}}A_{\vec{p}} - B^\dagger_{\vec{p}}B_{\vec{p}}) + \text{constant}$$

Consider the commutator

$$[A^\dagger_{\vec{k}}, Q] = \int_{\vec{p}} A^\dagger_{\vec{p}}[A^\dagger_{\vec{k}}, A_{\vec{p}}] = -A^\dagger_{\vec{k}}$$

$$[B^\dagger_{\vec{k}}, Q] = \int_{\vec{p}} B^\dagger_{\vec{p}}[B^\dagger_{\vec{k}}, B_{\vec{p}}] = B^\dagger_{\vec{k}}$$

The one-particle states have the following charges

$QA^\dagger_{\vec{k}}|\Omega\rangle = (A^\dagger + A^\dagger Q)|\Omega\rangle = A^\dagger|\Omega\rangle :$ +positive charged single particle state

$QB^\dagger_{\vec{k}}|\Omega\rangle = (B^\dagger + B^\dagger Q)|\Omega\rangle = -B^\dagger|\Omega\rangle :$ −negative charged single anti-particle state

Similar to the case of the real field, we have

$$HA^\dagger_{\vec{k}}|\Omega\rangle = (E_{\vec{p}}+E_0)A^\dagger|\Omega\rangle;\ HB^\dagger_{\vec{k}}|\Omega\rangle = (E_{\vec{p}}+E_0)B^\dagger|\Omega\rangle$$

Note states $A^\dagger_{\vec{k}}|\Omega\rangle,\ B^\dagger_{\vec{k}}|\Omega\rangle$ both have energy $E_{\vec{p}}+E_0$, with first state being a single particle state, with $Q=+1$ and the second state being an anti-particle state, with $Q=-1$.

Hence $A^\dagger_{\vec{k}}$ creates a particle state, $B^\dagger_{\vec{k}}$ create an anti-particle state.

### 4.12.1 Charge conjugation

The complex field scalar consists of two charged scalar particles which are a particle and antiparticle of each other, the particle carrying a positive charge and the antiparticle carrying a negative charge. This property of the complex scalar field is brought out by the charge conjugation operator, that interchanges particle with antiparticle.

Charge conjugation is a unitary operator $\mathcal{C}$ that is defined as follows

$$\mathcal{C}\varphi\mathcal{C}^\dagger = \eta\varphi^\dagger \;\; ; \;\; \mathcal{C}\varphi^\dagger\mathcal{C}^\dagger = \eta^*\varphi \;\; ; \;\; |\eta|^2 = 1$$

Repeating the transformation twice yields the following

$$\mathcal{C}\mathcal{C}^\dagger = \mathbb{I} \;\; \Rightarrow \;\; \mathcal{C}^2 = \mathbb{I}$$

The Fourier expansion given in Eqs. 4.12.1 and 4.12.2 yields the expected result

$$\mathcal{C}A_{\vec{k}}\mathcal{C}^\dagger = \eta B_{\vec{k}} \;\; ; \;\; \mathcal{C}B^\dagger_{\vec{k}}\mathcal{C}^\dagger = \eta A^\dagger_{\vec{k}}$$

Since charge conjugation interchanges particle with antiparticle, the charge operator for the complex scalar field must also change it sign from plus to minus. A straight forward calculation shows that under charge conjugation, the transformation yields the expected result given by

$$\mathcal{C}\mathcal{Q}\mathcal{C}^\dagger = -\mathcal{Q}$$

For a free complex scalar field, there is no coupling to charge, but in the presence of a gauge field, the charge allows the complex scalar field to be coupled to the gauge field. A field theory that is symmetric under charge conjugation does not have any invariant distinction between particle and the antiparticle.

## 4.13 Gaussian Integration

The functional formulation of quantum fields involves the integration theory of infinitely many integration variables. The study of free quantum fields, and of weakly interacting quantum fields, is based on an expansion that utilizes Gaussian functional integration.

Gaussian integration plays a key role in studying path integrals in quantum mechanics and in quantum field theory. To introduce the concepts of Gaussian functional integration, a few of its key properties the Gaussian, or the normal random variable, are reviewed.

The basic Gaussian integral is given by

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}\lambda x^2 + jx} = \sqrt{\frac{2\pi}{\lambda}} e^{\frac{1}{2\lambda} j^2} \tag{4.13.1}$$

*N-dimensional Gaussian integration*

The moment generating function for the $N$-dimensional Gaussian random variable is given by

$$\begin{aligned} Z[j] &= \int_{-\infty}^{+\infty} dx_1 \cdots dx_n e^S \\ \text{with} \quad S &= -\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_i J_i x_i \end{aligned} \tag{4.13.2}$$

Let $A_{ij}$ be a symmetric and positive definite matrix that has only positive eigenvalues. $A_{ij}$ can be diagonalized by an orthogonal matrix $M$

$$A = M^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} M \quad ; \quad M^T M = \mathbb{I}$$

Define new variables

$$\begin{gathered} z_i = M_{ij} x_j \quad ; \quad x_i = M^T_{ij} z_j \\ \prod_{i=1}^{N} dz_i = \det M \prod_{i=1}^{N} dx_i = \prod_{i=1}^{N} dx_i \equiv Dx \end{gathered}$$

Hence

$$Z[j] = \prod_i^N \int dz_i e^{-\frac{1}{2}\lambda_i z_i^2 + (JM^T)_i z_i} = \prod_{i=1}^{N} \left[ \sqrt{\frac{2\pi}{\lambda_i}} e^{\frac{1}{2\lambda_i}(JM^T)_i (JM^T)_i} \right]$$

In matrix notation

$$\sum_i \frac{1}{\lambda_i} (JM^T)_i (JM^T)_i = JA^{-1}J \quad ; \quad \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = (2\pi)^{N/2} \frac{1}{\sqrt{\det A}}$$

Hence

$$Z[J] = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2} JA^{-1}J} \tag{4.13.3}$$

All the moments of the coupled Gaussian random variables can be determined by the generating function given in Eq. 4.13.2; namely

$$E[x_1 x_2..x_N] = \frac{\partial^N}{\partial J_1 \partial J_2..\partial J_N} Z[J]|_{J=0}$$

Let $t = n\epsilon$, $n = 0, \pm 1, \pm 2 \cdots \pm N$. Taking the limit of $N \to \infty$, $\epsilon \to 0$ yields

$$S_0 = -\frac{1}{2} \int_{-\infty}^{+\infty} dt dt' x_t A_{tt'} x_{t'}$$

$$Z = \int Dx e^S$$

The generating functional for the simple harmonic oscillator is given by[2]

$$Z[j] = \frac{1}{Z} \int Dx e^{S_0 + \int dt j(t) x_t} = \exp\left\{ \frac{1}{2} \int_{-\infty}^{+\infty} dt dt' j_t A^{-1}_{tt'} j_{t'} \right\} \quad (4.13.4)$$

where

$$\int A^{-1}_{tt'} A_{t't''} dt' = \delta(t - t'') \quad (4.13.5)$$

## 4.14 Gaussian white noise

The properties of white noise are analyzed as this constitutes the simplest form of Gaussian functional integration; it also shows how the Dirac-delta functions for the correlation functions emerge from functional integration.

The fundamental properties of Gaussian white noise are that

$$E[R(t)] = 0 \quad ; \quad E[R(t)R(t')] = \delta(t - t') \quad (4.14.1)$$

Figure 4.4 shows that there is an independent (Gaussian) random variable $R(t)$ for each instant of time $t$.

Discretize time, namely $t = n\epsilon$, , with $n = 1, 2....N$, and with $R(t) \to R_n$. The probability distribution function of white noise is given by

$$P(R_n) = \sqrt{\frac{\epsilon}{2\pi}} \mathrm{e}^{-\frac{\epsilon}{2} R_n^2} \quad (4.14.2)$$

Hence, $R_n$ is a Gaussian random variable with zero mean and $1/\sqrt{\epsilon}$ variance, and is denoted by $N(0, 1/\sqrt{\epsilon})$. The following result is essential in deriving the rules of Ito calculus

$$R_n^2 = \frac{1}{\epsilon} + \quad \text{random terms of } 0(1) \quad (4.14.3)$$

[2] The term generating functional is used instead as generating function as in Eq. 4.13.3 to indicate that one is considering a system with infinitely many variables.

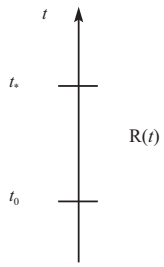


Figure 4.4 One random variable $R(t)$ for each instant of time.

To write the probability measure for $R(t)$, with $t_1 \leq t \leq t_2$ discretize $t \to n\epsilon$. White noise $R(t)$ has the probability distribution given Eq. 4.14.2. The probability measure for the white noise random variables in the interval $t_1 \leq t \leq t_2$ is the given by

$$\mathcal{P}[R] = \prod_{n=1}^{N} P(R_n) = \prod_{n=1}^{N} e^{-\frac{\epsilon}{2}R_n^2} \tag{4.14.4}$$
$$\int dR = \prod_{n=1}^{N} \sqrt{\frac{\epsilon}{2\pi}} \int_{-\infty}^{+\infty} dR_n$$

Taking the continuum limit of $\epsilon \to 0$ yields, for $t_1 < t < t_2$

$$\mathcal{P}[R, t_1, t_2] \to e^{S_0} \quad ; \quad S_0 = -\frac{1}{2}\int_{t_1}^{t_2} dt R^2(t) \tag{4.14.5}$$
$$Z = \int DR e^{S_0} \quad ; \quad \int dR \to \int DR$$

The action functional $S_0$ is ultra-local with all the variables being decoupled. Gaussian integration, given in Eq. 4.13.3, yields

$$Z[j, t_1, t_2] = \frac{1}{Z}\int DR e^{\int_{t_1}^{t_2} dt j(t) R(t)} e^{S_0[} = e^{\frac{1}{2}\int_{t_1}^{t_2} dt j^2(t)} \tag{4.14.6}$$

The correlation functions are given by

$$E[R(t)] = 0$$
$$E[R(t)R(t')] = \frac{1}{Z}\int DR\ R(t)R(t')e^{S_0} = \frac{\partial^2}{\partial j(t)\partial j(t')} Z[j]\Big|_{j=0} = \delta(t - t')$$

and yields the result given in Eq. 4.14.1.

The result given in Eqs. 4.14.5 and 4.14.6 show that white noise is represented by a path integral with an ultra local action $S_0$.

## 4.15 Free Scalar Field: Path integral

To rigorously define the functional integral, spacetime has to be discretized a finite lattice, with total number of lattice sites being finite and equal to $N^d$. Instead, a heuristic deviation is given of a result that can be obtained rigorously based on a lattice spacetime.

The path integral for the scalar quantum field is given by

$$Z = \int D\varphi \exp\{iS\} \quad ; \quad S = \int_{-\infty}^{\infty} dt \int d^3x \mathcal{L}(t, \vec{x})$$

with the Lagrangian for the free scalar quantum field given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2}m^2\varphi^2 + j\varphi$$

The external current $j(x)$ is a classical function and is introduced so that the generating functional for the free field can be evaluated. Since $S$ is quadratic in the field, Gaussian integration is sufficient to perform the functional integral.

To diagonalize the action, define the Fourier transform

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu x_\mu} \phi_k = \phi^*(x) \Rightarrow \phi_k^* = \phi_{-k}$$

Define

$$\phi_k = \phi_k^R + i\phi_k^I \quad \Rightarrow \quad \phi_{-k} = \phi_k^R - i\phi_k^I$$

Hence, the independent integration variables are only the positive momentum degrees of freedom and yields

$$\prod_x \int_{-\infty}^{\infty} d\phi(x) = \prod_{k>0} \int_{-\infty}^{\infty} d\phi_k^R d\phi_k^I$$

Note

$$S = \frac{1}{2}\int d^4x[(\partial_0\phi)^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}[k_0^2 - \vec{k}^2 - m^2]|\phi_k|^2$$

To make the Gaussian integrations convergent, displace $k_\mu k^\mu - m^2$ to $k_\mu k^\mu - m^2 + i\epsilon$; this yields

$$\exp\{iS\} \to \exp\{iS - \epsilon\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}|\phi_k|^2\} \tag{4.15.1}$$

Hence, $\epsilon$ provides the regulator to render the Gaussian integrations convergent. Using the rules of the Gaussian integrations yields

$$e^{W(j)} = \frac{1}{Z}\int D\varphi e^{\frac{1}{2}i\int(k_\mu k^\mu - m^2 + i\epsilon)|\phi_k|^2 + i\int j_{-k}\phi_k} = \exp(-\frac{1}{2}i\int j_{-k}D_k j_k)$$

where

$$D_k = \frac{1}{k_\mu k^\mu - m^2 + i\epsilon}$$

The correlator is given by

$$E[\varphi_p \varphi'_p] = \frac{1}{Z}\int D\varphi \varphi_p \varphi_{p'} e^{iS} = \frac{\partial^2}{i^2 \partial j_{-p} \partial j_{-p'}} e^W|_{j=0}$$
$$= -\frac{\partial}{\partial j_{-p}}(-iD_{p'} j_{-p'}) e^W = iD_p \delta(p+p')$$

The Feynman propagator is given by

$$D(x,y) = \frac{1}{Z}\int D\varphi \varphi(x)\varphi(y) e^{iS} = \frac{1}{Z}\int D\varphi \{\int_{p,p'} e^{ipx+ip'y} E[\varphi_p \varphi'_p]\} e^{iS}$$
$$= i\int_{p,p'} e^{ipx+ip'y} \frac{\delta(p+p')}{p^2 - m^2 + i\epsilon}$$
$$= i\int_p \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} \tag{4.15.2}$$

The deviation shows that Feynman's $+i\epsilon$ prescription for $D(x,y)$ is the result of rendering the path integral convergent.

## 4.16 Continuation to Euclidean time

To make the path integral a rigorously defined expression, one analytically continues Minkowski time to Euclidean time.

Let *Euclidean coordinates and momentum* be given by

$$x^E = (x_0^E, x_i^E) = (\tau, x_i^E) \quad ; \quad p^E = (p_0^E, p_i^E)$$

Minkowski coordinates and momentum $x^\mu, p^\mu$ are analytically continued to Euclidean space by the following[3]

$$x^0 = x_0 = -ix_0^E = -i\tau \quad , \quad x_i = x_i^E \tag{4.16.1}$$
$$p^0 = p_0 = ip_0^E \quad , \quad -p^i = p_i = p_i^E \tag{4.16.2}$$

Note upper and lower indices are the same for Euclidean space. The continuation to Euclidean space yields the result

$$p_0 x_0 = -i^2 p_0^E x_0^E = p_0^E x_0^E$$

[3] The continuation of Minkowski to Euclidean momentum is a result that follows from the definition of canonical momentum in Hamiltonian mechanics – and is discussed by Baaquie (2014).

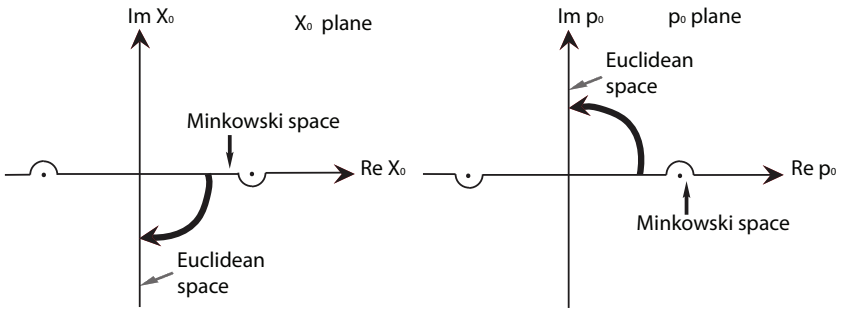


Figure 4.5 Rotation of the time $x_0$ and energy axis $p_0$ in going from Minkowski to Euclidean space.

In particular, we have

$$p^\mu x_\mu = p^0 x_0 + \sum_i p^i x_i = p_0 x_0 - \sum_i p_i x_i = p_0^E x_0^E - \sum_i p_i^E x_i^E \quad (4.16.3)$$

But in taking the Fourier transform of a function $f(p^2)$ the following substitution is made for Euclidean space [**?**]

$$p_0^E x_0^E - \sum_i p_i^E x_i^E \Rightarrow p_0^E x_0^E + \sum_i p_i^E x_i^E \quad (4.16.4)$$

The continuation yields

$$p_\mu p^\mu = p_0^2 - p_i p_i = -(p_0^E)^2 - (p_i^E)^2 = -p_\mu^E p_\mu^E \;\; ; \;\; x_\mu x^\mu = -x_\mu^E x_\mu^E \quad (4.16.5)$$

Both $x_0$ and $p_0$ are considered to be complex variables , and the $x_0$ and $p_0$ integrals in Minkowski space are considered to be contour integrals. The analytic continuation of the $x_0$ and $p_0$ contour integrals from Minkowski to Euclidean space is given in Figure 4.5. Note the direction of rotation of the time $x_0$ and momentum $p_0$ axis is defined unambiguously by the regularization of the functional integral given in Eq. 4.15.1. The time axis rotates through $-\pi/2$ to avoid crossing the singularity in the contour integral and the energy axis $E = p_0$ rotates through $+\pi/2$, as shown in Figure 4.5. Hence, we have the following deformation of the spacetime and four-momentum contour integrals.

- The continuation of time yields

$$dx_0 = -i dx_0^E \;\; ; \;\; d^3 x_i = d^3 x_i^E$$

The rotation clockwise of the contour integral for Euclidean time is given

in Figure 4.5 and yields

$$\int_{-\infty}^{+\infty} dx_0 = -i \int_{+i\infty}^{-i\infty} dx_0^E$$

- For the momentum integral

$$dp_0 = i dp_0^E \quad ; \quad d^3p_i = d^3p_i^E$$

The deformation of the momentum contour for Euclidean energy $p_0^E$ is given in Figure 4.5 and yields

$$\int_{-\infty}^{+\infty} dp_0 = i \int_{-i\infty}^{+i\infty} dp_0^E$$

### *4.16.1 Euclidean quantum fields*

For Euclidean time defined by $t = -i\tau$, the Euclidean field is defined by

$$\varphi(t, \vec{x}) = \varphi(\tau, \vec{x}) \quad ; \quad \frac{\partial \varphi}{\partial t} = i \frac{\partial \varphi}{\partial \tau}$$

To make the analytical continuation, $\tau = it$ means that $t = \pm\infty$ goes to $\tau = \pm i\infty$. Hence, as shown in Figure 4.5, the time integral is rotated counter-clockwise by $\pi/2$.

The Euclidean Lagrangian is given by

$$\mathcal{L}(\tau, \vec{x}) = \mathcal{L}_M(-i\tau, \vec{x}) = -\frac{1}{2}(\frac{\partial \varphi}{\partial \tau})^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - V(\varphi) < 0$$

Hence, the action is negative $S < 0$ and the path integral is well defined

$$Z = \int D\varphi e^S < \infty$$

The generating functional no longer needs the $+i\epsilon$ term to be well-defined and is given by

$$Z(J) = e^{W(J)} = \int D\varphi e^{S + \int j\varphi}$$

For the free scalar $S = S_0$ is quadratic and

$$Z[J] = \exp[\frac{1}{2} \int_{x,y} \varphi(x) D(x-y) \varphi(y)] = \exp[\frac{1}{2} \int_p j_{-p}(p^2 + m^2) j_p]$$

The Euclidean propagator is given by

$$D_E(x-y) = E[\varphi(x)\varphi(y)] = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip^E(x-y)}}{(p^E)^2 + m^2} \tag{4.16.6}$$

Time ordering continues to hold in Euclidean with $\tau_1 < \tau_2 \cdots \tau_k$. Hence

$$\langle\Omega|T[\varphi(t_1,\vec{x}_1)\cdots\varphi(t_k,\vec{x}_k)]|\Omega\rangle = \langle\Omega|T[\varphi(\tau_1,\vec{x}_1)\cdots\varphi(\tau_k,\vec{x}_k)]|\Omega\rangle$$

The Heisenberg time dependent for the field operator for the free scalar field for Minkwoski spacetime, from Eq. 4.6.5, is given by

$$\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{-iE_{\vec{p}}t+i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{iE_{\vec{p}}t-i\vec{p}\cdot\vec{x}})$$

with

$$E_{\vec{p}}t - \vec{p}\cdot\vec{x} = p_\mu x^\mu \quad ; \quad p^\mu = (p^0,\vec{p})$$

The analytic continuation given in Eq. 4.16.5, with the momentum $\vec{p}$ flipping its sign as in Eq. 4.16.4, yields the following expansion for the free Euclidean scalar quantum field

$$\phi_E(\tau,\vec{x}) = \int \frac{d^3p^E}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}e^{iE_{\vec{p}}\tau+i\vec{p}^E\cdot\vec{x}^E} + a^\dagger_{\vec{p}}e^{-iE_{\vec{p}}\tau-i\vec{p}^E\cdot\vec{x}^E}) \quad (4.16.7)$$

with

$$p_0^E = E_{\vec{p}} = \sqrt{\vec{p}^2+m^2}$$

The Hamiltonian for both Minkowski and Euclidean spacetime is given by the same expression; dropping the superscript on the Euclidean momentum variable, Eq. 4.6.2 yields

$$H_E = H = \int \frac{d^3p}{(2\pi)^3}E(\vec{p})a^\dagger_{\vec{p}}a_{\vec{p}} + \delta^{(3)}(0)\int d^3pE(\vec{p}) \quad (4.16.8)$$

A similar result holds for the complex scalar field discussed in Section 4.12.

### *4.16.2 Euclidean Feynman propagator*

The continuation of the Minkowski propagator – given in Eq. 4.15.2 – to Euclidean time yields, from Eq. 4.16.5, the Euclidean propagator

$$\begin{aligned} D_E(x) &= i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip_\mu x^\mu}}{p^2-m^2+i\epsilon} = i\int\frac{d^3p_i dp_0}{(2\pi)^4}\frac{e^{-ip_\mu^E x_\mu^E}}{((p_0)^2-p_ip_i)-m^2+i\epsilon} \\ &= i^2\int\frac{d^3p_i^E dp_0^E}{(2\pi)^4}\frac{e^{ip_\mu^E x_\mu^E}}{-p_\mu^E p_\mu^E - m^2+i\epsilon} \\ &= \int\frac{d^4p^E}{(2\pi)^4}\frac{e^{ip^E x^E}}{(p^E)^2+m^2} \end{aligned}$$

and we recover the result given in Eq. 4.16.6.

# 5

# The Dirac spinor field

The Dirac equation was obtained in 1928 by P. A. M. Dirac with objective of obtaining a relativistic version of the Schrodinger equation. What Dirac discovered, instead, was a relativistic field equation for a *new* type of particles, namely, relativistic fermions – which includes the electron as its most well known example. The relativistic electron is a *spinor field*, carrying spin and constituted by an electron and the antielectron.

The Dirac equation is the field equation for the spinor field and is analogous to the Maxwell equation, which governs the classical behavior of the (relativistic) electromagnetic field. Freeman Dyson once said about Dirac that 'His discoveries were like exquisitely carved statues falling out of the sky, one after another. He seemed to be able to conjure laws of nature from pure thought.'

The Dirac equation predicts that relativistic particles come in pairs of a particle and its antiparticle, which is equally physical and fundamental like its particle. In particular, the Dirac equation predicts that the electron is a spin half particle that comes together with a companion spin half anti-electron, namely the positron. The existence of an antiparticle was, at that time, completely unknown and beyond the concepts known to physics. The discovery of the anti-electron in 1932 by Carl Anderson is considered to be one of the greatest triumphs of modern theoretical physics. Since the antiparticle is a salient and outstanding feature of the Dirac equation, the main focus of this Chapter is to explicate the specific features of the antiparticle and its relation to the corresponding particle.

The canonical formulation of the Dirac equation is discussed in this Chapter and many standard results are derived. Spacetime is taken to be a four dimensional Minkowski manifold, unlike most of the other Chapters where spacetime is taken to be a Euclidean manifold. There are two reasons for this; firstly, since the path integral is not discussed, there is no particular

advantage in going to Euclidean time. Secondly, using Minkowski spacetime allows one to introduce Minkowski Dirac gamma matrices and which in turn allows the reader to connect with the vast literature on the Dirac equation that almost always use the Minkowski formulation.

This Chapter prepares the reader for Chapter **??** in which the Dirac equation is analyzed using the formalism of fermionic calculus.

## 5.1 The Dirac equation

In Minkowski space the relativistic wave equation is given by

$$\left( \frac{\partial^2}{\partial (ct)^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \left(\frac{mc}{\hbar}\right)^2 \right) \psi(t, \vec{x}) = 0$$

The natural length that enters the wave equation, consistent with relativity and quantum mechanics, is called the Compton wavelength,and for an electron with mass $m_e$ is given by

$$\frac{\hbar}{m_e c} = 2.426 \times 10^{-12} \text{ meters} \quad : \quad \text{Electron's Compton wavelength}$$

The scale of the Compton wavelength is much smaller than the typical size of an atom, which is about $10^{-10}$ meters. Hence, one expects that a relativistic and quantum mechanical particle, since it is much smaller than the atom, is a candidate for being a fundamental particle; the leading example of a fundamental particle is the (relativistic) electron.

In covariant notation

$$\left(\hbar^2 \partial_\mu \partial^\mu + m^2 c^2\right) \psi(x) = 0 \tag{5.1.1}$$

where, for four vectors $A_\mu, B_\mu$, with $\mu = 0, 1, 2, 3$, one has

$$x_\mu = (ct, \vec{x}) \quad ; \quad A_\mu B^\mu \equiv A_\mu B_\nu \eta^{\mu\nu} \quad ; \quad \eta^{\mu\nu} = diag(1, -1, -1, -1) = \eta_{\mu\nu}$$

Dirac felt that the second order time derivative $\partial^2/\partial t^2$ in the relativistic wave equation spoils the main mathematical beauty of the Schrödinger equation – which in Dirac's view is that it has only a first order time derivative. To address this issue, Dirac chose to factor the wave operator in the following manner

$$\hbar^2 \partial_\mu \partial^\mu + m^2 c^2 = (+i\hbar \gamma_\mu \partial^\mu + mc)\,(-i\hbar \gamma_\mu \partial^\mu + mc) \tag{5.1.2}$$

The quantities $\gamma_\mu$ are not real numbers but are, instead, 4×4 matrices with the following property

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \equiv \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

There are infinitely many equivalent representations of the Dirac matrices. A representation particularly suited for studying the Hilbert space of the Dirac field is, in block 2×2 notation, the following

$$\vec{\gamma} = \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \;\; ; \;\; \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.1.3}$$

where $\vec{\sigma}$ are $2 \times 2$ Pauli matices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;\; ; \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;\; ; \;\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence, from Eq. 5.1.2

$$\begin{aligned} &(+i\hbar\gamma_\mu\partial^\mu + mc)\,(-i\hbar\gamma_\mu\partial^\mu + mc) \\ &= \hbar^2\gamma_\mu\partial^\mu\gamma_\nu\partial^\nu + m^2c^2 = \hbar^2\frac{1}{2}\{\gamma_\mu,\gamma_\nu\}\partial^\mu\partial^\nu + m^2c^2 \\ &= \hbar^2\eta_{\mu\nu}\partial^\mu\partial^\nu + m^2c^2 = \hbar^2\partial_\mu\partial^\mu + m^2c^2 \end{aligned}$$

and one has recovered the left hand side of Eq. 5.1.2.

The function $\psi(t,\vec{x})$ given in Eq. 5.1.1 is seen, from Eq. 5.1.2, to satisfy the following

$$(-i\hbar\gamma_\mu\partial^\mu + mc)\,\psi(t,\vec{x}) = 0 \quad : \quad \text{Dirac equation} \tag{5.1.4}$$

Setting $\hbar = 1 = c$ and writing out all the components of the Dirac equation yields

$$\sum_{\beta=1}^{4}\left(-i\gamma_{0,\alpha\beta}\frac{\partial}{\partial t} + i\vec{\gamma}_{\alpha\beta}\cdot\vec{\partial} + m\delta_{\alpha\beta}\right)\psi_\beta(t,\vec{x}) = 0 \tag{5.1.5}$$

The Dirac equation is not a relativistic Schroedinger equation since $\psi(t,\vec{x})$ is not a probability amplitude. Instead, the Dirac equation is a fundamental *field equation* that describes electrons and positrons – analogous to Maxwell's field equation for the electromagnetic field $A_\mu(t,\vec{x})$.

The Dirac field has four components, namley $\psi_\alpha(t,\vec{x})$ , $\alpha = 1,..,4$. Every electron has spin 1/2 and hence needs two independent fermionic variables for its description; its antiparticle, namely the positron is also spin 1/2 and needs another two independent fermionic variables. Hence, as expected, a Dirac fermion $\psi_\alpha(t,\vec{x})$ has in total four fermionic degrees of freedom and is described by the four-component complex fermionic field.

Furthermore, since a spin 1/2 field describes fermions obeying the Pauli exclusion principle, the Dirac field $\psi(t,\vec{x})$ is an anticommuting fermionic field variable – in contrast to the electromagnetic field $A_\mu(t,\vec{x})$, which is described a bosonic (real) field variable.

## 5.2 Dirac Lagrangian and Hamiltonian

The Dirac Lagrangian can be obtained from the Dirac equation. The Dirac equation given in Eq. 5.1.4 can be considered to be the result of varying a classical action. Considering $\bar{\psi}$ and $\psi$ to be independent fields, the Lagrangian $\mathcal{L}(\bar{\psi}, \psi)$ is defined by

$$\begin{aligned} &\frac{\delta}{\delta\bar{\psi}} \int d^4x \mathcal{L}(\bar{\psi}, \psi) = 0 \\ &\Rightarrow \frac{\delta\mathcal{L}(\bar{\psi}, \psi)}{\delta\bar{\psi}} = 0 = (-i\gamma_\mu\partial^\mu + m)\,\psi(t, \vec{x}) \end{aligned} \tag{5.2.1}$$

which yields

$$\mathcal{L} = -\bar{\psi}(-i\gamma_\mu\partial^\mu + m)\psi = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi \tag{5.2.2}$$

The negative sign for the Dirac Lagrangian is taken to be consistent the rules of fermion Gaussian integration given in Section 6.12.1. The Dirac action is given by

$$S = \int d^dx\mathcal{L} = \int d^dx\bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi \tag{5.2.3}$$

The Dirac Hamiltonian is obtained in a manner similar to the bosonic case using the procedure of Lagrangian mechanics. Consider a boson degree of freedom $q$ with Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - V(q)$$

The canonical momentum $p$ and Hamiltonian $H$ for the system are defined by

$$\begin{aligned} p &= \frac{\partial\mathcal{L}}{\partial\dot{q}} = m\dot{q} \\ H &= \dot{q}p - \mathcal{L} = \frac{1}{2}\dot{q}^2 + V(q) \end{aligned}$$

The canonical momentum of the Dirac field $\Pi_\alpha$, for $\dot{\psi} = \partial\psi/\partial t$, is defined by

$$\Pi_\alpha = \frac{\delta\mathcal{L}}{\delta\dot{\psi}_\alpha} = i[\bar{\psi}\gamma_0]_\alpha \tag{5.2.4}$$

where the anticommuting property has been used to obtain $\Pi_\alpha$. Define the conjugate field by

$$\psi^\dagger \equiv \bar{\psi}\gamma_0$$

The Dirac Hamiltonian density, using the anticommuting property of the Dirac field is given by

$$\begin{aligned}\mathcal{H}(\vec{x}) &= \frac{\partial\psi_\alpha}{\partial t}\Pi_\alpha - \mathcal{L} = \psi^\dagger(i\gamma_0\vec{\gamma}\cdot\vec{\partial} + m\gamma_0)\psi \\ \Rightarrow \mathcal{H}(\vec{x}) &= \psi^\dagger(-i\vec{\alpha}\cdot\vec{\partial} + \beta m)\psi\end{aligned}$$

where, in block 2×2 notation

$$\vec{\alpha} = -\gamma_0\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad ; \quad \beta = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the canonical formalism, to obtain the Dirac Hamiltonian density from the Lagrangian, the only change is to remove the time derivative term from the Dirac Lagrangian and reverse the sign of the remaining term. The canonical formalism reflects the fact that the time derivative term in the Dirac Lagrangian, is due to the fermionic Hilbert space and is not a part of the Hamiltonian. A similar dropping of the time derivative term was required for obtaining the transfer matrix for a simpler fermionic case, discussed in Baaquie (2014).

The Dirac Lagrangian is written in phase space variables and is analogous to the formulation of bosonic system in phase space, with a derivation given in Baaquie (2014).

The Hamiltonian for the Dirac field is given by

$$\begin{aligned}H &= \int d^3x\mathcal{H}(\vec{x}) = \int d^3x\psi^\dagger(-i\vec{\alpha}\cdot\vec{\partial} + \beta m)\psi \\ &= \int d^3x\psi^\dagger h\psi \end{aligned} \tag{5.2.5}$$

where the matrix $h$ is given by

$$h = i\gamma_0\vec{\gamma}\cdot\vec{\partial} + \gamma_0 m = -i\vec{\alpha}\cdot\vec{\partial} + \beta m \tag{5.2.6}$$

The fields $\psi^\dagger(\vec{x}), \psi(\vec{x})$ are considered to be time independent Schrödinger *operators*. To quantize the Dirac Hamiltonian given in Eq. 5.2.5, the matrix $h$ has to be diagonalized so that one can ascertain the normal modes of $H$. To quantize $H$ canonical anticommutation relations are imposed on the normal modes.

## 5.3 Euclidean Dirac Lagrangian

The formulation of the Dirac field in Euclidean time is more suitable for studies based on the path integral and for studying the lattice Dirac field.

The Euclidean formulation is briefly outlined. Let $\tau$ denote Euclidean time and $\gamma_\mu^E$ denote Euclidean gamma matrices. Define

$$(t,\vec{x}) \to (\tau,\vec{x}) \;\; ; \;\; t = -i\tau \;\; ; \;\; \gamma_0^E = \gamma_0 \;\; ; \;\; \gamma_i^E = i\gamma_i \; ; \; i = 1,2,3$$

Eq. 5.2.2 yields

$$\begin{aligned}
&\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi = \bar{\psi}(i\gamma_0\partial^0 - i\gamma_i\partial^i - m)\psi \\
&\mathcal{L} \to \mathcal{L}_E = -\left(\gamma_0^E\frac{\partial}{\partial\tau} + \sum_{i=1}^{3}\gamma_i^E\frac{\partial}{\partial x_i} + m\right)\psi(\tau,\vec{x}) \\
&\Rightarrow \mathcal{L}_E = -\bar{\psi}(\gamma_\mu^E\partial_\mu + m)\psi \quad : \text{Euclidean Dirac Lagrangian} \qquad (5.3.1)
\end{aligned}$$

The Euclidean Dirac action is given by

$$\mathcal{S}_E = i\mathcal{S} = \int d\tau d\vec{x}\mathcal{L}_E(\tau,\vec{x}) = -\int d^4x\bar{\psi}(\gamma_\mu^E\partial_\mu + m)\psi \qquad (5.3.2)$$

The gamma matrices obey the anticommutation equation

$$\{\gamma_\mu^E,\gamma_\nu^E\} = 2\delta_{\mu\nu} \;\; ; \;\; (\gamma_\mu^E)^\dagger = \gamma_\mu^E \quad : \text{Hermitian}$$

An explicit representation of the Euclidean gamma matrices, from Eq. 5.1.3, is given by

$$\vec{\gamma}^E = i\begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \;\; ; \;\; \gamma_0^E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (5.3.3)$$

## 5.4 Plane Wave Solutions

To diagonalize the matrix $h$ given in Eq. 5.2.6, it's eigenstates have to be computed. Similar to non-interacting bosonic systems, the Dirac equation also has plane waves as eigenstates. Due to the particle and antiparticle content of the Dirac field, and the fact that they both have spin 1/2, the plane wave solutions are a four dimensional column vectors.

The eigenstates of $h$ have the form

$$u(\vec{p})e^{i\vec{p}\cdot\vec{x}} \qquad (5.4.1)$$

where

$$hu(\vec{p}) = (\vec{\alpha}\cdot\vec{p} + m\beta)u(\vec{p}) = Eu(\vec{p}) \qquad (5.4.2)$$

Note

$$h = \vec{\alpha}\cdot\vec{p} + m\beta = \begin{pmatrix} m & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -m \end{pmatrix} \qquad (5.4.3)$$

and

$$h^2 = \begin{pmatrix} m^2 + \vec{p}^2 & 0 \\ 0 & m^2 + \vec{p}^2 \end{pmatrix} \tag{5.4.4}$$

Hence

$$h^2 u_0 = E^2 u_0 \Rightarrow E^2 = m^2 + \vec{p}^2 \tag{5.4.5}$$

and yields two branches for energy given by

$$E = \pm\sqrt{m^2 + \vec{p}^2} \tag{5.4.6}$$

The two branches of energy are shown in Figure 5.1.

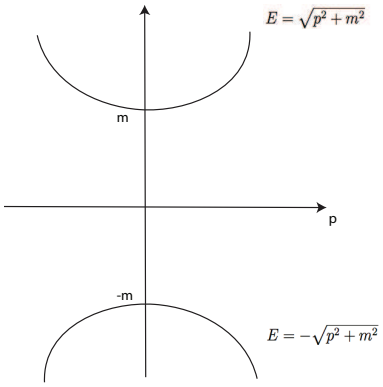


Figure 5.1 Two branches of the energy of a single particle with momentum $p$.

Define $\omega$ to be the positive energy solution given by

$$\omega = +\sqrt{m^2 + \vec{p}^2} \tag{5.4.7}$$

Four linearly independent four-dimensional vectors are the following

$$|u_0^{(1)}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |u_0^{(2)}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|u_0^{(3)}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |u_0^{(4)}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.8}$$

The positive and negative energy plane wave solutions can be obtained by

the following procedure. Define

$$u_{\pm}(\vec{p}) = (h \pm \omega)u_0 \tag{5.4.9}$$

Then

$$\begin{aligned} hu_+(\vec{p}) &= (h^2 + \omega h)u_0 = \omega(\omega + h)u_0 = \omega u_+(\vec{p}) \\ &\Rightarrow E = \omega \end{aligned} \tag{5.4.10}$$

Similarly

$$\begin{aligned} hu_-(\vec{p}) &= (h^2 - \omega h)u_0 = \omega(\omega - h)u_0 = -\omega u_-(\vec{p}) \\ &\Rightarrow E = -\omega \end{aligned} \tag{5.4.11}$$

$(h \pm \omega)u_0$ yields eight eigenstates, of which only four are linearly independent. Two eigenstates are taken with positive energy $E = +\omega$ and another two with negative energy $E = -\omega$.

To explicitly determine $u_+$ note that

$$u_+ = (h + \omega)u_0 = \begin{pmatrix} m + \omega & \vec{p} \cdot \vec{\sigma} \\ \vec{p} \cdot \vec{\sigma} & \omega - m \end{pmatrix} u_0 \tag{5.4.12}$$

and

$$\vec{p} \cdot \vec{\sigma} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \tag{5.4.13}$$

The following choice (with $N$ being a normalization) is made in the literature. Let

$$|u^{(i)}(\vec{p})\rangle = N \left( \frac{h + \omega}{\omega + m} \right) |u_0^{(i)}\rangle \qquad , \qquad i = 1, 2$$

The components are the following

$$|u^{(1)}(\vec{p})\rangle = N \begin{pmatrix} 1 \\ 0 \\ p_3/(m + \omega) \\ (p_1 + ip_2)/(m + \omega) \end{pmatrix} \quad ; \quad |u^{(2)}(\vec{p})\rangle = N \begin{pmatrix} 0 \\ 1 \\ (p_1 - ip_2)/(m + \omega) \\ -p_3/(m + \omega) \end{pmatrix}$$

Similarly, the negative energy solutions are given by

$$|u^{(i)}(\vec{p})\rangle = N \left( \frac{h - \omega}{m + \omega} \right) |u_0^{(i)}\rangle \qquad , \qquad i = 3, 4$$

and yields

$$|u^{(3)}(\vec{p})\rangle = N \begin{pmatrix} p_3/(m+\omega) \\ (p_1+ip_2)/(m+\omega) \\ -1 \\ 0 \end{pmatrix} \; ; \; |u^{(4)}(\vec{p})\rangle = N \begin{pmatrix} (p_1-ip_2)/(m+\omega) \\ -p_3/(m+\omega) \\ 0 \\ -1 \end{pmatrix} \quad (5.4.14)$$

Note that the four eigenvectors are orthogonal, namely

$$\langle u^{(i)}|u^{(j)}\rangle \sim \delta_{i-j} \quad ; \quad i,j = 1,2,3,4$$

Two different normalization are commonly used for the $u^{(i)}(\vec{p})$.

1. The eigenfunctions are taken to be orthonormal and yield

$$\langle u^{(i)}|u^{(i)}\rangle = 1 \qquad \Rightarrow N_1 = \sqrt{(\omega+m)/2\omega}$$

With this normalization

$$S = (u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}) = \begin{pmatrix} N_+ & -N_-\vec{p}\cdot\vec{\sigma} \\ N_-\vec{p}\cdot\vec{\sigma} & N_+ \end{pmatrix} \quad ; \quad N_\pm = \sqrt{\frac{\omega \pm m}{2\omega}}$$

$N_1$ is the normalization that will be used in a later analysis of the Dirac field using fermion calculus.

2. For obtaining a relativistically invariant four-vector, the norm of the eigenvector, namely $u^*u$, is treated as the 0-th component of a four-vector; to achieve one needs

$$\langle u^{(i)}_{\vec{p}}|u^{(i)}_{\vec{p}}\rangle = \frac{\omega}{m} \quad \Rightarrow \quad N_2 = \sqrt{(\omega+m)/(2m)}$$

$N_2$ is the normalization is used in this Chapter.

In summary, for $|E| = \omega > 0$, taking $V = \int d^3x$ to be the finite volume of space, for the system with normalization $N_2$ one obtains the eigenfunctions of the $h$ matrix as follows

$$U^{(i)}_+ = \sqrt{\frac{m}{\omega V}}\; u^{(i)}(\vec{p})e^{i\vec{p}\cdot\vec{x}} \quad ; \quad i = 1,2 \qquad (5.4.15)$$

and

$$U^{(i)}_- = \sqrt{\frac{m}{\omega V}}\; u^{(i)}(\vec{p})e^{i\vec{p}\cdot\vec{x}} \quad ; \quad i = 3,4 \qquad (5.4.16)$$

with normalization for all the plane waves given by

$$\int_V U^*_i(\vec{x})U(\vec{x})_j d^3x = \delta_{i-j} \qquad (5.4.17)$$

For $V \to \infty$, the allowed spectrum for $E^2$ is given by $\sqrt{\vec{p}^2+m^2}$, with

$0 < \vec{p}^2 < \infty$. Hence, the positive-energy free Dirac particle have $m \leq E \leq \infty$, whereas for the negative energy solution have $-\infty \leq E \leq -m$

## 5.5 Quantization of the Dirac Field

Consider the normal mode expansion of the Dirac field operators

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{i=1}^{4} \sqrt{\frac{m}{\omega_{\vec{p}}}} e^{i\vec{p}\cdot\vec{x}} u^{(i)}(\vec{p}) b_{\vec{p}}^{(i)}$$

$$\psi^\dagger(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{i=1}^{4} \sqrt{\frac{m}{\omega_{\vec{p}}}} e^{-i\vec{p}\cdot\vec{x}} u^{(i)*}(\vec{p}) b_{\vec{p}}^{(i)\dagger} \tag{5.5.1}$$

where $b_{\vec{p}}^{(i)}, b_{\vec{p}}^{(i),\dagger}$ are the normal modes. If the values of the coefficients $b_{\vec{p}}^{(i)}, b_{\vec{p}}^{(i),\dagger}$ are fixed to have some definite fixed values, the fields $\psi^\dagger(\vec{x}), \psi(\vec{x})$ can be thought of as classical fields having some fixed (fermionic) value.

Since the plane wave solutions $u^{(i)}(\vec{p})$ diagonalize the Hamiltonian density, one obtains the following diagonal form for the Dirac Hamiltonian

$$H = \sum_{\vec{p}} \left[ \sum_{i=1,2} \omega b_{\vec{p}}^{(i)\dagger} b_{\vec{p}}^{(i)} - \sum_{j=3,4} \omega b_{\vec{p}}^{(j)\dagger} b_{\vec{p}}^{(j)} \right] \tag{5.5.2}$$

In the old 'naive' interpretation of Dirac, the positive part of the $H$ refers to electron creation and destruction operators with positive energy $\omega$. The negative part of $H$ was thought as coming from the creation of negative energy $-\omega$ states by creation operator $b_{\vec{p}}^{(j)\dagger}$, and $b_{\vec{p}}^{(j)\dagger}|\Omega\rangle = 0$ was taken to mean that all states with energy $-\omega = -\sqrt{p^2 + m^2}$ were occupied. A hole in the sea could be thought of as an antielectron. However, with the introduction of the idea of the positron as a particle in its own right, the idea of the negative sea of electron can be dispensed with and $H$ is rewritten entirely in terms of the electron and antielectron operators.

In the Hamiltonian and state space formulation, the field $\psi(t, \vec{x})$, for each value of $t, \vec{x}$ is a Heisenberg *operator* on state space; equivalently the coefficients $b_{\vec{p}}^{(i)}$ in Eq. 5.5.1 become operators for each $t, \vec{p}$ and each $i$.

The Dirac field $\psi(t, \vec{x})$ is an anticommuting field satisfying the Pauli exclusion principle. The **canonical quantization** of the Dirac field is based on imposing equal time anti-commutation relations for the normal mode coefficients, namely

$$\{b_{\vec{p}}^{(i)}, b_{\vec{p}'}^{(i')\dagger}\} = \delta_{i-i'} \delta_{\vec{p}-\vec{p}'} \tag{5.5.3}$$
$$\{b, b\} = 0 = \{b^\dagger, b^\dagger\}$$

From the anticommutation given above and Eq. 5.5.2 for the Hamiltonian, one would naively think, similar to the case of bosons, that the vacuum (ground state) $|\Phi_0\rangle$ of the Dirac field should be defined so that it is annihilated by all the destruction operators $b^{(i)}_{\vec{p}}$, namely

$$b^{(i)}_{\vec{p}}|\Phi_0\rangle = 0 \quad : \quad \text{False vacuum}$$

The problem with this vacuum is that it is unstable since the (second) negative energy term in the Hamiltonian in Eq. 5.5.2 implies that the field can keep on lowering it's energy by having more and more negative energy electrons – leading to a vacuum state with infinite negative energy. The proper Dirac vacuum is defined in the next Section.

The time dependent Dirac field $\psi(t,\vec{x})$ is considered to be a Heisenberg time dependent operator. The normal mode expansion is given by[1]

$$\psi(t,\vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{p}}\sum_{i=1}^{4}\sqrt{\frac{m}{\omega_{\vec{p}}}}e^{i\vec{p}\cdot\vec{x}}u^{(i)}(\vec{p})b^{(i)}_{\vec{p}}(t) \tag{5.5.4}$$

The Dirac Heisenberg field operators are quantized by considering the fields $\psi^\dagger(t,\vec{x}), \psi(t,\vec{x})$ to be Heisenberg operators with a plane wave expansion given by considering the normal mode coefficients $b^{(i)}_{\vec{p}}(t)$ to be time-dependent *Heisenberg operators*.

The Heisenberg equations of motion yield

$$\frac{\partial b^{(i)}_{\vec{p}}(t)}{\partial t} = i[H, b^{(i)}_{\vec{p}}(t)] = \begin{cases} -i\omega b^{(i)}_{\vec{p}}(t) & i = 1,2 \\ i\omega b^{(i)}_{\vec{p}}(t) & i = 3,4 \end{cases} \tag{5.5.5}$$

and similarly for $b^{(i)\dagger}_{\vec{p}}(t)$. Since the Heisenberg equations of motion are linear for the Dirac Hamiltonian given in Eq. 5.5.2, they can be solved using exponential functions and yield

$$b^{(i)}_{\vec{p}}(t) = \begin{cases} e^{-i\omega t}b^{(i)}_{\vec{p}} & i = 1,2 \\ e^{i\omega t}b^{(i)}_{\vec{p}} & i = 3,4 \end{cases} \tag{5.5.6}$$

where $b^{(i)}_{\vec{p}}$ are time-independent Schrödinger operators. Hence

$$\psi(t,\vec{x}) = \frac{1}{V}\sum_{\vec{p}}\sqrt{\frac{m}{\omega}}\left(\sum_{i=1,2} b^{(i)}_{\vec{p}}u^{(i)}_{\vec{p}}e^{i(\vec{p}\cdot\vec{x}-\omega t)} + \sum_{j=3,4} b^{(j)}_{\vec{p}}u^{(j)}_{\vec{p}}e^{i(\vec{p}\cdot\vec{x}+\omega t)}\right) \tag{5.5.7}$$

[1] The field $\psi(t,\vec{x})$ is a **quantum field**. In the path integral formulation the quantum field $\psi(t,\vec{x})$ has **all possible fermionic values** and, for each value of $t,\vec{x}$, the field $\psi(t,\vec{x})$ is an independent fermionic integration variable [Baaquie (2014)].

The plane wave solution that were obtained for diagonalizing the Dirac Hamiltonian can be shown to be the following

$$\begin{aligned}\psi_{\text{plane wave}}(t,\vec{x}) &= \langle\Phi_0|\psi(t,\vec{x})|b^{(i)\dagger}_{\vec{p}}\Phi_0\rangle \\ &= \sqrt{\frac{m}{\omega}}u^{(i)}(\vec{p})e^{i(\vec{p}\cdot\vec{x}-\omega t)}\end{aligned} \tag{5.5.8}$$

where $|\Phi_0\rangle$ is the Dirac vacuum.

The charge and momentum operators for the Dirac field are given by

$$Q = e\int \psi^\dagger\psi d^3x = e\sum_{\vec{p}}\sum_{i=1}^{4} b^{(i)\dagger}_{\vec{p}}b^{(i)}_{\vec{p}} \tag{5.5.9}$$

and

$$\vec{P} = -i\int \psi^\dagger\vec{\nabla}\psi = \sum_{\vec{p}}\sum_{i=1}^{4}\vec{p}\,b^{(i)\dagger}_{\vec{p}}b^{(i)}_{\vec{p}} \tag{5.5.10}$$

## 5.6 Electron and positron operators

When Dirac discovered his equation, the concept of antiparticles was absent in physics. The 'negative energy' states were interpreted by Dirac to be negative energy electron states. No negative energy electrons have ever been observed; in fact, for a particle that is moving with only kinetic energy, a negative energy state means that it must have imaginary velocity and hence is physically forbidden. Dirac postulated that the vacuum state consists of a sea of all negative energy electron states being occupied; Dirac then identified the concept of an antiparticle state with the *absence* – from the vacuum state – of a negative energy electron state, called a hole.

*Positron (Hole) Theory*

In the hole theory of the positron, as originally conceived by Dirac, the *absence* of electron with $-\omega, \vec{p}, s$ is equivalent to the *presence* of positron with $\omega, -\vec{p} - s$. The Table 5.6 given below summarizes the hole theory of positrons.

| | $\mathcal{Q}$ | E | P | Spin |
|---|---|---|---|---|
| Electron state $E<0$ | $-\lvert e\rvert$ | $-\lvert E\rvert$ | $\vec{p}$ | $\pm\frac{\hbar}{2}$ |
| Positron | $\lvert e\rvert$ | $\lvert E\rvert$ | $-\vec{p}$ | $\mp\frac{\hbar}{2}$ |

Thus charge conjugation takes the plane wave solution with $|\omega|$, $\vec{p}$ to a solution with $-|\omega|$ and $-\vec{p}$. The hole-theory states that the absence of $\psi^c$ is equal to hole(positron) with energy $+\omega$ and momentum $\vec{p}$. On quantizing the Dirac field,it will be seen that under charge conjugation the electron $\rightleftarrows$ positron.

### *5.6.1 Modern interpretation*

In the modern interpretation of the Dirac field there are no negative energy electron states and one dispenses with the hole-theory that requires the negative sea of electron. Instead, the Dirac field is understood to consist of two types of particles, namely electrons and positron(antielectrons). All particles have positive energy and the vacuum simply consists of no electrons and no positrons.

To precisely define electrons and positrons in the Dirac Hamiltonian, the following redefinitions are made of the creation and destruction operators that appear in the system. Define the *electron* creation and destruction operators to be

$$i = 1, 2 \;\; ; \;\; b^{(i)\dagger}_{\vec{p}} \;\; ; \;\; b^{(i)\dagger}_{\vec{p}} \qquad \delta_{s-s'}\delta_{\vec{p}-\vec{q}} = \{b^{(s)\dagger}_{\vec{p}}, b^{(s')}_{\vec{q}}\} \tag{5.6.1}$$

The *positron* creation and destruction operators are defined as

$$d^{(1)\dagger}_{\vec{p}} = -b^{(4)}_{-\vec{p}} \;\; ; \;\; d^{(2)\dagger}_{\vec{p}} = b^{(3)}_{-\vec{p}} \;\; ; \;\; \{d^{(s)\dagger}_{\vec{p}}, d^{(s')}_{\vec{q}}\} = \delta_{s-s'}\delta_{\vec{p}-\vec{q}} \tag{5.6.2}$$

The equal time anticommutation relations, from Eq. 5.5.3 are the following

$$\{b^{(s)}_{\vec{p}}, b^{(s')\dagger}_{\vec{p'}}\} = \delta_{s-s'}\delta_{\vec{p}-\vec{p'}} = \{d^{(s)}_{\vec{p}}, d^{(s')\dagger}_{\vec{p'}}\} \;\; ; \;\; s, s' = 1, 2 \tag{5.6.3}$$
$$\{d, d\} = 0 = \{d^\dagger, d^\dagger\} \;\; ; \;\; \{b, d\} = 0 = \{b, d^\dagger\} \;\; ; \;\; \{b, b\} = 0 = \{b^\dagger, b^\dagger\}$$

To identify the electron and positron creation and destruction we need to examine the charge operator $\mathcal{Q}$. In anticipation of future result, note that

$b^{(i)\dagger}_{\vec{p}}$ : electron creation operator with positive energy $+\,\omega$
$b^{(i)}_{\vec{p}}$ : electron destruction operator with positive energy $+\,\omega$
$d^{(i)\dagger}_{\vec{p}}$ : positron creation operator with positive energy $+\,\omega$
$d^{(i)}_{\vec{p}}$ : positron destruction operator with positive energy $+\,\omega$

Hence, the vacuum state $|0\rangle$ is taken to have *no* electrons or positrons; that is

$$b^{(i)}_{\vec{p}}|0\rangle = 0 = d^{(i)}_{\vec{p}}|0\rangle \tag{5.6.4}$$

### 5.6.2 Hamiltonian, Charge and Momentum Operators

In terms of the electron and positron creation and annihilation operators

$$\begin{aligned} H &= \sum_{\vec{p}} \sum_{s} \omega_{\vec{p}} \left( b_{\vec{p}}^{(s)\dagger} b_{\vec{p}}^{(s)} - d_{-\vec{p}}^{(s)} d_{-\vec{p}}^{(s)\dagger} \right) \\ &= \sum_{\vec{p},s} \omega_{\vec{p}} \left( b_{\vec{p}}^{(s)\dagger} b_{\vec{p}}^{s} + d_{\vec{p}}^{(s)\dagger} d_{\vec{p}}^{(s)} - 1 \right) \end{aligned} \tag{5.6.5}$$

The Dirac vacuum, as defined in Eq. 5.6.4, is a state with no electrons and no positrons, namely

$$b_{\vec{p}}^{(s)}|0\rangle = 0 = d_{\vec{p}}^{(s)}|0\rangle$$

The energy $E_0$ of the Dirac vacuum, from Eq. 5.6.5, is given by

$$\begin{aligned} & H|0\rangle = E_0|0\rangle \\ \Rightarrow\ & E_0 = -\sum_{\vec{p},s} \omega_{\vec{p}} = -2 \sum_{\vec{p}} \omega_{\vec{p}} \end{aligned} \tag{5.6.6}$$

In terms of electron and positron operators, the charge $Q$ operator is given by

$$\begin{aligned} Q &= e \sum_{\vec{p}} \sum_{i=1}^{4} b_{\vec{p}}^{(i)\dagger} b_{\vec{p}}^{(i)} \\ &= e \sum_{\vec{p}} \left( \sum_{i=1}^{2} b_{\vec{p}}^{(i)\dagger} b_{\vec{p}}^{(i)} + \sum_{j=3}^{4} b_{\vec{p}}^{(i)\dagger} b_{\vec{p}}^{(i)} \right) \\ &= e \sum_{\vec{p}} \sum_{s=1,2} (b_{\vec{p}}^{(s)} b_{\vec{p}}^{(s)\dagger} + d_{-\vec{p}}^{(s)} d_{-\vec{p}}^{(s)\dagger}) \\ &= e \sum_{\vec{p}} \sum_{s=1,2} (b_{\vec{p}}^{(s)} b_{\vec{p}}^{(s)\dagger} - d_{\vec{p}}^{(s)\dagger} d_{\vec{p}}^{(s)} + 1) \end{aligned}$$

Defining $N_{\vec{p},s}^{-} = b_{\vec{p}}^{(s)\dagger} b_{\vec{p}}^{(s)}$ for electrons and similarly $N_{\vec{p},s}^{+} = d_{\vec{p}}^{(s)+} d_{\vec{p}}^{(s)}$ for positrons yields, for $H, Q$ and momentum $\vec{P}$ operators, the following

$$\begin{aligned} H &= \sum_{p,s} \omega_{\vec{p}} \left( N_{p,s}^{-} + N_{p,s}^{+} - 1 \right) \\ Q &= -|e| \sum_{p,s} \left( N_{ps}^{-} - N_{ps}^{+} + 1 \right) \\ \vec{P} &= \sum_{\vec{p},s} \vec{p} \left( N_{ps}^{-} + N_{ps}^{+} \right) \end{aligned} \tag{5.6.7}$$

### 5.6.3 Field operators

The spinors for electrons are column vectors

$$u_{\vec{p}}^{(1)} = u^{(1)}(\vec{p}) \;\; ; \;\; u_{\vec{p}}^{(2)} = u^{(2)}(\vec{p}) \tag{5.6.8}$$

and the spinors for the positron are

$$v_{\vec{p}}^{(1)} = -u^{(4)}(-\vec{p}) \;\; ; \;\; v_{\vec{p}}^{(2)} = u^{(3)}(-\vec{p}) \tag{5.6.9}$$

The definition are chosen so that the annihilation of a negative-energy electron of $-\vec{p}$ and spin-down appears as the creation of a positron with $\vec{p}$ and spin-up. The normalization of the spinors is

$$\begin{aligned} u_{\vec{p}}^{(s)\dagger} u_{\vec{p}}^{(s)} &= \delta_{s-s'}(\frac{\omega}{m}) = v_{\vec{p}}^{(s)\dagger} v_{\vec{p}}^{(s')} \\ v^{(s')\dagger}(-\vec{p}) u^{(s')}(\vec{p}) &= 0 = u^{(s')\dagger}(-\vec{p}) v^{(s)}(\vec{p}) \end{aligned} \tag{5.6.10}$$

The expansion of the Dirac field is given by

$$\begin{aligned} \psi(t,\vec{x}) &= \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{s=1,2} \sqrt{\frac{m}{\omega_{\vec{p}}}} \left[ b_{\vec{p}}^{(s)} u^{(s)}(\vec{p}) e^{i(\vec{p}\cdot\vec{x}-\omega t)} + d_{\vec{p}}^{(s)\dagger} v^{(s)}(\vec{p}) e^{-i(\vec{p}\cdot\vec{x}-\omega t)} \right] \\ &\quad : \text{annihilates electrons and creates positrons} \\ \bar{\psi}(t,\vec{x}) &= \psi^\dagger \gamma_0 \\ &= \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{s=1,2} \sqrt{\frac{m}{\omega_{\vec{p}}}} \left[ d_{\vec{p}}^{(s)} \bar{v}^{(s)}(\vec{p}) e^{i(\vec{p}\cdot\vec{x}-\omega t)} + b_{\vec{p}}^{(s)\dagger} \bar{u}^{(s)}(\vec{p}) e^{-i(\vec{p}\cdot\vec{x}-\omega t)} \right] \\ &\quad : \text{annihilates positrons and creates electrons} \end{aligned} \tag{5.6.11}$$

The quantization of the Dirac field yields, from Eq. 5.6.2, the following canonical equal time anti-commutation equation

$$\{\psi_\alpha(t,\vec{x}), \psi_\beta^\dagger(t,\vec{x}')\} = \delta_{\alpha\beta}\delta_{\vec{x}-\vec{x}'} \tag{5.6.12}$$

The unequal time anti-commutator can be computed from the expansion of the fields $\psi(t,\vec{x})$ and $\psi^+(t',\vec{x}')$ and yields

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\} = 0 \;\; ; \;\; x = (t,\vec{x}) \; ; \; x' = (t,\vec{x}') \tag{5.6.13}$$

if

$$(x-x')^2 = (\vec{x}-\vec{x}') - c^2(t-t')^2 > 0 \qquad : \qquad \text{spacelike} \tag{5.6.14}$$

Note for spacelike separation of $x, x'$ of the fermion fields one has

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')] \neq 0 \tag{5.6.15}$$

One may naively think that this would imply violation of causality since

physical operators with spacelike separation, being causally independent, must commute. However, this is not the case as Eq. 5.6.15 does not imply a violation of relativity since $\psi$ and $\bar{\psi}$ are not directly measurable. What can be measured is the charge and current density of electrons and positrons given by

$$j_\mu(x) = ie\bar{\psi}\gamma_\mu\psi - ie\langle\Phi_0|\bar{\psi}\gamma_\mu\psi|\Phi_0\rangle \tag{5.6.16}$$

Using

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

it can be shown that

$$[j_\mu(x), j_\nu(x^{'})] = 0 \qquad \text{if} \qquad (x - x')^2 > 0$$

and hence operators representing physically measurable properties of a system are indeed causally independent.

## 5.7 Charge Conjugation

The Dirac field contains both particles and antiparticles. Since one of the unique and remarkable feature of the Dirac field is the concept of antiparticles, it is of great interest to understand this new construction of theoretical physics in some detail. Hence, we now take a closer look at the properties of antiparticles using the concept of charge conjugation.

Charge conjugation is the operation that exchanges electrons and positrons. In the Dirac field, there is no explicit difference between the particle and it's antiparticle. Hence one expects that the properties of the Dirac field be should be unchanged if one exchanges particles with antiparticles, and which indeed is the case.

The appearance of charge $e$ occurs in the Dirac equation only when the electron is coupled to the electromagnetic field

$$A_\mu = (A_0, \vec{A}) \tag{5.7.1}$$

If one couples the Dirac field to the electromagnetic field, charge is able to differentiate electrons from the positrons since they are oppositely charged. Hence, under charge conjugation, one expects that the transformed field should have the same coupling to the electromagnetic field as before, but with charge $e$ replaced by the opposite charge $-e$.

To study charge conjugation, the Dirac field is coupled to the electromagnetic field using the procedure of minimal coupling, which is effected by

replacing $\partial_\mu$ by $\partial_\mu - ieA_\mu$. Minimal coupling maintains local gauge invariance for the coupled theory. Hence, from Eq. 5.1.4, the Dirac equation with coupling to the electromagnetic field is given by

$$-i(\frac{\partial}{\partial x_\mu} - ieA_\mu)\gamma^\mu\psi + m\psi = 0 = (\gamma_\mu(-i\partial^\mu - eA^\mu) + m)\,\psi \quad (5.7.2)$$

One expects that the electron in presence of $A_\mu$ gauge field with charge $e$ should be equivalent to a positron with charge $-e$.

To study this question, the Dirac equation is analyzed to ascertain whether there is an equivalent 'charge conjugated' field $\psi^c$ – that satisfies Eq. 5.7.2 but with $e$ replaced by $-e$, as given below

$$\Big(-i\frac{\partial}{\partial x_\mu} + eA_\mu\Big)\gamma^\mu\psi^c + m\psi^c = 0 \quad (5.7.3)$$

Complex conjugating Eq. 5.7.2 yields

$$\left(\gamma^*_\mu(i\partial^\mu - eA^\mu) + m\right)\psi^* = 0 \quad (5.7.4)$$

Let

$$\psi^c = C\psi^* \quad : \quad \text{charge conjugation} \quad (5.7.5)$$

We verify that Eq. 5.7.4 is obtained from the conjugate equation Eq. 5.7.3 by choosing charge conjugation matrix $C$, given in Eq. 5.7.5, that satisfies

$$(C)^{-1}\gamma_\mu C = -\gamma^*_\mu \;\; ; \;\; \mu = 0, 1, 2, 3 \quad (5.7.6)$$

From Eqs. 5.7.3 and 5.7.5

$$\Big(-i\partial_\mu + eA_\mu\Big)\gamma^\mu C\psi^* + mC\psi^* = 0 = \Big(-i\partial_\mu + eA_\mu\Big)C^{-1}\gamma^\mu C\psi^* + m\psi^*$$
$$\Rightarrow \left(\gamma^*_\mu(i\partial^\mu - eA^\mu) + m\right)\psi^* = 0$$

and we have obtained Eq. 5.7.4 as required.

In the representation we are working with it can be shown that

$$C = i\gamma_2 = C^* = C^{-1} = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \;\; ; \;\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\Rightarrow C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (5.7.7)$$

The definitions introduced in Eq. 5.6.9 for the spinors of the positron can be shown to be obtained by charge conjugation, namely

$$v^{(s)}(\vec{p}) = Cu^{(s)*}(\vec{p}) = i\gamma_2 u(\vec{p})^{(s)*} \;\; ; \;\; s = 1, 2 \quad (5.7.8)$$

The positron spinors $v^{(s)}(\vec{p})$ have a similar relation with the electron spinors. Since $C^{*-1} = C$, Eq. 5.7.8 yields

$$u^{(s)}(\vec{p}) = C^{*-1}v^{(s)*}(\vec{p}) = Cv^{(s)*}(\vec{p}) \tag{5.7.9}$$

showing that charge conjugation carried out twice yields the original system.

### *5.7.1 Example*

Consider $u^{(1)}(t, \vec{x})$, the positive energy $E = \omega > 0$ electron spinor. Applying charge conjugation to this state should yield the antiparticle state with positive energy $\omega$. From Eq. 5.7.8, the positron spinor is given by

$$v^{(1)}(\vec{p}) = Cu^{(1)*}(\vec{p}) = i\gamma_2 u^{(1)*}(\vec{p})$$

From Eqs. 5.4.14 and 5.7.7

$$\begin{aligned} v^{(1)}(\vec{p}) &= \sqrt{\frac{\omega+m}{2m}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ p_3/(\omega+m) \\ (p_1 - ip_2)/(\omega+m) \end{pmatrix} \\ &= \sqrt{\frac{\omega+m}{2m}} \begin{pmatrix} (p_1 - ip_2)/(\omega+m) \\ -p_3/(\omega+m) \\ 0 \\ 1 \end{pmatrix} \\ &= -u^{(4)}(-\vec{p}) \end{aligned} \tag{5.7.10}$$

where the last equation follows from Eq. 5.4.14 and we have obtained the definition given earlier in Eq. 5.6.9 for $v^{(1)}(\vec{p})$. Hence, under charge conjugation particle state with spin up, energy $\omega$ and momentum $\vec{p}$, namely $u^{(1)}(\vec{p})$, is mapped into $-u^{(4)}(-\vec{p}) = v^{(1)}(\vec{p})$, which is the state of an antiparticle with spin down, momentum $-\vec{p}$ and energy $\omega$.

Similarly, it can be shown by an explicit calculation that

$$v^{(2)}(\vec{p}) = Cu^{(2)*}(\vec{p}) = u^{(3)}(-\vec{p})$$

### *5.7.2 Charge conjugation of the Dirac field*

Charge conjugation defined in Eq. 5.7.5 is interpreted as an operator equation for carrying out charge conjugation for the quantized Dirac field. Charge conjugation is analyzed to show that for the quantized Dirac field, it is a symmetry that inter-changes particle with antiparticle, namely that under the operation of $\mathcal{C}$ electrons and positrons are exchanged.

Note that we have the following operator equation

$$\psi^\dagger = (\psi^*)^T \quad ; \quad \psi^* = (\psi^\dagger)^T$$

Hence, the operator interpretation of Eq. 5.7.5 is that the charge conjugated field is expressed follows

$$\psi^c = C\psi^* \quad \Rightarrow \quad \psi^c = C\psi(^\dagger)^T \tag{5.7.11}$$

From Eq. 5.6.11, Hermitian conjugation for the Dirac quantum field is defined on the operators $b^{(s)}_{\vec{p}}$ and $d^{(s)\dagger}_{\vec{p}}$. The spinors are complex conjugated and transposed by Hermitian conjugation and taking the transpose in Eq. 5.7.11 restores the spinors to column vectors. Hence, Eqs. 5.7.11 and 5.6.11 yield

$$\begin{aligned}\psi^c(t,\vec{x}) &= \frac{1}{V}\sum_{\vec{p},s}\sqrt{\frac{m}{\omega}}\left[b^{(s)\dagger}_{\vec{p}}Cu^{(s)*}_{\vec{p}}e^{-i(\vec{p}\cdot\vec{x}-\omega t)} + d^{(s)}_{\vec{p}}Cv^{(s)*}_{\vec{p}}e^{i(\vec{p}\cdot\vec{x}-\omega t)}\right] \\ &= \frac{1}{V}\sum_{\vec{p},s}\sqrt{\frac{m}{\omega}}\left[d^{(s)}_{\vec{p}}u^{(s)}_{\vec{p}}e^{i(\vec{p}\cdot\vec{x}-\omega t)} + b^{(s)\dagger}_{\vec{p}}v^{(s)}_{\vec{p}}e^{-i(\vec{p}\cdot\vec{x}-\omega t)}\right] \qquad (5.7.12) \\ &: \text{ annihilates positron and creates electrons}\end{aligned}$$

where Eqs. 5.7.8 and 5.7.9 have been used to obtain Eq. 5.7.12.

In the charge conjugated field $\psi^c(t,\vec{x})$, the spinor $v^{(s)}_{\vec{p}}$ is carried by the electron creation operator $b^{(s)\dagger}_{\vec{p}}$ and spinor $u^{(s)}_{\vec{p}}$ is carried by the positron destruction operator $d^{(s)}_{\vec{p}}$. Hence, as expected, the charge conjugated Dirac field $\psi^c(t,\vec{x})$, given in Eq. 5.7.12, has exchanged the particle and antiparticle content of the original Dirac field $\psi(t,\vec{x})$ given in Eq. 5.6.11.

From Eq. 5.7.12 one can also read off the effect of charge conjugation on the creation and annihilation operators. Denoting by $\mathcal{C}$ the unitary charge conjugation operator that acts on the operators of the Dirac field, we have the following results

$$\mathcal{C}: \qquad \mathcal{C}b^{(s)}_{\vec{p}}\mathcal{C}^{-1} = d^{(s)}_{\vec{p}} \qquad ; \qquad \mathcal{C}b^{(s)\dagger}_{\vec{p}}\mathcal{C}^{-1} = d^{(s)\dagger}_{\vec{p}}$$

Hence, it follows that

$$\begin{aligned}\mathcal{C}H\mathcal{C}^{-1} = H \quad &: \quad \text{invariant} \\ \mathcal{C}Q\mathcal{C}^{-1} = -Q \quad &: \quad \text{as expected.}\end{aligned}$$

## 5.8 Casimir Force for Fermions

The Casimir force is a result of the zero-point energy, which is the eigenenergy of the vacuum state of a quantum field. One can of course normal order

the Hamiltonian to remove the zero-point energy, but the effect of the energy of the vacuum reappears if one compares the difference of the energy between different vacuum states. One of the simplest way of studying the Casimir effect is to change the boundary conditions on the quantum field. The change in energy in going from ona vacuum state to another is a finite and measurable quantity, which in fact dominates physics at the nanoscale.

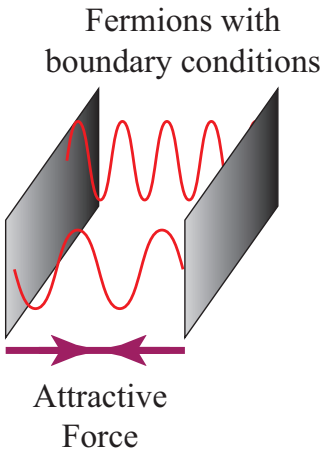


Figure 5.2 Casimir effect: fermions confined to a slab bounded by $z = 0$ and $z = d$.

Consider the Dirac field that is defined for infinite range of say two of the space co-ordinates, that is $-\infty \leq x,\ y \leq +\infty$ and with the $z$-co-ordinate taking values in a finite and open range, that is $0 \leq z \leq d$, as shown in Figure 5.2. This case has been studied by Milonni (1994). The case of a Dirac fermions defined on a cylinder have different boundary conditions and is considered later in Section 5.9.

What are the boundary conditions on the fermions in the ?

The current normal to the conducting plate should be zero. Let $\vec{n}(z = 0) = (0, 0, -1)$ and $\vec{n}(z = d) = (0, 0, 1)$. Then

$$\vec{n} \cdot \vec{j} = n_i \bar{\psi} \gamma_i \psi \tag{5.8.1}$$

It is shown below in Eq. 5.8.5, that these boundary conditions follow from requiring that the Dirac Lagrangian on the semi-infinite domain have no boundary terms on being varied, that is $\delta S = 0$ should not yield any boundary terms. The reason for demanding that no boundary terms appear is because only then does the quantum system have a (well-defined) Hamiltonian and state space.

Consider the Dirac action; suppressing the $x, y$-coordinates and writing out only the $z$-coordinate yields

$$S = \int dz\mathcal{L}$$

$$\mathcal{L} = -i\bar{\psi}(\partial^{\mu}\gamma_{\mu} + m)\psi = -i\bar{\psi}\Big(\partial_0\gamma_0 - \sum_{i=1}^{2}\partial_i\gamma_i - \partial_z\gamma_z\Big)\psi \tag{5.8.2}$$

where

$$\gamma_z = \gamma_3 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \tag{5.8.3}$$

For a finite range for $z \in [0, d]$ $\mathcal{L}$ is written more symmetrically as

$$\mathcal{L} = -\frac{i}{2}(\bar{\psi}\partial_{\mu}\gamma_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma_{\mu}\psi) - im\bar{\psi}\psi \tag{5.8.4}$$

In the variation of the action only the gradient term in the $z$-direction yields a boundary term given by the following

$$\delta S = .... - \frac{i}{2}\int_0^d dz\partial_z(\delta\bar{\psi}\gamma_z\psi) + ...$$

Performing the integration over $dz$ yield the following *boundary term*

$$\begin{aligned}\delta S_{\text{Boundary Term}} &= \frac{i}{2}\delta\bar{\psi}\gamma_z\psi\Big|_0^d \\ &= \frac{i}{2}\left[\delta\bar{\psi}\gamma_z\psi(d) - \delta\bar{\psi}\gamma_z\psi(0)\right]\end{aligned}$$

A boundary condition is imposed on the Dirac fermions so that the boundary term is zero, namely

$$\delta S_{\text{Boundary Term}} = 0 = \frac{i}{2}\left[\delta\bar{\psi}\gamma_z\psi(d) - \delta\bar{\psi}\gamma_z\psi(0)\right] = 0 \tag{5.8.5}$$

One can see that the boundary condition given in Eq. 5.8.5 is a special case of the more general boundary condition given by Eq. 5.8.1.

Each boundary term must cancel by itself due to locality. In block 2×2 notation

$$\begin{aligned}\delta\bar{\psi}\gamma_z\psi &= i\begin{pmatrix} \delta\bar{\psi}_u & \delta\bar{\psi}_\ell \end{pmatrix}\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\begin{pmatrix} \psi_u \\ \psi_\ell \end{pmatrix} \\ &= -i\delta\bar{\psi}_u\sigma_3\psi_\ell + i\delta\bar{\psi}_\ell\sigma_3\psi_u \end{aligned} \tag{5.8.6}$$

Impose constraint

$$i\gamma_3\psi(0) = \psi(0) \tag{5.8.7}$$

In terms of the components of the field, constraint given in Eq. 5.8.7, suppressing the argument of the boundary fermions, yields

$$\begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} \begin{pmatrix} \psi_u \\ \psi_\ell \end{pmatrix} = \begin{pmatrix} \psi_u \\ \psi_\ell \end{pmatrix} \quad \Rightarrow \psi_\ell = i\sigma_3\psi_u \tag{5.8.8}$$

There is only one constraint from Eq. 5.8.8 since the other component yields

$$\psi_u = -i\sigma_3\psi_\ell = -i^2\sigma_3^2\psi_u = \psi_u \quad \Rightarrow \quad \delta\bar{\psi}_u = \bar{\psi}_u$$

Eq. 5.8.8 yields the following

$$\bar{\psi}_\ell = -\psi_\ell^+ = -\psi_u^+(-i\sigma_3) = i\bar{\psi}_u\sigma_3 \quad \Rightarrow \quad \delta\bar{\psi}_\ell = i\delta\bar{\psi}_u\sigma_3 \tag{5.8.9}$$

Hence, applying Eq. 5.8.6 for $z = 0$ and using above equations yields

$$i\delta\bar{\psi}\gamma_3\psi = -i\delta\bar{\psi}_u\sigma_3\psi_\ell + i\delta\bar{\psi}_\ell\sigma_3\psi_u = -i^2\delta\bar{\psi}_u\sigma_3\sigma_3\psi_u + i^2\delta\bar{\psi}_u\sigma_3^2\psi_u = 0$$

Similar to Eq. 5.8.7 and keeping in mind that fermions are antiperiodic at the two ends of an open interval, one imposes another boundary condition

$$i\gamma_3\psi(d) = -\psi(d) \tag{5.8.10}$$

In summary, the boundary conditions imposed on the boundary value of the fermions are the following

$$i\gamma_3\psi(0) = \psi(0) \quad ; \quad i\gamma_3\psi(d) = -\psi(d) \tag{5.8.11}$$

Consider only the $z$-axis that has the open boundary condition, ignoring the other space directions. The Fourier expansion of the Dirac field, as discussed by Milonni (1994), is given by

$$\psi(z) = \sum_{p_z}(e^{ip_z z} + ie^{-ip_z z}\gamma_3)\psi_n \tag{5.8.12}$$

Boundary condition given in Eq. 5.8.11 requires

$$e^{ip_z d} = -e^{-ip_z d} \quad \Rightarrow \quad p_z = \frac{\pi n}{2d} \quad : \quad n = 1, 3, 5, \cdots \tag{5.8.13}$$

Hence, from Eq. 5.8.13, the boundary conditions in Eqs. 5.8.11 and 5.8.13 are satisfied by

$$\psi(z) = \sum_{n=1,3,5,\cdots}(e^{\frac{i\pi n}{2d}z} + ie^{-\frac{i\pi n}{2d}z}\gamma_3)\psi_n \tag{5.8.14}$$

The vacuum energy of the Dirac field is given by Eq. 5.6.6

$$E_0 = -2\sum_{\vec{p}}\sqrt{\vec{p}^2 + m^2} \simeq -2\sum_{\vec{p}}\sqrt{\vec{p}^2}$$

since for the Casimir effect, the mass is irrelevant since the contribution comes only from the momentum that are much higher than $m$.

For the open boundary condition, $p_z$ has discrete values as given in Eq. 5.8.13; using $\vec{p}^2 = \vec{p}_\perp^2 + p_z^2$ yields the following energy of the vacuum

$$E(d) = -2 \int \frac{d^2 p_\perp}{(2\pi)^2} \sum_{n=1,3,5,\cdots} [\vec{p}_\perp^2 + \frac{n^2\pi^2}{4d^2}]^{1/2} \tag{5.8.15}$$

Consider a change of variables $d^2 p_\perp = 2\pi\xi d\xi$; furthermore, let

$$\sqrt{\xi^2 + \frac{n^2\pi^2}{4d^2}} = x \quad \Rightarrow \quad \xi d\xi = x dx \tag{5.8.16}$$

Hence

$$E(d) = -\frac{1}{\pi} \sum_{n=1,3,5,\cdots} \int_{n\pi/2d}^{\infty} dx x^2 \tag{5.8.17}$$

Regularizing the sum with $e^{-\alpha x}$ yields

$$\begin{aligned} E(d) &= -\frac{1}{\pi} \frac{\partial^2}{\partial\alpha^2} \sum_{n=1,3,5,\cdots} \int_{n\pi/2d}^{\infty} dx e^{-\alpha x} \\ &= -\frac{1}{\pi} \frac{\partial^2}{\partial\alpha^2} (\frac{1}{\alpha} \sum_{n=1,3,5,\cdots} e^{-n\pi\alpha/2d}) \end{aligned} \tag{5.8.18}$$

But

$$\sum_{n=1,3,5,\cdots} e^{-n\pi\alpha/2d} = \frac{1}{2\sinh(\frac{\pi\alpha}{2d})} \tag{5.8.19}$$

and[2]

$$\begin{aligned} E(d) &= -\frac{1}{2\pi} \frac{\partial^2}{\partial\alpha^2} \left[ \frac{2d}{\pi\alpha^2} - \frac{\pi}{12d} + \frac{7\pi^3}{360} \frac{\alpha^2}{8d^3} + 0(\alpha^3) \right] \\ &= -\frac{7\pi^2}{2880 d^3} \qquad : \text{Attractive !} \end{aligned} \tag{5.8.20}$$

Note that although the zero point energy of the fermion field in negative unlike the case for the photon where it is positive, the force in both cases is attractive! The reason is that the sum for fermion over the odd modes effectively switches the sign of the (regularized) vacuum energy.

[2] Note that the energy of the vacuum state for infinite volume, that is, without any boundary conditions, is given by

$$E(d) = -\frac{1}{2\pi} \frac{\partial^2}{\partial\alpha^2} \left[ \frac{2d}{\pi\alpha^2} \right]$$

## 5.9 Casimir force in a (anti-)periodic geometry

We will now repeat the analysis for the Casimir force but with periodic and antiperiodic boundary conditions in the $z$-direction. Consider the Dirac field confined to a infinite slab in the $x, y$-directions and with cylindrical boundary conditions (periodic or antiperiodic) in the $z$-direction. The fermions are defined on $S^1$ with radius $R$ and $d = 2\pi R$. The path integral and state space both are a sum of two contributions, one from fermions periodic in the $z$-direction and the other being antiperiodic in the $z$-direction. These yield the following boundary conditions on the fermions.[3]

- Periodic – termed as the R (Ramond sector)

$$\psi(z) = \psi(z+d) \quad ; \quad \bar{\psi}(z) = \bar{\psi}(z+d)$$

  represent the periodic boundary condition, which means that the fermions at $z = 0$ is equals to fermions at $z = d$.
- Antiperiodic – termed as the NS (Neveu-Schwarz sector)

$$\psi(z) = -\psi(z+d) \quad ; \quad \bar{\psi}(z) = -\bar{\psi}(z+d)$$

  represent the anti-periodic boundary condition, which means that the fermions at $z = 0$ is equals to the negative of the fermions at $z = d$.

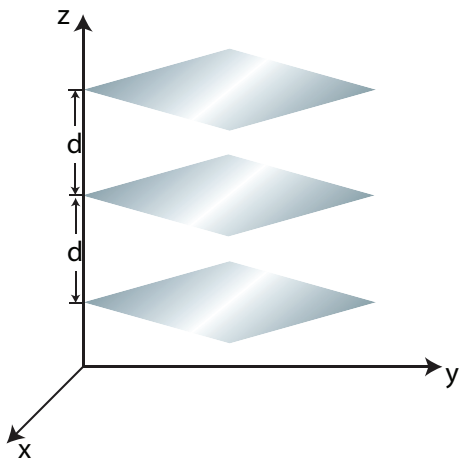


Figure 5.3 Casimir effect: fermions in an infinite slab in the $x, y$-directions, with periodic or antiperiodic boundary conditions in the $z$-direction.

[3] This is similar to the world-sheet boundary conditions in superstring theory. The periodic and antiperiodic boundary conditions are the Ramond and Neveu-Schwarz boundary conditions respectively, and we use adopt this terminology [J.Polchinski (1998)].

In particular, the Hamiltonian for with periodic and antiperiodic boundary conditions is given by

$$H = H_R \oplus H_{NS} \tag{5.9.1}$$

For the open boundary case, discussed in Section 5.8, the ends are open. This implies that the fermions at the boundaries are distinct. So we had to introduce boundary conditions given in Eq. 5.8.11 so as to obtain the classical field equation. However, for the cylindrical geometry, the classical field equation is automatically satisfied by the periodicity and anti-periodicity of the fermions and is shown below.

- Periodic. The periodic boundary condition is

$$\psi(0) = \psi(d) \ , \ \bar{\psi}(0) = \bar{\psi}(d) \tag{5.9.2}$$

The classical field equation (ignoring the $x, y, t$ dependence) is

$$\delta S = \ldots \delta\bar{\psi}(d)\gamma_z\psi(d) - \delta\bar{\psi}(0)\gamma_z\psi(0)\ldots \tag{5.9.3}$$

For the periodic boundary condition,

$$\begin{aligned} \delta S &= \ldots \delta\bar{\psi}(d)\gamma_z\psi(d) - \delta\bar{\psi}(0)\gamma_z\psi(0)\ldots \\ &= \ldots \delta\bar{\psi}(0)\gamma_z\psi(0) - \delta\bar{\psi}(0)\gamma_z\psi(0)\ldots = 0 \end{aligned} \tag{5.9.4}$$

- Antiperiodic. The anti-periodic boundary condition is

$$\psi(0) = -\psi(d) \ , \ \bar{\psi}(0) = -\bar{\psi}(d) \tag{5.9.5}$$

For the anti-periodic boundary condition,

$$\begin{aligned} \delta S &= \ldots \delta\bar{\psi}(d)\gamma_z\psi(d) - \delta\bar{\psi}(0)\gamma_z\psi(0)\ldots \\ &= \ldots - \delta\bar{\psi}(0)\gamma_z(-\psi(0)) - \delta\bar{\psi}(0)\gamma_z\psi(0)\ldots = 0 \end{aligned} \tag{5.9.6}$$

Hence the classical field equation is satisfied by the periodicity and the anti-periodicity.

We assume the fermion degree of freedom has the similar form as that in the previous case of open boundary conditions.

$$\psi(z) = e^{ip_3 z}\psi_n \tag{5.9.7}$$

The difference of the above Eq. 5.9.7 from the previous section is the constraint in $p_3$ and the absence of the $\gamma_z$ term.

To determine $p_3$ impose the boundary condition

$$\psi(0) = \psi_n \ ; \ \psi(d) = e^{ip_3 d}\psi_n \tag{5.9.8}$$

- Periodic Considering the periodic case: $\psi(0) = \psi(d)$

$$\begin{aligned} e^{ip_3 d} &= 1 = \cos p_3 d + i \sin p_3 d \\ \Rightarrow p_3 &= \frac{n\pi}{d} \quad n = 0, 2, 4, \ldots : \text{even} \end{aligned} \tag{5.9.9}$$

Hence, for the periodic boundary condition

$$\psi(z) = e^{i\frac{n\pi}{d}z}\psi_n \quad n = 0, 2, 4, \ldots : \text{even} \tag{5.9.10}$$

- Antiperiodic. Considering the anti-periodic case: $\psi(0) = -\psi(d)$

$$\begin{aligned} e^{ip_3 d} &= -1 = \cos p_3 d + i \sin p_3 d \\ p_3 &= \frac{n\pi}{d} \quad : n = 1, 3, 5, \ldots, \text{odd} \end{aligned} \tag{5.9.11}$$

Hence, for the anti-periodic boundary condition,

$$\psi(z) = e^{i\frac{n\pi}{d}z}\psi_n \quad : n = 1, 3, 5, \ldots, \text{odd} \tag{5.9.12}$$

### *5.9.1 Zero-point energy*

We will now calculate the zero-point casimir energy. Recall for the open boundary condition, from Eq. 5.8.18

$$E(d) = -\frac{1}{\pi} \lim_{\alpha \to 0} \frac{\partial^2}{\partial \alpha^2} \frac{1}{\alpha} \sum_{n=1,3,..} e^{-\frac{n\pi\alpha}{2d}}$$

The above equation is used for evaluating the zero-point energy, by replacing $2d$ by $d$in the equation above and summing over even $n$ or odd $n$ for periodic and antiperiodic cases, respectively.

- For the periodic case, the zero-point casimir energy $E_R$ is

$$E_R = -\frac{1}{\pi} \lim_{\alpha \to 0} \frac{\partial^2}{\partial \alpha^2} \frac{1}{\alpha} \sum_{n=0,2,4,..} e^{-\frac{n\pi\alpha}{d}} \tag{5.9.13}$$

The summation yields

$$\begin{aligned} \sum_{n=0,2,4,..} e^{-\frac{n\pi\alpha}{d}} &= e^{\frac{\pi\alpha}{d}} \left(2 \sinh \frac{\pi\alpha}{d}\right)^{-1} \\ &= \ldots - \frac{1}{90} \frac{\pi^3 \alpha^3}{d^3} \ldots \quad \text{(we are interested in the } \alpha^3 \text{ term)} \end{aligned}$$

The zero-point casimir energy for periodic case becomes

$$E_R = \frac{1}{45} \frac{\pi^2}{d^3} \tag{5.9.14}$$

- For anti-periodic case, the zero-point casimir energy $E_{NS}$ is

$$E_{NS} = -\frac{1}{\pi}\lim_{\alpha\to 0}\frac{\partial^2}{\partial\alpha^2}\frac{1}{\alpha}\sum_{n=1,3,5,..} e^{-\frac{n\pi\alpha}{d}} \tag{5.9.15}$$

The summation term

$$\sum_{n=1,3,5,..} e^{-\frac{n\pi\alpha}{d}} = \sum_m e^{-\frac{(2m+1)\pi\alpha}{d}} = e^{-\frac{\pi\alpha}{d}}\sum_m e^{-\frac{2m\pi\alpha}{d}}$$
$$= \ldots\frac{7}{720}\frac{\pi^3\alpha^3}{d^3}\ldots \text{ (we are interested in the } \alpha^3 \text{ term)}$$

For zero-point casimir energy for anti-periodic case becomes

$$E_{NS} = -\frac{7}{360}\frac{\pi^2}{d^3} \tag{5.9.16}$$

The Casimir force is given by

$$F = -\frac{\partial E}{\partial d}$$

Hence the Casimir forces for periodic case is

$$F_R(d) = \frac{\pi^2}{15d^4} \tag{5.9.17}$$

and the Casimir forces for antiperiodic case is

$$F_{NS}(d) = -\frac{7\pi^2}{120d^4} \tag{5.9.18}$$

We can see that the Casimir force for periodic boundary condition is repulsive and the Casimir force for anti-periodic boundary condition is attractive.

# 6

# The Photon vector field

The photon field is a *vector quantum field* that describes quantized electromagnetic field. In the classical limit, it is the Maxwell electromagnetic field. The photon field is a gauge field, with its lowest energy excitation above the vacuum state being the massless photon.

The photon field is ubiquitous in Nature and serves as an exemplar of an Abelian gauge field. The key symmetry of the photon field is that of gauge invariance. All the interactions of the Standard Model, carried by the electro-weak bosons and colored gluons, obey non-Abelian gauge symmetry.

The main thrust of this Chapter is to study the main features of gauge symmetry, and to derive the implications of gauge fixing – for both the path integral and Hamiltonian formulations of the photon field. The Fadeev-Popov quantization scheme is discussed for the Abelian gauge field, with both the path integral and the BRST state space approach being studied in some detail.

## 6.1 Gauge symmetry

Consider the Abelian gauge field $A_\mu$ with gauge transformation given by

$$A_\mu \to A_\mu(\phi) = A_\mu + \partial_\mu \phi \quad ; \quad \delta A_\mu \equiv A_\mu(\phi) - A_\mu = \partial_\mu \phi$$

Consider the gauge invariant field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Under a gauge transformation, the field tensor is invariant since

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = \partial_\mu \partial_\nu \phi - \partial_\nu \partial_\mu \phi = 0$$

Hence a gauge invariant Lagrangian in Euclidean space is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 \;\; ; \;\; S = \int d^4x\mathcal{L}$$

The quantum field theory is defined by

$$Z = \int DAe^S \tag{6.1.1}$$

The prartition function $Z$ is divergent, that is, $Z = \infty$. The divergence of $Z$ is a consequence of gauge invariance.

Noteworthy 6.1: Jacobian

Consider the following change of independent variables

$$x_i,\ i = 1, 2, \cdots, N \rightarrow y_i = y_i(x),\ i = 1, 2, \cdots, N$$

Then

$$dx_i = \sum_j \frac{\partial x_i}{\partial y_j} dy_j = \sum_j J_{ij} dy_j$$

where $J_{ij}$ is the Jacobian of the transformation. The measure has the following transformation

$$\prod_i dx_i = J \prod_i dy_i \;\; ; \;\; J = \det(J_{ij}) \tag{6.1.2}$$

Let the eigenvalues of the matrix $J_{ij}$ be given by $\lambda_n$; then the determinant is given by

$$\det J = \prod_{n=1}^{N} \lambda_n \tag{6.1.3}$$

The divergence of $Z$ can be seen by considering a change of variables

$$A_i, A_0 \rightarrow \tilde{A}_i, \phi$$

that is encoded in the following gauge transformation

$$\tilde{A}_i = A_i - \partial_i\phi \;\; ; \;\; A_0 = \partial_0\phi$$

Hence, from Eq. 6.1.2

$$DA_i = D\tilde{A}_i \;\; ; \;\; DA_0 = \det(\partial_0) D\phi = \text{const} \times D\phi$$

The action $S$ is gauge invariant and hence does not depend on $\phi$ and $S[A] = S[\tilde{A}]$. The change of variables yields, upto irrelevant constants

$$\begin{aligned} Z &= \int DAe^{S[A]} = \int D\tilde{A}D\phi e^{S[\tilde{A}]} = [\int D\phi] \times [\int D\tilde{A}e^{S[\tilde{A}]}] \\ &= \infty \times \text{finite}. \end{aligned} \tag{6.1.4}$$

Note that although the path integral

$$\int D\tilde{A}e^{S[\tilde{A}]} \quad : \quad \text{finite}$$

is finite, the path integral breaks Lorentz invariance since in this gauge $\tilde{A}_0 = 0$, and is called the temporal gauge. Hence this path integral is not suitable for perturbation theory using Feynman diagrams. Later, in Section 6.5 where the gauge symmetry is considered from the Hamiltonian point of view, the temporal gauge will be appropriate. This is because the Hamiltonian singles out one direction as time and apparently breaks Lorentz symmetry.[1]

## 6.2 Gauge Fixing the Action

To obtain a finite and convergent partition function $Z$, the infinite term $\int D\phi = \infty$ needs to be factored out of the path integral. This procedure is called gauge-fixing or choosing a gauge. The pioneering work of Faddeev and Slavnov (1980) showed how to gauge-fix the path integral for gauge fields.

Consider a gauge-fixing term that necessarily breaks gauge-invariance. A Lorentz invariant choice of gauge is given by

$$s = \partial_\mu A_\mu = 0$$

Define

$$s(\phi) = \partial_\mu A_\mu(\phi) = \partial_\mu A_\mu - \partial^2\phi$$

Consider the identity, for $t = t(x)$

$$1 = \frac{\int D\phi\delta(s(\phi) - t)}{\int D\phi'\delta(s(\phi') - t)} \quad ; \quad \delta(s) \equiv \prod_x \delta(s(x)) \tag{6.2.1}$$

The partition function can be written as

$$Z = \int DAe^{S[A]} \frac{\int D\phi\delta(s(\phi) - t)}{\int D\phi'\delta(s(\phi') - t)} \tag{6.2.2}$$

[1] Lorentz symmetry is preserved in the Hamiltonian formulation by a combination of gauge and Lorentz transformations.

Do an inverse gauge transformation given by

$$A_\mu \to A_\mu(\phi^{-1}) = A_\mu + \partial_\mu \phi \tag{6.2.3}$$

Then the gauge fixing term in the numerator decouples from the gauge transformation

$$s(\phi \cdot \phi^{-1}) = \partial_\mu A_\mu(\phi^{-1}) - \partial^2 \phi = \partial_\mu(A_\mu + \partial_\mu \phi) - \partial^2 \phi = \partial_\mu A_\mu$$

and the denominator yields

$$s(\phi^{'}) = \partial_\mu A_\mu(\phi^{-1}) - \partial^2 \phi^{'} = \partial_\mu A_\mu - \partial^2(\phi^{'} - \phi) \tag{6.2.4}$$

Define the Fadeev-Popov *counter-term* by

$$e^{S_c} = 1/\int D\phi \delta(s(\phi)) \quad : \quad \text{gauge-invariant} \tag{6.2.5}$$

The partition function simplifies to

$$Z = \int DA \left[\frac{\int D\phi \delta(s)}{\int D\phi' \delta(s(\phi' - \phi))}\right] e^S = \underbrace{\{\int D\phi\}}_{\text{drop}} \underbrace{\int DA e^S \delta(s) e^{S_c}}_{\text{convergent}}$$

Dropping the overall irrelevant constant yields

$$Z = \int DA e^S \delta(s) e^{S_c} \tag{6.2.6}$$

After gauge-fixing, the partition function is convergent. Note in the functional integral, all the four components of the gauge field are integration variables. The action $S$ is Lorentz invariant and so is the Fadeev-Popov counter-term $S_{FP}$ – hence leading to a Lorentz invariant action and a symmetric perturbation expansion.

The gauge-fixed partition function is a particular case of the fact that gauge-fixing leaves all the gauge invariant expectation values unchanged. Let $\hat{O}[A]$ be a gauge invariant functions of the fields. The gauge invariant expectation value is given by

$$E[\hat{O}[A]] = \int DA \hat{O}[A] e^S \tag{6.2.7}$$

The expectation value of $\hat{O}[A]$, after gauge fixing, is given by

$$E[\hat{O}[A]] = \int DA \hat{O}[A] e^S \delta(s) e^{S_{FP}} \tag{6.2.8}$$

To prove result given above in Eq. 6.2.8 is equal to the gauge invariant expression in Eq. 6.2.7, consider a gauge-transformation $A \to A(\phi)$ and

integrate over all gauge-transformations by $\int D\phi$. Using the fact that $\hat{O}[A], S$ and $S_{FP}$ are all gauge invariant, the expectation value is given by

$$E[\hat{O}[A]] = \int D\phi E[\hat{O}[A(\phi)]] = \int DA\hat{O}[A]e^{S}e^{S_{FP}} \int D\phi\delta(s(\phi))$$
$$= \int DA\hat{O}[A]e^{S}$$

and we recover the result given in Eq. 6.2.7.

The partition function is given by

$$Z = \det(-\partial^2) \int DA \prod_x \delta(s(x) - t(x))e^{S} \tag{6.2.9}$$

Since $Z$ is independent of $t(x)$, the following is valid

$$Z = \text{const.} \int \prod_x dt(x) \exp\{-\frac{\alpha}{2} \int d^4x t^2(x)\} Z$$
$$= \det(-\partial^2) \int DA e^{-\frac{\alpha}{2}\int d^4x s^2(x)} e^{S} \tag{6.2.10}$$

Hence, the gauge-fixed action $S_{GF}$ is given by

$$S_{GF} = -\frac{1}{4} \int d^4x F_{\mu\nu}^2 - \frac{\alpha}{2} \int d^4x s^2 + \ln(\det(-\partial^2)) \tag{6.2.11}$$

The Fadeev-Popov term $\det(-\partial^2)$ is usually ignored for the Abelian gauge field since it is independent of the gauge field $A_\mu$, but plays a crucial role for non-Abelian gauge fields.

Making all the four components of the gauge field independent degrees of freedom comes at a price, which is the gauge-fixing term that is inserted as a delta function constraint $\delta(s)$, and which necessarily breaks gauge invariance. The compensation for introducing the gauge-fixing term for leaving the gauge-invariant sector invariant is the appearance of the Fadeev-Popov counter-term. For the Abelian gauge field, the Fadeev-Popov counter-term does not depend on the gauge field $A_\mu$, but this is not the case for the non-Abelian gauge fields.

The great utility of gauge-fixing the theory and using the gauge-fixed path integral to evaluate the expectation values is that gauge-fixing leaves the gauge invariant sector. Since it is Lorentz invariant, the gauge-fixed theory can be used for a Feynman expansion of all gauge-invariant correlation functions.

### 6.3 The ghost term: finite time path integral

Consider the partition function for finite time – denoted by $\tau$. The finite time path integral, denoted by $Z(\tau)$, is a logically necessary part of the path integral formulation of quantum fields. For example, in Section 7.6, the finite time partition function $Z(\tau)$ is used for determining all the eigenenergies of two dimensional quantum electrodynamics.[2]

$$Z(\tau) = tr(e^{-\tau \hat{H}}) \tag{6.3.1}$$

In computing the trace of the operator $\exp\{-\tau \hat{H}\}$, with the defining Equation 6.3.1 for $Z$, the trace must be taken over a complete set of states tht are *periodic* functions in the time direction.

Since $S = -\frac{1}{4}\int d^4x F_{\mu\nu}^2$ is gauge invariant, the path integral representation of $Z$ is given by

$$Z(\tau) = \int DAe^S = \det(-\partial^2)\Big|_\tau \int DAe^{S+S_\alpha}$$

For infinite time path integral the ghost-fields decouple from the gauge field. However, for finite time, the Fadeev-Popov term needs to be kept and we examine this aspect. Since they are defined by the determinant of bosonic fields, the ghost field are *periodic* functions for a finite time $\tau$ – unlike space-time fermions which would be anti-periodic over period $\tau$.

Hence, the determinant $\det(-\partial^2)$ is taken over only periodic functions. In the Feynman gauge $\alpha = 1$, and the gauge fixed action for finite temperature is

$$S_{GF} = -\frac{1}{2}\int_0^\tau dt \int d^3x A_\mu(-\partial^2)A_\mu$$

Hence, ignoring irrelevant constants[3]

$$Z(\tau) = \frac{\det_\tau(-\partial^2)}{[\det_\tau(-\partial^2)]^{4/2}} = \frac{1}{\det_\tau(-\partial^2)}$$

The gauge-fixed action $S_{GF}$ has four components for the gauge field $A_\mu$, whereas there are only two physical degrees of freedom. This is because the photon is massless and has only two polarizations. The Faddeev-Popov

[2] It turns out that in Euclidean time, $Z(\tau)$ is formally equivalent to the partition function of statistical mechanics with $\tau = 1/(k_B \times \text{temperature})$, but this connection is irrelevant for our discussion.

[3] Eqs. 4.13.2 and 4.13.3

$$Z[J] = \prod_{n=1}^{N}\int_{-\infty}^{+\infty} dx_n \exp\{-\frac{1}{2}\sum_{i,j=1}^{N} x_i A_{ij} x_j\} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

ghost determinant precisely cancels the extra factors of determinant that arise from the gauge-fixed action $S_{GF}$; the final answer obtained above is what one expects.

To evaluate $\det_\tau(-\partial^2)$, one needs to find all the eigenvalues and then use the infinite dimensional generalization of Eq. 6.1.3. In the momentum basis the eigenfunctions $\psi_{np}$, which are periodic in interval $[t, t+\tau]$, are given by

$$-\partial^2(\psi_{np}) = (-\partial_t^2 - \vec{\partial}^2)e^{i\frac{2\pi nt}{\tau}}e^{i\vec{p}\cdot\vec{x}} = \left[(\frac{2\pi n}{\tau})^2 + \vec{p}^2\right]e^{i\vec{p}\cdot\vec{x}}$$

Hence using $\det J = \prod_n \lambda_n$ yields the determinant[4]

$$\begin{aligned}\det_\tau(-\partial^2) &= \prod_{n=-\infty}^{+\infty}\prod_{\vec{p}}\left[(\frac{2\pi n}{\tau})^2 + \vec{p}^2\right] = \text{const.}\prod_{\vec{p}}\prod_{n=1}^{\infty}\left[1 + (\frac{p\tau}{2\pi n})^2\right]^2 \\ &= \text{const.}\prod_{\vec{p}}\left[2\sinh(\frac{p\tau}{2})\right]^2 \qquad (6.3.2)\end{aligned}$$

and we obtain the finite time partition function[5]

$$Z(\tau) = \prod_{\vec{p}}\frac{1}{\left[2\sinh(\frac{p\tau}{2})\right]^2} = e^{-\frac{\tau}{2}\sum_{\vec{p}}|\vec{p}|\times 2}\frac{1}{\prod_{\vec{p}}(1-e^{-\tau|\vec{p}|})^2}$$

with

$$E_0 = 2\times\frac{1}{2}\sum_{\vec{p}}|\vec{p}| \quad : \quad \text{zero-point energy}$$

The factor of 2 in the vacuum energy comes from the two polarizations of the photon.

## 6.4 Feynman and Landau gauges

The action in the Lorentz gauge has $s = \partial_\mu A_\mu$ and is given by

$$\begin{aligned}S_{GF} &= -\frac{1}{4}\int F_{\mu\nu}^2 - \frac{\alpha}{2}\int(\partial_\mu A_\mu)^2 \\ &= -\frac{1}{2}\int A_\mu(-\partial_\mu^2\partial_\nu + \partial_\mu\partial_\nu)A_\nu - \frac{\alpha}{2}\int A_\mu(-\partial_\mu\partial_\nu)A_\nu \\ &= -\frac{1}{2}\int A_\mu(-\partial^2\delta_{\mu\nu} + (1-\alpha)\partial_\mu\partial_\nu)A_\nu \\ &= -\frac{1}{2}\int A_\mu M_{\mu\nu}A_\nu \qquad (6.4.1)\end{aligned}$$

[4] $\prod_{n=1}^{\infty}[1+(\frac{x}{\pi n})^2] = \sinh(x)/x$.

[5] If one identifies $\tau = 1/(k_B \times \text{temperature})$, then the following equation yields the distribution function for Bose-Einstein statistics.

The generating functional for the gauge-fixed action is given by

$$Z[J] = \int DA e^{S_{GF}+\int j_\mu A_\mu}$$
$$= \exp(\frac{1}{2}\int j_\mu D_{\mu\nu} j_\nu) \tag{6.4.2}$$

where $D_{\mu\nu}$ is the propagator of the photon given by

$$D_{\mu\nu} = M^{-1}_{\mu\nu} \tag{6.4.3}$$

Note that the propagator is not gauge invariant, and is given by

$$D_{\mu\nu}(x,x') = E[A_\mu(x)A_\nu(x')] = \frac{1}{Z}\int DA e^{S_{GF}} A_\mu(x)A_\nu(x')$$
$$= \langle\Omega|A^H_\mu(\vec{x},t)A^H_\nu(\vec{x}',t')|\Omega\rangle \quad : \quad t > t' \tag{6.4.4}$$

Define $|p\rangle$ by

$$\langle\mu|p\rangle = \partial_\mu$$

The inverse of the propagator is written as

$$M_{\mu\nu}(x,x') = \Big((-\partial^2)(\delta_{\mu\nu} + (\alpha-1)\frac{\partial_\mu\partial_\nu}{\partial^2})\Big)\delta(x-x') \tag{6.4.5}$$
$$M = (-\partial^2)e^{\beta|p\rangle\langle p|}\delta(x-x') \tag{6.4.6}$$

with $e^\beta = \alpha$. Hence

$$M^{-1} = \frac{1}{(-\partial^2)}(e^{-\beta|p\rangle\langle p|})\delta(x-x')$$
$$= \frac{1}{(-\partial^2)}(1+(e^{-\beta}-1)|p\rangle\langle p|)\delta(x-x') \tag{6.4.7}$$

and the propagator is given by

$$D_{\mu\nu}(x,x') = M^{-1}_{\mu\nu}(x,x') = (\delta_{\mu\nu} + (\frac{1}{\alpha}-1)\frac{\partial_\mu\partial_\nu}{\partial^2})\frac{1}{(-\partial^2)}\delta(x-x')$$
$$= (\delta_{\mu\nu} + (\frac{1}{\alpha}-1)\frac{\partial_\mu\partial_\nu}{\partial^2})D(x-x')$$

where

$$D(x) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ikx}}{k^2}$$

Note that the propagator is divergent for $\alpha = 0$, as expected, since the gauge-fixing term is then zero.

1. Feynman gauge is defined by $\alpha = 1$ and yields

$$D_{\mu\nu}(x-y) = \delta_{\mu\nu}D(x-y) \qquad \text{: diagonal propagator} \tag{6.4.8}$$

2. Landau gauge is defined by $\alpha = \infty$ and yields

$$D_{\mu\nu}(x-y) = (\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}) D(x-y) \tag{6.4.9}$$
$$\partial_\mu D_{\mu\nu} = 0 \quad : \text{transverse propagator}$$

## 6.5 Hamiltonian and gauge invariant state space

The Hamiltonian and state space is obtained for the photon field. The derivation is similar to the one for a scalar field, with the additional feature of gauge invariance leading to new features in the derivation of the Hamiltonian.

Consider the gauge invariant Lagrangian

$$\mathcal{L} = -\frac{1}{4}\sum_{\mu\nu} F_{\mu\nu}^2 = -\frac{1}{2}\sum_i F_{0i}^2 - \frac{1}{4}\sum_{ij} F_{ij}$$

Note

$$\sum_{ij} F_{ij}^2 = \sum_{ij} (\partial_i A_j - \partial_j A_i)^2 = 2\sum_{i=1}^{3} (\epsilon_{ijk}\partial_j A_k)^2$$

Define magnetic field by

$$\vec{B} = \vec{e}_i B_i \quad ; \quad B_i = \epsilon_{ijk}\partial_j A_k$$

Furthermore, for $A_0 = \partial_0 \phi$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = \partial_0 A_i - \partial_i \partial_0 \phi = \partial_0 (A_i - \partial_i \phi) = \partial_0 A_i(\phi)$$

Note the far reaching fact that the time component $A_0$ is not a dynamical degree of freedom, but instead, appears in the action to ensure gauge invariance of the kinetic term $\partial_0 A_i$.

To obtain the Hamiltonian, the partition function needs to be considered as the product of $e^{-TH} = e^{-\epsilon H} \dots e^{-\epsilon H}$. Discretize time $t = n\epsilon$; spacetime coordinate is then $(n, \vec{x})$. Suppressing all space indices, the action yields

$$\begin{aligned} S &= -\frac{1}{2}\int dt \int_{\vec{x}} \{[\partial_0 A_i(\phi)]^2 + \vec{B}^2\} \\ &\simeq -\frac{\epsilon}{2}\sum_n \int_{\vec{x}} \frac{1}{\epsilon^2}\Big\{[A_{n+1,i}(\phi_{n+1}) - A_{ni}(\phi_n)]^2 + \vec{B}_n^2\Big\} \end{aligned} \tag{6.5.1}$$

The partition function is given by

$$\begin{aligned} Z = \prod_{ni} \int dA_{ni} \int d\phi_n \cdots \langle A_{n+2,i}(\phi_{n+2}) | e^{-\epsilon H} | A_{n+1,i}(\phi_{n+1}) \rangle \\ \langle A_{n+1,i}(\phi_{n+1}) | e^{-\epsilon H} | A_{n,i}(\phi_n) \rangle \langle A_{n,i}(\phi_n) | e^{-\epsilon H} | A_{n-1,i}(\phi_{n-1}) \rangle \end{aligned}$$

In other words, since all $\phi_n$'s occurs only in the combination of $A_{ni}-\partial_i\phi_n$, the $\int d\phi_n$ integrations occurs only in the state vectors $|A_{ni}(\phi_n)\rangle$ and $\langle A_{ni}(\phi_n)|$ respectively. Hence we define the *gauge invariant* Hamiltonian by

$$\langle A_i|e^{-\epsilon H}|A_j^{'}\rangle \equiv \int D\phi D\phi^{'}\langle A(\phi)|e^{-\epsilon H}|A^{'}(\phi^{'})\rangle \tag{6.5.2}$$

Equivalently, the completeness equation for the photon field is

$$\mathbb{I} = \int DAD\phi|A(\phi)\rangle\langle A(\phi)| \tag{6.5.3}$$

where

$$|A(\phi)\rangle \equiv \prod_{\vec{x}}\prod_{i}|A_i(\vec{x})-\partial_i\phi(\vec{x})\rangle \tag{6.5.4}$$

Consider the matrix element of the Hamiltonian $H$ between only gauge-invariant states, say $\Phi^{'}[A_i]$, $\Phi[A_i]$ such that

$$\Phi[A_i(\phi)] = \Phi[A_i] \quad ; \quad \Phi'[A_i(\phi)] = \Phi'[A_i] \tag{6.5.5}$$

Then

$$\begin{aligned}\langle\Phi^{'}|e^{-\epsilon H}|\Phi\rangle &= \int DA^{'}DA\langle\Phi^{'}|A^{'}\rangle\langle A^{'}|e^{-\epsilon H}|A\rangle\langle A|\Phi\rangle \\ &= \int DA^{'}DA\int D\phi^{'}D\phi\Phi^{'}[A^{'}]\langle A^{'}(\phi^{'})|e^{-\epsilon H}|A(\phi)\rangle\Phi[A]\end{aligned} \tag{6.5.6}$$

Doing inverse gauge transformation on $A_i$ and $A_i^{'}$

$$\begin{aligned}A_i(\phi) &\to A_i(\phi\cdot\phi^{-1}) \equiv A_i \\ \Phi[A] &\to \Phi[A(\phi^{-1})] = \Phi[A]\end{aligned} \tag{6.5.7}$$

results in the decoupling of the $\int D\phi D\phi'$ integrations from the matrix element. Hence, from Eq. 6.5.6

$$\langle\Phi^{'}|e^{-\epsilon H}|\Phi\rangle = \int DA^{'}DA\Phi'[A']\langle A'|e^{-\epsilon H}|A\rangle\Phi[A] \tag{6.5.8}$$

For gauge-invariant states, from Eq. 6.5.8

$$\langle A|e^{-\epsilon H}|A'\rangle = e^{-\frac{1}{2\epsilon}\int_{\vec{x}}(A_i-A_i')^2}e^{-\frac{\epsilon}{2}\int_{\vec{x}}\vec{B}^2} \simeq e^{-\epsilon\hat{H}}\langle A|A'\rangle$$

and yields

$$\hat{H} = -\frac{1}{2}\int_{\vec{x}}\frac{\delta^2}{\delta A_i^2(\vec{x})} + \frac{1}{2}\int_{\vec{x}}\vec{B}^2(\vec{x}) \equiv \frac{1}{2}\int_{\vec{x}}(\vec{E}^2+\vec{B}^2) \tag{6.5.9}$$

where

$$E_j(\vec{x}) = \frac{\delta}{i\delta A_j(\vec{x})} \qquad : \text{ electric field operator}$$

The Hamiltonian $\hat{H}$ given in Eq. 6.5.9 represents the photon field only on gauge invariant states $\Phi[A]$ such that

$$\Phi[A(\phi)] = \Phi[A] \tag{6.5.10}$$

Gauge invariance in turn implies the following

$$\begin{aligned}\Phi[A(\phi)] &\equiv \Phi[A_i - \partial_i\phi] \\ &= \exp\left\{\int_{\vec{x}} -\partial_i\phi(\vec{x})\frac{\delta}{\delta A_i(\vec{x})}\right\}\Phi[A] \\ &= e^{i\int_{\vec{x}}\phi(\vec{x})\partial_i E_i(\vec{x})}\Phi[A]\end{aligned} \tag{6.5.11}$$

Since $\phi(\vec{x})$ is arbitrary, Eq. 6.5.10 yields the constraint of Gauss's law that

$$\partial_i E_i(\vec{x})|\Phi\rangle = \vec{\nabla}\cdot\vec{E}(\vec{x})|\Phi\rangle = 0 \tag{6.5.12}$$

In summary, no gauge was chosen for $Z$ or the action $S$ in deriving the Hamiltonian, and $H$ was obtained that acts only on gauge-invariant states. The procedure followed is seen to be equal to choosing the temporal gauge

$$A_0 = 0 \quad \Rightarrow \quad E_i = F_{0i} = \partial_0 A_i$$

Canonical quantization yields

$$[A_i^{\vec{x}}, E_j^{\vec{x}'}] = i\delta_{\vec{x}-\vec{x}'}\delta_{ij}$$

with the simultaneous constraint that

$$\vec{\nabla}\cdot\vec{E}(\vec{x})|\Phi\rangle = 0$$

## 6.6 Gauge-fixing the Hamiltonian: Coulomb Gauge

Gauge symmetry and gauge fixing are analyzed in the Hamiltonian formulation. The formalism is quite distinct from the path integral and hinges on gauge-fixing the electric field operator, which is a differential operator for the photon quantum field.

The state space of the photon field is gauge invariant, with the state vectors obeying

$$\Phi[A(\phi)] = \Phi[A] \tag{6.6.1}$$

Gauge invariance of the state space shows that of the three component of

the gauge field $A_i(\vec{x})$ only *two* are independent–the third component being a redundant gauge degree of freedom. The two independent components of $A_i$ reflect the fact that the photon has only two polarizations.

To eliminate the gauge degree of freedom from the state space one has to impose one constraint at every point $\vec{x}$ on the gauge field $A_i(\vec{x})$ by gauge transforming it to a new constrained gauge field $C_i(\vec{x})$ ; the commonly chosen constraint condition is the Coulomb gauge given by

$$\partial_i C_i = 0 \tag{6.6.2}$$

The constraint on $C_i(\vec{x})$ in turn requires that the electric field operator be modified since the three differential operators $E_i = \delta/i\delta A_i$ are no longer independent.

The gauge transformed $C_i$ yields the constrained electric field $\tilde{E}_i^T$ operator; we will show in this Section that in the Coulomb gauge, the electric field $\tilde{E}_i^T$ satisfies the *transverse* commutation equation

$$[C_i, \tilde{E}_j^T] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2})\delta(\vec{x} - \vec{x}') \quad \Rightarrow \quad [\partial_i C_i, \tilde{E}_j^T] = 0 = [C_i, \partial_j \tilde{E}_j^T]$$

as required by $\partial_i C_i = 0$. It can be shown that

$$[\tilde{B}_i(t, \vec{x}), \tilde{E}_j(t', \vec{x}')] = 0 \;\text{ if } (\vec{x} - \vec{x}')^2 - (t - t')^2 > 0 \qquad : \text{space-like}$$

We choose the Coulomb gauge for a more general configuration where there is an electric charge density $\rho(\vec{x})$. The charge density can come from charged complex scalar fields or charge carrying fermions. In presence of charges, Gauss's law is modified to

$$(\vec{\nabla} \cdot \vec{E} - \rho)|\Phi\rangle = 0 \;\; ; \;\; E_i(\vec{x}) = \frac{\delta}{i\delta A_i(\vec{x})} \tag{6.6.3}$$

In the Coulomb gauge, the redundant gauge degree of freedom is completely eliminated. To do so, perform a gauge transformation from $A_i$ to $(C_i, \phi)$ such that

$$C_i = A_i - \partial_i \phi \tag{6.6.4}$$

and choose the Coulomb gauge

$$\nabla_i C_i = 0 \tag{6.6.5}$$

The magnetic field is unchanged

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{C} \tag{6.6.6}$$

The kinetic part of the Hamiltonian, namely $\vec{E}^2$ has to expressed in terms

of $C_i$ and $\phi$. Gauss's law also has to re-expressed in terms of $C_i$ and $\phi$. For both these cases, one needs to express $\delta/\delta A_i$ in terms of the new variables.

The chain rule yields

$$\frac{\delta}{\delta A_i(\vec{x})} = \int d^3y \sum_{j=1}^{3} \frac{\delta C_j(\vec{y})}{\delta A_i(\vec{x})} \frac{\delta}{\delta C_j(\vec{y})} + \int d^3y \frac{\delta \phi(\vec{y})}{\delta A_i(\vec{x})} \frac{\delta}{\delta \phi(\vec{y})} \tag{6.6.7}$$

To evaluate the coefficients $\delta c_j/\delta A_i$ and $\delta\phi/\delta A_i$ one needs to analyze the change of variables (gauge transformation). Note

$$\partial_i C_i = 0 = \partial_i A_i - \vec{\partial}^2 \phi \;\Rightarrow\; \vec{\partial}^2 \phi = \partial_i A_i \tag{6.6.8}$$

Hence, from Eq. 6.6.8

$$\vec{\partial}_y^2 \frac{\delta \phi(\vec{y})}{\delta A_i(\vec{x})} = \frac{\delta}{\delta A_i(\vec{x})} \partial_j^y A_j(\vec{y}) = \partial_i^y \delta(\vec{x} - \vec{y})$$

and which yields

$$\frac{\delta \phi(\vec{y})}{\delta A_i(\vec{x})} = \frac{1}{\partial_x^2} \partial_i^y \delta(\vec{x} - \vec{y}) \tag{6.6.9}$$

Also, from Eq. 6.6.4

$$\frac{\delta C_j(\vec{y})}{\delta A_i(\vec{x})} = \delta_{ij}\delta(\vec{x} - \vec{y}) - \partial_j^y \frac{\delta \phi(\vec{y})}{\delta A_i(\vec{x})} = (\delta_{ij} - \frac{\partial_i^y \partial_j^y}{\partial^2})\delta(\vec{x} - \vec{y})$$

Hence, from Eq. 6.6.7

$$\frac{\delta}{\delta A_i(\vec{x})} = \int_y \langle x|(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2})|y\rangle \frac{\delta}{\delta C_j(\vec{y})} + \int_y \frac{1}{\partial_x^2} \partial_i^y \frac{\delta}{\delta \phi(\vec{y})}$$

More explicitly

$$\frac{\delta}{\delta A_i(\vec{x})} = \int_y G_{ij}(x - y) \frac{\delta}{\delta C_j(\vec{y})} - \int_y \partial_i D(x - y) \frac{\delta}{\delta \phi(\vec{y})} \tag{6.6.10}$$

where

$$G_{ij}(x - y) = \langle x|(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2})|y\rangle = \delta ij \delta(x - y) + \partial_i \partial_j D(x - y)$$

and with the Coulomb potential given by

$$D(\vec{x}) = \langle \vec{x}| \frac{1}{-\partial^2} |\vec{y}\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2} = \frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|}$$

In compact notation

$$\frac{\delta}{\delta A_i} = \frac{\delta}{\delta C_i^T} + \frac{\partial_i}{\partial^2} \frac{\delta}{\delta \phi}$$

and the transverse electric field operator is given by

$$E_i^T = \frac{\delta}{i\delta C_i^T}$$

The transformed electric field operator is transverse since

$$\begin{aligned}\partial_i^x \frac{\delta}{\delta C_i^T(\vec{x})} &= \int_{\vec{y}} \partial_i^x (\delta_{ij} - \frac{\partial_i^x \partial_j^x}{\partial^2}) \delta(\vec{x}-\vec{y}) \frac{\delta}{\delta C_j(\vec{y})} \\ &= \int_{\vec{y}} (\partial_i^x - \frac{\partial^2 \partial_i^x}{\partial^2}) \delta(\vec{x}-\vec{y}) \frac{\delta}{\delta C_j(\vec{y})} = 0 \end{aligned} \tag{6.6.11}$$

The transverse electric operator is orthogonal to the generator of gauge transformations since

$$\int_{\vec{x}} \frac{\delta}{\delta C_i^T(\vec{x})} \frac{\partial_i}{\partial^2} \frac{\delta}{\delta \phi(\vec{x})} = \int_{\vec{x}} \Big[ - \partial_i \frac{\delta}{\delta C_i^T(\vec{x})} \cdot \frac{1}{\partial^2} \frac{\delta}{\delta \phi(\vec{x})} \Big] = 0$$

Hence, the electric field yields

$$\int_{\vec{x}} \vec{E}^2 = \int_{\vec{x}} \left[ \frac{\delta}{i\delta A_i(\vec{x})} \right]^2 = \int_{\vec{x}} \left[ \frac{\delta}{i\delta C_i^T(\vec{x})} \right]^2 + \int_{\vec{x}} \sum_i \left[ \frac{\partial_i}{\partial^2} \frac{\delta}{i\delta \phi(\vec{x})} \right]^2$$

Note the gauge term gives

$$\begin{aligned}\int_{\vec{x}} \sum_i \left[ \frac{\partial_i}{\partial^2} \frac{\delta}{i\delta\phi(\vec{x})} \right]^2 &= \int_{\vec{x},\vec{y},\vec{y}'} \partial_i^x D(\vec{x}-\vec{y}) \frac{\delta}{i\delta\phi(\vec{y})} \cdot [\partial_i^y D(\vec{x}-\vec{y}')] \frac{\delta}{i\delta\phi(\vec{y}')} \\ &= \int D(\vec{x}-\vec{y}) \partial^2 D(\vec{x}-\vec{y}') \frac{\delta^2}{i\delta\phi(\vec{y}) i\delta\phi(\vec{y}')} \\ &= \int D(\vec{x}-\vec{y}) \frac{\delta^2}{i\delta\phi(\vec{x}) i\delta\phi(\vec{y})} \end{aligned}$$

The kinetic term of the photon Hamiltonian is given by

$$\int_{\vec{x}} \vec{E}^2 = \int_{\vec{x}} (\vec{E}^T)^2 + \int \frac{\delta}{i\delta\phi_{\vec{x}}} \frac{1}{4\pi} \frac{1}{|\vec{x}-\vec{y}|} \frac{\delta}{i\delta\phi(\vec{y})}$$

Gauss's law yields the following

$$\partial_i E_i(\vec{x}) = \partial_i^x \frac{\delta}{\delta A_i(\vec{x})}$$

The constraint from Gauss's law completely factorizes from $C_i$, the physical degrees of freedom, and is expressed purely in terms of the gauge transformation. From Eq. 6.6.7, using the transversality condition given in Eq. 6.6.11, yields

$$\partial_i E_i(\vec{x}) = \partial_i^x \int \frac{\delta\phi(\vec{y})}{\delta A_i(x)} \frac{\delta}{i\delta\phi_{\vec{y}}} = \int \left[ \frac{\partial^2}{\partial^2} \delta(\vec{x}-\vec{y}) \right] \frac{\delta}{i\delta\phi_{\vec{y}}} = \frac{\delta}{i\delta\phi(\vec{x})}$$

Hence Gauss's law in the presence of a charge density, from Eq. 6.6.3, is given by

$$\left[\frac{\delta}{i\delta\phi(\vec{x})} - \rho(\vec{x})\right]|\Phi\rangle = 0$$

and which yields

$$\Phi[A] = e^{i\int\rho(\vec{x})\phi(\vec{x})}\Phi[C]$$

where $\Phi[A] = \Phi[C]$ is gauge invariant

For any arbitrary matrix element of the Hamiltonian between gauge invariant states

$$\begin{aligned}\langle\Phi|H|\Phi\rangle &= \langle\Phi|e^{-i\int\rho\phi}\left[H + \frac{1}{2}\int\frac{\delta}{i\delta\phi_x}\frac{1}{4\pi(\vec{x}-\vec{y})}\frac{\delta}{i\delta\phi_{\vec{y}}}\right]e^{i\int\rho\phi}|\Phi\rangle \\ &= \langle\Phi|\tilde{H}|\Phi\rangle\end{aligned}$$

with

$$\tilde{H} = \frac{1}{2}\int_{\vec{x}}\{(\vec{E}^T)^2 + \vec{B}^2\} + \frac{1}{2}\int_{\vec{x},\vec{y}}\rho(\vec{x})\frac{1}{4\pi|\vec{x}-\vec{y}|}\rho(\vec{y}) \qquad (6.6.12)$$

: Hamiltonian in the Coulomb gauge

The instantaneous Coulomb potential arises from gauge fixing. It is not retarded and may raise question about Lorentz invariance. The theory in fact gives the correct expressions and the electric field is

$$\vec{E} = \vec{E}^T + \frac{1}{\partial^2}\vec{\nabla}\frac{\delta}{i\delta\phi} \qquad (6.6.13)$$

### *6.6.1 Coulomb gauge normal mode expansion*

In Minkowski space, the expansion of the gauge field $\vec{C}(t,\vec{x})$ in Coulomb gauge can be written by considering the gauge field as a collection of free scalar fields that has an normal mode expansion similar to the one given in Eq. 4.5.4. This yields the following expansion in terms of the creation and destruction operators

$$\vec{C}(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}(\sum_{\lambda=1}^{2}\{a_{\vec{k}\lambda}e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}\lambda}e^{i\omega_k t - i\vec{k}\cdot\vec{x}}\}\vec{\epsilon}_{\vec{k}\lambda} \qquad (6.6.14)$$

where $\omega_{\vec{k}} = |\vec{k}|$ and $\vec{\epsilon}_{\vec{k}\lambda}$ are the polarization vectors. The Coulomb gauge condition is given by

$$\nabla\cdot\vec{C}(t,\vec{x}) = 0 \quad\Rightarrow\quad \vec{k}\cdot\vec{\epsilon}_{\vec{k}\lambda} = 0$$

The photon creation and annihilation operators have commutation equations

$$[a_{\vec{k}'\lambda}, a^{\dagger}_{\vec{k}\lambda'}] = \delta_{\lambda\lambda'}\delta_{\vec{k}-\vec{k}'} \tag{6.6.15}$$

The electric field operator is given by $\tilde{E}_i = \partial_0 C_i$ with an expansion similar to Eq. 4.5.5 and yields

$$\partial_0 \vec{C} = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \sum_{\lambda} \Big( a_{\vec{k}\lambda} e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} - a^{\dagger}_{\vec{k}\lambda} e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \Big) \vec{\epsilon}_{\vec{k}\lambda}$$

The Hamiltonian, in the Coulomb gauge, is given by

$$\hat{H} = \frac{1}{2} \int d^3x \Big[ (\vec{E}^T)^2 + \vec{B}^2 \Big] = \frac{1}{2} \int d^3x \Big[ (\partial_0 \vec{C})^2 + (\nabla \times \vec{C})^2 \Big] \tag{6.6.16}$$

The normal mode expansion of the Hamiltonian is given by

$$\hat{H} = \sum_{\lambda=1}^{2} \int_{\vec{k}} \omega_{\vec{k}} a^{\dagger}_{\vec{k}\lambda} a_{\vec{k}\lambda} + [\frac{1}{2} \cdot 2 \int_{k} \omega_{\vec{k}}] \; \delta(\vec{0}) \tag{6.6.17}$$

The spin operator of the photon field is given by

$$S^I = \epsilon^{Iij} \int d^3x \partial_0 C^j C^i$$

The normal mode expansions given in Eqs. 6.6.14 and 6.6.16 yield, after some simplifications and dropping a normal ordering infinite constant, the following

$$\vec{S} = \frac{i}{2} \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^3} \Big( a^{\dagger}_{\vec{k}\lambda} a_{\vec{k}\lambda'} - a^{\dagger}_{\vec{k}\lambda'} a_{\vec{k}\lambda} \Big) \vec{\epsilon}_{\vec{k}\lambda'} \times \vec{\epsilon}_{\vec{k}\lambda}$$

Let $\vec{k} = |k|\vec{e}_3$; choose polarization vectors

$$\vec{\epsilon}_{\vec{k}1} = \vec{e}_1(\vec{k}) \;\; ; \;\; \vec{\epsilon}_{\vec{k}2} = \vec{e}_2(\vec{k})$$

Then

$$\vec{S} = i \int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}}{|k|} \Big( a^{\dagger}_{\vec{k}2} a_{\vec{k}1} - a^{\dagger}_{\vec{k}1} a_{\vec{k}2} \Big) = \int \frac{d^3k}{(2\pi)^3} \frac{\vec{k}}{|k|} \Big( a^{\dagger}_{\vec{k}+} a_{\vec{k}+} - a^{\dagger}_{\vec{k}-} a_{\vec{k}-} \Big)$$

where the helicity basis is defined by

$$a^{\dagger}_{\vec{k}\pm} = \frac{1}{\sqrt{2}} (a^{\dagger}_{\vec{k}1} \pm i a^{\dagger}_{\vec{k}2})$$

The spin operator $\vec{S}$ is diagonal in the helicity basis, with each momentum degree of freedom having a spin along and in the opposite direction to the direction of momentum.

## 6.7 Faddeev-Popov quantization

From Eq. 6.2.5, the Faddeev-Popov counter-term is given by

$$e^{S_c} = 1/\int D\phi\delta(s(\phi))$$

Consider

$$s(\phi) = \partial_\mu A_\mu - \partial^2\phi \quad \Rightarrow \quad Ds = \det(-\partial^2)D\phi$$

Hence

$$e^{S_c} = \det(-\partial^2)\frac{1}{\int Ds\delta(s)} = \det(-\partial^2) \tag{6.7.1}$$

The representation of the determinant $\det(-\partial^2)$ using fermion integration, reviewed in Section 6.12 and discussed in detail in Baaquie (2014), is fundamental to the Faddeev-Popov formulation of constrained systems, and of gauge fields in particular. The full power of the Faddeev-Popov formulation comes to the fore in the study of Yang-Mills nonabelian gauge fields. We foreground the case of nonabelian gauge fields by developing discussing the essential ideas in the more simple case of the photon field. Fermion integration is reviewed in Appendic 6.12.

The gauge-fixed Euclidean action $S_{GF}$, from Eq. 6.2.11, is given by

$$S_{GF} = -\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{\alpha}{2}\int d^4x s^2 + \ln(\det(-\partial^2))$$

Let $\bar{c}, c$ be two complex fermion scalar fields, called ghost fields for reasons discussed below. The result of fermion integration yields

$$\det(-\partial^2) = \int D\bar{c}Dc\exp\{S_{FP}\} \tag{6.7.2}$$

The Faddeev-Popov determinant requires the ghost field $\bar{c}$, $c$ to be complex fermionic fields due to the result for fermion Gaussian integration given in Eq. 6.12.9.[6]

The Faddeev-Popov ghost field action, using Eq. 6.12.9, is given by

$$S_{FP} = -\int d^4x\bar{c}(-\partial^2)c = \int d^4x\partial_\mu\bar{c}\partial_\mu\ c \tag{6.7.3}$$

The gauge-fixed action, with a covariant gauge-fixing term, is given by

$$\begin{aligned} S_{GF} &= -\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{\alpha}{2}\int d^4x(\partial_\mu A_\mu)^2 + \int d^4x\partial_\mu\bar{c}\partial_\mu\ c \\ &= S + S_\alpha + S_{FP} \end{aligned} \tag{6.7.4}$$

[6] Real fermionic variables would yield $\sqrt{M}$ instead of $\det M$, and has been discussed in detail in **?**.

The fermionic ghost field have spin zero and hence are not physical – since all physical spacetime fermions must have half integer spin. Nevertheless, the ghost field plays a key role in the study of gauge field theories and string theory and their full significance in gauge field theory is mysterious has not yet been understood. Since the fields $\bar{c}$, $c$ do not obey the spin-statistic rule that requires that all fermions must have half-integer spin, they are for this reason called ghost fields.

### 6.8 Ghost state space and Hamiltonian

The state space of the quantum field is determined by the time derivative terms in the action; the reason being that the time derivative couples the gauge field at two different instants and at each instant, the gauge field is a coordinate of the underlying state space. The ghost action is similar to the action for the complex scalar field discussed in Section 4.12. Hence, the state space has the fermion coordinate eigenstates given by

$$|\bar{c}, c\rangle = |\bar{c}\rangle \otimes |c\rangle \equiv \prod_{\vec{x}} |\bar{c}(\vec{x})\rangle \otimes |c(\vec{x})\rangle$$

The completeness equation is given by

$$\int D\bar{c}Dc|\bar{c}, c\rangle\langle\bar{c}, c| = \mathbb{I}$$

The inner product is given by

$$\langle\bar{c}, c|\bar{c}', c'\rangle = \delta(\bar{c} - \bar{c}')\delta(c - c')$$

Due to the rules of fermion calculus, we have

$$\delta(\bar{c} - \bar{c}')\delta(c - c') = -(\bar{c} - \bar{c}')(c - c') \tag{6.8.1}$$

To prove above statement, consider an arbitrary function $f(\bar{c}, c)$ of $\bar{c}, c$ with the following Taylor expansion

$$f(\bar{c}, c) = \alpha + \beta\bar{c} + \gamma c + \omega\bar{c}c$$

Using the rules of fermion integration given in Section 6.12 yields

$$\int d\bar{c}dc f(\bar{c}, c)(\bar{c} - \bar{c}')(c - c') = \int d\bar{c}dc \left(\alpha\bar{c}c - \beta\bar{c}\bar{c}'c - \gamma c\bar{c}' + \omega\bar{c}c\bar{c}'c' + \cdots\right)$$
$$= \int d\bar{c}dcc\bar{c} \left(\alpha + \beta\bar{c}' + \gamma c' + \omega\bar{c}'c' + \cdots\right) = -f(\bar{c}', c')$$

where $\cdots$ refers to terms that go to zero. Hence we have verified Eq. 6.8.1. In general, for $N$ complex fermions, one has

$$\prod_{n=1}^{N} \delta(\bar{c}_n - \bar{c}'_n)\delta(c_n - c'_n) = (-1)^N \prod_{n=1}^{N} (\bar{c}_n - \bar{c}'_n)(c_n - c'_n)$$

The gauge-fixed action in the Feynman gauge, with $\alpha = 1$, from Eq. 6.7.4 is given by

$$S_{GF} = -\frac{1}{2}\int d^4x (\partial_\mu A_\nu)^2 + \int d^4x \partial_\mu \bar{c} \partial_\mu \, c \tag{6.8.2}$$

In the covariant gauge, the gauge field state space requires all four components of the gauge field the completeness equation is given by

$$\mathbb{I} = \prod_{\vec{x}} \int dA_\mu(\vec{x}) |A_\mu(\vec{x})\rangle\langle A_\mu(\vec{x})| \equiv \int DA_\mu |A_\mu\rangle\langle A_\mu|$$

The completeness equation for the gauge plus ghost field is given by

$$\mathbb{I} = \int DA_\mu D\bar{c} Dc |\bar{c}, c; A_\mu\rangle\langle \bar{c}, c; A_\mu| \tag{6.8.3}$$

The connection of the state space and action for the quantum mechanical case in given in Eq 2.4.1; the Dirac-Feynman formula for Euclidean time is given by

$$\langle \bar{c}, c; A_\mu | e^{-\epsilon H} | \bar{c}', c'; A'_\mu \rangle = \mathcal{N}(\epsilon) e^{\epsilon \mathcal{L}(\bar{c}, \bar{c}', c, c'; A'_\mu \epsilon)} \tag{6.8.4}$$

where $\mathcal{N}(\epsilon)$ is a normalization.

The action consists of two decoupled free fields; the Hamiltonian is the sum of the gauge field and ghost Hamiltonians and given by

$$H = H_A + H_G \tag{6.8.5}$$

To obtain the Hamiltonian of the gauge field the steps in Section 4.3 can be repeated and yields, from Eq. 4.3.3, the Hamiltonian

$$\begin{aligned} H_A &= -\frac{1}{2}\int d^3x \frac{\delta^2}{\delta A_\mu^2(\vec{x})} + \frac{1}{2}\int d^3x (\partial_i A_\nu(\vec{x}))^2 \quad ; \quad i = 1, 2, 3 \\ &= \frac{1}{2}\int d^3x [\pi_\mu^2 + (\partial_i A_\nu)^2] \quad ; \quad \pi_\mu(\vec{x}) = -i\frac{\delta}{\delta A_\mu(\vec{x})} \end{aligned} \tag{6.8.6}$$

To obtain the ghost field Hamiltonian, note that the ghost Lagrangian density, from Eq. 6.8.2, is given by

$$\epsilon \mathcal{L}_G = \frac{1}{\epsilon}(\bar{c} - \bar{c}')(c - c') + \epsilon \partial_i \bar{c} \partial_i \, c \tag{6.8.7}$$

The fermion identity

$$\langle \bar{c}, c| \exp\{-\epsilon \frac{\delta^2}{\delta \bar{c} \delta c}\} |\bar{c}', c'\rangle = \exp\{-\epsilon \frac{\delta^2}{\delta \bar{c} \delta c}\} \delta(\bar{c} - \bar{c}') \delta(c - c')$$
$$= -(1 - \epsilon \frac{\delta^2}{\delta \bar{c} \delta c})(\bar{c} - \bar{c}')(c - c') = -\Big[(\bar{c} - \bar{c}')(c - c') + \epsilon\Big]$$
$$= -\epsilon \exp\{\frac{1}{\epsilon}(\bar{c} - \bar{c}')(c - c')\} \tag{6.8.8}$$

yields, from Eqs. 6.8.4, 6.8.7 and 6.8.8, the ghost field Hamiltonian

$$H_G = \int d^3x \Big[\frac{\delta^2}{\delta \bar{c}(\vec{x}) \delta c(\vec{x})} - \partial_i \bar{c}(\vec{x}) \partial_i c(\vec{x})\Big] \tag{6.8.9}$$

### *6.8.1 BRST cohomology and physical state space*

The Coulomb gauge, discussed in Section 6.6, is manifestly non-covariant and explicitly breaks Lorentz invariance. The gauge condition is $A_0 = 0$ supplemented by imposing the transversality condition on the state space given by $\partial_i A_i = 0$.

One can also obtain a state space description of the photon field using a c*ovariant gauge* that respects Lorentz symmetry. For example, consider the Lorentz gauge defined by $\partial_\mu A_\mu = 0$; one can consistently quantize the photon field using the Gupta-Bleuler formalism. However, this approach cannot be generalized to Yang-Mills non-Abelian gauge fields.

The BRST method of quantization is a modern formulation of the Gupta-Bleuler approach that is equally valid for the Yang-Mills fields. The ghost fields are used to remove the extra degrees of freedom for the case of a covariant (Lorentz invariant) gauge that can be generalized to the Yang-Mills case.

The BRST method has the following ingredients. The gauge-fixed action has a BRST symmetry with a fermionic BRST charge operator $Q_B$. The BRST symmetry yields a conserved charge such that $[H, Q_B] = 0$, where $H$ is the Hamiltonian of the gauge fixed theory given in Eq. 6.8.5. The state space is enlarged to include the ghost field $\mathcal{V}_A \otimes \mathcal{V}_{\bar{c},c}$ with completeness on this state space given by Eq. 6.8.3. The physical state space is defined, for $|\Phi\rangle \in \mathcal{V}_A \otimes \mathcal{V}_{\bar{c},c}$.

BRST cohomology given below defines the physical gauge invariant state space.

- The operator $Q_B$ is nilpotent in the sense that $Q_B^2 = 0$.

- The exact states are of the form $Q_B|\chi\rangle$ that are automatically annihilated by $Q_B$. These exact states are states that correspond to pure gauge transformations such that $A_\mu = \partial_\mu \phi$.
- Physical states $|\Phi\rangle$ are *constrained* to be annihilated by $Q_B$ and are said to closed under $Q_B$ and obey $Q_B|\Phi\rangle = 0$. Since charge $Q_B$ is conserved, the constraint is conserved over time.
- Physical states $|\Phi\rangle$ are not exact, that is $|\Phi\rangle \neq Q_B|\chi\rangle$.
- States that can be written as $|\Phi\rangle + Q_B|\chi\rangle$ are all equivalent to $|\Phi\rangle$ and can be shown to differ from it by only a gauge transformation.
- Physical states $|\Phi\rangle$ are precisely the gauge invariant states of the photon field.

The BRST formalism is shown in Section 6.10.1 to reduce to the Gupta-Bleuler scheme for the case of the Abelian gauge field.

## 6.9 BRST charge $Q_B$

The gauge fixed action given in Eq. 6.8.2 has a so-called BRST invariance, which is the result of choosing a gauge and its compensating Fadeev-Popov counter term. The Noether current due to BRST invariance yields the BRST charge $Q_B$ [Das (2006), Zinn-Justin (1993)].

It is convenient to write the gauge-fixed action in terms of an auxiliary field as this simplifies the derivation. From Eq. 6.8.2, in the Feynman gauge with $\alpha = 1$, the gauge fixed action is given by

$$S_{GF} = -\frac{1}{4}\int d^4x F_{\mu\nu}^2 - \frac{1}{2}\int d^4x (\partial_\mu A_\mu)^2 + \int d^4x \partial_\mu \bar{c}\partial_\mu\, c \quad (6.9.1)$$

and is re-written using an auxiliary field $G$ as follows

$$e^{S_{GF}} = \int DG e^{\tilde{S}} \;\; ; \;\; \tilde{S} = \int d^4x \mathcal{L} \quad (6.9.2)$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}G^2 + i\partial_\mu G A_\mu + \partial_\mu \bar{c}\partial_\mu\, c \quad (6.9.3)$$

Consider a fermionic parameter $\lambda$ such that

$$\lambda^2 = 0 \;\; ; \;\; \{\lambda, \bar{c}\} = 0 = \{\lambda, c\}$$

The BRST transformation consists of an infinitesimal gauge transformation for $A_\mu$ together with the following transformation of the ghost and auxiliary fields

$$\delta A_\mu = \lambda \partial_\mu c \;\; ; \;\; \delta \bar{c} = -i\lambda G \;\; ; \;\; \delta c = 0 = \delta G \quad (6.9.4)$$

Note one can define a BRST transformation with $\delta\bar{c} = 0$ and $\delta c \neq 0$; this gives rise to the same BRST charge.

The BRST variation of Eq. 6.9.2 is given by

$$\delta\tilde{S} = i\lambda \int \partial_\mu G \partial_\mu c + i\lambda \int G\partial^2 c = 0$$

BRST invariance yields a conserved fermionic charge $Q_B$ using the Noether current discussed in Section 3.2. From Eq. 3.3.2 the BRST current is given by the variation of the Lagrangian given in Eq. 6.9.3. Using

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}(-\frac{1}{4}F_{\mu\nu}^2) = F_{\nu\mu} = -F_{\mu\nu}$$

the variation of the Lagrangian in Eq. 6.9.2 under the BRST symmetry given in Eq. 6.9.4 yields

$$j_\mu = \delta A_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} + \delta\bar{c}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{c})} = \lambda\partial_\nu c F_{\nu\mu} - i\lambda G\partial_\mu c \tag{6.9.5}$$

Since the BRST current is conserved $\partial_\mu j_\mu = 0$, dropping the parameter $\lambda$ yields the conserved BRST charge

$$\begin{aligned} Q_B &= \int d^3x j_0 = \int d^3x \left(\partial_i c F_{i0} - iG\partial_0 c\right) \\ &= \int d^3x \left(-c\partial_i F_{i0} - iG\partial_0 c\right) \end{aligned}$$

The Euler-Lagrange equation for $A_\mu$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial\mathcal{L}}{\partial A_\nu} \quad \Rightarrow \quad \partial_\mu F_{\nu\mu} = i\partial_\nu G$$

leads to the following

$$Q_B = \int d^3x \left(ic\partial_0 G - iG\partial_0 c\right)$$

The Euler-Lagrange equation for $G$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu G)}\right) = \frac{\partial\mathcal{L}}{\partial G} \quad \Rightarrow \quad i\partial_\mu A_\mu = -G$$

yields the final expression for BRST charge

$$Q_B = \int d^3x \Big(c(\vec{x})\partial_0(\partial_\mu A_\mu(\vec{x})) - (\partial_\mu A_\mu(\vec{x}))\partial_0 c(\vec{x})\Big) \tag{6.9.6}$$

## 6.10 $Q_B$ and state space

The conserved charge is an operator $Q_B$ that acts on the state space of the gauge fixed action. To obtain an explicit representation of the operator $Q_B$ the operator representation of the quantum fields $A_\mu$ and $\bar{c}, c$ are written in terms of creation and destruction operators.

Since the auxiliary field has been removed in Eq. 6.9.6, $Q_B$ follows from the gauge fixed action given by Eq. 6.9.1

$$S_{GF} = -\frac{1}{2}\sum_{\mu,\nu}\int d^4x(\partial_\mu A_\nu)^2 + \int d^4x \partial_\mu \bar{c}\partial_\mu c \tag{6.10.1}$$

The gauge field consists of four free scalar fields and the ghost field is the fermionic version of the free complex scalar field. Hence we can use the results of Chapter 4 on free fields with some minor modifications.

The normal mode expansion in Euclidean time $t = -i\tau$ of the gauge field, similar to the case for free field given in Eq. 4.16.7, is the following

$$A_\mu(\tau,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}(e^{iE_{\vec{p}}\tau+i\vec{p}\cdot\vec{x}}a_{\vec{p}\mu} + e^{-iE_{\vec{p}}\tau-i\vec{p}\cdot\vec{x}}a^\dagger_{\vec{p}\mu}) \; ; E_{\vec{p}} = |\vec{p}| \tag{6.10.2}$$

The equal time canonical commutation equations given in Eq. 4.3.5 yields

$$[a_{\vec{p}\mu}, a^\dagger_{\vec{k}\nu}] = (2\pi)^3\delta^3(\vec{p}-\vec{k})\delta_{\mu-\nu},$$

with the rest of the commutators being zero.

The ghost field are similar to the complex scalar fields, as discussed in Section 4.12, and one has the expansion for Euclidean time given by

$$c(\tau,\vec{x}) = \int_{\vec{p}}\frac{1}{\sqrt{2E_{\vec{p}}}}(e^{iE_{\vec{p}}\tau+i\vec{p}\cdot\vec{x}}A_{\vec{p}} + e^{-iE_{\vec{p}}\tau-i\vec{p}\cdot\vec{x}}B^\dagger_{\vec{p}}) \; ; \; \int_{\vec{p}} = \int\frac{d^3p}{(2\pi)^3} \tag{6.10.3}$$

$$\bar{c}(\tau,\vec{x}) = \int_{\vec{p}}\frac{1}{\sqrt{2E_{\vec{p}}}}(e^{-iE_{\vec{p}}\tau-i\vec{p}\cdot\vec{x}}A^\dagger_{\vec{p}} + e^{iE_{\vec{p}}\tau+i\vec{p}\cdot\vec{x}}B_{\vec{p}})$$

The equal time canonical anticommutation equations

$$\{\frac{\partial c(\tau,\vec{x})}{\partial\tau}, c(\tau,\vec{x}')\} = \delta^3(\vec{x}-\vec{x}') = \{\frac{\partial\bar{c}(\tau,\vec{x})}{\partial\tau}, \bar{c}(\tau,\vec{x}')\}$$

yield

$$\{A_{\vec{p}}, A^\dagger_{\vec{p}'}\} = \{B_{\vec{p}}, B^\dagger_{\vec{p}'}\} = \delta^3(\vec{p}-\vec{p}')$$

and with all the other anticommutators being zero.

The gauge field Hamiltonian is given by Eq. 4.16.8 and the ghost Hamiltonian is similar to the complex scalar field given in Eq. 4.12.4. Hence[7]

$$\begin{aligned} H &= H_A + H_G \\ &= \int_{\vec{p}} E_{\vec{p}} \Big[ a^\dagger_{\vec{p}\mu} a_{\vec{p}\mu} + A^\dagger_{\vec{p}} A_{\vec{p}} + B^\dagger_{\vec{p}} B_{\vec{p}} \Big] + E_0 \end{aligned} \tag{6.10.4}$$

Substituting Eqs. 6.10.3 and 6.10.3 into the expression for $Q_B$ given in Eq. 6.9.6, and after some algebra, yields

$$Q_B = -\int_{\vec{p}} (p_\mu a^\dagger_{\vec{p}\mu} A_{\vec{p}} + p_\mu a_{\vec{p}\mu} B^\dagger_{\vec{p}}) \tag{6.10.5}$$

To verify that $Q_B^2 = 0$ consider the following

$$\begin{aligned} Q_B^2 &= \frac{1}{2}\{Q_B, Q_B\} \\ &= \int_{\vec{p},\vec{k}} \Big\{ p_\mu a^\dagger_{\vec{p}\mu} k_\nu a^\dagger_{\vec{k}\nu} A_{\vec{p}} A_{\vec{k}} + p_\mu a_{\vec{p}\mu} k_\nu a_{\vec{k}\nu} B^\dagger_{\vec{p}} B^\dagger_{\vec{k}} \Big\} + \int_{\vec{p},\vec{k}} p_\mu k_\nu [a_{\vec{p}\mu}, a^\dagger_{\vec{k}\nu}] B^\dagger_{\vec{p}} A_{\vec{k}} \\ &= \int_{\vec{p}} p_\mu p_\mu B^\dagger_{\vec{p}} A_{\vec{p}} \end{aligned} \tag{6.10.6}$$

Since all state vectors are defined for Minkowski spacetime, we need to analytically continue time back from Euclidean to Minkowski spacetime.

Note that $p_\mu$ is a four-vector in Euclidean spacetime; let $p^M_\mu$ be the momentum four vector in Minkowski spacetime that corresponds to $p_\mu$. Then from Eq. 4.16.1

$$E_{\vec{p}} = |\vec{p}| = p_0 = i p^M_0 \quad ; \quad p_i = p^M_i \quad : i = 1, 2, 3$$

From Eq. 6.10.2 that $E_{\vec{p}} = |\vec{p}|$ and hence

$$p_\mu p_\mu \to -p^{M\mu} p^M_\mu = -(p_0)^2 + \vec{p}^2 = -|\vec{p}|^2 + \vec{p}^2 = 0$$

Hence, from Eq. 6.10.6

$$p^{M\mu} p^M_\mu = 0 \quad \Rightarrow \quad Q_B^2 = \int_{\vec{p}} p^{M\mu} p^M_\mu B^\dagger_{\vec{p}} A_{\vec{p}} = 0$$

Hence we have the important result that as an *operator* $Q_B$ is nilpotent, namely that $Q_B^2 = 0$.

[7] $E_0 = 2\delta^{(3)}(0) \int d^3p E_{\vec{p}}$.

To show that $Q_B$ is conserved, consider the following.

$$[Q_B, H_A] = -\int_{\vec{p}} (-p_\mu a^\dagger_{\vec{p}\mu} A_{\vec{p}\mu} + p_\mu a_{\vec{p}\mu} B^\dagger_{\vec{p}}) \tag{6.10.7}$$

$$[Q_B, H_G] = -\int_{\vec{p}} (p_\mu a^\dagger_{\vec{p}\mu} A_{\vec{p}\mu} - p_\mu a_{\vec{p}\mu} B^\dagger_{\vec{p}}) \tag{6.10.8}$$

From above and Eq. 6.10.4

$$[Q_B, H] = [Q_B, H_A + H_G] = 0$$

and hence we have confirmed that for the quantized theory BRST charge is conserved.

As discussed in Section 6.8.1, every vector $|\Phi\rangle$ in the physical gauge invariant state space is annihilated by $Q_B$, that is

$$Q_B|\Phi\rangle = 0 \tag{6.10.9}$$

### *6.10.1 Gupta-Bleurel condition*

It is shown how the definition of state space given by BRST quantization reduces to the Gupta-Bleurel constraint on state space when the gauge field state space is considered by itself, without the presence of the ghost field.

From Eqs. 6.10.6 and 6.10.9

$$0 = Q_B|\Phi\rangle = -\int_{\vec{p}} (p_\mu a^\dagger_{\vec{p}\mu} A_{\vec{p}} + p_\mu a_{\vec{p}\mu} B^\dagger_{\vec{p}})|\Phi\rangle$$

Since the gauge and ghost field are decoupled, the physical state vector is taken to be a tensor product

$$|\Phi\rangle = |\Phi_A\rangle|\Phi_G\rangle$$

Ghost number is conserved and hence $|\Phi_G\rangle$ is taken to have zero ghost number. From the expression for $Q_B$ given in Eq. 6.10.6, the ghost state is taken to be the ground state $|\Phi_G\rangle = |\Omega_G\rangle$ that is annihilated by $A_{\vec{p}}$ and yields[8]

$$A_{\vec{p}}|\Omega_G\rangle = 0 = B_{\vec{p}}|\Omega_G\rangle$$

[8] It can be verified by using the ghost Hamiltonian $H_G$ given in Eq. 6.8.9 that

$$H_G|\Omega_G\rangle = 0 \quad \Rightarrow \quad \langle \bar{c}, c|\Omega_G\rangle = \mathcal{N} \exp\{-\int \frac{d^3p}{(2\pi)^3} |p| \bar{c}(\vec{p}) c(\vec{p})\}$$

where $\bar{c}(\vec{p}); c(\vec{p})$ are the Fourier transform of the coordinate basis $\bar{c}(\vec{x}); c(\vec{x})$.

Hence the BRST constraint reduces to

$$Q_B|\Phi\rangle = -i\int_{\vec{p}} p_\mu a_{\vec{p}\mu}|\Phi_A\rangle B^\dagger_{\vec{p}}|\Omega_G\rangle = 0$$

Since $B^\dagger_{\vec{p}}|\Omega_G\rangle \neq 0$, to achieve $Q_B|\Phi\rangle = 0$ the following constraint is imposed on the gauge field state vectors

$$Q_B|\Phi\rangle = 0 \quad \Rightarrow \quad p_\mu a_{\vec{p}\mu}|\Phi_A\rangle = 0$$

This constraint can be written more transparently in real space; from Eq. 6.10.2

$$\partial_\mu A_\mu(\vec{x}) = \partial_\mu A^{(-)}_\mu(\vec{x}) + \partial_\mu A^{(+)}_\mu(\vec{x})$$

where $A^{(-)}_\mu$ has all the annihilation operators and $A^{(+)}_\mu$ has all the creation operators. The BRST constraint yields the following

$$p_\mu a_{\vec{p}\mu}|\Phi_A\rangle = 0 \quad \Rightarrow \quad \partial_\mu A^{(-)}_\mu(\vec{x})|\Phi_A\rangle = 0 \ : \ \text{Gupta-Bleuler condition}$$

Note that, since

$$\langle\Phi_A|\partial_\mu A^{(+)}_\mu = 0$$

the Gupta-Bleuler condition implies that for any physical state $|\Phi_A\rangle$

$$\langle\Phi_A|\partial_\mu A_\mu|\Phi_A\rangle = \langle\Phi_A|\left(\partial_\mu A^{(+)}_\mu + \partial_\mu A^{(-)}_\mu\right)|\Phi_A\rangle = 0$$

In other words, instead of imposing the gauge condition on the operator, namely $\partial_\mu A_\mu = 0$, in the Gupta-Bleuler approach the weaker condition is imposed that the expectation value of the operator $\partial_\mu A_\mu$ for any gauge invariant (physical) state has to be zero.

In conclusion, the physical gauge invariant state space is given by the constraint

$$Q_B|\Phi\rangle = 0 \ ; \ \ |\Phi\rangle = |\Phi_A\rangle|\Omega_G\rangle$$

The physics of the BRST constraint on state space has been discussed in Peskin and Schroeder (1995).

## 6.11 Discussion

The quantum theory of the Abelian gauge field revolves around the concept of gauge invariance and the main focus of the discussion was on analyzing the various ramifications of gauge symmetry. Although the photon is a free quantum field, a number of complexities arise in addressing the concept of gauge invariance.

It seems that only gauge invariant *vector fields* can be consistently defined as a quantum field theory. Gauge invariance entails extra non-physical degrees of freedom and the gauge-fixing removes the redundant degrees of freedom in a vector field.

There are many ways of gauge-fixing the theory. One can remove the unphysical degrees of freedom by for example, choosing one component, such as $A_0$ in the temporal gauge and set it to zero. This choice of gauge leaves a very asymmetric action that is suitable for studying the Hamiltonian and state space – but is not amenable to Feynman perturbation expansion. The Hamiltonian in the Coulomb gauge was derived to show the complexity of choosing a gauge for a differential operator.

A symmetric way of removing the unphysical degrees of freedom is to choose a covariant gauge. This choice of gauge is compensated by the Fadeev-Popov counter term, which in turn introduces the concept of the ghost field. Fadeev-Popov quantization is most suitable for using Feynman perturbation theory. A Hamiltonian and gauge invariant state space – using cohomology of the nilpotent BRST charge operator $Q_B$ – was derived for the covariant gauge.

Although the ghost field is an artifact of gauge fixing, it seems to have mathematical significance that is yet to be fully understood since it also appears in superstring theory. The ghost field brings about a marvelously compact re-organization of the mathematics of both the photon field's path integral as well its state space.

## 6.12 Appendix: Fermion calculus

Fermionic calculus has been discussed in detail in **?**. The results here are focused are on the Gaussian path integrals for fermionic variables.

An independent and self-contained formalism for realizing all the defining properties of fermions is provided by a set of *anti-commuting fermionic variables* $\psi_1, \psi_2, \ldots \psi_N$ and its complex conjugate fermionic variables $\bar{\psi}_1, \bar{\psi}_2, \ldots \bar{\psi}_N$ defined by the following properties

$$\{\psi_i, \psi_j\} = -\{\psi_i, \psi_j\} \quad ; \quad \{\bar{\psi}_i, \psi_j\} = -\{\psi_j, \bar{\psi}_i\} \quad ; \quad \{\bar{\psi}_i, \bar{\psi}_j\} = -\{\bar{\psi}_j, \bar{\psi}_i\}$$

Hence, it follows that

$$\psi_i^2 = 0 = \bar{\psi}_i^2$$

Noteworthy 6.2: Real and complex fermions

Fermions, like ordinary real variables, can be real and complex. Consider a real fermion $\chi$; its defining property is

$$\chi^2 = 0 \quad \Rightarrow \quad f(\chi) = a = b\chi \quad \text{with} \quad a, b : \text{real numbers}$$

A complex fermion $\psi$ and its complex conjugate $\hat{\psi}$ is given by real fermions $\chi, \xi$ as follows

$$\psi = \chi + i\xi \quad ; \quad \bar{\psi} = \chi - i\xi$$

Fermionic differentiation is defined by

$$\frac{\delta}{\delta\psi_i}\psi_j = \delta_{i-j} \quad ; \quad \frac{\delta}{\delta\psi_i}\bar{\psi}_j = 0$$

and

$$\frac{\delta^2}{\delta\psi_i\delta\psi_j} = -\frac{\delta^2}{\delta\psi_j\delta\psi_i} \quad \Rightarrow \quad \frac{\delta^2}{\delta\psi_i^2} = 0 = \frac{\delta^2}{\delta\bar{\psi}_i^2}$$

Similarly all the fermionic derivative operators $\delta/\delta\psi_i$,$\delta/\delta\bar{\psi}_i$ anti-commute.

Similar to the case of $\int_{-\infty}^{+\infty} dx f(x)$ which is invariant under $x \to x + a$, that is $\int_{-\infty}^{+\infty} dx f(x) = \int_{-\infty}^{+\infty} dx f(x + a)$, *define* fermion integration by the following

$$\int d\bar{\psi} f(\bar{\psi}) = \int d\bar{\psi} f(\bar{\psi} + \bar{\eta}) \tag{6.12.1}$$

Since $\bar{\psi}^2 = 0$, Taylors expansion shows that the most general function of the variable $\bar{\psi}$ is given by

$$f = a + b\bar{\psi}$$

It follows that rules of fermion integration that yield Eq. 6.12.1 are given by the following

$$\begin{gathered} \int d\bar{\psi} = 0 = \int d\psi \\ \int d\bar{\psi}\bar{\psi} = 1 = \int d\psi\psi \\ \int d\bar{\psi}\bar{\psi} d\psi\psi = 1 = -\int d\bar{\psi} d\psi \bar{\psi}\psi \end{gathered} \tag{6.12.2}$$

For $N$ fermionic variables $\psi_i$, with $i = 1, 2, ..N$, one has the generalization

$$\left[\prod_{n=1}^{N} \int d\psi_n\right] \psi_{i_1}\psi_{i_2}...\psi_{i_n} = \epsilon_{i_1,i_2,..i_n} \tag{6.12.3}$$

where $\epsilon_{i_1,i_2,..i_n}$ is the completely antisymmetric epsilon tensor.

Consider a change variables for a single variable, namely

$$\psi = a\chi + \zeta$$

where $a$ is a constant and $\zeta$ is a constant fermion. From Eq. 6.12.2, the non-zero fermion integral yields

$$1 = \int d\psi\psi = \int d\psi(a\chi + \zeta) = \int d\psi a\chi \quad \Rightarrow \quad d\psi = \frac{1}{a} d\chi \tag{6.12.4}$$

Note this is the inverse for the case of real variables, since $x = ay$ yields $dx = ady$.

For the case of $N$ fermions, the anti-symmetric matrix $M_{ij}) = -M_{ji}$ yields the following change of variables

$$\psi_i = \sum_{j=1}^{N} M_{ij}\chi_j \quad \Rightarrow \quad \psi = M\chi$$

Similar to Eq. 6.12.4, it follows that

$$\prod_{i=1}^{N} d\psi_i = \frac{1}{\det M} \prod_{j=1}^{N} d\chi_j \quad \Rightarrow \quad D\psi = \frac{1}{\det M} D\chi \tag{6.12.5}$$

where $D\psi = \prod_{i=1}^{N} d\psi_i$ and so on.

### 6.12.1 Gaussian integration: Complex fermions

Consider the $N$-dimensional Gaussian integral for *complex fermions* $\psi_n$ and $\bar{\psi}_n$

$$Z[J] = \prod_{n=1}^{N} \int d\bar{\psi}_n d\psi_n \exp\{-\bar{\psi}_n M_{nm}\psi_m + \bar{J}_n\psi_n + \bar{\psi}_n J_n\} \tag{6.12.6}$$

where $M_{nm} = -M_{mn}$ is an antisymmetric matrix.[9]

[9] For real fermions $\psi = \psi^*$. For complex fermions $\psi = \psi_1 + i\psi_2$.

An antisymmetric matrix $M = -M^T$ can be diagonalized by a *unitary* transformation

$$M = U^\dagger \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} U \ , U^\dagger U = 1$$

In matrix notation

$$M = U^\dagger \Lambda U \qquad , \qquad UU^\dagger = 1, \qquad \det(UU^\dagger) = 1 \tag{6.12.7}$$

where $\Lambda = diag(\lambda_1, ...\lambda_N)$. Since the fermions $\psi,\ \bar{\psi}$ are complex, define the change of variables using the unitary matrix $U$, and from Eq. 6.12.5[10]

$$\begin{aligned} \bar{\psi}U^\dagger = \bar{\eta} \qquad , \qquad \eta = U\psi \\ D\eta D\bar{\eta} = \frac{1}{\det(U)} D\psi \times \frac{1}{\det(U^\dagger)} D\bar{\psi} = D\psi D\bar{\psi} \end{aligned} \tag{6.12.8}$$

Hence, the fermion integrations completely factorize and yields, for $\bar{J}_n = 0 = J_n$, the following

$$\begin{aligned} Z[0] &= \int D\bar{\psi}D\psi \exp\{-\bar{\psi}M\psi\} = \prod_n \int d\bar{\eta}_n d\eta_n \exp\{-\sum_{n=1}^{N} \lambda_n \bar{\eta}_n \eta_n\} \\ &= \prod_n \left[ \int d\bar{\eta}_n d\eta_n e^{-\lambda_n \bar{\eta}_n \eta_n} \right] = \prod_n \lambda_n = \det M \end{aligned} \tag{6.12.9}$$

The fermion Gaussian integration obtained in Eq. 6.12.9 can be directly done using the rules of fermion integration. On expanding the exponential term $\exp\{-\bar{\psi}M\psi\}$, only *one term* – namely $(-1)^N \left(\bar{\psi}M\psi\right)^N / N!$ containing the product of all the fermion variables – is non-zero inside the integrand. Using the notation of summing over repeated indexes, Eq. 6.12.3 yields

$$\begin{aligned} &\int D\bar{\psi}D\psi e^{-\bar{\psi}M\psi} = \int D\bar{\psi}D\psi \left[ \frac{(-1)^N}{N!} \left(\bar{\psi}M\psi\right)^N \right] \\ &= \frac{(-1)^N}{N!} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_N j_N} \int D\bar{\psi}D\psi \bar{\psi}_{i_1} \psi_{j_1} \bar{\psi}_{i_2} \psi_{j_2} \cdots \bar{\psi}_{i_N} \psi_{j_N} \\ &= \frac{1}{N!} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_N j_N} \epsilon_{i_1 i_2 \cdots i_N} \epsilon_{j_1 j_2 \cdots j_N} = \det M \end{aligned}$$

The partition function with an external source given in Eq. 6.12.6 is evaluated by a shift of fermion integration variables. Write the partition function

[10] Note for real fermions one cannot use a unitary transformation for a change of fermionic variables as this would lead to the transformed fermions being complex.

as

$$Z[J] = \int D\bar{\psi} D\psi \exp\{-(\bar{\psi} - \bar{J}M^{-1})M(\psi - M^{-1}J) + \bar{J}M^{-1}J\}$$

Using the fundamental property of fermion integration that it is invariant a constant shift of fermion variables – as given in Eq. 6.12.1 – yields

$$\bar{\psi} \to \bar{\psi} + \bar{J}M^{-1} \;\; ; \;\; \psi \to \psi + M^{-1}J$$

and hence

$$Z[J] = \int D\bar{\psi} D\psi \exp\{-\bar{\psi}M\psi + \bar{J}M^{-1}J\} = (\det M) \exp\{\bar{J}M^{-1}J\} \quad (6.12.10)$$

The correlator, using Eqs. 6.12.6 and 6.12.10, is given by

$$\begin{aligned} G_{mn} = E[\psi_m \bar{\psi}_n] &= \frac{1}{Z[0]} \int D\bar{\psi} D\psi \bar{\psi}_n \psi_m \exp\{-\bar{\psi}M\psi\} \\ &= \frac{1}{Z[0]} \frac{\delta^2}{\delta J_n \delta \bar{J}_m} Z[J]\Big|_{J=0=\bar{J}} \\ \Rightarrow G_{mn} = E[\bar{\psi}_m \bar{\psi}_n] &= M^{-1}_{mn} = -M^{-1}_{nm} \end{aligned} \quad (6.12.11)$$

For the case multi-component complex fermions $\bar{\psi}_\alpha(x), \psi_\beta(x)$, the action is given by

$$S = -\int d^d x \bar{\psi}_\alpha(x) M_{\alpha\beta}(x-y) \psi_\beta(y)$$

The propagator in coordinate space, from Eq. 6.12.11, is given by

$$G_{\beta\alpha}(x,y) = E[\psi_\beta(x)\bar{\psi}_\alpha(y)] = -M^{-1}_{\beta\alpha}(x-y) \quad (6.12.12)$$

In the Fourier representation, define the Fourier transformed field

$$\psi_\beta(x) = \int \frac{d^d p}{(2\pi)^d} e^{ipx} \psi_{p\beta} \;\; ; \;\; \bar{\psi}_\alpha(x) = \int \frac{d^d p}{(2\pi)^d} e^{-ipx} \bar{\psi}_{p\alpha}$$

and which yields the Minkowski action

$$iS = -\int \frac{d^d p}{(2\pi)^d} \bar{\psi}_{p\alpha} M_{\alpha\beta}(p) \psi_\beta(p) \;\; ; \;\; M_{\alpha\beta}(x) = \int d^d x e^{ipx} M_{\alpha\beta}(p)$$

The propagator in Fourier space is given by

$$G_{\beta\alpha}(p,p') = E[\psi_{p\beta}\bar{\psi}_{p'\alpha}] = -(2\pi)^d \delta^d(p-p') M^{-1}_{\beta\alpha}(p) \quad (6.12.13)$$

Foregrounding the path integral quantization of the Dirac field discussed in

Chapter **??**, consider the action given by Eq. 5.2.3

$$iS = i \int d^d x \bar{\psi}_\alpha(x)[i\gamma_\mu \partial^\mu - m]\psi_\beta(x) \tag{6.12.14}$$

$$= -i \int \frac{d^d p}{(2\pi)^d} \bar{\psi}_{p\alpha}[\gamma_\mu p^\mu + m]\psi_{p\beta} \tag{6.12.15}$$

Hence, from Eq. 6.12.13, the propagator is given by

$$G_{\beta\alpha}(p, p') = E[\psi_{p\beta}\bar{\psi}_{p'\alpha}] = (2\pi)^d \delta^d(p - p')G_{\beta\alpha}(p)$$
$$\Rightarrow G(p) = \frac{i}{\gamma_\mu p^\mu + m} \tag{6.12.16}$$

# 7

# Two dimensional quantum electrodynamics

## 7.1 Introduction

Free quantum fields can be reduced to studying only a finite number of degrees of freedom. The study of free quantum fields is a preparation for studying interacting quantum fields, that in general have infinitely many coupled degrees of freedom. One of the simplest interacting quantum field theory is massless quantum electrodynamics in two dimensions, also called the Schwinger model: the theory of massless fermions coupled to photons in two spacetime dimensions. Studying the Schwinger model is a preparation for analyzing more complicated nonlinear quantum field theories.

The Schwinger model can be solved exactly and is a toy model that exhibits many non-perturbative features that one expects in four dimensional quantum field theories. Some of these are the following.

- The role of regularization in breaking classical symmetries can be studied exactly.
- The model spontaneously breaks the chiral symmetry of the theory. The axial vector current that is classically conserved is not conserved after quantization.
- The breaking of chiral symmetry generates a mass for the gauge field.
- The model exhibits confinement of the fundamental fermions: the eigenstates of the interacting theory do not have any fermions.
- The Wilson loop is evaluated exactly to examine if it is the appropriate order parameter for confinement of the fundamental fermions.

The solutions generally take Schwinger's(1962) approach of using the field equations with a suitably regularized gauge-invariant current to solve the model [Lowenstein and Swieca (1971), Hetrick et al. (1995)]. The model has been studied using Feynman diagrams [Falck and Kramer (1988)]. The

solution discussed in this Chapter is based on the path integral [Baaquie (1982)].

The action is regularized in a manner that preserves gauge invariance. Note that there is no need to introduce a cut-off since the theory is super-renormalizable. The gauge field is the decomposed into a sum of a gauge-invariant and gauge-variant pieces. The gauge-variant piece is removed by a gauge transformation and the coupling of the fermions to the gauge-invariant piece is regularized. The fermion path integral is performed to all orders in perturbation theory using the axial-vector current anomaly [Bell and Jackiw (1969), Adler (1969)].

The confinement criterion for the interacting theory is studied using the Wilson loop integral [Wilson (1974)], which shows the exp(-area) dependence for the pure gauge field but not for the interacting theory. The behavior of the Wilson loop for the interacting theory is explained by studying the eigenstates of the Schwinger model [Baaquie (1983)].

## 7.2 The Euclidean action

For simplicity, the model is studied in two dimensional Euclidean spacetime. The Euclidean action, from Eq. 5.3.2, is defined for two dimensional spacetime using two-component spinors $\psi$ and $\bar{\psi}$ and gauge field $A_\mu$. Dropping the superscript $E$, the action is given by

$$A = -\frac{1}{4g^2}\int_x F_{\mu\nu}^2 - \int_x \bar{\psi}(\partial_\mu + iA_\mu)\gamma_\mu\psi \tag{7.2.1}$$

$$= A_B + A_F + A_I \tag{7.2.2}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.2.3}$$

and

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \int_x \equiv \int_{-\infty}^{\infty} d^2x$$

Note that $g$ is the bare coupling constant and carries the dimension of mass.

Noteworthy 7.1: Two dimensional Euclidean Gamma matrices

The following concrete representation is chosen for the Euclidean gamma matrices.

$$\gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.2.4}$$

Define the $\gamma_5$ matrix by

$$\gamma_5 \equiv i\gamma_1\gamma_2 \;\; ; \;\; \{\gamma_5, \gamma_\mu\} = 0 \;\; \Rightarrow \;\; \epsilon_{\mu\nu}\gamma_\nu = i\gamma_\mu\gamma_5 \tag{7.2.5}$$

where $\epsilon_{01} = -\epsilon_{10} = 1$. A concrete representation of $\gamma_5$ is given by

$$\gamma_5 = i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.2.6}$$

---

Consider a *global chiral transformation* given by

$$\psi_x \to e^{\zeta\gamma_5}\psi_x \qquad \bar{\psi}_x \to \bar{\psi}_x e^{\zeta\gamma_5} \;\; : \;\; \zeta \;\; : \text{constant} \tag{7.2.7}$$

The action given in Eq. 7.2.1 is invariant under Eq. 7.2.7

$$A \to A \;\; : \;\; \text{Classical chiral invariance}$$

Two dimensional massless quantum electrodynamics is *classically* a chirally invariant theory.

The gauge field is decomposed as follows[1]

$$A_\mu(x) = \epsilon_{\mu\nu}\partial_\nu s(x) + \partial_\mu\varphi(x) \tag{7.2.8}$$

where $\epsilon_{\mu\nu}$ is the antisymmetric tensor and $s(x)$ and $\varphi(x)$ are pseudoscalar and scalar fields respectively. Using Eq. 7.2.8 yields

$$F_{\mu\nu} = -\epsilon_{\mu\nu}\partial^2 s \;\; ; \;\; \epsilon_{\mu\nu}F_{\mu\nu} = -2\partial^2 s \tag{7.2.9}$$

Note that $s(x)$ is gauge invariant and $\varphi(x)$ is gauge dependent. Using Eq. 7.2.8 gives

$$\begin{aligned} A_1 &= A_F + A_I \\ &= -\int_x [\bar{\psi}\partial\psi + i\bar{\psi}\gamma_\mu\psi(\epsilon_{\mu\nu}\partial_\nu s + \partial_\mu\phi)] \;\; ; \;\; \partial = \sum_\mu \gamma_\mu\partial_\mu \end{aligned}$$

Performing the *local* gauge transformation on the fermions

$$\psi \to e^{i\varphi}\psi \qquad \bar{\psi} \to \bar{\psi}e^{-i\varphi}, \tag{7.2.10}$$

yields

$$A_1 = -\int_x (\bar{\psi}\partial\psi + i\bar{\psi}\gamma_\mu\psi\epsilon_{\mu\nu}\partial_\nu s) \tag{7.2.11}$$

In eliminating the field $\varphi(x)$, we have chosen no gauge for the gauge for the gauge field; the only restriction we have is that we can evaluate only

[1] In fluid mechanics, $\varphi$ is the velocity potential and $s$ the stream function.

gauge-invariant expectation values for the fermions. Given the ansatz given in Eq. 7.2.8 for $A_\mu$ it is most suitable to work in the Landau gauge

$$\partial_\mu A_\mu = 0 \tag{7.2.12}$$

and which implies that for each $x$

$$\varphi(x) = 0 \tag{7.2.13}$$

in the path integral. Using Eq. 7.2.5 yields

$$A_1 = -\int_x (\bar{\psi}\partial\psi + \bar{\psi}\gamma_\mu\gamma_5\psi\partial_\mu s) \tag{7.2.14}$$

Consider the local *chiral transformation* given by

$$\psi_x \to e^{s(x)\gamma_5}\psi_x \qquad \bar{\psi}_x \to \bar{\psi}_x e^{s(x)\gamma_5} \tag{7.2.15}$$

Applying Eq. 7.2.15 to Eq. 7.2.14 yields

$$A_1 = -\int_x \bar{\psi}\partial\psi \tag{7.2.16}$$

and the fermions completely decouple from the gauge field.

The classical symmetry of gauge invariance will be preserved on quantizing the fermion and gauge field, but it will turn out that the classical chiral symmetry of the theory is broken. The reason being that the short distance fluctuations that are absent in the classical theory but fundamental for quantum fields do not respect chiral symmetry.

## 7.3 Point-split regularization

The action is regularized by point-splitting the axial-vector current. When the fermions in the axial current are point-split, an average over all possible angles has to be performed to recover spherical symmetry. This averaging also requires that the matrices $\gamma$ and $\partial_\mu s$ are also projected in the direction in which the fermions are point-split. We pick an arbitrary direction $\theta$ in the two-dimensional plane. Let $\hat{e}_1$ and $\hat{e}_2$ be the unit basis vectors defining the operator $\partial$; let

$$\begin{aligned}\hat{e}_1(\theta) &= \cos\theta\hat{e}_1 + \sin\theta\hat{e}_2 \\ \hat{e}_2(\theta) &= -\sin\theta\hat{e}_1 + \cos\theta\hat{e}_2\end{aligned} \tag{7.3.1}$$

Then define

$$\begin{aligned}\gamma_\mu(\theta) &= \hat{e}_\mu(\theta)\cdot\gamma \\ \partial_\mu(\theta) &= \hat{e}_\mu(\theta)\cdot\partial\end{aligned} \tag{7.3.2}$$

Note

$$\sum_\mu \gamma_\mu(\theta)\partial_\mu(\theta) = \sum_\mu \gamma_\mu \partial_\mu \quad ; \quad \gamma_5(\theta) = \gamma_5$$

and

$$\{\gamma_\mu(\theta), \gamma_\nu(\theta)\} = 2\delta_{\mu\nu} \tag{7.3.3}$$

Let $a$ be an infinitesimal number; then the (point-split) regularized action is defined by

$$A_1[s, a, \theta] = -\int_x \left(\bar{\psi}_x \partial\psi_x + \bar{\psi}_x \gamma_\mu(\theta)\gamma_5 \psi_{x+ae_\mu(\theta)} \partial_\mu(\theta) s(x)\right) \tag{7.3.4}$$

The regularization of the axial current does not generate any gauge-dependent terms for the boson sector since the field $s(x)$ is gauge invariant; however, this regularization does not respect gauge invariance for the fermion-boson coupling sector. In spite of using a non-explicitly gauge-invariant fermion-boson coupling, it is shown by Baaquie (1982) that complete gauge invariance is recovered for the fermions in the limit $a \to 0$.

Note that there is no reason for putting the line integral $\exp\left(i\int_x^{x+a} A_\mu dl_\mu\right)$ between the point-split fermions as it would not restore gauge invariance for the fermions.

The quantum field theory is defined by the Feynman path integral of $e^A$ over all values of the fermions and the gauge field; in the Landau gauge the path integral is given by

$$\begin{aligned} Z &= \prod_x \int ds(x) d\bar{\psi}(x) d\psi(x) e^{A_B[s] + A_1[s,a,\theta]} \\ &\equiv \int Ds D\bar{\psi} D\psi e^{A_B[s] + A_1[s,a,\theta]} \end{aligned}$$

where we have ignored certain (infinite and irrelevant) constants.

The path integral over the fermion field is performed giving

$$e^{A'[s,a,\theta]} = \int D\bar{\psi} D\psi e^{A_F + A_I} \tag{7.3.5}$$

Before taking the limit $a \to 0$, $A'$ is symmetrized by integrating over all possible directions of the angle $\theta$. The final result is

$$A'[s] = \lim_{a\to 0} \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} A'[s, a, \theta] \tag{7.3.6}$$

The action $A'[s]$ is evaluated in Section 7.4.

The gauge field sector is given by

$$Z = \int Dse^{A_B[s]+A'[s]}$$

### *7.3.1 Free fermion propagator*

The fermion path integral is evaluated by a perturbation expansion; for carrying out this expansion, the free fermion propagator is required. Define the propagator by

$$G_{\alpha\beta}(x) = \int D\bar{\psi}D\psi\psi_\alpha(x)\bar{\psi}_\beta(0)\exp\{-\int_x \bar{\psi}\partial\psi\}$$

From Eq. 6.12.16

$$G_{\alpha\beta}(x-y) = E[\psi_\alpha(x)\bar{\psi}_\beta(y)] = \left(\frac{1}{\partial_\mu\gamma_\mu}\right)_{\alpha\beta}\delta(x-y)$$

In bra and ket matrix notation

$$G(x) = E\Big[|\psi(x)\rangle\langle\bar{\psi}(0)|\Big] = \frac{1}{\partial_\mu\gamma_\mu}\delta(x) = \frac{\partial_\mu\gamma_\mu}{\partial_\nu\partial_\nu}\delta(x) \tag{7.3.7}$$

Consider

$$\frac{1}{-\partial_\nu\partial_\nu+\alpha^2}\delta(x) = \int_{-\infty}^{+\infty}\frac{d^2k}{(2\pi)^2}\frac{e^{ikx}}{k^2+\alpha^2} = \frac{1}{2\pi}K_0(\alpha|x|)$$

Hence

$$\begin{aligned} G(x) &= -\partial_\mu\gamma_\mu\lim_{\alpha\to 0}\frac{1}{2\pi}K_0(\alpha|x|) = \partial_\mu\gamma_\mu(\frac{1}{4\pi}\ln(x^2)) \\ \Rightarrow G(x) &= \frac{1}{2\pi}\frac{x_\mu\gamma_\mu}{x^2} \end{aligned} \tag{7.3.8}$$

## 7.4 Fermion path integral

Performing an integration by parts gives for the regularized action

$$A_1 = -\int_x \bar{\psi}_x\partial\psi_x + \int_x s(x)\partial j^5(x) \tag{7.4.1}$$

where the divergence of the regularized axial-vector current is given by

$$\partial j^5(x) \equiv \partial_\mu(\theta)(\bar{\psi}_x\gamma_\mu(\theta)\gamma_5\psi_{x+ae_\mu(\theta)}) \tag{7.4.2}$$

For $a \neq 0$, performing the local chiral transformation as given in Eq. 7.2.15 can no longer decouple the fermions from the gauge field – unlike the result obtained in Eq. 7.4.12. Hence, the coupling of the gauge field to the fermions

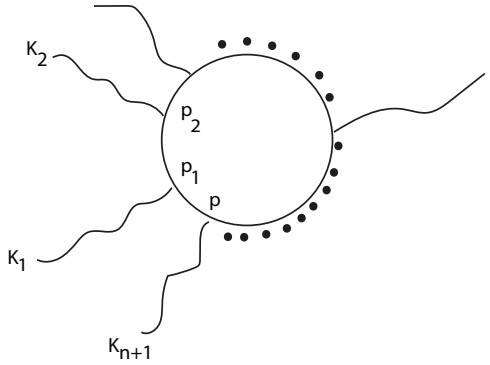


Figure 7.1 A one-loop diagram with arbitrary number of external bosonic lines.

is via the anomaly in axial current and the regularization of the axial current shows this.

If one carries out the chiral rotations after setting $a = 0$, the fermions apparently decouple from the gauge field in the Lagrangian as in 7.4.12. However, the chiral anomaly re-appears since the fermion integration measure is not invariant under a chiral rotation, as has been discussed by Fujikawa (1984).

Performing the fermion path integral yields

$$\begin{aligned} e^{A'[s,a,\theta]} &= \int D\bar{\psi}D\psi e^{A_1} \\ &= \int D\bar{\psi}D\psi \exp\{-\int_x \bar{\psi}_x \partial\psi_x\}[1 + \int_x s(x)\partial j^5(x) + \frac{1}{2}\left(\int_x s(x)\partial j^5(x)\right)^2 + ..] \\ &= Z \times \left[1 + \int_x s(x)E[\partial j^5(x)] + \frac{1}{2}\int_{x,y} s(x)s(y)E[\partial j^5(x)\partial j^5(x)] + ...\right] \quad (7.4.3) \end{aligned}$$

where

$$E[\mathcal{O}] \equiv \frac{1}{Z}\int D\bar{\psi}D\psi \exp\{-\int_x \bar{\psi}_x \partial\psi_x\}\mathcal{O}$$

and

$$Z = \int D\bar{\psi}D\psi \exp\{-\int_x \bar{\psi}_x \partial\psi_x\}$$

For infinite time $Z$ is an irrelevant constant, but will be important for finite time.

Eq. 7.4.3 yields the connected diagrams

$$A'[s,a,\theta] = \sum_{l=1}^{\infty} \frac{1}{l} \int_{x_1,\ldots,x_l} s(x_1)\ldots s(x_l) E[\partial j^5(x_1)\ldots\partial j^5(x_l)]_c \quad (7.4.4)$$

where $E[\ldots]_c$ denotes the connected fermion loop using the free massless fermion propagator. Graphically, $A'$ is given by an arbitrary number of fields $s(x_l)$ attached to a single fermion loop, and is shown in Figure 7.1.

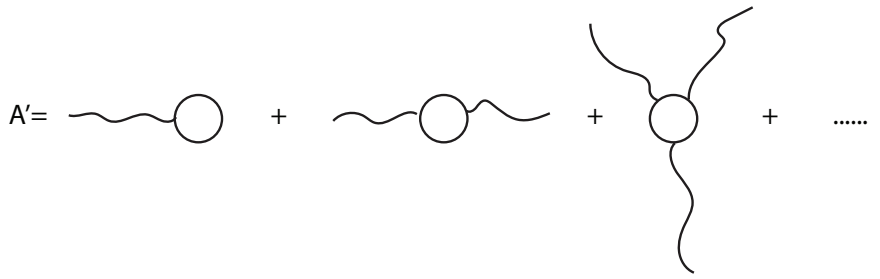


Figure 7.2 Finite diagrams, in the limit of $a \to 0$, with one, two and three external boson lines.

It is shown in Baaquie (1982) that in the limit $a \to 0$ only the first two terms of Eq. 7.4.4 are non-zero, that is

$$A'[s,a,\theta] = \int_x s(x) E[\partial j^5(x)]_c + \frac{1}{2}\int_{x,y} s(x)s(y) E[\partial j^5(x)\partial j^5(y)]_c + 0(s^3) \quad (7.4.5)$$

and this is shown graphically in Figure 7.2.[2]

The finite terms are evaluated in momentum space. The Fourier transform of the boson and fermion fields are defined as usual and

$$j^5_\mu(x) = \int_p e^{ipx} j^5_{p\mu} \qquad p_\mu(\theta) = \hat{e}(\theta)\cdot p$$
$$\int_p \equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d^2p \qquad \delta(p) = (2\pi)^2 \prod_\mu \delta(p_\mu)$$

The tadpole diagram, in bra and ket notation of Eq. 7.3.7, yields

$$\begin{aligned} E[\partial j^5(x)] &= -\partial_\mu(\theta) tr\left(\gamma_\mu(\theta)\gamma_5 E[|\psi_{x+ae_\mu(\theta)}\rangle\langle\bar{\psi}_x|]\right) \\ &= -\partial_\mu(\theta) tr\left(\gamma_\mu(\theta)\gamma_5 G(ae_\mu(\theta))\right) \\ &= \frac{-1}{2\pi}\partial_\mu(\theta) tr\Big(\gamma_\mu(\theta)\gamma_5 \frac{\gamma\cdot e_\mu(\theta)}{a}\Big) = 0 \end{aligned}$$

[2] The cubic term $O(s^3)$ is zero since the one-loop fermion yields an anti-symmetric function.

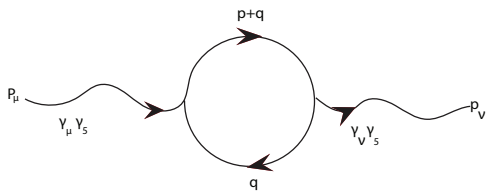


Figure 7.3 Finite diagram, in the limit of $a \to 0$, with two external boson lines.

where the free fermion propagator, from Eq. 7.3.8, is given by

$$G(x) = \frac{1}{2\pi}\frac{x_\mu\gamma_\mu}{x^2} = \frac{1}{2\pi}\frac{x\cdot\gamma}{x^2}$$

Hence, from Eq. 7.4.5

$$A' = \frac{1}{2}\int_{p,p'} s_{-p}s_{-p'}ip_\mu(\theta)ip'_\nu(\theta)E[j^5_{p\mu}(\theta)j^5_{p'\nu}(\theta)]_c \tag{7.4.6}$$

The diagram with two external bosonic lines is evaluated in momentum space. Performing the fermion path integration yields

$$\Gamma_p\delta(p+p') = ip_\mu(\theta)ip'_\nu(\theta)E[j^5_{p\mu}(\theta)j^5_{p'\nu}(\theta)]_c \tag{7.4.7}$$

where

$$\Gamma_p = -p_\mu(\theta)p_\nu(\theta)e^{iap_\nu(\theta)}\int_q e^{ia(q_\mu(\theta)+q_\nu(\theta))}tr\left(\gamma_\mu(\theta)\gamma_5\frac{1}{q}\gamma_\nu(\theta)\gamma_5\frac{1}{p+q}\right)$$

The Feynman diagram for the above expression is given in Figure 7.3. $\Gamma_p$ is evaluated in Appendix I and we obtain

$$\Gamma_p = -\frac{1}{\pi}\big(p_1^2(\theta) + p_2^2(\theta) + p_1(\theta)p_2(\theta)\big) + 0(a) \tag{7.4.8}$$

$$= -\frac{1}{\pi}(p^2 + p_1(\theta)p_2(\theta)) + 0(a) \tag{7.4.9}$$

Therefore, from 7.4.6 and 7.4.8 we obtain, for $a \to 0$

$$A'[s,\theta] = -\frac{1}{2\pi}\int_p s_{-p}s_p\big(p^2 + p_1(\theta)p_2(\theta)\big) \tag{7.4.10}$$

Taking the symmetric limit by integrating over $\theta$ gives

$$\begin{aligned} A'[s] &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta A'[s,\theta] \\ &= -\frac{1}{2\pi}\int_p p^2 s_{-p}s_p = -\frac{1}{2\pi}\int_x (\partial s)^2 \end{aligned} \tag{7.4.11}$$

The effective action for the boson sector has been obtained by integrating out the fermions and yields the final result

$$\begin{aligned} A &= A_B + A' \\ &= -\frac{1}{2g^2}\int_x \left((\partial^2 s)^2 + \frac{g^2}{\pi}(\partial s)^2\right) \\ \Rightarrow A &= -\frac{1}{2g^2}\int_p p^2(p^2+m^2)s_{-p}s_p \;\; ; \; m^2 = \frac{g^2}{\pi} \end{aligned} \tag{7.4.12}$$

The action given by Eq. 7.4.12 for the boson field sector can be obtained from the massless free boson field by integrating out all the momentum modes with momentum higher than $m$ [Shalloway (1979)]. In effect, 7.4.12 is the action for the massless free field for distances much larger than $1/m$.

### *7.4.1 Fermion generating functional*

For completeness we evaluate the generating functional for the fermions. Recall from 7.3.4 that we have for the regularized action

$$A_1 = -\int_x \left(\bar{\psi}_x \partial \psi_x + \bar{\psi}_x \gamma_\mu(\theta)\gamma_5 \psi_{x+ae_\mu(\theta)} \partial_\mu(\theta) s(x)\right) \tag{7.4.13}$$

$$\equiv -\int_{x,y} \bar{\psi}_x M(x,y)\psi_y \tag{7.4.14}$$

where

$$M(x,y) = \partial\delta(x-y) + \sum_\mu \delta(x+ae_\mu(\theta)-y)\gamma_\mu(\theta)\gamma_5\partial_\mu(\theta)s(x)$$

The result of Section 7.4 can be written in terms of the rules of fermion integration discussed in Section 6.12. Using Eq. 6.12.9, we write Eq. 7.4.10 as follows

$$e^{A'[s,\theta]} = \int D\bar{\psi}D\psi \exp\{-\int \bar{\psi}M\psi\} = \det M \tag{7.4.15}$$

Let $\bar{\eta}(x), \eta(x)$ be fermionic sources. Then the generating functional is given by

$$e^{F[\bar{\eta},\eta]} = \int D\bar{\psi}D\psi \exp\left(-\int \bar{\psi}M\psi + \int(\bar{\eta}\psi + \bar{\psi}\eta)\right) \tag{7.4.16}$$

Using the invariance of the fermion (path) integration under the transformation (in matrix notation)

$$\begin{aligned} \psi &\to \psi - M^{-1}\eta \\ \bar{\psi} &\to \bar{\psi} - \bar{\eta}M^{-1} \end{aligned} \tag{7.4.17}$$

we have, using Eqs. 6.12.10 and 7.4.15

$$\begin{aligned} e^{F[\bar{\eta},\eta]} &= \exp\left(\int_{x,y} \bar{\eta}(x) M^{-1}(x,y)\eta(y)\right) \det M \\ &= \exp\left(\int_{x,y} \bar{\eta}(x) M^{-1}(x,y)\eta(y)\right) \exp\left(A'[s,\theta]\right) \end{aligned} \tag{7.4.18}$$

Therefore, from Eqs. 7.4.11 and 7.4.18 and averaging over $\theta$, yields

$$F[\bar{\eta},\eta] = \int_{x,y} \bar{\eta}(x) M^{-1}(x,y)\eta(y) - \frac{1}{2\pi}\int_x (\partial s)^2 \tag{7.4.19}$$

It can easily be verified that

$$M^{-1}(x,y) = e^{s(x)\gamma_5} G(x-y) e^{s(y)\gamma_5} \tag{7.4.20}$$

From the form of $F[\bar{\eta},\eta]$ we see that by performing the local chiral rotation on the fermions as given by Eq. 7.2.15, the fermions are effectively decoupled from the gauge field and behave like free massless fermions. The regulator affects only $\det M$, with $M^{-1}$ not requiring a regulator.

As an example of the effect of the regulator, the propagator for the gauge-invariant fermion-antifermion state is calculated. Using the bra and ket notation, as in Eq. 7.3.7, yields

$$\begin{aligned} K(x) &= E[\bar{\psi}_x \psi_x \bar{\psi}_0 \psi_0]_c = E[\bar{\psi}_x e^{2s(x)\gamma_5} \psi_x \bar{\psi}_0 e^{2s(0)\gamma_5} \psi_0]_c \\ &= -tr\left(E\left[e^{2s(x)\gamma_5} |\psi_x\rangle\langle\bar{\psi}_0| e^{2s(0)\gamma_5} |\psi_0\rangle\langle\bar{\psi}_x|\right]\right) \end{aligned}$$

After the chiral rotation, the fermions are free and performing the fermion path integral yields

$$\begin{aligned} K(x) &= -tr\left(E\left[e^{2s(x)\gamma_5} G(x) e^{2s(0)\gamma_5} G(-x)\right]\right) \\ &= G^2(x)\frac{1}{Z}\int Ds\ tr\left(e^{2(s(x)-s(0))\gamma_5}\right) e^{A_B + A'} \\ &= 2G^2(x)\frac{1}{Z}\int Ds e^{2(s(x)-s(0))} e^{A_B + A'} \\ &= \frac{1}{2\pi^2}\frac{1}{x^2} e^{P(x)} \end{aligned} \tag{7.4.21}$$

For details of the boson path integration see Section 7.5. The final result is ($m^2 = g^2/\pi$)

$$P(x) = 2g^2 \int_p \frac{|1 - e^{ipx}|^2}{p^2(p^2+m^2)} = 2(\gamma + \ln(\frac{1}{2}mx) + K_0(mx))$$

where $\gamma$ is Euler's constant and $K_0$ the associated Bessel function. The function $P$ has the following asymptotic expansions

$$P \sim \begin{cases} 2m^2x^2 & x \to 0 \\ 2\ln(mx) & x \to \infty \end{cases} \tag{7.4.22}$$

Therefore, for $x \to \infty$, we have from equations 7.4.21 and 7.4.22

$$K(x) = \frac{m^2}{8\pi^2}e^{2\gamma} + 0(e^{-mx}) = \frac{g^2}{8\pi^3}e^{2\gamma} \tag{7.4.23}$$

The cluster decomposition for $\bar{\psi}\psi$ is violated since $K(x)$ does not go to zero exponentially for large $x$ but, instead, tends to a constant; this is taken to indicate the breakdown of chiral symmetry for the vacuum state of the quantum field. The exact result obtained in Eq. 7.4.23 has been used for verifying numerical studies of the Schwinger model by Marinari et al. (1981).

### *7.4.2 Axial-vector current chiral anomaly*

Recall that from Eqs. 7.2.9 and 7.4.2

$$A = -\frac{1}{2g^2}\int_x (\partial^2 s)^2 - \int_x (\bar{\psi}_x \partial \psi_x + j^5_\mu(x)\partial_\mu(\theta)s(x)) \tag{7.4.24}$$

where

$$j^5_\mu(x) \equiv \bar{\psi}_x \gamma_\mu(\theta)\gamma_5 \psi_{x+ae_\mu(\theta)} \tag{7.4.25}$$

The Ward identities for the axial current are violated, and to see the effect of the anomaly the divergence of the vector and axial-vector currents is calculated. Performing only the fermion integration and using 7.4.20 yields

The tadpole diagram, in bra and ket notation of Eq. 7.3.7, yields

$$\begin{aligned} E[\partial j^5(x)] &= -\operatorname{Tr}\left(e^{-a\partial_\mu(\theta)s(x)\gamma_5}\gamma_\mu(\theta)\gamma_5 E[|\psi_{x+ae_\mu(\theta)}\rangle\langle\bar{\psi}_x|]\right) \\ &= -\operatorname{Tr}\left(e^{-a\partial_\mu(\theta)s(x)\gamma_5}\gamma_\mu(\theta)\gamma_5 G(ae_\mu(\theta))\right) \\ &= -\frac{1}{2\pi a}\operatorname{Tr}\left(e^{-a\partial_\mu(\theta)s(x)\gamma_5}\gamma_\mu(\theta)\gamma_5\gamma_\mu(\theta)\right) \end{aligned}$$

since

$$G(ae_\mu(\theta)) = \frac{1}{2\pi a}\gamma \cdot e_\mu(\theta) = \frac{1}{2\pi a}\gamma_\mu(\theta)$$

Using $\gamma_5\gamma_\mu = i\epsilon_{\mu\nu}\gamma_\nu$ yields

$$E[j^5_\mu(z)] = -\frac{1}{2\pi a}\operatorname{Tr}\left(\gamma_\mu(\theta)\gamma_5\gamma_\mu(\theta)\right) = -\sum_\nu \frac{i\epsilon_{\mu\nu}}{2\pi a}\operatorname{Tr}(\gamma_\mu(\theta)\gamma_\nu(\theta)) = 0$$

and hence

$$E[j_\mu^5(x)] = -\frac{1}{2\pi a}\lim_{a\to 0}\mathrm{Tr}(e^{-a\partial_\mu(\theta)s(x)\gamma_5}\gamma_\mu(\theta)\gamma_5\gamma_\mu(\theta))$$
$$= -\frac{1}{\pi}\partial_\mu(\theta)s(x) \tag{7.4.26}$$

Therefore, using Eq. 7.2.9

$$E[\partial_\mu(\theta)j_\mu^5(x)] = -\frac{1}{\pi}\partial^2 s(x) = \frac{1}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu} \tag{7.4.27}$$

Furthermore

$$E[j_\mu(x)] = i\epsilon_{\mu\nu}\langle j_\nu^5(x)\rangle = -\frac{i}{\pi}\epsilon_{\mu\nu}\partial_\nu(\theta)s(x) \tag{7.4.28}$$

and hence

$$E[\partial_\mu(\theta)j_\mu(x)] = 0 \tag{7.4.29}$$

We see from the above that the anomaly breaks conservation of the axial-vector current but leaves the conservation of the vector current intact. This is the reason why the vector-current Ward identities are preserved in the limit $a \to 0$.

## 7.5 The Wilson loop integral

The Wilson loop was introduced by Wilson (1974) as a criterion for the confinement of quarks. The Wilson loop depends only on the topology of the manifold and is independent of the geometry of the underlying manifold. Witten (1989) showed that the expectation value of Wilson loops in the Chern-Simon gauge theory yield the Jones polynomials, which arise in the classification of knots and links.

The Wilson loop is evaluated exactly for the Schwinger model and it is seen that its behavior depends on the coupling of the theory, exhibiting the 'area law' of decaying exponentially as the area of the loop for pure gauge fields and depending of the perimeter of the loop for large coupling.

Let $C$ denote a circular contour of radius $L$ that encloses the (unique) area $\Gamma$, shown in Figure 7.4. The Wilson loop integral is defined by the gauge field as follows

$$\exp\left(i\oint_c A_\mu dl_\mu\right)$$

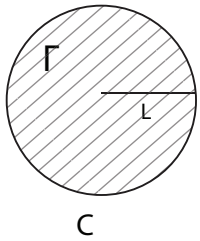


Figure 7.4 Wilson loop integral.

From equation Eq. refdecompose using Stokes theorem

$$\oint_c A_\mu dl_\mu = -\int_\Gamma \partial^2 s \tag{7.5.1}$$
$$= 2\pi L \int_p |p| J_1(L|p|) s_p$$

where $J_1$ is the Bessel function. Therefore, in the Landau gauge, the Wilson loop integral is

$$e^W = E\big[\exp\big(i \oint_c A_\mu dl_\mu\big)\big] \tag{7.5.2}$$
$$= \frac{1}{Z}\int Ds \exp\big(-i\int_\Gamma \partial^2 s\big) e^{A_B + A'} \tag{7.5.3}$$

Performing the boson integration using 7.5.1 and 7.4.12 gives

$$W = -\pi(gL)^2 \int_0^\infty dp \frac{p}{p^2+m^2} (J_1(Lp))^2 \tag{7.5.4}$$
$$= -\pi(gL)^2 I_1(mL) K_1(mL) \tag{7.5.5}$$

where $I_1$ and $K_1$ are the associated Bessel functions of the first and second kind. The expression for $W$ is exact; $W$ is a monotonically decreasing function of $L$ and has the following asymptotic behavior

$$\lim_{L\to\infty} W \sim \begin{cases} -L^2 & g^2 = 0 \\ -L & g^2 > 0 \end{cases} \tag{7.5.6}$$

We see from 7.5.6 that for $g > 0$, the large gauge field loops are not suppressed as exp(-area), whereas for $g^2 = 0$ they do show the area behavior. This brings us to the question of how we interpret the connection between the Wilson loop integral and confinement. The following interpretation is consistent with our result. If the loop integral show exp(-area) behavior for the pure gauge field (no coupling to fermions, i.e. $g^2 = 0$), then, when this gauge field is coupled to the fermions, the fermions are confined. In this

interpretation of the loop integral, we do not expect exp(-area) behavior in the presence of fermions; since we 'know' that the fermions are confined in the Schwinger model, the above interpretation is consistent with the results of this model.

## 7.6 Energy eigenvalues

The energy eigenvalues of the Schwinger model are exactly evaluated; due to a nontrivial cancellation the fermion eigenvalues are eliminated. The eigenspectrum of the interacting theory reduces to that of a free massive boson field.The energy of the fermion-string-antifermion state and the string tension can be exactly evaluated. The relation of the string tension to the Wilson criterion of confinement for the case of the interacting theory is analyzed. The discussion is based on Baaquie (1983).

Let $H$ be the Hamiltonian operator for Schwinger QED, $E_n$ the eigenenergies, and $|\Phi_n\rangle$ the orthonormal eigenfunctions. That is,

$$\langle \Phi_n | H | \Phi_m \rangle = E_n \delta_{nm} \tag{7.6.1}$$

Fermion eigenfunctions using the anticommuting variables has been discussed in Baaquie (2014).

Consider the 'partition function' for finite time

$$Z(\tau) = \mathrm{Tr}\left[e^{-\tau H}\right] \tag{7.6.2}$$

$$= \sum_n e^{-\tau E_n} \tag{7.6.3}$$

where the sum is over all the eigenenergies. The time $\tau$ plays the role of inverse temperature and $Z(\tau)$ can be evaluated using finite-temperature methods [Kapusta (1993)].

To evaluate $Z(\tau)$, we consider the finite-time action for Schwinger QED in two-dimensional Euclidean space, with the boson (fermion) variables being periodic (antiperiodic) with period $\tau$. In effect the field theory is defined on a cylinder of infinite length.

The finite-time Euclidean action is defined using the two-component spinors $\bar{\psi}$,$\psi$ and the gauge field $A_\mu$ , and with the coupling constant $g$. The notation $\mathcal{S}$ is used for the action to differentiate from the action $A$ used for infinite time in Section 7.2. Similar to the infinite time action given in Eq. 7.2.1, define the finite time action by the following

$$\begin{aligned} S &= -\frac{1}{4g^2}\int_0^\tau dt \int dx (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \int_0^\tau dt \int dx \bar{\psi}\gamma_\mu(\partial_\mu + iA_\mu)\psi \\ &\equiv S_B + S_F + S_I \end{aligned}$$

The boundary condition for bosons are periodic in the time direction and anti-periodic for the fermions. This gives

$$A_\mu(t,x) = A_\mu(t+\tau,x) \tag{7.6.4}$$

$$\psi(t,x) = -\psi(t+\tau,x) \quad ; \quad \bar{\psi}(t,x) - \bar{\psi}(t+\tau,x) \tag{7.6.5}$$

and

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \int dx \equiv \int_{-\infty}^{+\infty} dx \tag{7.6.6}$$

The finite time path integral yields the partition function

$$Z(\tau) = \prod_{t=0}^{\tau}\prod_{x\mu}\int dA_\mu(t,x)d\bar{\psi}(t,x)d\psi(t,x)\exp(S) \tag{7.6.7}$$

We repeat the calculation of Section 7.4 to perform the finite time fermion path integration. As in Eq. 7.2.8, consider the change of variables for finite time

$$A_\mu = \epsilon_{\mu\nu}\partial_\nu s + \partial_\mu\phi \tag{7.6.8}$$

The path integral measure of the gauge field for finite time transforms as

$$\prod_{t=0}^{\tau}\prod_{x\mu}\int dA_\mu(t,x) = \det(\partial^2)\prod_{t=0}^{\tau}\prod_{x}\int ds(t,x)d\phi(t,x) \tag{7.6.9}$$

where

$$\partial^2 = \partial_\mu\partial_\mu$$

The determinant of $\partial^2$ is defined on functions periodic in $t$ with period $\tau$.

Similar to Eq. 7.2.14, the action is given by

$$S = -\frac{1}{2g^2}\int_0^\tau dt\int dx(\partial^2 s)^2 - \int_0^\tau dt\int dx(\bar{\psi}\gamma_\mu\partial\psi_\mu + \bar{\psi}\gamma_\mu\gamma_5\psi\partial_\mu s)$$

To evaluate the fermion path integral, note that the finite-time fermion propagator is given by

$$G_\tau(t,x) = \frac{1}{(\gamma_\mu\partial_\mu)}\sum_{n=-\infty}^{+\infty} e^{in\pi}\delta(t-n\tau)\delta(x) \tag{7.6.10}$$

$$= \sum_{n=-\infty}^{+\infty} e^{in\pi}G(t-n\tau,x) \tag{7.6.11}$$

where, from Eq. 7.3.8

$$G(t,x) = \frac{1}{2\pi}\frac{t\gamma_0 + x\gamma_1}{x^2+t^2} \tag{7.6.12}$$

Hence, for $t^2 + x^2 = a^2 \to 0$, we have

$$G_\tau(t,x) = G(t,x) + O(a) \tag{7.6.13}$$

which shows that the short-distance behavior of $G_\tau$ is the same as the $\tau = \infty$ case.

As in Section 7.3, the regularization of the action is done by point-splitting the axial-vector current $\bar{\psi}\gamma_\mu\gamma_5\psi\partial_\mu s$. The only property of the fermion propagator that enters in the calculation is zero-distance behavior of the fermion propagator, which is unchanged for finite time as in Eq. 7.6.13.

The finite time fermion path integral is evaluated; the point-split regularized is not written explicitly as the result derived earlier in Section 7.4 is used. For finite time, the partition function is given by

$$Z = \prod_{t=0}^{\tau}\prod_x \int d\bar{\psi}d\psi \exp\{-\int \bar{\psi}\gamma_\mu\partial_\mu\psi\} = \det{}_F(\gamma_\mu\partial_\mu)$$

Hence, from Eq. 7.4.3, the finite time result, up to irrelevant constants, is given by

$$\begin{aligned}\exp(S') &\equiv \prod_{t=0}^{\tau}\prod_x \int d\bar{\psi}d\psi \exp(S_F + S_I) \\ &= Z\exp\left[-\frac{1}{2\pi}\int_0^\tau dtdx(\partial_\mu s)^2\right] \\ &= \det{}_F(\gamma_\mu\partial_\mu)\exp\left[-\frac{1}{2\pi}\int_0^\tau dtdx(\partial_\mu s)^2\right]\end{aligned}$$

Note that the fermion determinant $\det_F(\gamma_\mu\partial_\mu)$ has to be evaluated over the functions antiperiodic in time with period $\tau$.

To perform the boson integration, we drop the redundant integration over

$\phi$ (which is equivalent to choosing the Landau gauge), and have

$$\begin{aligned} Z(\tau) &= (\det{}_F(\gamma_\mu\partial_\mu))(\det\partial^2)\times \\ &\quad \prod_{t=0}^{\tau}\prod_x \int ds(t,x)\exp\left[-\frac{1}{2g^2}\int_0^\tau dtdx[(\partial^2 s)^2+m^2(\partial s)^2]\right] \\ &= \frac{(\det{}_F(\gamma_\mu\partial_\mu)(\det\partial^2)}{\{\det[\partial^2(\partial^2+m^2)]\}^{1/2}} \\ &= \frac{\det{}_F(\gamma_\mu\partial_\mu)(\det\partial^2)^{1/2}}{[\det(\partial^2+m^2)]^{1/2}} \quad ; \quad m^2=g^2/\pi \end{aligned}$$

From Eq. 6.3.2, for $d=1+1$, we have

$$[\det(\partial^2+m^2)]^{1/2}=\prod_p \sinh\left[\frac{\tau}{2}(p^2+m^2)^{1/2}\right], \tag{7.6.14}$$

The fermion determinant is given by

$$\det{}_F(\gamma_\mu\partial_\mu)=\sqrt{\det{}_F(\gamma_\mu\partial_\mu)\det{}_F(\gamma_\mu\partial_\mu)} \tag{7.6.15}$$

$$=\sqrt{\det{}_F(\partial^2)\mathbb{I}_{2\times 2}}=\det{}_F(\partial^2) \tag{7.6.16}$$

To evaluate the fermion determinant the antiperiodic boundary conditions are realized by functions that are antiperiodic in interval $[t,t+\tau]$. In the momentum basis the eigenfunctions and eigenergies are given by

$$-\partial_\mu\partial_\mu(e^{i\frac{\pi(2n+1)t}{\tau}}e^{ipx})=\left[(\frac{\pi(2n+1)}{\tau})^2+p^2\right](e^{i\frac{\pi(2n+1)t}{\tau}}e^{ipx})\ ;\ n=0,\pm1,\pm2,..$$

The fermion determinant is given by the product of all the eigenvalues and yields

$$\begin{aligned} \det{}_F(\gamma_\mu\partial_\mu)=\det{}_F(\partial^2) &= \prod_p\prod_{n=-\infty}^{+\infty}[(\frac{\pi(2n+1)}{\tau})^2+p^2] \\ &= \text{const.}\prod_p\prod_{n=0}^{+\infty}\left[1+(\frac{|p|\tau}{\pi(2n+1)})^2\right]^2 \end{aligned}$$

The fermion determinant, up to a constant, is hence[3]

$$\det{}_F(\gamma_\mu\partial_\mu)=\det{}_F(\partial^2)=\prod_p\cosh^2\left[\frac{|p|\tau}{2}\right] \tag{7.6.17}$$

To have well-defined products over the momentum $p$, let $x\in[0,N]$ with periodic boundary conditions. Then $p=(2\pi/N)l, l=0,1,...,N-1$.

[3] From Gradshteyn and Ryzhik (1980) $\prod_{n=0}^{+\infty}\left[1+(\frac{|p|\tau}{\pi(2n+1)})^2\right]=\cosh(\frac{|p|\tau}{2})$.

In the product $(\det \partial^2)^{1/2} \det_F(\gamma_\mu \partial_\mu)$, there is a remarkable cancellation that removes the fermion eigenenergies. Note that

$$(\det \partial^2)^{1/2} \det_F(\gamma_\mu \partial_\mu) = \prod_p \sinh\left[\frac{\tau}{2}|p|\right] \cosh^2\left[\frac{\tau}{2}|p|\right] \tag{7.6.18}$$

$$= \exp\left[\sum_p [\ln(1 - e^{-\tau|p|}) + 2\ln(1 + e^{-\tau|p|})] + \frac{3\tau}{2}\sum_p |p|\right] \tag{7.6.19}$$

We have for the first two terms in (7.6.19), for $N \to \infty$, a remarkable cancellation

$$N \int_{-\infty}^{+\infty} \frac{dp}{2\pi} [\ln(1 - e^{-\tau|p|}) + 2\ln(1 + e^{-\tau|p|})]$$
$$= \frac{N}{\pi\tau}\left[-\frac{\pi^2}{6} + 2\frac{\pi^2}{12}\right] = 0 \tag{7.6.20}$$

and hence

$$(\det \partial^2)^{1/2} \det_F(\gamma_\mu \partial_\mu) = \exp\left[\frac{3\tau}{2}\sum_p |p|\right] \tag{7.6.21}$$

where the term linear in the exponential is simply the zero-point energy. Note that the fermion determinant with its negative-energy solutions has canceled and the fermions have been eliminated from the eigenspectrum.

Eqs. 7.6.14 and 7.6.21 yield

$$Z(\tau) = \frac{\exp\left[\frac{3\tau}{2}\sum_p |p|\right]}{[\det(\partial^2 + m^2)]^{1/2}}$$
$$= \exp\left[-\frac{\tau}{2}\sum_p \left[(p^2 + m^2)^{1/2} - 3|p|\right] - \sum_p \ln\left\{1 - \exp[-\tau(p^2 + m^2)^{1/2}]\right\}\right]$$

Dropping the vacuum energy yields[4]

$$Z(\tau) = \prod_p \ln\left\{1 - \exp[-\tau(p^2 + m^2)^{1/2}]\right\}$$
$$= \prod_p \sum_{n_p=0}^{\infty} \exp\left[-\tau \sum_p n_p (p^2 + m^2)^{1/2}\right] \tag{7.6.22}$$

Comparing Eqs. 7.6.22 and 7.6.3 yields the exact eigenenergies

$$E[n] = \sum_p n_p (p^2 + m^2)^{1/2}, \quad n_p = 0, 1, ... \tag{7.6.23}$$

[4] Note that $\ln(1 - x) = -\sum_{n=0}^{\infty} x^n$.

The eigenspectrum is the equally spaced energy levels of the free massive boson field. The integers $n_p$ denote the number of particles $n_p$ with momentum $p$ that have been created in the system. These massive excitations of the field are the bound states of the fermion-antifermion pair interacting via the gauge field.

The eigenfunctions of the interacting theory are

$$|\Phi_n\rangle \equiv \otimes_p |n_p\rangle \quad \text{with eigenenergy} \quad \sum_p n_p (p^2 + m^2)^{1/2}$$

and form a complete basis for the Hilbert space of states.

The fermionlike eigenenergies have been eliminated from the spectrum due to its interaction with the gauge field, and the cancellation in Eq. 7.6.20 reflects this. The absence of fermionic energy levels is the first indication that the fermions are permanently confined.

## 7.7 The gauge invariant states

Consider a fermion-gauge field-antifermion state that is gauge-invariant. Before calculating the energetics of theis state, we briefly review the definition of energy for a field theory.

Recall from quantum mechanics the energy of an unnormalized state $|\Phi\rangle$ is given by

$$E = \frac{\langle \Phi | H | \Phi \rangle}{\langle \Phi | \Phi \rangle} \tag{7.7.1}$$

where $H$ is the Hamiltonian. To use the above definition for a quantum field, a suitable limiting procedure has to be used since the states are generally not normalizable. Also, since we are using the path integral, we derive the Hamiltonian from the action functional. Let $|\chi_\epsilon\rangle$ be a state close to $|\Phi\rangle$ such that

$$\lim_{\epsilon \to 0} |\chi_\epsilon\rangle = |\Phi\rangle \tag{7.7.2}$$

Then

$$E = \lim_{\epsilon \to 0} \frac{\langle \chi | H | \Phi \rangle}{\langle \chi | \Phi \rangle} = \lim_{\epsilon \to 0} \left[ \lim_{t \to 0} -\frac{\partial}{\partial t} \frac{\langle \chi | e^{-tH} | \Phi \rangle}{\langle \chi | \Phi \rangle} \right] \tag{7.7.3}$$

The amplitude $\langle \chi | \exp(-tH) | \Phi \rangle$ can be evaluated using the path integral, and hence the energy can be obtained using (7.7.3).

We are interested in the energy of the gaugeinvariant fermion-antifermion

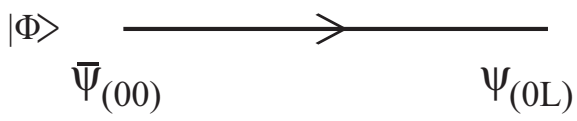


Figure 7.5 The gauge invariant fermion-antifermion state.

state separated by distance $L$. In the Schrödinger representation,

$$|\Phi\rangle = \bar{\psi}_{(0L)} \exp\left[i \int_0^L A_1(0,x)dx\right] \psi_{(00)}|\Omega\rangle \tag{7.7.4}$$

where $|\Omega\rangle$ is the vacuum state shown in Figure 7.5. The state $|\chi\rangle$ is obtained by infinitesimally displacing the fermion and antifermion in $|\Phi\rangle$. For notational simplicity, we will treat $|\chi\rangle$ as identical to $|\Phi\rangle$ and introduce $\epsilon$ at the end. Let

$$Q = \langle\Phi|e^{-tH}|\Phi\rangle \tag{7.7.5}$$

Then, in the Heisenberg representation, we have

$$\begin{aligned} Q = \langle\Omega|\bar{\psi}_{(t0)} \exp\left[-i \int_0^L A_1(t,x)dx\right] \psi_{(tL)}\bar{\psi}_{(0L)} \\ \times \exp\left[i \int_0^L A_1(0,x)dx\right] \psi_{(00)}|\Omega\rangle \end{aligned} \tag{7.7.6}$$

where the Heisenberg operators are appropriately time ordered for $t > 0$. We can represent $Q$, shown in Figure 7.5, as a path integral using the infinite-time action and have

$$\begin{aligned} Q = \frac{1}{Z}\langle\bar{\psi}_{(t0)} \exp\left[-i \int_0^L A_1(t,x)dx\right] \psi_{(tL)}\bar{\psi}_{(0L)} \\ \times \exp\left[i \int_0^L A_1(0,x)dx\right] \psi_{(00)}e^S\rangle \end{aligned} \tag{7.7.7}$$

The action $S$ in (7.7.7) is the infinite-time action obtained from (7.6.10) and $Z$ is also the infinite-time partition function; the brackets denote integration over all boson and fermion field variables, and $\bar{\psi}$, $\psi$, and $A_\mu$ in (7.7.7) are the field variables.

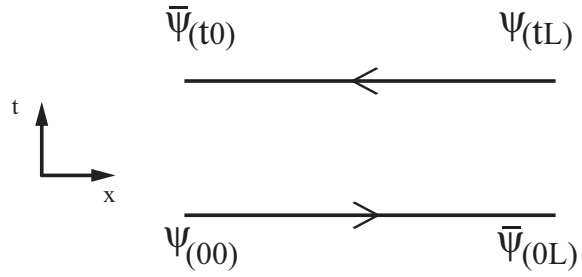


Figure 7.6 Time evolution of the gauge invariant fermion-antifermion state.

Introducing the regularizer $\epsilon$ and performing the path integration gives

$$Q_\epsilon = \langle\chi|e^{-tH}|\Phi\rangle \tag{7.7.8}$$

$$= \frac{1}{2\pi^2}\frac{1}{t^2+\epsilon^2}\exp[R(t,L)+P(t)] \tag{7.7.9}$$

where, using $p^2 = p_0^2 + p_1^2$,

$$R(t,L) = -\frac{g^2}{2}\int\frac{d^2p}{(2\pi)^2}\frac{|1-\exp(ip_0t)|^2|1-\exp(ip_1L)|^2}{p_1^2(p^2+m^2)} \tag{7.7.10}$$

$$P(t) = 2g^2\int\frac{d^2p}{(2\pi)^2}\frac{|1-\exp(ip_0t)|^2}{p^2(p^2+m^2)} \tag{7.7.11}$$

$$= 2[\gamma + \ln\left[\frac{mt}{2}\right] + K_0(mt)] \tag{7.7.12}$$

and $\gamma$ is Euler's constant. The term $R(t,L)$ comes from the gauge-field variables, and the term $P(t)$ from a combination of the gauge-field and fermion variables. Note for $L \to 0$, $R(t,L) \to 0$ and we recover the result given in Eq. 7.4.21.

We have the following exact properties of $R(t,L)$ and $P(t)$:

$$R(0,L) = 0, \quad P(0) = 0 \tag{7.7.13}$$

$$-\frac{\partial R(t,L)}{\partial t}|_{t=0} = \frac{g^2L}{2}, \quad \frac{\partial P(t)}{\partial t}|_{t=0} = 0 \tag{7.7.14}$$

Using (7.7.3) for the energy, and taking the limit of $\epsilon \to 0$ gives

$$E = -\frac{\partial R(t,L)}{\partial t}|_{t=0} = \frac{g^2L}{2} \tag{7.7.15}$$

The expression for energy is exact and is a gauge-invariant quantity. We

see that energy increases linearly with distance, and the string tension, i.e., the energy per unit length, is $g^2/2$.

It is obvious that the state $|\Phi\rangle$ is not an eigenstate, given its complicated time dependence. To understand how the state $|\Phi\rangle$ is constructed, we expand it in the energy eigenfunction basis. Let

$$|\Phi\rangle = \prod_p \sum_{n_p} C_{n_p} |n_p\rangle \tag{7.7.16}$$

Then,

$$Q = \langle\Phi|e^{-tH}|\Phi\rangle = \sum_{(n)} |C_n|^2 \exp(-tE_n)$$

From (7.7.9) we have for $mL \gg 1$ ($\alpha = g^2 L/2m$)

$$\begin{aligned} Q &= \frac{e^{2\gamma}}{8\pi^2} \exp[-\alpha + \alpha e^{-mt} + 1 - mtK_1(mt) + 2K_0(mt) + O(e^{-mL})] \\ &\simeq \frac{e^{2\gamma}}{8\pi^2} \exp(-\alpha + \alpha e^{-mt}) \end{aligned} \tag{7.7.17}$$

In obtaining (7.7.17) we have kept only the leading-order terms in $mL$, and in effect have disregarded eigenfunctions with nonzero momentum $p$. We call this the static approximation, and a detailed study shows that this approximation has the leading-order effects.

Ignoring the overall constant, we have

$$Q \simeq e^{-\alpha} \sum_{n=0}^{\infty} \left[\frac{\alpha^n}{n!}\right] e^{-nmt} \tag{7.7.18}$$

Hence, from (7.7.17) and (7.7.18),

$$E_n = nm \tag{7.7.19}$$

and

$$|C_n|^2 = e^{-\alpha} \left[\frac{\alpha^n}{n!}\right], \quad \sum_n |C_n|^2 = 1 \tag{7.7.20}$$

As expected, only the static eigenstates $|n_{p=0}\rangle \equiv |n\rangle$ with energy $nm$ contribute to $|\Phi\rangle$; note $|\Phi\rangle$ is a Poisson distribution in $|n\rangle$ and

$$|\Phi\rangle \simeq \sum_{n=0}^{\infty} C_n |n\rangle \tag{7.7.21}$$

We obtain

$$E = \langle\Phi|H|\Phi\rangle = m\sum_n n|C_n|^2 = m\alpha = \frac{g^2L}{2} \tag{7.7.22}$$

Note the static approximation gives the correct expression for the energy. Using the properties of the Poisson distribution, we have for the energy dispersion

$$\Delta E = \langle\Phi|\hat{H}^2|\Phi\rangle - E^2 = m^2\sum_n n^2|C_n|^2 - E^2 = mE \tag{7.7.23}$$

The dispersion in the energy is large and is proportional to the energy of the state. Hence in any numerical calculation it would be difficult to separate out the energy of the state from the background statistical fluctuations. We can see the origin of this large fluctuation by expanding the $|C_n|^2$ about its maxima, and assuming all the coefficients are real, we obtain, up to a normalization constant,

$$|\Phi\rangle \simeq \sum_{n=0}^{\infty} \exp\left[-\frac{1}{4\alpha}(n-\alpha)^2\right]|n\rangle \tag{7.7.24}$$

We see that $|\Phi\rangle$ is peaked at the state $|N\rangle$ with integer $N = \alpha$, and has a spread of $2\alpha$ which gives rise to the large dispersion for $E$.

Identifying the eigenstates $|N\rangle$ as $n$ pairs of fermion-antifermion bound states, we have the interpretation of (7.7.24) that the excited gauge invariant state is, with the largest amplitude, a state of $N \sim mL$ pairs. That is, for large L the gauge string 'breaks' instantaneously into a number of pairs proportional to the length of the string. See Fig. 7.7.

Hence, if we view the state $|\Phi\rangle$ as a case where the quarks (fermions) are well separated and attempt to see the single quark, we will end up observing the bound-state pairs and not the isolated fermion. Once the fermions constituting the gauge invariant state are separated by a distance larger than $m^{-1}$, pair production takes place. Hence we conclude that the fermions are permanently confined within a distance of $m^{-1}$, and no isolated single fermion can be seen for separations much larger than $m^{-1}$.

This view of string breakup has been discussed by other authors and the Schwinger model provides a quantitative basis for this.

## 7.8 String tension and fermion confinement

We define the string tension $\mu$ as the change in the energy of the string when the length is varied.

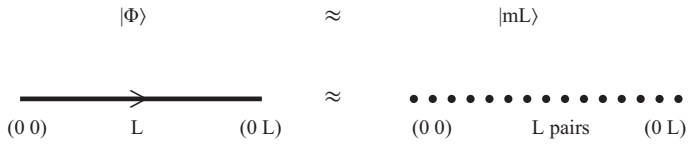


Figure 7.7 Eigenfunction expansion of the gauge invariant fermion-antifermion state.

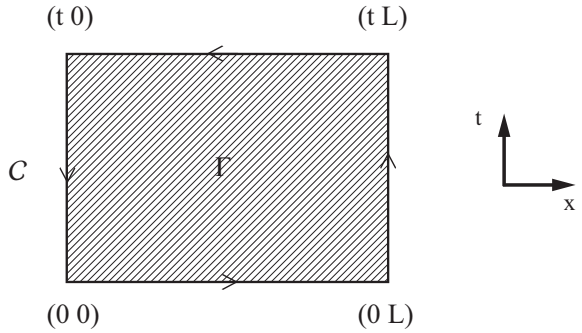


Figure 7.8 The Wilson loop.

That is,

$$\mu = \frac{\partial E}{\partial L} \tag{7.8.1}$$

where $E$ is the energy and $L$ the length of the string. For the gauge invariant fermion-antifermion state $|\Phi\rangle$ we have from (7.7.15)

$$\mu = \frac{g^2}{2} \tag{7.8.2}$$

The connection of $\mu$ with the Wilson loop integral can now be obtained. Consider a square contour of length $L$ and width $t$ and with enclosed area $\Gamma$ as shown in Figure 7.8. Similar to Eq. 7.5.1, the Wilson loop is given by

$$W = \langle \exp\left[ i \oint_C A_\mu dl_\mu \right] \rangle = \langle \exp\left[ i \int_\Gamma \partial^2 s \right] \rangle \tag{7.8.3}$$

Performing the bosonic path integral yields

$$W = \exp[R(t, L) + R(L, t)] \tag{7.8.4}$$

where $R(t, L)$ is given by (7.7.10).

For large loops and large time, we have

$$W = \exp\left[ -\frac{g^2}{2m}(t + L) + O(e^{-mt}, e^{-mL}) \right] \tag{7.8.5}$$

We see that $W$ does not have exp(-area) Wilson behavior, and the string tension $\mu$ cannot be extracted as the coefficient of the area term. The reason for this is the string breaks into $(t + L)$ pairs and gives the exp(-perimeter) behavior for the loop; in other words, fermion pair creation removes the Wilson behavior for the loop.

The Wilson loop in the absence of the fermions can be obtained by setting $m = 0$ in $R(t, L)$, and we obtain the exact result,

$$W = \exp\left[-\frac{g^2}{2}tL\right] \tag{7.8.6}$$

$$= \exp\left[-\mu tL\right] \tag{7.8.7}$$

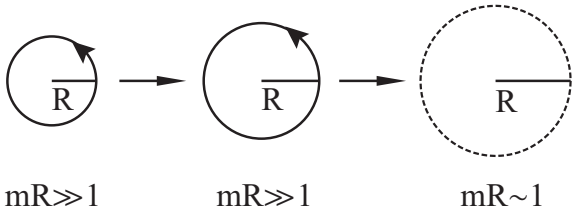


Figure 7.9 Breaking of the string.

and the string tension is the coefficient of the area term. Hence $\mu$ can be obtained by studying the large gauge-field loops in the absence of fermions, and the introduction of fermions does not change the value of $\mu$, at least in Schwinger QED. Note that in the *presence* of the fermions, for small loops, i.e., $t, L \ll 1$, we again have

$$W \cong \exp[-\mu tL + O(t^2, L^2)] \tag{7.8.8}$$

We hence have the following picture for confinement in Schwinger QED. The small gauge-field loops show the Wilson exp(-area) behavior. As the loops are made larger and larger, due to the Wilson behavior of the pure gauge loops, large energy is required to make these loops, and instead the system produces fermion-antifermion pairs, breaking the string and giving the exp(-perimeter) behavior. The string breaks, i.e., crosses over from the area to perimeter behavior at the characteristic length scale of the system, which for Schwinger QED is $m^{-1}$. See Figure 7.9.

In summary, the behavior of the large gauge-field loops in the pure gauge theory determine whether or not there is confinement of the fundamental fermions, and the Wilson criterion is appropriate. However, in the presence of the fermions, confinement can be equally studied by looking at the energy of the fermion-string-antifermion state, which involves looking at the

short-time behavior of the system. Both these approaches can be used for evaluating the string tension and give the same result in Schwinger QED.

## 7.9 Discussion

We solved the Schwinger model exactly using a point-spit current, and showed that the axial-current anomaly is responsible for generating the mass term for the boson field. A study of the Wilson loop integral for the interacting theory indicated confinement according to an interpretation of the Wilson criterion. In the presence of fermions, the loop integral breaks due to the creation of fermion-antifermion pairs and does not show the area law dependence. A detailed study of the eigenstates however confirmed that there are no fermions eigenstates for the interacting theory.

## 7.10 Appendix I

We prove 7.4.8, that is

$$\Gamma_p = -\frac{1}{\pi}(p^2 + p_1(\theta)p_2(\theta)) + 0(a) \tag{7.10.1}$$

where

$$\Gamma_p = p_\mu(\theta)p_\nu(\theta)\int_l e^{ia(l_\mu(\theta)+l_\nu(\theta))}\,\mathrm{Tr}\left(\gamma_\mu(\theta)\frac{1}{l}\gamma_\nu(\theta)\frac{1}{p+l}\right) \tag{7.10.2}$$

We use the identity

$$\sum_\mu e^{ial_\mu(\theta)}p_\mu(\theta)\gamma_\mu(\theta) = e^{ial_1(\theta)}p + (e^{ial_2(\theta)} - e^{ial_1(\theta)})p_2(\theta)\gamma_2(\theta) \tag{7.10.3}$$

to rewrite $\Gamma_p$, and use the familiar identity(used in current conservation) that

$$\frac{1}{l}p\frac{1}{p+l} = \frac{1}{l} - \frac{1}{p+l} \tag{7.10.4}$$

to perform the momentum space integration. That is

$$\Gamma_p = \Gamma_1 + \Gamma_2 + \Gamma_3 \tag{7.10.5}$$

where

$$\Gamma_1 = \int_l e^{2ial_1(\theta)} \operatorname{Tr}\left(p\frac{1}{l}p\frac{1}{p+l}\right) \tag{7.10.6}$$

$$= (1 - e^{2iap_1(\theta)}) \int_l \operatorname{Tr}(pl^{-1}) e^{2ial_1(\theta)} \tag{7.10.7}$$

$$= -\frac{1}{\pi} p_1^2(\theta) + 0(a) \tag{7.10.8}$$

$$\Gamma_2 = 2p_2(\theta) \int_l e^{ial_1(\theta)} (e^{ial_2(\theta)} - e^{ial_1(\theta)}) \operatorname{Tr}\left(\gamma_2(\theta)\frac{1}{l}p\frac{1}{p+l}\right) \tag{7.10.9}$$

$$= -\frac{1}{\pi} p_2(\theta)(p_1(\theta) + p_2(\theta)) + 0(a) \tag{7.10.10}$$

and

$$\Gamma_3 = p_2^2(\theta) \int_l (e^{ial_1(\theta)} - e^{ial_2(\theta)})^2 \operatorname{Tr}\left(\gamma_2(\theta)\frac{1}{l}\gamma_2(\theta)\frac{1}{p+l}\right) \tag{7.10.11}$$

$$= p_2^2(\theta) \int_l (e^{ial_1(\theta)} - e^{ial_2(\theta)})^2 \operatorname{Tr}\left(\gamma_2(\theta)\frac{1}{l}\gamma_2(\theta)\frac{1}{l}\right) + 0(a) \tag{7.10.12}$$

$$= p_2^2(\theta) \int_l (e^{ial_1(\theta)} - e^{ial_2(\theta)})^2 \frac{(2l_2^2(\theta) - l^2(\theta))}{l^4} \tag{7.10.13}$$

$$= 0 + 0(a) \tag{7.10.14}$$

Hence, collecting our results, we have from 7.10.8, 7.10.10 and 7.10.14

$$\Gamma_p = -\frac{1}{\pi}(p^2 + p_1(\theta)p_2(\theta)) + 0(a) \tag{7.10.15}$$

which is the result required.

# 8

# Bosonic string path integral

## 8.1 Introduction

A quantum particle consists of a point-like degree of freedom, as shown in Figure 8.1. In contrast, a quantum string is a one-dimensional object with the string degree of freedom extended in space, as shown in Figures 8.1 and 8.2.

Since the string is an extended object, and the evolution of a string spans out a two-dimensional world-sheet embedded in $d$-dimensional spacetime, as shown in Figure 8.3. The degrees of freedom of the string on the world-sheet constitute a *two-dimensional quantum field theory* and the techniques developed so far can be applied to the study this quantum field theory. The approach of Polyakov (1987) is adopted for studying the covariant bosonic path integral.

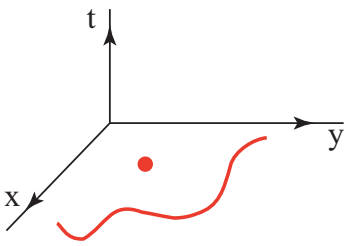


Figure 8.1 From point to a string.

The bosonic string theory is invariant under the reparametrization of the string world-sheet, and to have a finite path integral, similar to gauge theories, a gauge for the metric on the world-sheet has to be chosen. The Faddeev-Popov procedure of gauge fixing, discussed in Section 6.7 for the photon, can be applied to choosing a gauge for the bosonic string.

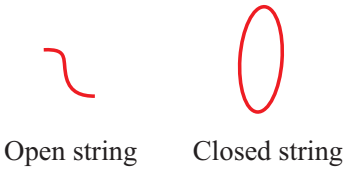


Figure 8.2 Open and closed strings.

The bosonic string theory discussed in this Chapter is the *quantum mechanics* of an open string since the evolution of only a *single* string is considered. The fission and fusion of strings can be studied using the path integral, and is briefly addressed later. The analysis of bosonic string quantum mechanics, including arriving at the critical spacetime dimension of $d_c = 26$, can be carried out exactly using free quantum fields defined on the underlying two dimensional world-sheet of the bosonic string.

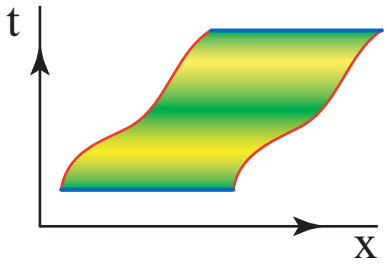


Figure 8.3 Worldsheet of string

One of the most striking feature of the quantum mechanics of a single string is that, unlike classical strings, it is only in certain critical spacetime dimensions – $d_c = 26$ for bosonic strings and $d_c = 10$ for superstrings – that the theory is consistent. This is shown in Figure 8.4. The critical dimension is required to preserve conformal invariance and it has been shown by Polyakov (1987) that the Faddeev-Popov ghost fields play a crucial role in determining the critical dimension of spacetime.

Noteworthy 8.1: Superstring Theory

Superstring theory is discussed in many standard texts suchas Polchinski (1998) – and which have being written for specialists. These texts are very technical, extensive and exhaustive – and require a major effort to master. The book by Zwiebach (2009) is an excellent introductory text on superstrings written for physics undergraduates.

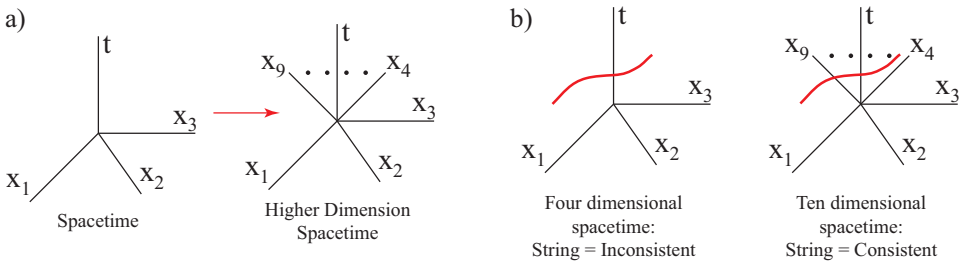


Figure 8.4 The superstring is inconsistent with quantum mechanics in four dimensional spacetime but is consistent in ten dimensional spacetime.

The mathematics of superstring theory is a natural continuation of quantum field theory. Without introducing any fundamentally new mathematics, one can start to study the formalism of superstring theory.

In this Chapter, the quantum mechanics of the covariant bosonic string is studied as an example of free quantum fields in two dimensions. This Chapter is meant as exemplar of how quantum mathematics can be extended into greater and more complex domains – as is the case with the formalism of superstring theory. String theory illustrates new features of free quantum fields, which forms one of the cornerstones of superstring theory.

## 8.2 Nambu-Goto String Action

Consider a relativistic point particle of mass m moving in a flat $d$-dimensional spacetime with metric $\eta^{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$; the trajectory of the particle is given by $X^\mu(\tau)$, where $\tau$ is some parameter ($\mu = 1, 2, \ldots, d$), and shown in Figure 8.5. The action $S$ is given by the proper distance covered by the particle in going from the initial to final position, i.e.

$$S = -m \int_a^b ds \tag{8.2.1}$$

$$= -m \int_a^b \left( \frac{dX}{d\tau} \frac{dX}{d\tau} \right)^{1/2} d\tau \tag{8.2.2}$$

The action is invariant under reparameterizations, that is

$$\tau = \tau(\tau') \tag{8.2.3}$$

$$X(\tau) = X'(\tau') \tag{8.2.4}$$

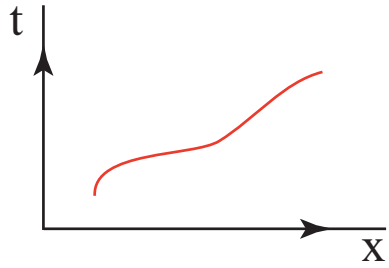


Figure 8.5 Worldline of particle

yields

$$S = -m \int_a^b \left( \frac{dX'}{d\tau'} \frac{dX'}{d\tau'} \right)^{1/2} d\tau' \tag{8.2.5}$$

A string on any spacelike slice is specified by the string coordinates $X^\mu(\sigma)$, where $\sigma$ is a parameter; as it evolves in time, the string sweeps out a two dimensional surface given by coordinates $X(\sigma, \tau)$, as shown in Figure 8.3.

In analogy with the relativistic particle, the Nambu-Goto action is proportional to the area swept out by the string, and string tension $T$ (with units of energy/length) is the analog of mass.

$$S_{NG} = -\frac{T}{2\pi} \int d\sigma d\tau \sqrt{\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{8.2.6}$$

with $\alpha, \beta = 0, 1 = \tau, \sigma$.

The area of the string world-sheet is independent of reparametrizations, as shown in Figure 8.6. The string coordinates $X^\mu(\sigma, \tau)$ are vectors in spacetime but on the world-sheet they are scalars. One requires that the action is invariant be the infinitesimal reparametrization transformations given by

$$\tau \to \tau' = \tau + \xi^0(\sigma, \tau) \tag{8.2.7}$$

$$\sigma \to \sigma' = \sigma + \xi^1(\sigma, \tau) \tag{8.2.8}$$

$$X_\mu(\sigma, \tau) \to X_\mu(\sigma', \tau') = X(\sigma, \tau) \tag{8.2.9}$$

The reason that the spacetime coordinates of the bosonic string $X_\mu(\sigma, \tau)$ are invariant under world-sheet reparametrizations is because, as mentioned, they are world-sheet scalars.

The world-sheet has a Riemannian metric $g_{\alpha\beta}(\sigma, \tau)$ in term of which the Polyakov string action is ($g = \det g_{\alpha\beta}$)

$$S = -\frac{T}{2\pi} \int d\sigma d\tau \sqrt{g} g^{\alpha\beta} (\partial_\alpha X^\mu \partial_\beta X_\mu) \tag{8.2.10}$$

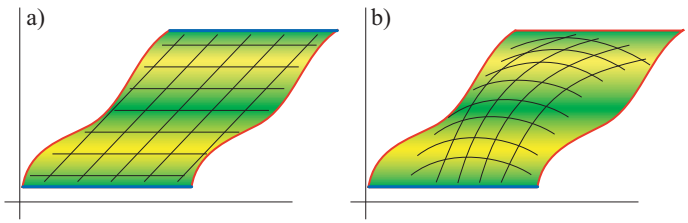


Figure 8.6 a) One choice of parametrization of the world-sheet using coordinates $\sigma, \tau$. b) Another choice of parametrization of the world-sheet using coordinates $\sigma', \tau'$.

Eliminating $g_{\alpha\beta}$ in S using the field equation for $g_{\alpha\beta}$ yields the Nambu-Goto action. Although the two actions are equivalent at the classical level, they are quite different at the quantum level. Most importantly, the Polyakov action can be given a reparametrization invariant regularization (for example the Pauli-Villars regularization) which reveals the quantum conformal anomaly of the string action.

Under the reparametrization (diffeomorphisms) given by Eq. 8.2.9, to $0(\xi)$ we have

$$\delta g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \tag{8.2.11}$$

$$\delta X^{mu} = \xi^\alpha \partial_\alpha X^\mu \tag{8.2.12}$$

where

$$\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta + \Gamma^\gamma_{\alpha\beta} \xi_\gamma \tag{8.2.13}$$

and $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel connection of the metric $g_{\alpha\beta}$.

The action is invariant under Weyl (conformal) rescalings, namely

$$\delta g_{\alpha\beta} = \lambda(\sigma, \tau) g_{\alpha\beta} \tag{8.2.14}$$

$$\delta X^\mu = 0 \tag{8.2.15}$$

Lastly, the action is invariant under space-time Poincare transformation

$$\delta X^\mu = a^\nu_\mu X^\nu + b^\nu \tag{8.2.16}$$

where $a^m u_\nu$ and $b^m u$ are constant.

In addition to $S$, the other terms which are reparametrization invariant are ($R^{(2)}$ is the world-sheet Rioci scalar)

$$S_1 = \mu^2 \int d\sigma d\tau \sqrt{g} \tag{8.2.17}$$

$$S_2 = \lambda \int d\sigma d\tau \sqrt{g} R^{(2)} \tag{8.2.18}$$

The action $S_1$ is necessary in spacetime dimension ¡ $d_c$, the $S_2$ yields the Euler number for world-sheet of non-trivial topology.

### *8.2.1 Light Cone Gauge*

Perform $dg_{\alpha\beta}$ integrations in Eq. 8.3.1 and recover the Nambu-Goto action of Eq. 8.2.6 and then fix a gauge for the $X^\mu(\sigma,\tau)$. Using light-like coordinates $\partial_\pm = \left(\frac{\partial}{\partial\tau} \pm \frac{\partial}{\partial\sigma}\right)$, we choose non-covariant constraints

$$Z = \int DX\delta(\partial_+ X\partial_+ X)\delta(\partial_- X\partial_- X)e^{iS_{NG}} \tag{8.2.19}$$

To linearize the constraints, choose the gauge for $X^\mu$ to be

$$X^\pm = X^l \pm X^d, \quad X^+ = a + p^+\tau \tag{8.2.20}$$

Integrating out $X^\pm$ from the path integral yields a Hamiltonian involving only the physical degrees of freedom. However, the light-cone formalism is only invariant under $SO(d-2)$ Lorentz transformations and not manifestly covariant. This poses severe problems in second quantizing the theory to obtain a field theory of strings.

## 8.3 Covariant Quantization

Following Polyakov, the string action is quantized using the Feynman path integral; let $i$ denote an initial $f$ the final string configuration; then the amplitude for the transition from $i$ to $f$ is given by

$$\begin{aligned} Z &= \int_i^f DgDX \exp(iS[g,X]) \\ Dg &= \prod_{\sigma,\tau} dg_{00}dg_{11}dg_{12} \\ DX &= \prod_{\sigma,\tau}\prod_{\mu} dX^\mu(\sigma,\tau) \end{aligned} \tag{8.3.1}$$

where recall from Eq. 8.2.10

$$S = -\frac{T}{2\pi}\int d\sigma d\tau\sqrt{g}g^{\alpha\beta}(\partial_\alpha X^\mu\partial_\beta x_\mu) \tag{8.3.2}$$

Due to invariance of $S$ under reparametrizations, the expression in Eq. 8.3.1 is divergent; to remove this divergence we have to gauge-fix the path integral. Usually, one of the following two gauges are chosen.

### 8.3.1 Conformal Gauge

The other choice is to choose a gauge for the metric $g^{\alpha\beta}$, with no constraints imposed on $X^\mu$; hence, in this approach the system is manifestly covariant even after gauge-fixing. The light-like coordinates are

$$\begin{bmatrix} \sigma^+ \\ \sigma^- \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} \tag{8.3.3}$$

Let

$$\Lambda^\alpha_\beta = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{8.3.4}$$

Also, for a two component vector

$$V^\pm = V^0 \pm V^1 \tag{8.3.5}$$

$$V_\pm = \frac{1}{2}(-V^0 \pm V^1) \tag{8.3.6}$$

In these coordinates, the metric $h$ is

$$h_{\alpha\beta} = \begin{bmatrix} h_{++} & h_{+-} \\ h-+ & h_{--} \end{bmatrix} = \Lambda^\delta_\alpha g_{\delta\gamma} \Lambda^\gamma_\beta \tag{8.3.7}$$

or

$$h_{++} = \frac{1}{4}(g_{11} + g_{22} + 2g_{12}) \tag{8.3.8}$$

$$h_{--} = \frac{1}{4}(g_{11} + g_{22} - 2g_{12}) \tag{8.3.9}$$

$$h_{+-} = h_{-+} = \frac{1}{4}(g_{11} - g_{22}) \tag{8.3.10}$$

The path integral can be written as

$$Z = \int dh_{++} dh_{--} dh_{+-} dX e^{iS[h,X]} \tag{8.3.11}$$

We choose the conformal gauge defined by

$$g_{\alpha\beta} = e^{\phi(\sigma,\tau)} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \equiv e^\phi \eta_{\alpha\beta} \tag{8.3.12}$$

or equivalently

$$h_{++} = 0 = h_{--} \tag{8.3.13}$$

$$h_{+-} = -e^\phi/2 \tag{8.3.14}$$

To implement the conformal gauge, the method developed for the Faddeev-Popov gauge- fixing procedure discussed in Section 6.7 is followed. The symmetry for strings is reparametrization invariance and a gauge is chosen to render the path integral finite. The following needs to be evaluated

$$\int D\xi_+ D\xi_- \delta(h_{++}(\xi_+))\delta(h_{--}(\xi_-)) \tag{8.3.15}$$

where $h_{\pm\pm}(\xi)$ is the value of $h_{\pm\pm}(\xi)$ after an infinitesimal coordinate transformation given by Eq. 8.2.9. From Eq. 8.2.11, it can be shown that

$$\delta(h_{++}(\xi)) = \nabla_+ \xi_+ = \partial_+ \xi_+ + (\partial_+ \phi)\xi_+ \tag{8.3.16}$$

$$\delta(h_{--}(\xi)) = \nabla_- \xi_- = \partial_- \xi_- - (\partial_- \phi)\xi_- \tag{8.3.17}$$

Changing variables from to $\xi_\pm$ to $h_{++}(\xi), h_{--}(\xi)$, we have

$$Dh_{++}(\xi) = \det\left(\frac{\delta(h_{++}(\xi))}{\delta\xi_+}\right) D\xi_+ \tag{8.3.18}$$

$$Dh_{--}(\xi) = \det\left(\frac{\delta(h_{--}(\xi))}{\delta\xi_-}\right) D\xi_- \tag{8.3.19}$$

Hence, from Eq. 8.3.15

$$\begin{aligned} &\int D\xi_+ D\xi_- \delta(h_{++}(\xi))\delta(h_{--}(\xi)) \\ &= (\det \nabla_+ \det \nabla_-)^{-1} \int Dh_{++} Dh_{--} \delta(h_{++})\delta(h_{--}) \end{aligned} \tag{8.3.20}$$

Since

$$\int Dh_{++} Dh_{--} \delta(h_{++})\delta(h_{--}) = 1$$

we obtain

$$1 = \det \nabla_+ \det \nabla_- \int D\xi_+ D\xi_- \delta(h_{++}(\xi))\delta(h_{--}(\xi)) \tag{8.3.21}$$

We have, from Eqs. 8.3.11 and 8.3.21

$$Z = \int Dh_{\pm\pm} Dh_{+-} DX D\xi_+ D\xi_- e^{iS[h,X]} (\det \nabla_+ \det \nabla_-)\delta(h_{++}(\xi))\delta(h_{--}(\xi))$$

We do the inverse transformation $h(\xi^{-1})$; since $S[h(\xi^{-1}), X] = S[h, X]$ due to reparametrization invariance – and similarly for $\det(\nabla_+)$ and $\det(\nabla_-)$ as they are reparametrization invariant). Hence the $\int D\xi_- D\xi_+$ integrations decouple from $Z$ and yield

$$Z = \int Dh_{++} Dh_{--} Dh_{+-} DX \det \nabla_+ \det \nabla_- \delta(h_{++})\delta(h_{--}) e^{iS[h,X]} \tag{8.3.22}$$

### 8.3.2 Weyl invariance

The variables $h_{++}$, $h_{--}$ have been eliminated in eq. 8.3.22 due to the delta-functions; only $h_{+-}$ is still coupled to the string coordinates $X_\mu$. At the classical level, due to Weyl invariance, $h_{+-}$ also decouples from $X^\mu$ as can be readily seen from the action Eq. 8.3.2 evaluated in the conformal gauge.

The Faddeev-Popov Jacobians $\det(\nabla_+)$, $\det(\nabla_-)$ can be represented using fermion integration as given in Eq. 6.12.9 and yields

$$\det(\nabla_+) = \int Dc^- Db_{--} e^{-\frac{1}{\pi}\int d\sigma d\tau c^- \nabla_+ b_{--}} \tag{8.3.23}$$

$$\det(\nabla_-) = \int Dc^+ Db_{++} e^{+\frac{1}{\pi}\int d\sigma d\tau c^+ \nabla_- b_{++}} \tag{8.3.24}$$

where $c^\pm$, $b_{\pm\pm}$ are the Faddeev-Popov anticommuting ghost fields. The ghost action is given by

$$S_{gh} = -\frac{1}{\pi}\int d\sigma d\tau \left\{c^- \nabla_+ b_{--} + c^+ \nabla^- b_{++}\right\} \tag{8.3.25}$$

Similar to the string coordinates $X_\mu$, the Weyl scaling transformation for the ghost fields, at the classical (non-regularized) level, from Eqs. 8.3.25, 8.3.17 and 8.3.24, yields

$$S_{gh} = -\frac{1}{\pi}\int d\sigma d\tau \left\{c^- e^{-\phi}\partial_+(e^\phi b_{--}) + c^+ e^{-\phi}\partial^-(e^\phi b_{++})\right\} \tag{8.3.26}$$

By performing the rescalings

$$c^\pm \to e^\phi c^\pm, \quad b_{\pm\pm} \to e^\phi b_{\pm\pm} \tag{8.3.27}$$

the field $\phi$ decouples from $S_{gh}$ and yields

$$S_{gh} = -\frac{1}{\pi}\int d\sigma d\tau \left\{c^- \partial_+ b_{--} + c^+ \partial^- b_{++}\right\} \tag{8.3.28}$$

### 8.3.3 String path in conformal gauge

It has been shown by Polyakov that Weyl invariance is broken due to quantum effects, and that for bosonic strings only in $d = 26$ spacetime dimensions does the first quantized string become conformally invariant and the conformal field $g_{ab}$ consequently decouples from the string field.

Henceforth, we will only consider $d = 26$. Let

$$c = c^+, c^- = \bar{c}, b = b_{--}, \bar{b} = b_{++}$$

The integration $Dh_{+-}$ decouples from the path integral in $d = 26$ and the conformal gauge we have chosen results in

$$g_{ab} = \eta_{ab}$$

We obtain, from Eqs. 8.2.10 and 8.3.28, the covariant gauge fixed action

$$S_{gf}[X, c, b] = -\frac{T}{\pi}\int d\sigma d\tau \partial_+ X^\mu \partial_- X_\mu - \frac{1}{\pi}\int d\sigma d\tau (\bar{c}\partial_+ b + c\partial_- \bar{b}) \quad (8.3.29)$$

Hence the path integral, from Eq. 8.3.22, is given by

$$\begin{aligned} Z &= \int DXDh_{\pm\pm}Dh_{+-} \det \nabla_+ \det \nabla_- \delta(h_{++})\delta(h_{--}) e^{iS[h,X]} \\ &= \int DXDcDbD\bar{c}D\bar{b} e^{iS_{gf}[X,c,b]} \end{aligned} \quad (8.3.30)$$

Eq. 8.3.30 is the sought for result of gauge-fixing and is the form of the string path integral for the remaining analysis.

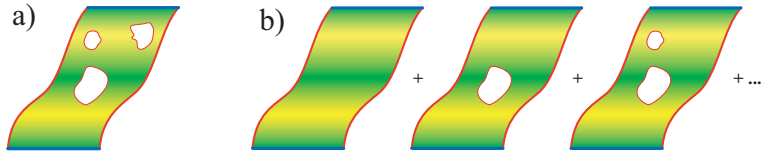


Figure 8.7 a) Open string with many holes. b) The string path integral given by a summation over distinct topologies.

To go beyond single string dynamics, according to Polyakov (1987) the fission and fusion of strings can be viewed as the world-sheet having non-trivial topology, 'handles' for closed strings, specified by the Euler number, and 'holes' for open strings. Figure 8.7(a) shows a string world-sheet with non-trivial topology. The full action is the $S + S_2$ and we have for the second quantized string field theory

$$z = \sum_{\text{Topologies}} e^{-4\pi\lambda(\text{Euler } \sharp)} \int dm \int DXDbD\bar{b}DcD\bar{c} e^{iS[X,\ldots]} \quad (8.3.31)$$

where $m = (m_1, m_2, \ldots)$ are the moduli parameters specifying the conformally inequivalent metrics for a given topology. The summation over all distinct open string topologies is shown in Figure 8.7(b).

The Faddeev-Popov ghosts $b, c, \bar{b}, \bar{c}$ are essential for decoupling the non-physical states; their role becomes even more important when fermions are introduced, since, unlike the case of bosonic degrees of freedom, the ghost fields are directly involved in the emission and absorption of fermionic degrees of freedom.

## 8.4 Virasoro Algebra

Recall the Polyakov action in conformal gauge is

$$S = -\frac{T}{\pi}\int d\sigma d\tau \partial_+ X^\mu \partial_- X_\mu - \frac{1}{\pi}\int d\sigma d\tau (c^+\partial_- b_{++} + c^-\partial_+ b_{--}) \quad (8.4.1)$$

This classical action must be taken together with field equations for $g_{\alpha\beta}$ yield the constraints on the classical string coordinates of vanishing and traceless energy-momentum tensor; that is from action given by Eq. 8.2.10

$$T_{\alpha\beta} = -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g\alpha\beta} = 0 \quad (8.4.2)$$

or

$$T_{\alpha\beta} = \partial_\alpha X \partial_\beta X - \frac{1}{2} g_{\alpha\beta}\partial X \partial X \quad (8.4.3)$$

In light-cone coordinates

$$T_{++} = \partial_+ X \partial_+ X = 0 \quad (8.4.4)$$

$$T_{--} = \partial_- X \partial_- X = 0 \quad (8.4.5)$$

$$T_{+-} = 0 = T^\alpha_\alpha \quad (8.4.6)$$

To obtain $T^{gh}_{\alpha\beta}$, note we can form, with respect to $\sigma, \tau$, a two vector $c^\alpha$ out of $c^\pm$ and a traceless symmetric tensor $b_{\alpha\beta}$ out of $b_{\pm\pm}$; then

$$S_{gh} = -\frac{1}{\pi}\int d\sigma d\tau c^\alpha \partial^\beta b_{\alpha\beta} \quad (8.4.7)$$

and on making the variation $\tau \to \tau + \delta\tau$ and $\sigma \to \sigma + \delta\sigma$, we obtain

$$T^{gh}_{++} = \frac{1}{2}c^+\partial_+ b_{++} + \partial_+ c^+ b_{++} \quad (8.4.8)$$

$$T^{gh}_{--} = \frac{1}{2}c^-\partial_- b_{--} - \partial_- c^- b_{--} \quad (8.4.9)$$

$$T^{gh}_{--} = 0 \quad (8.4.10)$$

We now limit all further derivations to the case of open strings. The open string coordinate is specified by parameter $\sigma\epsilon[0,\pi]$ with $X(0,\tau) \neq X(\pi,\tau)$. To obtain the field equation for $X(\sigma,\tau)$ we make a variation $X \to X + \delta X$, and obtain

$$\partial_+\partial_- X^\mu(\sigma,\tau) = 0 \quad (8.4.11)$$

together with the boundary condition

$$\delta X(\sigma,\tau)\partial_\sigma X(\sigma,\tau)|^\pi_{\sigma=0} = 0 \quad (8.4.12)$$

To satisfy the edge condition we have for the open string

$$\partial_\sigma X(0,\tau) = 0 = \partial_\sigma X(\pi,\tau) \tag{8.4.13}$$

Hence, from Eqs. 8.4.11 and 8.4.13

$$X^\mu(\sigma,\tau) = X^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \tag{8.4.14}$$

$$P^\mu(\sigma,\tau) = \partial_\tau X^\mu(\sigma,\tau) = \sum_{-\infty}^{+\infty}\alpha_n^\mu e^{-in\tau}\cos n\sigma \tag{8.4.15}$$

where $\alpha_0^\mu = p^\mu$

For quantization, we impose the equal commutation equation

$$[P^\mu(\sigma,\tau), X^\mu(\sigma',\tau)] = -i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{8.4.16}$$

which yields

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{n+m,0} \tag{8.4.17}$$

and

$$[p^\mu, X^\nu] = -i\eta^{\mu\nu} \tag{8.4.18}$$

Note $X_\mu$, is the position of the center of mass of the string and $p^\mu$ its total linear momentum. We also have

$$\partial_\pm X^\mu = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\alpha_n^\mu e^{in(\tau\pm\sigma)} \tag{8.4.19}$$

We finally obtain the normal mode expansion of the energy-momentum tensor operator

$$T_{\pm\pm} = (\partial_\pm X)^2 = \sum_{n=-\infty}^{+\infty} L_n e^{in(\tau\pm\sigma)} \tag{8.4.20}$$

Since the $\alpha_n$'s do not commute, we adopt the following definition for $L_n$ as

$$L_n = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\alpha_{m+n}^\mu\alpha_{-m\mu}, \quad n\neq 0 \tag{8.4.21}$$

$$L_n^+ = L_{-n} \tag{8.4.22}$$

$$L_0 = L_0^+ = \frac{1}{2}\alpha_0^2 + \sum_{m=1}^{\infty}\alpha_{-m}^\mu\alpha_{m\mu} \tag{8.4.23}$$

$L_n$ are the Virasoro generators which will express the constraint that $T_{\alpha\beta} =$

0 between physical states; they satisfy the Virasoro algebra with central extension given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{d}{12} n(n^2-1)\delta_{n+m,0} \tag{8.4.24}$$

where the last term is a c-nunber called central extension and reflects the breaking of Weyl invanance. The operator $L_0$ yields the masses of the string excitations and in effect is the Hamiltonian; to see this, note that Lo determines the time evolution of $X^\mu$ i.e.

$$\dot{X}^\mu(\sigma,\tau) = \partial_\tau X^\mu(\sigma,\tau) = i[X^\mu, L_0] \tag{8.4.25}$$

We now derive the operator expressions for the ghost for the fields: the invariance of the action under the BRST transformation Eq. 8.3.29 for the open string requires the boundary conditions

$$c^+(0) = c^-(0), \quad C^+(\pi) = c^-(\pi) \tag{8.4.26}$$

and

$$b_{--}(0) = b_{++}(0), \quad b_{--}(\pi) = b_{++}(\pi) \tag{8.4.27}$$

This yields the normal mode expansion consistent with the ghost field equations as

$$c^\pm(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} e^{in(\tau\pm\sigma)} c_{n+1} \tag{8.4.28}$$

$$b_{\pm\pm}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} e^{in(\tau\pm\sigma)} b_{n-2} \tag{8.4.29}$$

with

$$\{c_n, b_m\} = \delta_{n+m,0} \tag{8.4.30}$$

We also obtain from Eqs. 8.4.27 and 8.4.29

$$T^{gh}_{\pm\pm} = \sum_{n=-\infty}^{+\infty} e^{in(\tau\pm\sigma)} L^{gh}_{n-1} \tag{8.4.31}$$

with

$$T^{gh}_n = \sum_{m=-\infty}^{+\infty} (n-m) b_{n+m} c_{-m} \tag{8.4.32}$$

Note the unusual definition of the Fourier expansion in Eqs. 8.4.29 and 8.4.31 made for later convenience. The : : stand for normal ordering w.r.t.

an appropriate ghost state vector. The Virasoro algebra for the ghost sector is

$$[L_n^{gh}, L_m^{gh}] = (n-m)L_{n+m}^{gh} - \frac{13}{6}n(n^2-1)\delta_{n+m,0} \tag{8.4.33}$$

Hence, from Eqs. 8.4.24 and 8.4.33, we see that the generator of the world-sheet Virasoro generators

$$\mathcal{L}_n = L_n + L_n^{gh}$$

has the following algebra

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{1}{12}(d-26)n(n^2-1)\delta_{n+m,0} \tag{8.4.34}$$

Generator $\mathcal{L}_n$ has central extension

$$c = \frac{d-26}{12} = 0 \quad \text{for} \quad d = 26$$

and in $d = 26$ the bosonic string has the following Virasoro algebra

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} \quad ; \quad d = 26$$

The vanishing of central extension is required for Weyl conformal invariance and this is the reason that a bosonic string is consistent only in $d = 26$.

## 8.5 BRST invariance

Although $c$, $b$ and $\bar{c}, \bar{b}$ seem unrelated in the action Eq. 8.3.29, they are connected via the BRST symmetry transformation; the action is invariant under the BRST transformation given by

Then

$$\delta X^\mu = \epsilon(b\partial_- + \bar{b}\partial_+)X^\mu \equiv \epsilon \Delta X^\mu \tag{8.5.1}$$

$$\delta b = \epsilon \Delta b, \quad \delta \bar{b} = \epsilon \Delta \bar{b} \tag{8.5.2}$$

$$\delta c = \epsilon(c + 2\partial_- bc + \frac{T}{2}\partial_+ X \partial_- X) \tag{8.5.3}$$

$$\delta \bar{c} = \epsilon(\bar{c} + 2\partial_- \bar{b}c + \frac{T}{2}\partial_+ X \partial_- X) \tag{8.5.4}$$

From the BRST transformation given in Eqs. 8.5.1 we obtain a conserved current

$$\partial^\alpha j_\alpha^{\text{BRST}} = 0$$

which yields the conserved BRST charge

$$Q_{\rm BRST} = \frac{1}{2} : \int_0^\pi d\sigma c^-(T_{++} + T^{gh}_{++}) + c^+(T_{--} + T^{gh}_{--}) : \tag{8.5.5}$$

$$= c_0(L_0 - 1) + \sum_{n=1}^{\infty}(c_{-n}L_n + L_{-n}c_n) - \frac{1}{2}\sum_{n,m} : c_{m-n}c_{-m}b_n : \tag{8.5.6}$$

Classically, $Q^2_{\rm BRST} = 0$ for all dimensions, but on quantization due to normal ordering we have

$$Q^2_{\rm BRST} = \frac{d-26}{12}\mathcal{O} = 0 \quad \text{for} \qquad d = 26 \tag{8.5.7}$$

where $\mathcal{O}$ is an operator.

$Q_{\rm BRST}$ contains all the information on reparameterization invariance even for the gauge-fixed action. We will fix the physical space of states using the properties of $Q^2_{\rm BRST}$.

## 8.6 Physical Bosonic State Space

To obtain the physical states of the first quantized string, we need the string Hamiltonian together with the constraints on the Hilbert space resulting from reparametrization and conformal invariance.

In non-covariant light-cone gauge, the constraints introduced in the path integral in Eq. 8.2.19 are simply the vanishing of $T_{++}$ and $T_{--}$, and which lead to a Hilbert space with only physical states.

The classical constraints $T_{++} = 0 = T_{--}$ on covariant quantization cannot however be taken as operator equations since this would imply all $L_n = 0$ which is inconsistent with the Virasoro algebra. Hence, similar to the Gupta-Bleuler covariant formulation of QED discussed in Section 6.10.1, the constraints are expressed in the covariant quantum theory as the vanishing of the matrix elements of the energy-momentum tensor operator between any two physical states. We will see later how the vanishing of these matrix elements are just enough to decouple the negative norm states (that always result in covariant gauge-fixing) from the physical state space.

Let $\langle \text{phy}|$ be a physical state of the string; then, due to 8.4.6 we have

$$\langle \text{phy}|T_{++}|\text{phy}\rangle' = 0 \tag{8.6.1}$$

$$\langle \text{phy}|T_{--}|\text{phy}\rangle' = 0 \tag{8.6.2}$$

We will implement Eq. 8.6.2, completely ignoring for now the ghost fields.

From Eqs. 8.4.19, 8.6.2 implies that for any physical state $|\text{phy}\rangle$

$$L_n|\text{phy}\rangle = 0 = \langle\text{phy}|L_{-n} \qquad n \geq 1 \tag{8.6.3}$$

$$(L_0 - 1)|\text{phy}\rangle = 0 \tag{8.6.4}$$

Note that due to ordering ambiguities, $L_0$ is replaced by $L_0 - \alpha$ in $T_{++}$, and $\alpha = 1$ due to Lorentz invariance in $d = 26$ dimensions. It can be shown that if $|\text{neg}\rangle$ is any zero or negative norm state, then Eq. 8.6.3 implies that $\langle\text{neg}|\text{phy}\rangle = 0$ and hence, as required by the probabilistic interpretation of quantum mechanics, the physical states are all of positive definite norm.

The ground state of the string is denoted by $|0;k\rangle$, is defined by

$$\alpha_n^\mu|0;k\rangle = 0, \quad n \geq 1 \tag{8.6.5}$$

$$\alpha_n^\mu|0;k\rangle = k^\mu|0;k\rangle \tag{8.6.6}$$

The excited states of the string are given by the Fock basis

$$|\{n\};k\rangle = \alpha_{-n_1}^{\mu_1}\alpha_{-n_2}^{\mu_2}\ldots|0;k\rangle \tag{8.6.7}$$

An example of a negative norm state is $\alpha_{-n}^0|0;k\rangle$ since

$$\langle k';0|\alpha_n^0\alpha_{-n}^0|0;k\rangle = \eta^{00}\delta^{26}(k-k') = -\delta^{26}(k-k') \tag{8.6.8}$$

These type of negative norm states are all eliminated using the gauge conditions Eq. 8.6.3.

To evaluate the mass spectrum recall from Eq. 8.4.20, the mass of string excitation is

$$M^2 = -p^2 = -\alpha_0^2 \tag{8.6.9}$$

$$M^2 = 2\left(\sum_{n=1}^{\infty}\alpha_{-n}^\mu\alpha_{n\mu} - L_0\right) \tag{8.6.10}$$

Hence, on $|\text{phy}\rangle$, using Eq. 8.6.4

$$M^2|\text{phy}\rangle = 2\left(\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n - 1\right)|\text{phy}\rangle \tag{8.6.11}$$

The ground state Eq. 8.6.6 of the open string $|0;k\rangle$ satisfies conditions Eq. 8.6.3 for $M^2 = -2$; hence, the ground state contains a tachyon.

Consider the first excited state $\zeta_\mu(k)\alpha_{-1}^\mu|0;k\rangle$; for $L_n$, $n > 0$ to annihilate this state requires

$$\zeta_\mu k^\mu = 0 \tag{8.6.12}$$

which means there are only $d-1$ independent polarizations for $\zeta_\mu$. The norm of this state is positive only if

$$\zeta_\mu \zeta^\mu > 0 \tag{8.6.13}$$

One of the $(d-1)$ states allowed by Eq. 8.6.12 has null norm for $K^2 = 0 = M^2$, and consequently decouples from the physical subspace. Therefore from Eqs. 8.6.12 and 8.6.13 we see that for $M^2 = 0$, $\zeta_\mu$ has only $(d-2)$ polarization states. Hence, the first excited state consists of a massless vector particle having the expected $d-2$ states of polarization, and is interpreted as the gauge field quantum of the Yang-Mills field.

In summary we have the following spectrum states

| $(\text{Mass})^2$ | State |
|---|---|
| -1 | $\|\Omega\rangle = \|\text{vacuum}\rangle$ :scalar tachyon |
| 0 | $\alpha^\mu_{-1}\|\Omega\rangle$ |
| | :massless vector particle with $(d-2)$ states of polarization |
| 1 | $\alpha^\mu_{-2}\|\Omega\rangle, \alpha^\mu_{-1}\alpha^\nu_{-1}\|\Omega\rangle$ |
| | $(d-2) + (d-2)(d-1)/2 = \frac{1}{2}(d+1)(d-2)$ |
| | Polarizations : Massive particle |

The analysis of the string spectrum of states can be carried out level by level for all the higher energy states and shown to be free of negative norm states. The analysis for physical states made no reference to the Faddeev-Popov ghost fields; a more modern analysis of the spectrum of states uses $Q_{\text{BRST}}$ to define the physical subspace.

Consider an enlarged Hilbert space consisting of string states and ghost field states. A physical state $|\phi\rangle$ has the string coordinate field $X^\mu$ decoupled from the ghosts and hence will, in general, be of the form of a tensor product (similar to the factorization of the tranverse and longitudinal modes in Gupta-Bleuler QED)

$$|\phi\rangle = |\text{phy}\rangle|\Omega\rangle_{gh} \tag{8.6.14}$$

where $|\Omega\rangle_{gh}$ is the ghost vacuum. Since $Q^2_{\text{BRST}} = 0$ for $d = 26$, we consider two states $|\phi\rangle$ and $|\phi\rangle'$ to be equivalent if

$$|\phi\rangle' = |\phi\rangle + Q_{\text{BRST}}|\chi\rangle \tag{8.6.15}$$

The complete set of constraints on $|\phi\rangle$ due to reparametrization invariance is expressed by the single equation

$$Q_{\text{BRST}}|\phi\rangle = 0 \tag{8.6.16}$$

Note the crucial point that since $Q_{\rm BRST}$ is conserved, the condition Eq. 8.6.16 is valid for all time.

In addition to Eq. 8.6.16 we also demand that $|\phi\rangle$ be an eigenstate of the antihermetian ghost number operator with eigenvalue $-\frac{1}{2}$ that is

$$N_{gh}|\phi\rangle = -\frac{1}{2}|\phi\rangle \tag{8.6.17}$$

where

$$N_{gh} = \sum_{n=1}^{\infty}(b_{-n}c_n + c_{-n}b_n) + \frac{1}{2}[c_0, b_0] \tag{8.6.18}$$

Note fields $c, b$ have ghost number 1, - 1 respectively. The ghost vacuum is defined by

$$c_n|\Omega\rangle_{gh} = 0 = b_n|\Omega\rangle_{gh}, \quad n \geq 1 \tag{8.6.19}$$

The normal ordering for $L_n^{gh}$ is not defined w.r.t. $|\Omega\rangle_{gh}$ due to the ambiguity for the ordering of zero modes $c_0$ and $b_0$; instead normal ordering is defined w.r.t. the $\mathrm{SL}_2$ invariant state $|0\rangle_{gh}$ which is annihilated by $b_0$; more precisely

$$c_n|0\rangle_{gh} = 0 \quad ; \quad n \geq 2 \tag{8.6.20}$$

$$b_n|0\rangle_{gh} = 0, \quad ; \quad n \geq -1 \tag{8.6.21}$$

and

$$|\Omega\rangle_{gh} = c_1|0\rangle_{gh} \tag{8.6.22}$$

Note $|\Omega\rangle_{gh}$ is the lowest energy state for the ghost sector. With these properties of the ghost sector states, it follows that

$$Q_{\rm BRST}|\phi\rangle = (L_0 - 1)|\mathrm{phy}\rangle c_0|\Omega\rangle_{gh} + \sum_{n=1}^{\infty} L_n|\mathrm{phy}\rangle c_{-n}|\Omega\rangle_{gh} \tag{8.6.23}$$

Hence, requiring

$$Q_{\rm BRST}|\phi\rangle = 0$$

yields from Eq. 8.6.23 for the string sector

$$(L_0 - 1)|\mathrm{phy}\rangle = 0 \quad ; \quad L_n|\mathrm{phy}\rangle = 0, \quad n \geq 1 \tag{8.6.24}$$

which are precisely the gauge constraints Eq. 8.6.3 in the formulation of covariant string quantization.

## 8.7 Conclusion

There are many separate directions one can go from here. To remove the troublesome and non-physical tachyon state, fermions are introduced with supersymmetric coupling to the bosonic string. This yields a zero mass vacuum state and a critical dimension of 10.

The calculation for open strings can be repeated for closed strings, and the striking feature of closed strings is the complete decoupling of the left and right moving modes. To obtain the heterotic string, the left modes are considered as purely bosonic and moving in 26 dimensions while the right modes are considered to be supersymmetric and in 10 dimensions. The left moving bosonic string are compactified to 10 dimensions plus 16 flat (toral) internal dimensions.

Strings are an example of a conformally invariant quantum field theory in two world-sheet dimensional (parameter) space; powerful results from two dimensional conformal field theory can be fruitfully used in analyzing strings.

# PART THREE
# NONLINEAR QUANTUM FIELDS

Free quantum field theories, based on a quadratic Lagrangian, are linear and are similar to an infinite collection of (decoupled) quantum harmonic oscillators. There are no new features in the free field over and above quantum mechanics. Although there are interesting questions like causality, antiparticles and gauge fixing that one can study for free fields, viewed from the point of functional integration, free fields are essentially a collection of infinitely many *decoupled* degrees of freedom. The decoupled degrees of freedom can be solved, one by one, and consists of a finite collection of independent variables.

The key feature of quantum field theory that distinguishes it from quantum mechanics is that it consists of infinitely many coupled degrees of freedom. The coupling of the degrees of freedom is realized by their nonlinear interactions. In many instances, the fact that the system has infinitely many coupled independent variables (degrees of freedom) leads to apparent divergences and infinities in physically measurable quantities like charge and mass.

Part Three is largely based on studying the four-dimensional nonlinear scalar quantum field. The scalar degree of freedom is chosen since it is one the simplest case – without having additional features that are unnecessary for studying the infinities that arise in quantum field theory. For example, other quantum field theories, such nonlinear Yang-Mills gauge fields or Dirac fields coupled to other fields, also have infinities but are more complicated to analyze than the nonlinear scalar field.

The key idea for removing infinities that arise in quantum field theory is the concept of renormalization. The procedure of renormalization and the renormalization group is required to 'quench' these infinities and shows how to define and calculate physical parameters so that they are all finite. At a more theoretical level, the concept of renormalization leads to the idea of the renormalization group.

From the vantage point of the renormalization group, it becomes clear that quantum field theory describes a system with infinitely many distinct and coupled length scales, from zero to infinity. The formulation of the renormalization – based on a lattice or a momentum cutoff – pioneered byWilson (1983) clearly brings out this feature of quantum fields. The fundamental reason why apparent divergences arise in quantum field theory is because one is describing parameters at very large scales in terms of the parameters at the very small scale. Once one describes, for example, parameters at a given scale in terms of parameters of nearby scales, the theory is perfectly finite and there are no divergences.

A quantum field theory is equivalent to set of all of its correlation func-

tions. The free quantum field theory is the only case for which all the correlation functions can be written in terms of the correlation function of two fields. For all nonlinear quantum field theories, one needs to compute all of the correlation functions. One can carry out a perturbation expansion of the nonlinear quantum field about the free field; this perturbation requires a small dimensionless parameter, which is usually a coupling constant contained in the nonlinear interactions. For strongly coupled quantum fields, other techniques and approximation schemes, mostly based on numerical methods, have to be used.

The interpretation of the correlation functions depends on the theory being studied. In high energy physics, the correlation functions are related to the scattering cross-section of the underlying particles; for a system undergoing a phase transition, the correlation functions yield the critical indices; and in finance, the correlation functions describe the behavior of interest rates. In this book, no attempt is made to refer to a specific applications of the correlation functions unless necessary, since the aim is to view quantum fields as a mathematical entity not tied to any specific application.

Feynman diagrams provide a diagrammatic and graphical way of calculating the correlation functions of nonlinear quantum fields as a perturbation expansion about the free field. In some special cases, one can sum up infinite classes of Feynman diagrams, but in general one has a power series in the small expansion parameter. In addition to providing a direct physical intuition for understanding and describing the complex behavior of quantum fields, Feynman diagrams are also a powerful computational tool, since each diagram corresponds to a precise and well defined mathematical expression.

Feynman diagrams are ubiquitous in the description of quantum fields and will be used extensively in the later chapters. More complicated Feynman diagrams are used for describing quantum fields that appear in particle physics. In fact, it would be no exaggeration to state that the study of quantum fields would not have made the progress that it has made if not for the conceptual clarity and mathematical facility that Feynman diagrams have brought to the subject.

The nonlinear scalar field is used for introducing Feynman diagrams. In this book, only 'one-loop' calculations are carried out explicitly, and the Feynman diagrams for these can directly read off from the action: there is no need not write an elaborate set of rules. For this reason the rules of higher loop Feynman diagrams are not discussed – which are covered in most books on quantum field theory and need no repetition here.

The following Chapters elucidate the nonlinear properties of quantum fields. The procedure of renormalization is discussed. The renormalized ef-

fective action is calculated to illustrate how renormalization is used for a specific problem – such as symmetry breaking for quantum fields.

- In Chapter 9, the nonlinear scalar field is shown to have many divergences. The divergences are shown to be occurring in the apparent divergence in the physical mass, coupling constant and field variables. Feynman diagrams are introduced as the description of the various divergences.
- In Chapter 10 the procedure of renormalization is defined. A number of different renormalization schemes are shown to render the theory finite.
- In Chapter 11 the Callan-Symanzik formulation of the renormalization group is introduced. The key concept of the fixed point of the renormalization group is defined. It is shown how one can obtain non-perturbative results, including the scaling properties, for a quantum field near the fixed point.
- Chapter 12 is on the effective action, which is the generalization of the classical action. Symmetry breaking in quantum field theory is determined by the effective action. The procedure of renormalization is employed to compute the effective action to one-loop.
- Chapter 13 discusses the effective action for a massless scalar field coupled to the Maxwell Abelian gauge field. A renormalization group invariant symmetry breaking of the theory is explicitly calculated to one-loop.

# 9

# Nonlinear scalar field theory: Feynman diagrams

## 9.1 Introduction

To understand the essential content of quantum field theories, we study the scalar with infinitely many coupled degrees of freedom; this coupling is achieved by introducing a nonlinear potential of the fields in the Lagrangian. A cubic potential yields an unstable path integral, so the simplest nonlinear term is a potential that is a quartic power in the field, given or a real scalar field $\phi(x)$ by $\lambda\phi^4$. This theory is studied in detail in the following chapters since many of the key ideas of quantum fields, including the theory or renormalization, can be described by this nonlinear theory.

Nonlinear quantum field theories in general are almost impossible to solve exactly. The best one can often do is to study these theories using various approximation techniques. The nonlinear $\phi^4$ potential will be studied only for the weak coupling sector, and we can then carry out all calculations as a power series in $\lambda$. This expansion is a perturbation expansion about the free scalar quantum field. Furthermore, the terms in the perturbation expansion will be given a graphical interpretation using the formalism of Feynman diagrams.

## 9.2 Lagrangian of $\phi^4$ scalar field theory

The nonlinear $\phi^4$ Lagrangian, in Euclidean time, is given by the following

$$\mathcal{L} = -\Big[\frac{1}{2}\sum_i (\partial_i\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\Big]$$

The action is given by

$$S = \int d^4x\mathcal{L}$$

The action $S$ must be dimensionless since it is exponentiated to define $Z$. This defined the dimension of all the terms in the action. Consider the action in $d$ dimensional spacetime and let $a$ has the scale of length. The dimension of the field $\phi$ is $a^{d_\phi}$ and yields

$$1 = [\int d^d x(\partial_i\phi)^2] = a^d.\ \frac{1}{a^2}a^{2d_\phi} \quad \Rightarrow \quad d_\phi = \frac{1}{2}(2-d)$$

Under a scale transformation, we have the following

$$x \to x' = bx \quad \Rightarrow \quad \phi'(x') = b^{d_\phi}\phi(x)$$

and which yields

$$\int d^d x(\partial_i\phi)^2 \to \int d^d x'(\partial'_i\phi')^2 = b^{-2+d}b^{2d_\phi}\int d^d x(\partial_i\phi)^2 = \int d^d x(\partial_i\phi)^2$$

The dimension of the mass term $m^2$ is $a^{d_m}$; the mass term yields

$$1 = [m^2\int d^d x\phi^2] = a^{d_m}a^d a^{(2-d)} \quad \Rightarrow \quad d_m = -2$$

Mass has dimension of inverse of length since $c = 1$. This implies $[p] = [m], p = \frac{1}{a}$, and $px$ is dimensionless from $\hbar = 1$. Let the dimension of coupling constant $\lambda$ be $d_\lambda$. This yields

$$1 = [\lambda\int d^d x\phi^4] = a^{d_\lambda}a^d a^{4d_\phi} \quad \Rightarrow d_\lambda = d-4$$

---

Noteworthy 9.1: Noether's theorem: Scale transformation

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Recall from Eq. 3.2.7

$$0 = \delta\mathcal{S}_{\partial V} = \int_{\partial V} d\Sigma_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - [\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}]\delta x^\nu\Big) \tag{9.2.1}$$

where

$$x^\mu \to \tilde{x}^\mu = x^\mu + \delta x^\mu \ \ ; \ \ \Delta\phi = \tilde{\phi}(\tilde{x}) - \phi(x)$$

The scale transformation is given by

$$x^\mu \to \tilde{x}^\mu = e^\epsilon x^\mu$$

The Jacobian $J$, for $d$-dimensional spacetime is given by

$$\frac{\partial\tilde{x}^\mu}{\partial x^\nu} = e^\epsilon\delta^\mu_\nu \quad \Rightarrow \quad J = e^{d\epsilon}$$

From Section 9.2, under a scale transformation

$$\tilde{\phi}(\tilde{x}) = e^{\epsilon d_\phi}\phi(x) \ \ ; \ \ d_\phi = \frac{1}{2}(2-d)$$

For the self-interacting quartic scalar field the action is given by

$$\mathcal{L} = -\Big[\frac{1}{2}\sum_i (\partial_i\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\Big]$$

The action is given by

$$S = \int d^d x \mathcal{L}$$

Hence under a scale transformation, we have that, using $d_m = 2$, the following

$$\mathcal{L}(\tilde{\phi}, \tilde{x}) = -e^{(2d_\phi - 2)\epsilon}\Big[\frac{1}{2}\sum_i (\partial_i\phi)^2 + \frac{m^2}{2}e^{2d_\phi\epsilon}\phi^2 + e^{(4-d)\epsilon}\frac{\lambda}{4!}\phi^4\Big]$$

For $d = 4$, the field $\phi$ had dimension of $d_\phi = -1$, mass has dimension of $-2$ and $\lambda$ is a *dimensionless* coupling constant. The *massless* Lagrangian ($m = 0$) in $d = 4$ transforms as follows

$$\mathcal{L}(\tilde{\phi}, \tilde{x}) = e^{-4\epsilon}\mathcal{L}(\phi, x)$$

and the Jacobian is $J = e^{4\epsilon}$. Hence, in $d = 4$

$$\tilde{S} = \int d^4\tilde{x}\mathcal{L}(\tilde{\phi}, \tilde{x}) = \int d^4 x \mathcal{L}(\phi, x)$$

and the action is *scale invariant*.

For an infinitesimal scale transformation

$$\Delta\phi = -\epsilon\phi \;\; ; \;\; \tilde{x}^\mu = x^\mu + \delta x^\mu \;\; \Rightarrow \delta x^\mu = \epsilon x^\mu$$

Hence, from Eq. 9.2.1, the Noether current is given by

$$-\epsilon\mathcal{J}^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - [\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}]\delta x^\nu$$

Dropping the overall $\epsilon$ factor, the scaling current given by

$$\mathcal{J}^\mu = \phi\partial_\mu\phi + \Big[\partial_\mu\phi\partial^\nu\phi - \eta^{\mu\nu}(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi - \lambda\phi^4)\Big]x_\nu$$

---

Consider the partition function given by

$$Z = \int D\phi e^S = \int D\varphi e^{S_0 + S_I}$$

In terms of the Fourier transformed variables

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4}e^{ipx}\phi_p = \int_p e^{ipx}\phi_p \;\; ; \;\; \phi_p = \int d^4x e^{-ipx}\phi(x)$$

the quadractic term in the action is

$$S_0 = -\frac{1}{2}\int_p \varphi_{-p}(p^2+m^2)\varphi_(p)$$

and the quartic nonlinear term is

$$\begin{aligned} S_I &= -\frac{\lambda}{4!}\int d^4x\varphi^4(x) \\ &= -\frac{\lambda}{4!}\int_{p_1\cdots p_4} \delta(p_1+p_2+p_3+p_4)\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\varphi_{p_4} \end{aligned} \tag{9.2.2}$$

The nonlinear term couples all the degrees of freedom and in particular, transfers momentum across degrees of freedom with vastly different momenta.

Consider the case of weak coupling, with the dimensionless coupling constant being small, that is, $\lambda << 1$. To evaluate all the quantities, one can expand the nonlinear term in a power series

$$e^{S_I} = 1 + S_I + \frac{1}{2}S_I^2 + O(\lambda^3)$$

In particular, the partition function is given by

$$Z = \int D\varphi e^{S_0}[1 + S_I + \frac{1}{2}S_I^2 + \cdots] = Z_0(1 + Z_1 + Z_2 + \cdots) \tag{9.2.3}$$

The normalization, which gives an overall constant, is given by

$$Z_0 = \int D\varphi e^{S_0}$$

The first non-trivial term is

$$Z_1 = \frac{1}{Z_0}\int D\varphi e^{S_0}S_I = -\frac{\lambda}{4!}\int_{p_1\cdots p_4} \delta(\sum_i p_i)E_0[\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\varphi_{p_4}]$$

where we use the notation

$$E_0\big[\mathcal{O}[\varphi]\big] = \frac{1}{Z_0}\int D\varphi e^{S_0}\mathcal{O}[\varphi] \;\; ; \;\; Z_0 = \int D\varphi e^{S_0} \tag{9.2.4}$$

To evaluate this term requires the four field correlation function using the quadratic action and is the lowest order result. All the higher order terms in the expansion for $Z$ require the evaluation of the correlators of higher and higher order, and all these can be evaluated using the so called Wick's theorem.

## 9.3 Wicks Theorem

Wick's theorem is the result of Gaussian path integration and to foreground the derivation, the derivation for a finite number of independent degrees of freedom is first reviewed.

Recall from Eq. 4.13.3 all the correlators for Gaussian random variables can be evaluated from can be obtained from the generating function $Z[J]$.

$$\begin{aligned} Z[J] &= \frac{1}{Z_0}\int Dx \exp\{-\frac{1}{2}\sum_{mn} x_m M_{mn} x_n + \sum_n x_n J_n\} \\ &= \exp(\frac{1}{2}\sum_{nm} J_n M^{-1}_{nm} J_m) = \exp\{[W[J]\} \end{aligned}$$

with

$$S_0 = -\frac{1}{2}\sum_{mn} x_m M_{mn} x_n \quad ; \quad Z_0 = \int Dx \exp\{S_0\}$$

In particular, the correlator of two random variables is given by

$$\begin{aligned} E[x_i x_j] \equiv \overbrace{x_i x_j} &= \frac{1}{Z_0}\int Dx\ x_i x_j e^{S_0} = \frac{\partial^2}{\partial J_i \partial J_j} Z[J]\Big|_{J=0} \\ &= M^{-1}_{ij} \equiv D_{ij} \quad : \text{contraction} \end{aligned}$$

The correlation of odd number of variables is zero. The correlation of even number $N$ variables is given by *all possible* pair wise contraction of the random variables; this yields

$$E_0[x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_{N-1}} x_{i_N}] = \overbrace{x_{i_1} x_{i_2}} \cdots \overbrace{x_{i_{N-1}} x_{i_N}} + \text{all possible permutations}$$

Consider the example of

$$\begin{aligned} E_0[x_1 x_2 x_3 x_4] &= \frac{1}{Z}\int Dx e^{S_0} x_1 x_2 x_3 x_4 = \frac{\partial^4 e^{W[J]}}{\partial J_1 \cdots J_4} \\ &= D_{42} D_{31} + D_{41} D_{32} + D_{43} D_{21} \end{aligned} \tag{9.3.1}$$

The three terms are the result of making all possible pair-wise contractions.

The generalization of the result for a finite collection of Gaussian random variables to quantum fields is called the Wick's theorem. Consider the free field action

$$S_0 = -\frac{1}{2}\int d^4x [\sum_{i=1}^{4} (\partial_i \phi)^2 + m^2 \phi^2]$$

with the action in momentum space given by

$$S_0 = -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \phi_{-p}(p^2 + m^2)\phi_p$$

The generating functional is

$$e^{W[J]} = \exp\{\frac{1}{2}\int d^4x d^4y J(x) D(x-y) J(y)$$
$$= \exp\{\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} J_{-p} \frac{1}{p^2+m^2} J_p\} \tag{9.3.2}$$

where

$$J(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} J_p$$

The action is translational invariant, namely $S$ in invariant under $x \to x+a$. Hence, all the correlators have translational invariance. The propagator is given by

$$E_0[\phi(x)\phi(y)] = D(x-y) = \int_p e^{ip(x-y)} D_p \quad : \quad D_p = \frac{1}{p^2+m^2}$$

In momentum space, the propagator is given by

$$E_0[\phi_p \phi_{p'}] = \int d^4x d^4y e^{-i(xp+yp')} E_0[\phi(x)\phi(y)] = \delta(p+p') D_p$$
$$= \frac{\delta(p+p')}{p^2+m^2} \tag{9.3.3}$$

Feynman diagrams are a graphical method for representing the correlation functions of quantum fields. A fundamental ingredient of the Feynman diagram is the propagator, which is represented by a straight line given in Figure 9.1.


Figure 9.1 The Feynman diagram for the propagator is a single contraction and is represented by a straight line: a) In real space $D(x-y)$ b) In momentum space $D_p$.

Wick's theorem states that for a product of an even number of quantum fields, the correlator is given by all possible contractions; hence, similar to Eq. 9.3.1 the result is

$$E_0[\phi(x_1)\phi(x_2)\cdots\phi(x_i)\phi(x_j)\cdots\phi(x_{N-1})\phi(x_N)]$$
$$= D(x_1-x_2)\cdots D(x_i-x_j)\cdots D(x_{N-1}-x_N) + \text{all possible permutations}$$

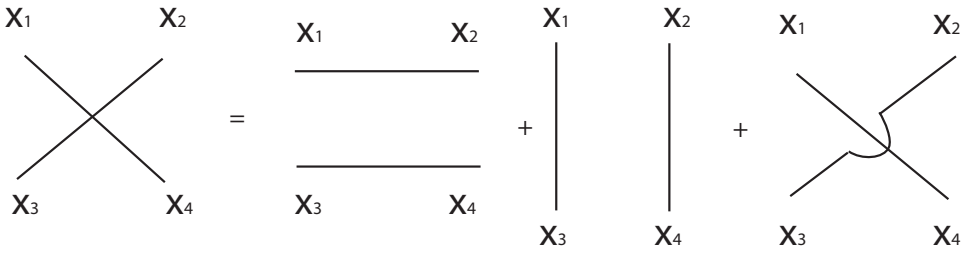


Figure 9.2 Free field expansion of the four-point correlation function.

The $n$-point correlation functions of the Gaussian theory are denoted by

$$D(x_1 \cdots x_N) \equiv E_0[\varphi(x_1) \cdots \varphi(x_N)]$$

and the four point correlator for the free field is denoted by

$$D(x_1, x_2, x_3, x_4)$$

For the free field, using Wick's theorem yields, similar to Eq. 9.3.1, the result

$$\begin{aligned} D(x_1, x_2, x_3, x_4) &= D(x_1 - x_2)D(x_3 - x_4) \\ &\quad D(x_1 - x_3)D(x_2 - x_4) + D(x_1 - x_4)D(x_2 - x_3) \end{aligned} \tag{9.3.4}$$

Figure 9.2 shows the pair wise contractions that yield the four-point correlator for a free quantum field. The various ways of contracting the four fields, all of which are in-equivalent, are shown in the Figure 9.2.

In the next Sections, the four-point correlator for a Lagrangian with nonlinear quartic interaction will be studied in detail and the Gaussian portion of the four point function, given above, will be subtracted to leave the purely nonlinear portion.

## 9.4 Partition Function

Recall we have an expansion for the partition function given by

$$Z = \int D\varphi e^{S_0}[1 + S_I + \frac{1}{2}S_I^2 + \cdots] = Z_0(1 + Z_1 + Z_2 + \cdots)$$

with the first non-trivial term given by

$$Z_1 = \frac{1}{Z_0}\int D\varphi e^{S_0} S_I = -\frac{\lambda}{4!}\int_{p_1 \cdots p_4} \delta(\sum_i p_i) E_0[\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\varphi_{p_4}]$$

The interaction action $S_I$ is represented by four line emanating from a single point, as shown in Figure 9.3.

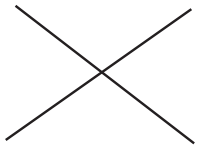

Figure 9.3 The vertex for $\phi^4$ interaction. Each line represents a field.

To evaluate $Z_1$, one needs to perform a Gaussian functional integral using Wick's theorem. We need to group the four field lines – represented in Figure 9.3 – pairwise and then make a *contraction* to connect pair of field lines. Each contraction results in a propagator that is also represented by a line, as in Figure 9.1.

Since all $\varphi_p$'s are equivalent; for the first contraction given in Figure 9.4(a), there are three choices and one choice for second contraction and that yields the diagram given in Figure 9.4 (b).


Figure 9.4 Evaluating the $\phi^4$ interaction vertex; each line represents a propagato.r a) Contracting a pair of fields. b) Contracting the remaining two fields.

Hence, the two contractions yield

$$Z_1 = -\frac{\lambda \cdot 3}{4!} \int_{p_1 \cdots p_4} \delta^4(\sum_i p_i) \frac{\delta(p_1 + p_2)}{p_1^2 + m^2} \frac{\delta(p_3 + p_4)}{p_3^2 + m^2}$$
$$= -\frac{\lambda}{8} \delta^4(0) \Big[ \int_p \frac{1}{p^2 + m^2} \Big]^2$$

The divergent term $(2\pi)^4 \delta^4(0)$ is a spacetime volume factor since

$$\int d^4x e^{ipx} = (2\pi)^4 \delta^4(p)$$

and hence $(2\pi)^4 \delta^4(0) = \int d^4x = V$. Putting the system in a finite volume $V$ replaces $(2\pi)^3 \delta^4(0)$ by $V$. The more serious divergence of the $Z_1$ term – for

a momentum cut-off $\Lambda$ – is given by

$$\int_p \frac{1}{p^2+m^2} \propto \int dp \frac{p^3}{p^2+m^2} \simeq \int_0^\Lambda \frac{p^3}{p^2+m^2} \sim \Lambda^2$$

Hence, $Z_1$ is divergent due to the infinitely many degrees of freedom and yields

$$Z_1 \simeq \Lambda^4 \quad : \text{divergent}$$

.

## 9.5 Connected correlation functions

Renormalization is a procedure for absorbing the divergences arising in the quantum field theory by redefining the coupling constant $\lambda$, mass $m$ and the field variable $\varphi$ such that the redefined theory is completely finite.

To renormalize the theory, it is sufficient to make the partition function finite – since it contains all the potential divergences. To organize the analysis of the divergences, a procedure has to be adopted that allows for the identification of all the divergent terms. Once these divergences have been renormalized, all the other divergences that could potentially appear in the theory are also rendered finite.

The partition function contains many combinations of the divergent terms, and the first step in the procedure of renormalization is to identify the so called connected correlation functions and is now discussed.

Recall from Eq. 9.2.3, the partition function is given by

$$Z = E_0[e^{S_I}] = \int D\varphi e^{S_0}[1 + S_I + \frac{1}{2}S_I^2 + \cdots] = Z_0(1 + Z_1 + Z_2 + \cdots)$$

The leading order term, of order $O(\lambda)$ – using the notation introduced in Eq. 9.2.4 – is given by

$$Z_1 = E_0[S_I] \equiv \frac{1}{Z_0}\int D\phi e^{S_0} S_I$$

The second term in the expansion for the partition function is given by

$$Z_2 = \frac{1}{2}E_0[S_I^2]$$

The computation for $Z_2$ yields a term that has the following form

$$\frac{1}{2}E_0[S_I]E_0[S_I] = \frac{1}{2}Z_1^2$$

and which is the result of only the fields from a given vertex being contracted

– independent of the fields of the other vertex. This term, which is order $O(\lambda^2)$, can be obtained by squaring the result of the earlier calculation for $Z_1$ and need not be calculated again.

The diagrammatic expansion of the partition function, to $O(\lambda^3)$ is shown in Figure 9.5. These diagrams are obtained by carrying all possible contractions on the product of fields that appear for a given order in the expansion.

Some of the diagrams that can be built up from the lower order diagrams are given by Figure 9.5 (c), which corresponds to $E_0[S_I]E_0[S_I]$ and Figure 9.5 (e), which corresponds to $E_0[S_I^2]E_0[S_I]$.

Figure 9.5 (c) and (e) are called *disconnected diagrams* and the remaining diagrams given by Figure 9.5 (a), (b) and (d) are *connected diagrams*.

Diagram are connected if all the lines, representing the contraction of a pair of fields, are connected. If some of the lines are *disjoint* – not being connected to another set of lines – then this diagram is disconnected and can be obtained from the lower order diagrams.

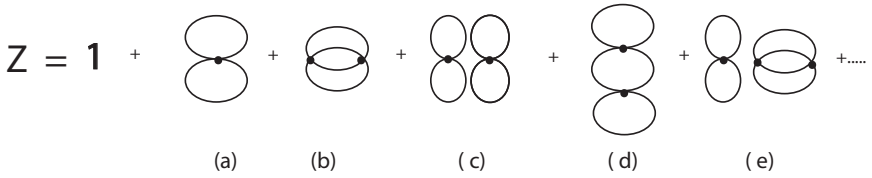


Figure 9.5 All the Feynman diagrams, both connected and disconnected, in the expansion of the partition function $Z$.

It can be shown (Zinn-Justin (1993)) that taking the logarithmic of $Z$ removes all the disconnected diagrams and yields

$$W = \ln(Z)$$

The generating function $W$, with a Feynman graph expansion shown in Figure 9.6, is given by a sum of only connected diagrams.

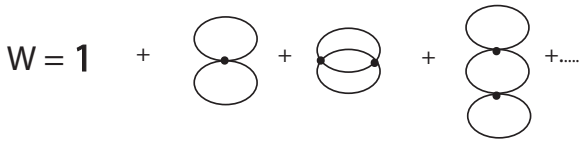


Figure 9.6 Only the connected Feynman diagrams are required in the expansion of the generating functional $W = \ln(Z)$.

The expansion of $e^W$ yields all connected and disconnected diagrams; all

the disconnected diagrams in Figure 9.5, are precisely generated, with the correct combinatorial factors, by expanding the exponential.

### *9.5.1* $W[J]$*: Generating functional*

A systematic way of studying the divergence of a quantum field is by analyzing the generating function $W$; in particular, all the correlation functions of the theory can be obtained from $W$.

For a quantum field, and for the $\phi^4$ theory in particular, the generating functional is given by

$$e^{W[J]} = \int D\varphi e^{S+\int J\varphi} \tag{9.5.1}$$

The exact $n$-point *connected* correlation functions of the full nonlinear quantum field theory, also called the Greens functions, are defined by

$$G(x_1 \cdots x_N) \equiv E[\varphi(x_1) \cdots \varphi(x_N)]_c = \frac{\delta^N W[J]}{\delta J(x_1) \cdots \delta J(x_N)}\Big|_{J=0}$$

A functional Taylor's expansion of the generating functional $W[J]$ for any quantum field theory is given by

$$W[J] = \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \cdots dx_N G(x_1 \cdots x_N) J(x_1) \cdots J(x_N) \tag{9.5.2}$$

The connected two-point correlation functions is given by

$$\begin{aligned} G(x_1, x_2) &= E[\varphi(x_1)\varphi(x_2)]_c = \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}|_{J=0} \\ &= \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\{\ln(\int D\varphi e^{S+J\varphi})\}|_{J=0} = \frac{\delta}{\delta J(x_1)}\{\frac{1}{Z(J)}\int D\varphi\varphi(x_2)e^{S+J\varphi}\} \\ &= \frac{1}{Z}\int D\varphi\varphi(x_1)\varphi(x_2)e^S - \{\frac{1}{Z}\int D\varphi\varphi(x_1)e^S\}\{\frac{1}{Z}\int D\varphi\varphi(x_2)e^S\} \\ &= E[\varphi(x_1)\varphi(x_2)] - E[\varphi(x_1)]E[\varphi(x_2)] \end{aligned} \tag{9.5.3}$$

For $\varphi^4$ theory

$$E[\varphi(x)] = 0$$

and hence

$$G(x_1, x_2) = E[\varphi(x_1)\varphi(x_2)]_c = E[\varphi(x_1)\varphi(x_2)]$$

The connected four-point correlator is given by

$$E[\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)]_c = \frac{\delta^4 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big|_{J=0}$$

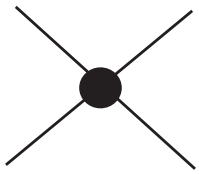

Figure 9.7 The complete four point correlator $E[\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)]$.

The connected correlation is given by subtracting all the disconnected Feynman diagrams and, for $\varphi^4$ theory, yields

$$G(x_1,x_2,x_3,x_4) = E[\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)]_c = E[\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)] - G(x_1,x_2)G(x_3,x_4) - G(x_1,x_3)]G(x_2,x_4) - G(x_1,x_4)G(x_2,x_3) \quad (9.5.4)$$

The connected correlator $G(x_1,x_2,x_3,x_4)$ is graphically shown in Figure 9.8.

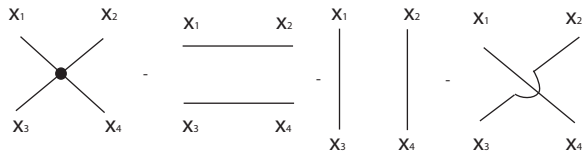


Figure 9.8 The connected four point correlator $G(x_1,x_2,x_3,x_4)$.

For the free quantum field the four point correlator is zero. The explicit expressions given in Eqs. 9.3.4 and 9.5.4 yield

$$G(x_1,x_2,x_3,x_4)\Big|_{\text{Gaussian}} = D(x_1,x_2,x_3,x_4) - D(x_1,x_2,x_3,x_4) = 0$$

where $D(x_1,x_2,x_3,x_4)$ is given by the Feynman diagrams in Figure 9.2.

For the free scalar field the generating functional $W_0[J]$ is given by

$$W_0[J] = \frac{1}{2}\int d^4x_1 d^4x_2 J(x_1) D(x_1-x_2) J(x_2)$$

The free field is linear and does not have any quartic vertex,and the connected correlator for all 4-point and higher cases are zero. This is because the generating functional for the free field $W_0[J]$ is quadratic in the external currents $J(x)$ and yields

$$\frac{\delta^n W_0[J]}{\delta J(x_1)\cdots\delta J(x_n)} = 0 \ ; \ n \geq 3 \quad : \text{Free field} \qquad (9.5.5)$$

In other words, the connected correlators pick out precisely the terms that are absent in the linear free field theory.

## 9.6 The two-point correlation function

The propagator, in momentum space, is given by two point correlation function

$$G_2(p_1, p_2) = E[\varphi(p_1)\varphi(p_2)]_c$$

For $\lambda << 1$, consider the following expansion

$$G_2(p_1, p_2) = \frac{1}{Z}\int D\varphi e^{S_0}\varphi(p_1)\varphi(p_2)[1 + S_I + \frac{1}{2!}S_I^2 + \cdots]$$

$$= G_2^0 + G_2^1 + G_2^2 + \cdots$$

The interaction term $S_I$ is given in Eq. 9.2.2.

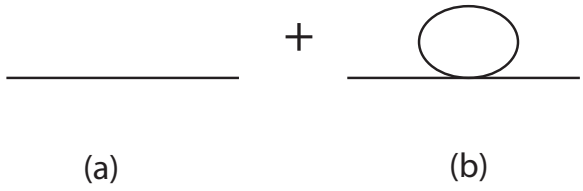


Figure 9.9 a) Tree diagram for the propagator. b) The bubble Feynman diagrams for the propagator to $O(\lambda)$.

The propagator has a Feynman expansion, with the first two terms given in Figure 9.9. The first term, given in Figure 9.9(a) is the free field propagator which, from Eq. 9.3.3, is given by

$$G_2^0(p_1, p_2) = \frac{\delta(p_1 + p_2)}{p_1^2 + m^2}$$

The $O(\lambda)$ contribution is given in Figure 9.9(b) and yields

$$G_2^1(p_1, p_2) = E_0[\varphi(p_1)\varphi(p_2)S_I]_c$$

$$= -\frac{\lambda}{4!}\int D\varphi \int_{k_1\cdots k_4} e^{S_0}\Big[\varphi(p_1)\varphi(p_2)\delta(\sum_i k_i)\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\Big]_c$$

There are $4 \cdot 3 = 12$ ways of obtaining the diagram and the one-loop yields

$$G_2^1(p_1, p_2) = -12 \cdot \frac{\lambda}{2}\int_{k_1\cdots k_4} \frac{\delta(p_1 + k_1)\delta(p_2 + k_2)\delta(k_3 + k_4)}{(p_1^2 + m^2)(p_2^2 + m^2)(k_3^2 + m^2)}$$
$$= -\frac{\lambda}{2}\frac{\delta(p_1 + p_2)}{(p_1^2 + m^2)^2}\int_k \frac{1}{(k^2 + m^2)}$$

where

$$J_1 = \int_k \frac{1}{k^2 + m^2}$$

Hence, to one loop

$$\begin{aligned} G_2(p_1, p_2) &= \delta(p_1 + p_2)[\frac{1}{(p_1^2 + m^2)} - \frac{\lambda}{2}\frac{J_1}{(p_1^2 + m^2)^2}] + O(\lambda^2) \\ &\simeq \frac{\delta(p_1 + p_2)}{p_1^2 + m^2 + \frac{\lambda}{2}J_1} + O(\lambda^2) \end{aligned} \tag{9.6.1}$$

Similar to the divergence in the partition function, the propagator is also divergent; introducing a momentum cut-off $\Lambda$ yields

$$J_1 = \int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 + m^2} \sim \Lambda^2$$

## 9.7 Four-point correlation function

The four point connected correlation, given by the Feynman diagrams to $O(\lambda^2)$ as shown in Figure 9.10, is defined by

$$G_4(p_1, \cdots, p_4) = E[\varphi(p_1)\varphi(p_2\varphi(p_3)\varphi(p_4)]_c$$

The Feynman diagrams required to evaluate the four point correlator is given in Figure 9.11. Every contraction of two fields $\varphi(p), \varphi(p')$ yields a propagator

$$\frac{\delta(p + p')}{p^2 + m^2}$$

Expanding the interaction term of the action $S_I$ yields

$$\begin{aligned} G_4(p_1, \cdots, p_4) &= \frac{1}{Z}\int D\varphi e^{S_0}\Big[\varphi(p_1)\varphi(p_2\varphi(p_3)\varphi(p_4)(1 + S_I + \frac{1}{2!}S_I^2 + \cdots)\Big]_c \\ &= \frac{1}{Z}\int D\varphi e^{S_0}\Big[\varphi(p_1)\varphi(p_2\varphi(p_3)\varphi(p_4)(S_I + \frac{1}{2!}S_I^2 + \cdots)\Big]_c \\ &= \frac{\delta(\sum_i p_i)}{\prod_i (p_i^2 + m^2)}\Big[\Gamma_4^0 + \Gamma_4^1 + O(\lambda^3)\Big] \end{aligned} \tag{9.7.1}$$

The momentum conserving $\delta$-function and the propagator for the external legs have been taken out.

As is clear from the Feynman diagrams, to $O(\lambda)$ the bubble diagrams do not affect interaction vertex. The reason being that these bubble diagrams modify the single lines, and which in effect is accounted for in the diagram given in Figure 9.9 for the propagator; it will be shown later that these

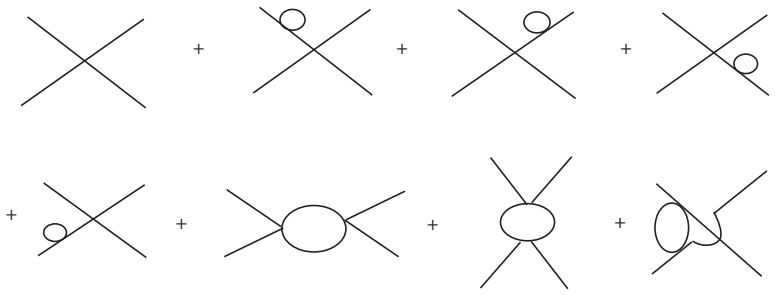

Figure 9.10 The four-point correlation function, to $O(\lambda^2)$.

diagrams can be absorbed into the redefinition of the mass parameter $m$, as in Eq. 9.6.1. In other words, the corrections to the propagator given by the bubble diagrams are taken into account by the complete two point correlation $G(p_1, p_2)$.

The first diagram in Figure 9.11 is the vertex function that occurs in the Lagrangian, and is called a tree diagram. There are 4! ways of contracting the external legs into the vertex and hence the Feynman diagram yields

$$\Gamma_4^0 = -\lambda$$

The three one-loop diagrams given in Figure 9.11 can be obtained from the diagram shown in Figure 9.14 by appropriately permuting its external momenta.

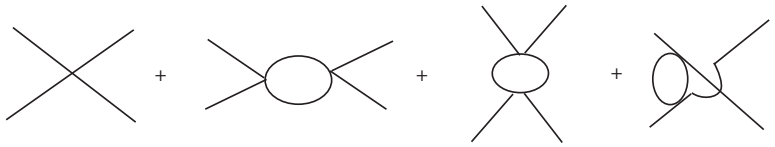

Figure 9.11 Feynman diagrams to $O(\lambda^2)$ for the four point correlator.

Hence, to $O(\lambda^2)$, there is only a single one-loop diagram, given in Figure 9.14, that needs to be evaluated and which has only a one loop momentum that needs to be integrated over. The diagram corresponds, up to a combinatorial factor that we will evaluate later, to the following expression

$$\frac{1}{Z_0}\int D\varphi e^{S_0}\left[\varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)S_I^2\right]_c \tag{9.7.2}$$

To $O(\lambda^2)$, the four-point function is given by the three topologically distinct

distinct one-loop diagrams given in Figure 9.11. Hence, from Eq. 9.7.1

$$G_4(p_1,\cdots,p_4) = \frac{\delta(\sum_i p_i)}{\prod_i (p_i^2+m^2)}\Big(-\lambda + \Gamma_4^1\Big)$$

Doing all the contraction in Eq. 9.7.2 yields the following for one of the topologically distinct diagrams given in Figure 9.11

$$\frac{1}{\prod_i (p_i^2+m^2)}\int_{k_1\cdots k_4}\int_{k_1'\cdots k_4'}\delta(\sum_i k_i)\delta(\sum_i k_i')$$
$$\times\delta(p_1+k_1)\delta(p_2+k_2)\delta(p_3+k_1')\delta(p_4+k_2')\frac{\delta(k_3+k_3')}{k_3^2+m^2}\frac{\delta(k_4+k_4')}{k_4^2+m^2}$$

The $\delta$- functions simplify the integral to the following[1]

$$\Gamma_4^1 = \frac{1}{2!}(-\frac{\lambda}{4!})^2 C_2\int_k \frac{1}{(k^2+m^2)((k-p_1-p_2)^2+m^2)} + \text{two more terms}$$

A power counting of the momentum dependence of the integration variables of the diagram for the propagators yields

$$\int^\Lambda \frac{d^4k}{k^4} \sim \ln\Lambda$$

Hence this diagram yields a logarithmic divergence.

## 9.8 Dimensional Regularization

From the analysis of the propagator and vertex function, we found that there are two divergent diagrams, namely a quadratic divergence arising from the higher order contributions to the propagator

$$\int^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} \sim \Lambda^2$$

and a logarithmic divergence arising from the higher order contributions to the vertex function given by

$$\int^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)((k+p)^2+m^2)} \sim \ln\Lambda$$

In fact, for $\phi^4$ theory, these are the only two type of divergent Feynman diagrams – with quadratic and logarithmic dependence on the momentum cut-off. These divergent terms receive contributions from all powers of the coupling constant $\lambda$.

[1] $C_2$ is a combinatorial factor that will be fixed later.

Making these divergent terms finite will also make the perturbation theory finite to all orders; the cut off $\Lambda$ can then be taken to infinity and one will then obtain a finite theory if the theory is renormalizable.

There are many ways to regulate a field theory, such as using momentum cut off or a lattice space time. One of the most widely used regularization scheme is that of dimensional regularization, in which the dimension $d$ of spacetime is taken to be a continuous parameter. This form of regularization preserves many symmetries including gauge invariance; it's main shortcoming is that it cannot regulate chiral fermions, which are one of the cornerstones of the weak interactions. Renormalization is the procedure for removing the regularization: obtaining a finite result is only possible for renormalizable quantum field theories.

The action for the real scalar field in $d$-spacetime dimensions is given by the following

$$\mathcal{S} = -\int d^d x \Big\{ \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\tilde{\lambda}}{4!}\phi^4 \Big\}$$

The dimensional analysis in Section 9.2 yields

$$d_\phi = \frac{2-d}{2} \quad ; \quad d_m = -1 \quad ; \quad d_{\tilde{\lambda}} = d - 4$$

Introduce a mass scale $\mu$ that has dimension of $1/a$, where $a$ has the dimension of length. The *dimensionless* coupling constant $\lambda$ is defined by

$$\tilde{\lambda} = \mu^{4-d}\lambda$$

and the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\mu^{4-d}}{4!}\lambda\phi^4$$

For $m^2 = 0$, the classical Lagrangian is invariant under a global scale transformation given by $x \to sx$, where $s$ is an arbitrary scale. The continuation to $d = 4 - \epsilon < 4$ dimensions, required by the need to regulate the quantum field theory, has introduced a length scale $\mu$. On taking the limit of $\epsilon \to 0$, the scale $\mu$ will continue to appear in the renormalized theory and this breaks scale invariance. In general, the breaking of scale invariance is a general feature of a renormalized quantum field theory.

### 9.8.1 Dimensionally regularized integrals

Consider the integral that appears at two-loops

$$J_2 = \int \frac{d^d k}{(2\pi)^d (k^2+\Delta)^2}$$

Note that

$$(\sqrt{\pi})^d = (\int dx e^{-x^2})^d = \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} = \frac{1}{2}\int d\Omega_d \Gamma(d/2)$$

and yields

$$\Omega_d = \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

The integral is then given by

$$J_2 = \int d\Omega_d k^{d-1} \frac{dk}{(k^2+m^2)^2} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dk \frac{k^{d-1}}{(k^2+\Delta)^2}$$

Let $k^2 = z$; we obtain

$$J_2 = \frac{1}{2}\int_0^\infty dz \frac{z^{(d-2)/2}}{(z+\Delta)^2} = \frac{1}{2}(\frac{1}{\Delta})^{2-d/2} \int_0^1 dx x^{1-d/2}(1-x)^{d/2-1}$$

where $x = \Delta/(z+\Delta)$.

Recall the definition of the beta function is given by

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Hence the final result is

$$J_2 = \int \frac{d^d k}{(2\pi)^d (k^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} (\frac{1}{\Delta})^{2-d/2} \tag{9.8.1}$$

The integral, for arbitrary $\alpha$, is given by

$$\begin{aligned}
&\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^\alpha} = \frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty du \frac{u^{d/2-1}}{(u+m^2)^\alpha} \\
&= \frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^d} (\frac{1}{m^2})^{\alpha-d/2} \int_0^\infty dy y^{d/2-1}(1+y)^{-\alpha} \\
&= \frac{\Gamma(\alpha-d/2)}{(4\pi)^{d/2}\Gamma(\alpha)} (\frac{1}{m^2})^{\alpha-d/2}
\end{aligned}$$

where

$$\int_0^\infty dy y^{d/2-1}(1+y)^{-\alpha} = \frac{\Gamma(d/2)\Gamma(\alpha-d/2)}{\Gamma(\alpha)}$$

Hence, in general

$$J_n = \int \frac{d^d k}{(2\pi)^d (k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} (\frac{1}{\Delta})^{n-d/2} \tag{9.8.2}$$

The singularities in the regularized integral $J_n$ are due to the singularities of the gamma function for negative integer arguments given by

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}[\frac{1}{\epsilon} + \varphi(n+1)] + O(\epsilon) \tag{9.8.3}$$

Noteworthy 9.2: Logarithmic and $\epsilon$ cutoff

The result above states that the Feynman bubble diagram given in Figure 9.9(b) is proportional to

$$\begin{aligned} J_1(\epsilon) &= \int \frac{d^d k}{(2\pi)^4} \frac{1}{(k^2+m^2)} = m^2 \frac{1}{m^\epsilon} \frac{(4\pi)^{\epsilon/2}}{(4\pi)^2} \Gamma(1-d/2) \\ &= \frac{m^2}{16\pi^2} (\frac{4\pi}{m^2})^{\epsilon/2} [-\frac{2}{\epsilon} + \varphi(2)] = -\frac{m^2}{8\pi^2} \frac{1}{\epsilon} + O(1) \end{aligned}$$

The divergent part of regularized $J_1$ is negative although the integral apparently looks positive since the integrand is positive. To understand how the regularized integral $J_1$ can be negative, we compare this result with the result obtained by using the momentum space regulator $\Lambda$; $J_1$ then is given by

$$\begin{aligned} J_1(\Lambda) &= \frac{1}{(2\pi)^4} \int_0^\Lambda dk k^3 \int d\Omega_d \frac{1}{k^2+m^2} = \frac{\lambda 2\pi^2}{2(2\pi)^4} \int_0^{\Lambda^2} dz \frac{z}{z+m^2} \\ &= \frac{\lambda}{16\pi^2} \int_0^{\Lambda^2} dz (1 - \frac{m^2}{z+m^2}) = \frac{\lambda}{16\pi^2} [\Lambda^2 - m^2 \ln(1+\Lambda^2/m^2)] \\ &= \frac{\lambda}{16\pi^2} [\Lambda^2 - m^2 \ln(\Lambda^2/m^2) + m^2 O(m^2/\Lambda^2)] \end{aligned}$$

We see that in dimensional regularization all the power low divergence in $\Lambda$, namely terms containing $\Lambda, \Lambda^2$...are all consistently set to zero. More precisely, for non-integer dimension $d$, we have Veltman's identity

$$\int \frac{d^d k}{|k|^n} = 0 \quad d : \text{fractional} \tag{9.8.4}$$

What remains are the logarithmic terms and

$$\ln(\Lambda^2/m^2) \text{ corresponds to } \frac{2}{\epsilon}$$

Hence we have the correspondence that

$$\frac{1}{\epsilon} \to \ln(\Lambda/m)$$

The above correspondence is only heuristic since there are many $\ln(\Lambda^2/m^2)$ terms generated by say breaking a symmetry such as gauge invariance that are absent in dimensional regularization.

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### *9.8.2 Propagator*

We now examine the regulated correlation functions. Recall, from the $O(\lambda)$ Feynman diagrams given in Figure 9.9

$$G_2(p_1,p_2) = G_2^0 + G_2^1 + O(\lambda^2) = \frac{\delta(p_1+p_2)}{p_1^2+m^2} + \frac{\delta(p_1+p_2)}{(p_1^2+m^2)^2} I_1$$

The Feynman bubble diagram for the propagator, given in Figure 9.9, yields

$$I_1 = (-\frac{\mu^\epsilon \lambda}{4!}) C_1 \int \frac{d^d k}{(2\pi)^d (k^2+m^2)} = -\frac{\mu^\epsilon \lambda}{2} \int \frac{d^d k}{(2\pi)^d (k^2+m^2)}$$

The first pre-factor comes from the interaction action $S_I$ and the second term $C_1 = 12$, as discussed in Eq. 9.6.1, is the combinatorial factor in the number of ways the diagram can be obtained.

For the mass term, the diagram yields, for $d = 4-\epsilon$, the following

$$I_1 = -\frac{1}{2} J_1 = -\frac{\lambda}{2} \mu^\epsilon \int \frac{d^d k}{(2\pi)^d (k^2+m^2)} = -\frac{\lambda \mu^\epsilon}{2} \frac{m^2}{m^\epsilon} \frac{(4\pi)^{\epsilon/2}}{(4\pi)^2} \Gamma(1-d/2)$$

From Eq. 9.8.3

$$\Gamma(1-d/2) = \Gamma(-1+(4-d)/2) = -[\frac{2}{\epsilon} + \varphi(2)]$$

Hence

$$I_1 = m^2 \frac{\lambda \mu^\epsilon}{16\pi^2} \frac{1}{m^\epsilon} \frac{1}{\epsilon} + \text{finite} \tag{9.8.5}$$

To order $O(1)$ in $\epsilon$, taking the limit of $\epsilon \to 0$ yields

$$\begin{aligned} I_1 &= -\frac{\lambda}{2} \mu^\epsilon \int \frac{d^d k}{(2\pi)^d (k^2+m^2)} \\ &= \frac{\lambda m^2}{2} \frac{1}{16\pi^2} (\frac{2}{\epsilon} + \varphi(2)) (\frac{4\pi\mu^2}{m^2})^{\epsilon/2} \\ &= \frac{\lambda m^2}{32\pi^2} [\frac{2}{\epsilon} + \varphi(2) + \ln(\frac{4\pi\mu^2}{m^2})] + O(\epsilon) \end{aligned} \tag{9.8.6}$$

Writing the regularized integral using the implicit scale $\mu$ yields the following

$$\int \frac{d^dk}{(2\pi)^d(k^2+m^2)} = -\mu^{-\epsilon}\frac{m^2}{16\pi^2}[\frac{2}{\epsilon}+\varphi(2)-\ln(\frac{m^2}{4\pi\mu^2})]+O(\epsilon) \quad (9.8.7)$$

The scale $\mu$ is arbitrary and is required by the dimension of the integral being $\mu^{-\epsilon}$; the scale $\mu$ always appears when divergences are canceled. As illustrated in Noteworthy 9.2, $\epsilon$ implicitly carries the scale $\mu$.

## 9.9 Two-loops regularized propagator

The next order term yields the $O(\lambda^2)$ contribution to the propagator and is given by

$$G(p_1,p_2) = G_2^0 + G_2^1 + G_2^2 + O(\lambda^3)$$

where

$$\begin{aligned}
G_2^2 &= E\Big[\varphi(p_1)\varphi(p_2)\frac{1}{2!}S_I^2\Big]_c \\
&= \frac{\lambda^2\mu^{2\epsilon}}{(4!)^2 2!}\int \delta(\sum_i k_i)\delta(\sum_i k_i')E[\varphi(p_1)\varphi(p_2)\varphi(k_1)\cdots\varphi(k_n)\varphi(k_1')\cdots\varphi(k_n')] \\
&= \frac{\delta(p+p')}{(p^2+m^2)^2}\Big[I_3+I_4\Big]
\end{aligned}$$

The $O(\lambda^2)$ contribution to $G_2$ consists of two diagrams. The bubble Feynman diagram that contributes to $G_2$ is denoted by $I_3$ and shown in Figure 9.12(a) and the connected sunset diagram is given in Figure 9.12(b).

The combinatorial factor for Figure 9.12(a) is given by

$$C_3 = 8\cdot 4\cdot 3\cdot 3$$

and yields the decoupled integrals

$$\begin{aligned}
I_3 &= \frac{\lambda^2\mu^{2\epsilon}}{2!(4!)^2}C_3\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{\delta(k_1+k_3)}{(k_1^2+m^2)(k_2^2+m^2)(k_3^2+m^2)} \\
&= \frac{\lambda^2\mu^{2\epsilon}}{4}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_1^2+m^2)(k_2^2+m^2)^2} = \frac{\lambda^2\mu^{2\epsilon}}{4}J_1J_2
\end{aligned}$$

Hence, from Eq. 9.8.2, the Feynman diagram yields

$$I_3 = \frac{\lambda^2\mu^{2\epsilon}}{4}\frac{1}{(4\pi)^d}\frac{m^2}{m^{2\epsilon}}(1-d/2)\Gamma^2(1-d/2)$$

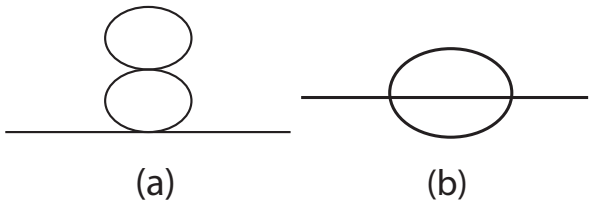


Figure 9.12 Two-loop contributions to the propagator. a) The two loops bubble diagram. b) The sunset diagram.

Simplifying, we obtain

$$I_3 = -m^2 \frac{\lambda^2 \mu^{2\epsilon}}{1024\pi^4} \frac{(4\pi)^\epsilon}{m^{2\epsilon}} [\frac{4}{\epsilon^2} + \frac{c}{\epsilon}] + \text{finite}$$

The loop-integration have completely factorized and have reduced to essentially one-loop diagrams. Note the appearance of $1/\epsilon^2$ is due to the two-loop integrations, In general, for a diagram of $O(\lambda^n)$, the most singular term is of order $1/\epsilon^n$, with all the lower orders poles also occurring.

### *9.9.1 Propagator: two-loops and sunset diagram*

The sunset diagram given in Figure 9.12(b). The combinatorial factor for this diagram is

$$C_4 = 8 \cdot 4 \cdot 3 \cdot 2$$

and yields

$$\begin{aligned} I_4 &= \frac{\lambda^2 \mu^{2\epsilon}}{6} \int \frac{d^d k_1 d^d k_2 d^d k_3 (2\pi)^d \delta(k_1 + k_2 + k_3 + p)}{(2\pi)^{3d} (k_1^2 + m^2)(k_2^2 + m^2)(k_3^2 + m^2)} \\ &= \frac{\lambda^2 \mu^{2\epsilon}}{6(2\pi)^{2d}} \int \frac{d^d k_1 d^d k_2}{(k_1^2 + m^2)(k_2^2 + m^2)((k_1 + k_2 - p)^2 + m^2)} \end{aligned}$$

This an example of a multi-loop Feynman diagram in that there are two momentum integral that are non-factorizable into separate integrations.

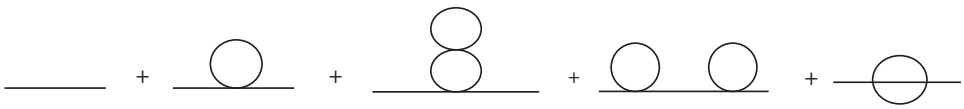


Figure 9.13 Feynman diagrams to $O(\lambda^2)$ for the propagator.

Counting the powers of momentum shows that the integral $I_4$ diverges as

$\Lambda^{2d-6}$, where $\Lambda$ is a momentum cut-off. Hence for $d = 4$

$$I_4 = \Lambda^2 + \Lambda + \ln\Lambda + ...$$

As is the case in general, dimensional regularization consistently sets the quadratic and linear divergence to zero, picking up only the $\ln\Lambda$ divergence. A long and tedious calculation yields for $p \simeq 0$, the following

$$I_4(p) = -\frac{\lambda^2}{6(16\pi^2)^2}\Big\{\frac{6m^2}{\epsilon^2} + \frac{6m^2}{\epsilon}\big[\frac{3}{2} + \varphi(1) + \ln(\frac{4\pi\mu^2}{m^2})\big] + \frac{p^2}{2\epsilon}\Big\} \qquad (9.9.1)$$

The $p^2/\epsilon$ term in $I_4(p)$ is of interest as it enters the field renormalization. The rest of the singular terms in $I_4(p)$ contribute to mass renormalization to $O(\lambda^2)$.

In summary, all the Feynman diagrams that contribute to the propagator to $O(\lambda^2)$, and determine the mass and field renormalization for $\varphi^4$ theory, are given in Figure 9.13. The propagator is given by

$$G_2(p) = \frac{1}{p^2+m^2} + \frac{1}{(p^2+m^2)^2}\frac{\lambda}{16\pi^2\epsilon}m^2 - \frac{1}{(p^2+m^2)^2}\frac{\lambda^2}{6(16\pi)^2\epsilon}p^2 + O(\lambda^2/\epsilon^2, p^4)$$

Define the two-point vertex function $\Gamma_2(p)$ to be the inverse of the propagator $G_2$ and given by

$$G_2^{-1}(p) = \Gamma_2(p) \qquad (9.9.2)$$

Then

$$\Gamma_2(p) = p^2 + m^2 - \frac{\lambda}{16\pi^2\epsilon}m^2 + \frac{\lambda^2}{6(16\pi^2)^2\epsilon}p^2 + O(\lambda^2/\epsilon^2, p^4) \qquad (9.9.3)$$

## 9.10 Vertex function

The four-point correlation function given by the three distinct topologies, with the different routing of the momentum variables as given in Figure 9.11.

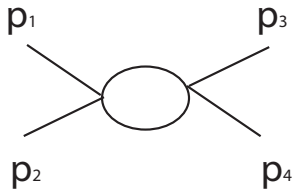


Figure 9.14 The $O(\lambda^2)$ Feynman diagram for the four point correlator.

Figure 9.14 shows one of the Feynman diagrams of Figure 9.11, with the external momenta labeled explicitly; this yields

$$I_2(p) = \frac{1}{2!}\Big(-\frac{\mu^\epsilon \lambda}{4!}\Big)^2 C_2 \int \frac{d^d k}{(2\pi)^d (k^2+m^2)((k+p)^2+m^2)} \quad ; \quad p = p_1 + p_2$$

and appears in the calculation of the vertex function. The combinatorial factor $C_2$ for the topology of the Feynman diagram given in Figure 9.14 yields

$$C_2 = 8 \cdot 4 \cdot 3 \cdot 3 \cdot 2$$

and hence

$$I_2(p) = \frac{\mu^{2\epsilon}\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d (k^2+m^2)((k+p)^2+m^2)}$$

Summing over the distinct topologies yields the fully symmetrized vertex function

$$G(p_1 \cdots p_4) = \frac{\delta(\sum_i pi)}{\prod_i (p_i^2+m^2)}[-\mu^\epsilon \lambda + I_2(p_1+p_2) + I_2(p_2+p_3) + I_2(p_3+p_4)] \tag{9.10.1}$$

$$\equiv \frac{\delta(\sum_i pi)}{\prod_i (p_i^2+m^2)} \Gamma_4(p_1 \cdots p_4) + O(\lambda^4) \tag{9.10.2}$$

To evaluate the integral $I_2(p)$, consider the identity

$$\frac{1}{\alpha\beta} = \int_0^1 dx \frac{1}{[\alpha x + \beta(1-x)]^2}$$

This yields

$$\frac{1}{(k^2+m^2)((k+p)^2+m^2)} = \int_0^1 dx \frac{1}{[k^2+m^2+2k\cdot p(1-x)+p^2(1-x)]^2}$$

Define new momentum $\tilde{k} = k + p(1-x)$; then

$$I_2(p) = \frac{\mu^{2\epsilon}\lambda^2}{2} \int_0^1 dx \frac{d^d \tilde{k}}{(2\pi)^d [\tilde{k}^2 + m^2 + p^2 x(1-x)]^2}$$

$$= \frac{\mu^{2\epsilon}\lambda^2}{2(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(\frac{1}{m^2 + p^2 x(1-x)}\right)^{2-d/2} \tag{9.10.3}$$

Eq. 9.8.3 yields

$$\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} + \varphi(1)$$

Hence

$$I_2(p) = \frac{\mu^\epsilon \lambda^2}{2(4\pi)^2} \int_0^1 dx[\frac{2}{\epsilon} + \varphi(1)][1 - \frac{\epsilon}{2} \ln(\frac{m^2 + p^2 x(1-x)}{4\pi\mu^2})]$$
$$= \frac{\mu^\epsilon \lambda^2}{32\pi^2}(\frac{2}{\epsilon} + \varphi(1)) - \frac{\mu^\epsilon \lambda^2}{32\pi^2} \int_0^1 dx \ln \frac{m^2 + p^2 x(1-x)}{4\pi\mu^2} \quad (9.10.4)$$

Recall the dimension of the Feynman diagram in Figure 9.14 is $\mu^\epsilon$ and which explicitly appears in the equation above for $I_2$. We have isolated the divergent pieces of the Feynman diagrams, which to lowest order in $\lambda$, appear as $1/\epsilon$.

The Feynman integral, with the dependence on $\lambda$ removed, is given by

$$\int \frac{d^d k}{(2\pi)^d (k^2 + m^2)((k+p)^2 + m^2)}$$
$$= \frac{\mu^{-\epsilon}}{16\pi^2}(\frac{2}{\epsilon} + \varphi(1)) - \frac{1}{16\pi^2} \int_0^1 dx \ln \frac{m^2 + p^2 x(1-x)}{4\pi\mu^2} + O(\epsilon)$$

The appearance of the scale $\mu$ is discussed after Eq. 9.8.7.

## 9.11 Generic divergences of Feynman diagrams

To foreground the discussion on renormalization in the next Chapter, the singular terms obtained from the Feynman diagrams are summarized below.

- **Mass divergence**

$$I_1 = \frac{\lambda \mu^\epsilon}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)}$$
$$= -\frac{\lambda m^2}{32\pi^2}(-\frac{2}{\epsilon} - \varphi(2))(\frac{4\pi\mu^2}{m^2})^{\epsilon/2} \approx \frac{\lambda m^2}{16\pi^2 \epsilon} + \text{finite}$$

- **Coupling constant divergence**

$$I_2(p) = \frac{\lambda^2 \mu^{2\epsilon}}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)[(k-p)^2 + m^2]}$$
$$= \frac{\lambda^2 \mu^{2\epsilon}}{16\pi^2}(\frac{1}{\epsilon} + \frac{1}{2}\varphi(1)) + \text{finite}$$

- **Field variable divergence**

  Diagrams that are given in Figures 9.12(a) and (b) are both $O(\lambda^2)$ and have terms of $(1/\epsilon^2)$ that are independent of momentum $p$ and will not enter the lowest order calculation for field renormalization that will be

considered. Hence, for $I_4$ we have from Eq. 9.9.1 the following expansion

$$\begin{aligned} I_4 &= \frac{\lambda^2 \mu^{2\epsilon}}{6(2\pi)^{2d}} \int \frac{d^d k_1 d^d k_2}{(k_1^2 + m^2)(k_2^2 + m^2)[(k_1 + k_2 - p)^2 + m^2]} \\ &= -\frac{\lambda^2}{12(16\pi^2)^2} [\frac{p^2}{\epsilon} + \text{constant} + O(p^4)] \end{aligned}$$

The constant term has terms of $O(1/\epsilon^2)$ that are independent of $p^2$.

## 9.12 Summary

The nonlinear $\phi^4$ scalar quantum field has divergences in its correlation functions. The divergent diagrams were first regularized, that is, rendered finite, by using the procedure of dimensional regularization, which entailed analytically continuing spacetime dimension to fractional dimension of $d = 4 - \epsilon$. The correlation functions were then computed by using the rules of dimensional regularization and were found to have terms such as $1/\epsilon, 1/\epsilon^2, \cdots$.

Feynman diagrams were introduced to represent the terms of the perturbation expansion for various quantities such as the partition function $Z$, $W = \ln Z$, the propagator, the four-point correlation function and so on. Feynman diagrams allowed us to represent higher order diagrams and discuss their properties.

What is worth noting is that the one-loop calculation for the case of renormalizable quantum field theories (this will become more clear in the next chapter) is sufficient to discuss the all the essential features of renormalization. The two-loop diagram was required for the field renormalization of the $\phi^4$ scalar quantum field; the result was quoted and not derived because such a derivation is not required for any of the later discussions.

This chapter puts into place all the ingredients that are required for the discussion on renormalization in the next chapter.

# 10

# Renormalization

## 10.1 Introduction

The divergences that are present in $\varphi^4$ nonlinear scalar quantum field theory are typical of all nonlinear quantum fields.

Relativistic invariance requires point-like interactions of the local quantum fields, dictated by the light cone structure of spacetime in special relativity; this requires that the field $\varphi^4(x)$, at each point $x$, be an independent degree of freedom and hence any finite volume of spacetime has infinitely many degrees of freedom. Furthermore, the correlation length of the field degrees of freedom is the inverse of the lowest mass in the system and for a nonlinear theory this leads to the coupling of infinitely many degrees of freedom.

Divergences appear in the study of phase transitions due to scale invariance since, at the critical point, the fluctuations of the degrees of freedom for all scales become coupled and require a consistent treatment of the infinitely coupled many degrees of freedom.

Renormalization is the procedure that removes all the divergences from a theory that is *renormalizable*. Only a special class of quantum field theories are renormalizable, with many theories, for which the divergences cannot be removed by any procedure, being non-renormalizable.

A large class of renormalized theories, and these are the ones that occur in many applications in high energy physics and critical phenomenon, are defined by the following multiplicative re-definition of the bare quantities in terms of the renormalized ones

$$\varphi_B = Z_\varphi^{1/2}\varphi_R \quad ; \quad m_B = Z_m m_R \quad ; \quad \lambda_B = Z_\lambda \lambda_R \mu^\epsilon \tag{10.1.1}$$

Note that the arbitrary scale $\mu$ enters only via the dimensional coupling constant $\lambda_B$.

The *dimensionless* renormalization constants $Z_\varphi, Z_m, Z_\lambda$, which we gener-

ically call $Z$, are functions only of the renormalized $m_R, \lambda_R$ – both of which are finite in the limit of $\epsilon \to 0$. The renormalized coupling constant $\lambda_R$ is *dimensionless*.

In mass independent renormalization schemes, that includes dimensional regularization, the renormalization constants $Z$ do not depend on the renormalized mass $m_R$.[1] Since all the $Z$'s are dimensionless, masses in perturbation theory can only appear in powers of $\ln(m_R/\mu)$. All the $Z$'s are well defined for massless theories; hence, to have a finite limit for the $Z$'s as $m_R \to 0$, all the mass terms must be absent. Hence, the reason that all the $Z$'s are well defined for massless theories is because they are independent of mass.

We consequently have that, in dimensional regularization, all the $Z$'s are independent of mass and

$$Z_\varphi = Z_\varphi(\lambda_R, \epsilon) \;\; ; \;\; Z_m = Z_m(\lambda_R, \epsilon) \;\; ; \;\; Z_\lambda = Z_\lambda(\lambda_R, \epsilon) \qquad (10.1.2)$$

The renormalization constants, ignoring terms of $O(\epsilon)$, are given in perturbation theory by the following expansion

$$Z = \lim_{\epsilon \to 0} Z(\lambda_R, \epsilon) = 1 + \sum_{n=1}^{\infty} (\lambda_R)^n \sum_{i=1}^{n} \frac{c_i}{\epsilon^i} = 1 + \sum_{n=1}^{\infty} \frac{Z_n(\lambda_R)}{\epsilon^n} \qquad (10.1.3)$$

The field variables $\phi_B, \phi_R$ cannot be observed. To obtain a finite and nontrivial renormalized theory, rescaling of the field variables is determined by the requirement that the quantum fluctuations for both the bare and renormalized theory must have the same magnitude.

In the perturbative formulation of a renormalizable quantum field theory, the procedure of multiplicative renormalization is sufficient to remove all the divergences for theories. Renormalization can be carried out using many different procedures, with a few of these are the following.

- Bare perturbation theory
- Renormalized perturbation theory
- Background field method
- Wilson renormalization theory

These procedures can be carried out to all orders in perturbation theory, but in practice the background field method is one that is most useful for one or two loop calculations.

[1] There are other regularization schemes, like momentum cut-off discussed in Section 10.11, for which, unlike dimensional regularization, $m_R$ can appear in the renormalization constants $Z$.

## 10.2 Renormalization schemes

The renormalization constants given in Eq. 10.1.1 require that a definition of the renormalized quantities be given, and only then can the bare coupling constants and masses be related to the renormalized ones. The renormalization conditions are defined by a scheme that is independent of mass so that it can be applied to massless theories as well.

We discuss an scheme that defines the subtractions required for defined the counter terms such that the renormalization constants are independent of mass. The first scheme is defined in terms of the vertex functions and is a physically direct manner; the second scheme is based on dimensional regularization and is simpler to carry, but more indirect in how the renormalized is connected to measurable quantities.

We specifically discuss $\phi^4$ theory; since there are three renormalization constants – given by $Z_m, Z_\varphi, Z_\lambda$ – we need three renormalization conditions; these conditions hold for any theory that has three three renormalization constants, and in particular includes quantum electrodynamics.

Recall the two point vertex function $\Gamma_2$ and four point vertex function $\Gamma_4$ have been defined in Eqs. 9.9.2 and 9.10.1 respectively. The *renormalized* vertex functions satisfy the following three renormalization conditions[2]

$$\begin{aligned} \Gamma_2^R(p)\Big|_{p^2=0} &= m^2_{\text{phy}} \quad ; \quad \frac{d^2}{dp^2}\Gamma_2^R(p)\Big|_{p^2=0} = 1 \\ \Gamma_4^R(p)\Big|_{p_i\cdot p_j=\mu^2(\delta_{ij}-1/4)} &= -\lambda_{\text{phy}} \end{aligned} \tag{10.2.1}$$

The physically measured parameters of the theory are $m_{\text{phy}}, \lambda_{\text{phy}}$ – all of which are finite and have no singularities. All schemes of renormalization must yield the same value for these parameters.

The choice of momentum scale $\mu$ for defining the renormalization counterterms is arbitrary and any fixed momentum can be chosen.[3] The arbitrariness of the momentum scale $\mu$ for evaluating the renormalized quantities is illustrated for the case of $\phi^4$ in Section.

For an analysis confined to one-loop diagrams, a simple but less physically transparent scheme is called *minimal subtraction*. Dimensional regularization, and in particular applied to minimal subtraction, is a mass-independent renormalization scheme, and is discussed in Section 10.5. In minimal subtraction, one singles out the $1/\epsilon$ terms and directly subtracts these terms by

[2] The vertex function are 1PI (one particle irreducible) and discussed in Section 12.5. The fact that of being 1PI irreducible does not affect any of the our discussions in this Chapter.

[3] For massless theories, setting all the momentum of all the external legs to zero can lead to infrared divergences; this will be addressed later in Section 12.9 in the analysis of the effective potential for massless $\phi^4$ theory.

defining the (divergent) bare coupling constants and masses in terms of the (finite) renormalized masses and coupling constants.

Fixing the renormalized Lagrangian to yield finite renormalized correlation functions by the various schemes are all equivalent. However, the different schemes for defining the renormalized mass and coupling constant, in general, have only an indirect connection to the physically measured parameters $m_{\text{phy}}, \lambda_{\text{phy}}$. Moreover, the renormalized masses and coupling constants for the different schemes have different definitions and are related by the renormalization group transformation – discussed later in Section 11.5. A momentum cut-off procedure is discussed in Section 10.11, the background field method is discussed in Section 10.12 and the Wilson renormalization theory is discussed in Section 11.10.

## 10.3 Bare perturbation theory

In dimensional regularization, the propagator and four point function are divergent as $\epsilon \to 0$, where $d = 4 - \epsilon$. To obtain finite results from $\phi^4$ theory, one can redefine the ingredients of the theory and then study if the theory is finite as $\epsilon \to 0$.

One starts from the 'original' Lagrangian, which is called the bare Lagrangian, with all the coupling constants and field appearing being called bare quantities; and from the bare quantities, the renormalized quantities, including the renormalized Lagrangian, is obtained.

The original Lagrangian is taken to be written in terms of the bare quantities, namely $\phi_B, \lambda_B,\ m_B$, and is given by

$$\mathcal{L}_B = -\frac{1}{2}\partial_\mu \varphi_B \partial_\mu \varphi_B - \frac{1}{2} m_B^2 \varphi_B^2 - \frac{\lambda_B}{4!}\varphi_B^4$$

This definition holds in all dimension $d$.

The bare Lagrangian defines a theory with *no cut-off*. As discussed earlier, the dimension of $\lambda_B$ is $[1/\mu^\epsilon]$,where $\mu$ has the dimension of momentum. The Feynman diagrams for the bare theory are expressed *without* the scale $\mu$.

## 10.4 Mass and field renormalization

The bare coupling constant $\lambda_B$ carries a dimension of $1/a^\epsilon$ and mass has dimension of $1/a$, where $a$ has the dimension of length. In the expressions for the Feynman diagrams, the term $\lambda\mu^\epsilon$ has to be replaced by $\lambda_B$, and the fractional power of the mass $m_B^\epsilon$ has to be kept for dimensional consistency. The dependence of the divergent integrals on the bare quantities is written

out explicitly, and yields

$$I_1 = \lambda_B m_B^2 \tilde{I}_1 \;\; ; \;\; I_2 = \lambda_B^2 \tilde{I}_2 \;\; ; \;\; I_3 = \lambda_B^2 \tilde{I}_3 \;\; ; \;\; I_4 = \lambda_B^2 \tilde{I}_4$$

The two point function is given by

$$G_2(p,p') = E[\varphi_B(p)\varphi_B(p')] = \delta(p+p')G_2(p)$$

Although the calculation will be limited to $O(\lambda)$, the $O(\lambda^2)$ term needs to be included in the propagator so as to obtain a non-trivial field renormalization. The tadpole and sunset diagram yields

$$\begin{aligned} G_2(p) = \frac{1}{p^2+m_B^2} - \frac{\lambda_B}{4!}\cdot 12 \cdot \frac{1}{(p^2+m_B^2)^2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{p^2+m_B^2} \\ + \frac{1}{(p^2+m_B^2)^2}\lambda_B^2\tilde{I}_4(p) \end{aligned} \tag{10.4.1}$$

Hence

$$\Gamma_2 = G_2^{-1} = p^2 + m_B^2 + \lambda_B m_B^2 \tilde{I}_1 - \lambda_B^2 \tilde{I}_4(p)$$

where, using Eq. 9.8.5

$$\tilde{I}_1 = \frac{1}{2m_B^2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m_B^2)} = -\frac{1}{16\pi^2}\frac{1}{m_B^\epsilon}\frac{1}{\epsilon} + \text{finite} \tag{10.4.2}$$

and from Eqs. 9.9.1 and 10.1.1

$$\tilde{I}_4(p) = -\frac{\mu^{-2\epsilon}}{12(16\pi^2)^2}[\frac{p^2}{\epsilon} + \text{constant} + O(p^4)]$$

The renormalized propagator is defined by

$$G_2^{(R)}(p,p') = E[\varphi_R(p)\varphi_R(p')]$$

The renormalized mass $m_R$ and the field rescaling constant $Z_\phi$ are defined by the following

$$\lim_{p\to 0} E[\varphi_R(p)\varphi_R(p')] = \frac{\delta(p+p')}{p^2+m_{\text{phy}}^2} = Z_\varphi^{-1} E[\varphi_B(p)\varphi_B(p')] \tag{10.4.3}$$

Note there are *two* renormalization conditions in Eq. 10.4.3 above

- The renormalized mass $m_R^2$ is defined to yield a finite value for the physically observable mass $m_{\text{phy}}$
- The coefficient of the $p^2$ is fixed to be exactly one

Eq. 9.9.3 gives $G_2^{-1}(p) = \Gamma_2(p)$; writing out the result in terms of the bare coupling constant $\lambda_B$ and the factors of $m^\epsilon$ that arise in the dimensionally regularized integrals yield the following

$$\Gamma_2^B(p) = p^2 + m_B^2 - \frac{\lambda_B}{m_B^\epsilon}\frac{1}{16\pi^2\epsilon}m_B^2 + \frac{\lambda_B^2\mu^{-2\epsilon}}{12(16\pi^2)^2\epsilon}p^2 + O(\lambda_B^2, p^4) \quad (10.4.4)$$

From Eq. 10.4.3, the result of the bare perturbation theory yields the following result

$$G_2^R = Z_\varphi^{-1}G_2^B = \frac{Z_\varphi^{-1}C^{-1}}{p^2 + C^{-1}(1+\lambda_B\tilde{I}_1)m_B^2} = \frac{1}{p^2 + m_{\text{phy}}^2}$$

where

$$C^{-1} = 1 - \frac{\lambda_R^2}{12(16\pi)^2\epsilon}$$

The field renormalization constant is chosen so that

$$Z_\varphi = C^{-1} = 1 - \frac{\lambda_R^2}{12(16\pi)^2\epsilon} \quad (10.4.5)$$

Hence, from above

$$m_{\text{phy}}^2 = (1+\lambda_B\tilde{I}_1)m_B^2 + O(\lambda_B^2) = m_B^2 - \lambda_B\frac{1}{16\pi^2\epsilon}\frac{1}{m_B^\epsilon}m_B^2 \quad (10.4.6)$$

## 10.5 Minimal subtraction

The mass term, from Eq. 10.4.4, is given by

$$m_B^2 - \frac{\lambda_B}{m_B^\epsilon}\frac{1}{16\pi^2\epsilon}m_B^2$$

The bare mass is chosen to cancel the divergence coming from the divergent term given by $1/\epsilon$. A *key* feature in choosing the definition of $m_B$ is that the divergent term is canceled using *only* the renormalized coupling constant $\lambda_R$ since all the renormalization constants $Z$'s depend only on $\lambda_R$, as in Eq. 10.1.1; furthermore, the cancellation of the divergent term must be independent of mass and momentum as required by the fact that renormalization constants $Z$'s are dimensionless as in Eq. 10.1.2.

Of course, canceling a divergent term is ambiguous up to a finite constant. We use the scheme of *MS (minimal subtraction)*, which entails canceling precisely the $1/\epsilon$ term.[4] Hence, from Eq. 10.4.6

$$m_B^2 = m_R^2 + \frac{\lambda_R}{16\pi^2\epsilon}m_B^2 + O(\lambda_B^2) \quad (10.5.1)$$

[4] There are more elaborate called $\bar{M}S$ and leads to the same renormalized theory.

The finite renormalized mass $m_R^2$ is given by

$$m_B^2 = (1 + \frac{\lambda_R}{16\pi^2\epsilon})m_R^2 + O(\lambda_R^2) \tag{10.5.2}$$

The physically observed mass $m_{\text{phy}}$, from Eq. 10.4.6, is given by the expression

$$\lim_{p\to 0} G_2^R(p) = \lim_{p\to 0} Z_\varphi^{-1} G_2^B(p) = \lim_{p\to 0} \frac{1}{p^2 + m_{\text{phy}}^2} = \frac{1}{m_{\text{phy}}^2} \tag{10.5.3}$$

Note the connection of the renormalized mass $m_R$ with the physical mass $m_{\text{phy}}$ is given by $G_2^R(p)$ as in Eq. 10.2.1. Instead, the result obtained above in Eq. 10.5.3 is the result of minimal subtraction, with the connection of $m_{\text{phy}}$ with the renormalized mass is indirect $m_R$ and is given below.

From Eq. 10.1.2, to leading order in $\lambda_B$, we have

$$\lambda_B = \mu^\epsilon \lambda_R + O(\lambda_R^2)$$

Hence, the mass term, from Eq. 10.5.3, is given by

$$\begin{aligned} m_{\text{phy}}^2 \equiv m_B^2 - \frac{\lambda_B}{m_B^\epsilon}\frac{1}{16\pi^2\epsilon} m_B^2 = m_R^2 - \frac{\lambda_R}{16\pi^2\epsilon} m_R^2 [1 + \frac{\mu^\epsilon}{m_R^\epsilon}] \\ \Rightarrow \quad m_{\text{phy}}^2 = m_R^2 \{1 - \frac{\lambda_R}{16\pi^2} \ln(\frac{\mu}{m_R})\} + O(\lambda_R^2) \end{aligned} \tag{10.5.4}$$

Note that for $\mu = m_R$, the physical mass $m_{\text{phy}}$ is equal to the renormalized mass $m_R$. The significance of physical mass $m_{\text{phy}}$ is discussed in Section 11.4

Noteworthy 10.1: Mass renormalization in quantum mechanics

An example from quantum mechanics illustrates the physics of mass renormalization. Consider the Hamiltonian with a $x^4$ potential given by

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m^2x^2 + \frac{\lambda}{4!}x^4 = H_0 + H_I$$

This yields the Euclidean Lagrangian given by

$$\mathcal{L} = -\Big[\frac{1}{2}(\frac{dx}{dt})^2 + \frac{1}{2}m^2x^2 + \frac{\lambda}{4!}x^4\Big]$$

The correlation function is given by

$$E[x_p x_p'] = \frac{1}{p^2 + m^2} - \frac{\lambda}{4!}.12.\frac{1}{(p^2+m^2)^2}\int \frac{dk}{2\pi}\frac{1}{k^2+m^2} = \frac{1}{p^2 + m^2 + \delta m^2}$$

Note that

$$\int \frac{dk}{2\pi}\frac{e^{ikx}}{k^2+m^2} = \frac{1}{2m}\exp\{-m|x|\}$$

Since the mass has no dependence on a momentum scale (such as $\mu$), the renormalized mass is defined as follows

$$m_R^2 = m^2 + \delta m^2 = m^2 + \frac{\lambda}{4m} = m^2(1 + \frac{\lambda}{4m^3})$$

$$m_R = m\sqrt{1 + \frac{\lambda}{4m^3}} = m + \frac{\lambda}{8m^2}$$

The energy levels of the simple harmonic oscillator $\lambda = 0$ are given by

$$E_n = E_0 + nm$$

with the ground state energy given by

$$E_0 = \frac{1}{2}m$$

The first order shift in the energy levels is given by

$$E_n + \Delta E_n$$

where

$$\Delta E_n = \langle n|H_I|n\rangle$$

and $|n\rangle$ is the oscillator eigenstates. In terms of the creation and annihilation operators

$$x = \frac{1}{\sqrt{2m}}(a + a^\dagger) \quad ; \quad [a, a^\dagger] = \mathbb{I}$$

The shift in the ground state energy $\Delta E_0$ yields the new ground state energy given by

$$E_\Omega = E_0 + \Delta E_0 = \frac{1}{2}m + \frac{\lambda}{32m^2}$$

Similarly the correlation to the first excited state is given by $\Delta E_1$. The shift in the energy is

$$E_I = E_1 + \Delta E_1 = \frac{3}{2}m + \frac{5\lambda}{32m^2}$$

and the energy gap to lowest order is given by

$$E_I - E_\Omega = m + \frac{\lambda}{8m^2} = m_R$$

Hence the renormalization of mass is the result of interactions changing the energy levels of the system. The renormalized mass $m_R$ is the energy of the first excited state above the ground state. The observed mass of a particle in quantum field theory is defined to be the renormalized mass and which

is the energy of the first excited state above the ground state. Hence, the result from quantum mechanics regarding the renormalized mass has a similar interpretation as the renormalized mass in quantum field theory.

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### 10.6 Coupling constant renormalization

To complete the renormalization procedure, $\lambda_B$ needs to be expressed in terms of the renormalized parameters $\lambda_R$ and $m_R$. The analysis is more complicated since the mass and momentum dependence of the four point function needs to be accounted for.

The renormalized four point function is given by

$$E[\varphi_R(p_1)\cdots\varphi_R(p_4)] = Z_\varphi^{-2} E[\varphi_B(p_1)\cdots\varphi_B(p_4)] = Z_\varphi^{-2} G_4^B(p_1,\cdots,p_4)$$

and, in terms of the connected vertex function $\Gamma_4$, is given by

$$G_4^B(p_1,\cdots,p_4) = \frac{\delta(\sum_i p_i)}{\prod_i (p_i^2+m_B^2)}\Gamma_4^B(p_1,\cdots,p_4)$$

The diagrams with a bubble diagrams are precisely accounted for by the renormalization of mass, replace $m_B^2$ by $m_R^2$. The one loop diagrams need to be analyzed due to the fact $\lambda_B$ has a dimension. Writing out the one loop vertex $\Gamma_4$ yields, from Eq. 9.10.2, the following

$$\Gamma_4^B(p_1,\cdots,p_4) = -\lambda_B + \lambda_B^2\{\tilde{I}_2(p_1+p_2)+\tilde{I}_2(p_2+p_3)+\tilde{I}_2(p_3+p_4)\} \quad (10.6.1)$$

with momentum conservation requiring that $p_1+p_2+p_3+p_4=0$.

The one-loop integral is given by

$$\tilde{I}_2(p) = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m_B^2)[(k-p)^2+m_B^2]}$$

Note that $I_2(p)$ carries the dimension of $1/\lambda_B$; this is necessary since the first term in $\Gamma_4$ is $\lambda_B$ and which has dimension of $\mu^\epsilon$.

The dimensionless renormalized coupling constant $\lambda_R$ is defined using minimal subtraction; hence, we need to isolate the $1/\epsilon$ term in $\Gamma_4^R$ and subtract it in defining $\lambda_B$ in terms of $\lambda_R$.

Recall from Eq. 9.10.3 that

$$\begin{aligned}\tilde{I}_2(p) &= \frac{1}{2(4\pi)^{d/2}}\int_0^1 dx \frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{m_B^2+p^2x(1-x)}\right)^{2-d/2} \\ &= \frac{1}{2(4\pi)^2}(\frac{2}{\epsilon}+\varphi(3))\int_0^1 dx\left(\frac{4\pi}{m_B^2+p^2x(1-x)}\right)^{\epsilon/2} \qquad (10.6.2)\end{aligned}$$

The renormalized vertex function, using $Z_\phi = O(\lambda_B^2)$, is given by

$$\Gamma_4^R(p_1,\cdots,p_4) = Z_\phi^{-2}\Gamma_4^B(p_1,\cdots,p_4) = \Gamma_4^B(p_1,\cdots,p_4) + O(\lambda_B^3)$$

Similar to the procedure followed for mass renormalization, we use the MS (minimal subtraction) scheme to choose the renormalized coupling constant $\lambda_R$ to precisely cancel the $1/\epsilon$ singular term in $\Gamma_4^R$. To identify the singular term, consider $p^2 = 0$ in Eqs. 10.6.1 and 10.6.2. Then, using $m_B = m_R + O(\lambda_B)$

$$\Gamma_4^R(p_1,\cdots,p_4)\Big|_{p_i=0} = -\lambda_B + \frac{3}{16\pi^2\epsilon}\frac{\lambda_B^2}{m_R^\epsilon} + O(\lambda_B^3) \tag{10.6.3}$$

To cancel the $1/\epsilon$ term choose the following definition of the renormalized coupling constant $\lambda_R$

$$\lambda_B = \Big(1 + \frac{3\lambda_R}{16\pi^2\epsilon}\Big)\lambda_R\mu^\epsilon + O\big(\lambda_R^3\big) \quad ; \quad \lambda_R \equiv \lambda_R(\mu) \tag{10.6.4}$$

We can then write Eq. 10.6.3 as follows

$$\begin{aligned}\Gamma_4^R(p_1,\cdots,p_4)\Big|_{p_i=0} &= -\Big(1 + \frac{3\lambda_R}{16\pi^2\epsilon}\Big)\lambda_R\mu^\epsilon + \lambda_R^2\mu^\epsilon\frac{3}{16\pi^2\epsilon}\frac{\mu^\epsilon}{m_R^\epsilon} + O(\lambda_R^3)\\ &= -\lambda_R\mu^\epsilon + \lambda_R^2\mu^\epsilon\frac{3}{16\pi^2\epsilon}\big(\frac{\mu^\epsilon}{m_R^\epsilon} - 1\big)\\ &= -\lambda_R\mu^\epsilon + \lambda_R^2\mu^\epsilon\frac{3}{16\pi^2}\ln(\frac{\mu}{m_R}) + O(\epsilon) \quad : \text{Finite}\end{aligned}$$

Hence, we see that the choice of $\lambda_R$ made in Eq. 10.6.4 leads to a finite $\Gamma_4^R$ for $p_i = 0$. For the general case of $p_i \neq 0$, in Eqs. 10.6.1 and 10.6.2, we replace $\lambda_B$ by $\lambda_R$, using Eq. 10.6.4, and obtain

$$\begin{aligned}\mu^{-\epsilon}\Gamma_4^R(p_1,\cdots,p_4) &= -\lambda_R - \frac{3\lambda_R^2}{16\pi^2\epsilon}\\ &+\frac{\lambda_R^2}{2(4\pi)^2}(\frac{2}{\epsilon} + \varphi(3))\int_0^1 dx\Big\{(\frac{4\pi\mu^2}{f(s)})^{\epsilon/2} + (\frac{4\pi\mu^2}{f(u)})^{\epsilon/2} + (\frac{4\pi\mu^2}{f(t)})^{\epsilon/2}\Big\}\end{aligned} \tag{10.6.5}$$

where, as shown in Figure 10.1, the variables $s, u, t$ are defined as follows

$$f(s) = m_R^2 + sx(1-x) \; ; \; s = (p_1+p_2)^2 \; ; \; u = (p_1+p_3)^2 \; ; \; t = (p_1+p_4)^2$$

Taking the limit of $\epsilon \to 0$ in Eq. 10.6.5 – and simplifying – yields the renormalized four point vertex function[5]

$$\Gamma_4^R(s,u,t) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\int_0^1 dx\Big\{\ln(\frac{f(s)}{c\mu^2}) + \ln(\frac{f(u)}{c\mu^2}) + \ln(\frac{f(t)}{c\mu^2})\Big\} \tag{10.6.6}$$

[5] There is a $\bar{MS}$, called minimal subtraction bar subtraction scheme in which the constant $c$ is absorbed in the subtraction along with the singular $1/\epsilon$ term.

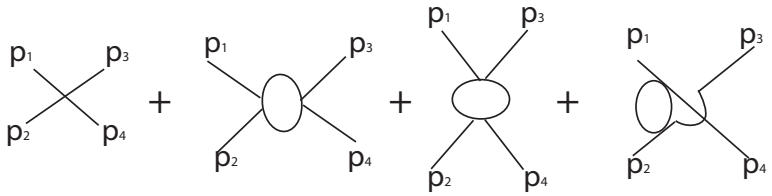


Figure 10.1 The one loop Feynman diagram with the permutations on the external legs of the one loop diagrams.

where

$$c = 4\pi e^{\varphi(3)}$$

Similar to the case of the renormalized mass, the renormalized coupling constant $\lambda_R$ is the value of the physically measured coupling constant for $\mu = m_R$ – and is discussed further in Section 10.6.1. The significance of the renormalized vertex function $\Gamma_4^R(p_1, \cdots, p_4)$ is discussed in Section 11.4.

### *10.6.1 Renormalized massless four-point vertex*

To make the procedure of subtractions more clear, we analyze the massless theory and the equations are more transparent. The massless four-point function is the result for the case when all momentum are much larger than the mass $m_R$, since due to dimensional reasons, mass always appears in the Feynman diagrams as $m_R/|p|$, where $|p|$ is the magnitude of a typical Euclidean momentum.

In the limit of $m_R \to 0$, the integral given in Eq. 10.6.6 yields

$$\int_0^1 dx \ln(\frac{f(s)}{c\mu^2}) \to \int_0^1 dx \Big\{ \ln(\frac{s}{c\mu^2}) + \ln(x(1-x)) \Big\} = \ln(\frac{s}{\tilde{c}\mu^2})$$

where $\tilde{c}$ is a constant. Hence, from Eq. 10.6.6, the massless renormalized four point function is given by

$$\Gamma_4^R(s,u,t) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2} \Big\{ \ln(\frac{s}{\tilde{c}\mu^2}) + \ln(\frac{u}{\tilde{c}\mu^2}) + \ln(\frac{t}{\tilde{c}\mu^2}) \Big\} \qquad (10.6.7)$$

Choose the symmetric external momenta $p_i^0$ that conserve momentum such that

$$p_i^0 \cdot p_j^0 = \tilde{c}\mu^2(\delta_{ij} - \frac{1}{4}) \quad \Rightarrow \quad s(p_i^0) = u(p_i^0) = t(p_i^0) = \tilde{c}\mu^2 \quad (10.6.8)$$

The renormalized four point function is then

$$\Gamma_4^R(p_i^0) = -\lambda_R(\mu) \tag{10.6.9}$$

where the renormalized coupling constant $\lambda_R = \lambda_R(\mu)$ is defined at the momentum scale of $\tilde{c}\mu$.

Eq. 10.6.9 shows that for the massless case, the scheme of minimal subtraction has a simple and direct equivalent in the vertex function renormalization conditions given in Eq. 10.2.1. However, this is not the case for the massive case, since as can be seen from Eq. 10.6.6, the connection of $\Gamma_4^R(p_i)$ with $\lambda_R(\mu)$ is quite indirect, and $-\lambda_R(\mu)$ cannot be made equal to $\Gamma_4^R(p_i)$ for any choice of momenta $p_i$.

## 10.7 Change of scale $\mu$

To illustrate the arbitrariness of the momentum scale $\mu$ for evaluating the renormalized quantities, consider the renormalized four point vertex function and renormalized coupling constant for massless $\phi^4$ theory given in Eq. 10.6.7

$$\begin{aligned}\Gamma_4^R(s,u,t) &= -\lambda_R(\mu) - \frac{\lambda_R^2(\mu)}{32\pi^2}\Big\{\ln(\frac{s}{\tilde{c}\mu^2}) + \ln(\frac{u}{\tilde{c}\mu^2}) + \ln(\frac{t}{\tilde{c}\mu^2})\Big\} \\ &\equiv -\lambda_R(\mu) - \frac{\lambda_R^2(\mu)}{32\pi^2}L(s,u,t)\end{aligned} \tag{10.7.1}$$

such that, from Eqs. 10.6.8 and 10.6.9

$$\Gamma_4^R(p^0) = -\lambda_R(\mu) \;\;;\;\; p_i^0 \cdot p_j^0 = \tilde{c}\mu^2(\delta_{ij} - 1/4)$$

One can define the renormalized coupling constant defined at some arbitrary (fixed) momentum and the scale $\tilde{\mu}$ is chosen as follows

$$s = s_0 \;\;;\;\; s = u_0 \;\;;\;\; t = t_0 \;\;;\;\; \tilde{c}^3\tilde{\mu}^6 = s_0 t_0 u_0$$

Eq. 10.7.1 yields

$$\begin{aligned}&\Gamma_4^R(s_0,u_0,t_0) \equiv -\lambda_R(\tilde{\mu}) = -\lambda_R(\mu) - \frac{\lambda_R^2(\mu)}{32\pi^2}L(s_0,u_0,t_0) + O(\lambda_R^3) \\ &\Rightarrow \lambda_R(\mu) = \lambda_R(\tilde{\mu}) - \frac{3\lambda_R^2(\tilde{\mu})}{16\pi^2}\ln(\frac{\tilde{\mu}}{\mu}) + O(\lambda_R^3)\end{aligned} \tag{10.7.2}$$

The two renormalized coupling constants $\lambda_R(\mu)$ and $\lambda_R(\tilde{\mu})$ in Eq. 10.7.2 are related by the renormalization group discussed in Section 11.5.

Hence, from Eqs. 10.7.1 and 10.7.2

$$\Gamma_4^R(s,u,t) = -\lambda_R(\tilde{\mu}) - \frac{\lambda_R^2(\tilde{\mu})}{32\pi^2}L(s_0,u_0,t_0) - \frac{\lambda_R^2(\tilde{\mu})}{32\pi^2}L(s,u,t)$$
$$\Rightarrow \Gamma_4^R(s,u,t) = -\lambda_R(\tilde{\mu}) - \frac{\lambda_R^2(\tilde{\mu})}{32\pi^2}[\ln\frac{s}{s_0} + \ln\frac{u}{u_0} + \ln\frac{t}{t_0}]$$

Eq. 10.7.2 is another expression of the $\beta$-function. To see this, let $\tilde{\mu} = \mu + \Delta\mu$; then, from Eq. 10.7.2

$$\lambda_R(\mu) = \lambda_R(\mu) + \Delta\mu\frac{\partial\lambda_R(\mu)}{\partial\mu} - \Delta\mu\frac{3\lambda_R^2(\mu)}{16\pi^2}\frac{1}{\mu} + O(\lambda_R^3)$$
$$\Rightarrow \beta(\lambda_R) \equiv \mu\frac{\partial\lambda_R(\mu)}{\partial\mu} = \frac{3}{16\pi^2}\lambda_R^2$$

and we have recovered the one-loop beta function of $\phi^4$ theory.

## 10.8 $O(N)$ symmetric scalar field

Consider the $N$-component scalar field $\phi_i; i = 1, 2\cdots N$ that is invariant under $O(N)$ rotations of the scalar field. We use bare perturbation theory to renormalize the model.

The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\sum_i(\partial\phi_i)^2 - \frac{m_B^2}{2}\sum_i\phi_i^2 - \frac{\lambda_B}{4!}(\sum_i\phi_i^2)^2$$

The $\phi^4$ term is symmetrized and yields

$$\frac{\lambda_B}{4!}(\sum_i\phi_i^2)^2 = \frac{1}{4!}\lambda_B\sum_{ijkl}C_{ijkl}\phi_i\phi_j\phi_k\phi_l$$

where

$$C_{ijkl} = \frac{1}{3}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{ik}\delta^{jk})$$

The propagator is given by

$$G_{IJ}(p,p') = E[\phi_I(p)\phi_J(p')]$$

The Feynman diagrams to one loop are given in Figure 9.9 and yields[6]

$$\begin{aligned} G_{IJ}(p,p') &= \delta(p+p')\{\frac{\delta_{IJ}}{p^2+m_B^2} + \lambda_B \frac{C_{IJ\alpha\alpha}}{(p^2+m_B^2)^2}\tilde{I}_1(p)\} \\ &= \delta(p+p')\delta_{IJ}\{\frac{1}{p^2+m_B^2} + \lambda_B \frac{N+2}{3}\frac{1}{(p^2+m_B^2)^2}\tilde{I}_1(p)\} \\ &= \delta(p+p')\delta_{IJ}\cdot\frac{1}{p^2+m_B^2+\delta m^2} \end{aligned}$$

and from Eq. 10.4.2

$$\tilde{I}_1 = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m_B^2)} = \frac{m_B^2}{16\pi^2}\frac{1}{m_B^\epsilon}\frac{1}{\epsilon} + \text{finite}$$

From Eq. 10.5.1

$$m_B^2 = m_R^2 + \frac{N+2}{3}\frac{\lambda_R}{16\pi^2\epsilon}m_B^2 + O(\lambda_B^2) \tag{10.8.1}$$

Hence the renormalized mass is given by

$$m_R^2 = m_B^2(1 - \frac{N+2}{3}\frac{\lambda_R}{16\pi^2\epsilon})$$

we obtain[7]

$$\gamma_m = \frac{1}{2}\frac{\mu}{m_R^2}\frac{\partial m_R^2}{\partial \mu} = \frac{N+2}{6}\frac{\lambda_R}{16\pi^2}$$

The vertex function to one loop is given in Figure 10.1 and yields

$$\Gamma_{4,IJKL} = \Gamma^0_{4,IJKL} + \Gamma^1_{4,IJKL}$$

To the lowest order, the vertex function is given by the first diagram in Figure 10.1

$$\Gamma^0_{4,IJKL} = -\lambda_B C_{IJKL}$$

The three one loop diagrams given in Figure 10.1 yield the vertex function

$$\Gamma^1_{4,IJKL} = \lambda_B^2 C_{IJ\alpha\beta}C_{KL\alpha\beta}\tilde{I}_2 + \text{permutations}$$

From Eq. 10.6.2, we have

$$\begin{aligned} \tilde{I}_2(p) &= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m_B^2)[(k-p)^2+m_B^2]} \\ &= \frac{1}{2(4\pi)^2}(\frac{2}{\epsilon}+\varphi(1))\int_0^1 dx\left(\frac{4\pi}{m_B^2+p^2x(1-x)}\right)^{\epsilon/2} \end{aligned}$$

[6] Repeated indices summed over.
[7] Note $\frac{1}{\mu}\frac{\partial\lambda_R}{\partial\mu} = -\epsilon\lambda_R + O(\lambda_R^2)$.

The vertex yields

$$C_{IJ\alpha\beta}C_{KL\alpha\beta} = \frac{1}{9}(\delta^{IJ}\delta^{\alpha\beta} + \delta^{I\alpha}\delta^{J\beta} + \delta^{I\beta}\delta^{J\alpha})(\delta^{KL}\delta^{\alpha\beta} + \delta^{K\alpha}\delta^{L\beta} + \delta^{K\beta}\delta^{L\alpha})$$

$$\Rightarrow \quad C_{IJ\alpha\beta}C_{KL\alpha\beta} = \frac{1}{9}(\delta^{IJ}\delta^{KL}(N+4) + 2\delta^{IK}\delta^{JL} + 2\delta^{IL}\delta^{JK})$$

Adding the three one loop diagrams given in Figure 10.1 yields

$$\Gamma^1_{4,IJKL} = \frac{\lambda_B^2}{3}(N+8)C_{IJKL}\tilde{I}_2$$

Hence, the vertex function, to one loop, is given by

$$\Gamma_{4,IJKL} = \Gamma^0_{4,IJKL} + \Gamma^1_{4,IJKL} = \{-\lambda_B + \frac{\lambda_B^2}{3}(N+8)\tilde{I}_2\}C_{IJKL}$$

We obtain, as in Eq. 10.6.4

$$\lambda_B = \mu^\epsilon \lambda_R(1 + \frac{N+8}{3}\frac{\lambda_R}{16\pi^2\epsilon})$$

From Eq. 10.6.6, we have the renormalized vertex function

$$\Gamma^R_{4,IJKL} = \Big\{ -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\int_0^1 dx \Big\{ \ln(\frac{f(s)}{c\mu^2}) + \ln(\frac{f(u)}{c\mu^2}) + \ln(\frac{f(t)}{c\mu^2}) \Big\} C_{IJKL}$$

Following the steps in obtaining the $\beta$ function given in Section 11.1.1 yields

$$\beta = \mu\frac{\partial\lambda_R}{\partial\mu} = -\epsilon\lambda_R + \frac{N+8}{3}\frac{\lambda_R^2}{16\pi^2} \tag{10.8.2}$$

We recover the earlier result given in Eq. 11.1.3 for the single scalar field by setting $N = 1$.

## 10.9 Renormalization constants of $\phi^4$ theory

The results for the renormalization constants are collected. From Eq. 10.6.4

$$\lambda_B = Z_\lambda \lambda_R \quad \Rightarrow \quad Z_\lambda = 1 + \frac{3\lambda_R}{16\pi^2\epsilon}$$

Recall from Eq. 10.4.5

$$Z_\varphi = 1 - \frac{\lambda_R^2}{12(16\pi^2)^2\epsilon} + O(\lambda_R^3)$$

Furthermore, from Eq. 10.5.2, mass renormalization is given by

$$m_B = (1 + \frac{\lambda_R}{16\pi^2\epsilon})^{1/2} m_R + O(\lambda_R^2) = (1 + \frac{\lambda_R}{32\pi^2\epsilon})m_R + O(\lambda_R^2)$$

and hence

$$m_B = Z_m m_R \quad \Rightarrow \quad Z_m = 1 + \frac{\lambda_R}{32\pi^2\epsilon}$$

In summary, to leading order in $O(\lambda_R)$ and in $1/\epsilon$, the dimensionless renormalization constants are given by

$$Z_m = 1 + \frac{\lambda_R}{32\pi^2\epsilon} \;\; ; \;\; Z_\lambda = 1 + \frac{3\lambda_R}{16\pi^2\epsilon} \;\; ; \;\; Z_\phi = 1 - \frac{\lambda_R^2}{12(16\pi^2)^2\epsilon} \quad (10.9.1)$$

Note that, as expected from the general features of dimensional regularization, the $Z$'s depend only on $\lambda_R$ and are independent of mass $m_R$; hence, this scheme is equally valid for massless quantum fields as well.

Both the bare quantities, $m_B, \lambda_B$ are divergent, but are tuned to diverge in a very precise manner, and leave a finite renormalized result. This result is similar to calculus in that infinitesimals are multiplied and divided to yield finite results.

This completes, to lowest non-trivial order, the renormalization of $\phi^4$ theory. The renormalized mass $m_R$ and coupling constant $\lambda_R$ are finite. All measurable results of $\phi^4$ theory are finite functions of $m_R$ and $\lambda_R$. The result are given by one loop Feynman diagrams for $m_R$ and $\lambda_R$ and unexpectedly, one had to go two-loops to obtain $Z_\varphi$.

## 10.10 Renormalized Perturbation Theory

The renormalization scheme using the bare Lagrangian shows that one can render the theory finite by introducing renormalized mass and coupling constants via the renormalization conditions. One then obtains an expansion of the bare quantities as a power serious in terms of the renormalized mass and coupling constant.

The lowest order result was obtained using bare perturbation Theory shows that the bare mass and coupling constant, to lowest order, are equal to the renormalized ones, with divergent terms being added at higher order to render the theory finite. This pattern is a general feature of the renormalization procedure. The bare Lagrangian can be completely replaced by a Lagrangian defined by the renormalized field variables and coupling constants.

The analysis of bare perturbation shows that the bare Lagrangian is defined in terms of the bare mass and coupling constant $m_B, \lambda_R$ and renormalized field variable $\varphi_R$. Another equivalent way of organizing the perturbation expansion is to work directly with only the renormalization quantities. The

renormalized Lagrangian is obtained in the following manner. Recall the Bare Lagrangian is defined by

$$\mathcal{L}_B = -\frac{1}{2}(\partial_\mu \varphi_B)^2 - \frac{1}{2}m_B^2\varphi_B^2 - \frac{\lambda_B}{4!}\varphi_B^4$$

Writing the bare Lagrangian in terms of the renormalized parameters and field yields

$$\begin{aligned}
&\mathcal{L}_B = -\frac{1}{2}Z_\varphi(\partial_\mu\varphi_R)^2 - \frac{Z_m^2 Z_\varphi}{2}m_R^2\varphi_R^2 - \frac{\lambda_R\mu^\epsilon}{4!}Z_\lambda Z_\varphi^2\varphi_R^4 \\
&= -\frac{1}{2}(\partial_\mu\varphi_R)^2 - \frac{1}{2}m_R^2\varphi_R^2 - \frac{\lambda_R\mu^\epsilon}{4!}\varphi_R^4 \\
&\quad -\frac{1}{2}(Z_\varphi - 1)(\partial_\mu\varphi_R)^2 - \frac{1}{2}(Z_m^2 Z_\varphi - 1)m_R^2\varphi_R^2 - \frac{\lambda_R\mu^\epsilon}{4!}(Z_\lambda Z_\varphi^2 - 1)\varphi_R^4 \\
&= \mathcal{L}_R + \mathcal{L}_{CT}
\end{aligned} \tag{10.10.1}$$

The Lagrangian $\mathcal{L}_R$ yields the renormalized theory. The *counter-term* Lagrangian $\mathcal{L}_{CT}$ is defined, order by order in $\lambda_R$, so that all the divergent terms that arise from $\mathcal{L}_R$ in perturbation theory are exactly canceled to each order by $\mathcal{L}_{CT}$.

There is clearly an ambiguity in the definition of $\mathcal{L}_{CT}$ since, in canceling a divergent term, one can always add a finite piece and still cancel the divergent piece. The procedure in the following derivation is called *minimal subtraction* and consists of canceling the divergent $1/\epsilon$ terms that appear to lowest order.

Recall from Eq. 10.9.1, bare perturbation theory yields

$$Z_m = 1 + \frac{\lambda_R}{32\pi^2\epsilon} \quad ; \quad Z_\lambda = 1 + \frac{3\lambda_R}{16\pi^2\epsilon} \quad ; \quad Z_\varphi = 1 - \frac{\lambda_R^2}{12(16\pi^2)^2\epsilon}$$

Hence

$$Z_\varphi = 1 - \frac{\lambda_R^2}{12(16\pi^2)^2}\frac{1}{\epsilon}; \quad Z_m^2 Z_\varphi = 1 + \frac{\lambda_R}{16\pi^2}\frac{1}{\epsilon}; \quad Z_\lambda Z_\varphi^2 = 1 + \frac{3\lambda_R}{16\pi^2}\frac{1}{\epsilon} \tag{10.10.2}$$

The result given in Eq. 10.10.2 refers to only the renormalized coupling constant and mass. The renormalization constants $Z_\varphi, Z_m, Z_\lambda$ are re-derived using only the renormalized parameters $m_R, \lambda_R$. The result is re-derived in this Section directly using renormalized perturbation theory and with no reference to the bare theory.

The three renormalization constants are fixed using the minimal subtraction scheme. The singularities that go as $1/\epsilon$ are directly subtracted by introducing suitable counter-terms in the Lagrangian.

Similar to the bare propagator given in Eq. 10.4.1, the renormalized Lagrangian $\mathcal{L}_R$ yields the following the tadpole and sunset diagram for the renormalized propagator

$$G_2^R(p) = \frac{1}{p^2+m_R^2} - \frac{\lambda_R}{4!}\cdot 12\cdot \frac{1}{(p^2+m_R^2)^2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{p^2+m_R^2} + \frac{1}{(p^2+m_R^2)^2}\lambda_R^2\tilde{I}_4(p)$$

and hence

$$G^R(p) = \frac{1}{p^2+m_R^2} + \frac{1}{(p^2+m_R^2)^2}\frac{\lambda_R m_R^2}{16\pi^2\epsilon}\left(\frac{\mu}{m_R}\right)^\epsilon - \frac{1}{(p^2+m_R^2)^2}\frac{\lambda_R^2 p^2}{12(16\pi^2)^2\epsilon} + O(\lambda^3)$$

To cancel the two divergent terms, using the $MS$ scheme of minimal subtraction, the $1/\epsilon$ terms are precisely canceled by adding the following terms to $\mathcal{L}_R$ as *counter-terms*

$$\mathcal{L}_R - \frac{1}{2}\frac{\lambda_R m_R^2}{16\pi^2\epsilon}\varphi_R^2 + \frac{1}{2}\frac{\lambda_R^2}{12(16\pi^2)^2\epsilon}(\partial_\mu\varphi)^2$$

Hence, from Eq. 10.10.1 and above equation

$$Z_m^2 Z_\varphi - 1 = \frac{\lambda_R}{16\pi^2\epsilon} \quad ; \quad Z_\lambda Z_\varphi - 1 = \frac{\lambda_R^2}{12(16\pi^2)^2\epsilon}$$

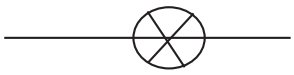

Figure 10.2 The mass counter-term in $\mathcal{L}_{CT}$ is a new term in the Lagrangian

These counter-terms give rise to the following Feynman diagrams given in Figure 10.2 and yields the following renormalized propagator

$$\begin{aligned} G^R(p) &= \frac{1}{p^2+m_R^2} + \frac{1}{(p^2+m_R^2)^2}\frac{\lambda_R m_R^2}{16\pi^2\epsilon}\left(\frac{\mu}{m_R}\right)^\epsilon - \frac{1}{(p^2+m_R^2)^2}\frac{\lambda_R^2 p^2}{12(16\pi^2)^2\epsilon} \\ &\quad - \frac{1}{(p^2+m_R^2)^2}\frac{\lambda_R m_R^2}{16\pi^2\epsilon} + \frac{1}{(p^2+m_R^2)^2}\frac{\lambda_R^2 p^2}{12(16\pi^2)^2\epsilon} \\ &= \frac{1}{p^2+m_R^2} + \frac{1}{(p^2+m_R^2)^2}\frac{\lambda_R m_R^2}{16\pi^2}\ln\left(\frac{\mu}{m_R}\right) \equiv \frac{1}{p^2+m_{\text{phy}}^2} \end{aligned} \tag{10.10.3}$$

where the physical mass is given by

$$m_{\text{phy}}^2 = \frac{\lambda_R}{16\pi^2}m_R^2\left(1-\ln\left(\frac{\mu}{m_R}\right)\right)$$

Note we have recovered the result given in Eq. 10.5.4 that was obtained using bare perturbation theory.

Consider the evaluation of the Feynman diagrams given in Figure 10.1

using the renormalized Lagrangian. From Eq. 10.6.5, the renormalized four point vertex is given by

$$\mu^{-\epsilon}\Gamma_4^R(p_1,\cdots,p_4) = -\lambda_R + \frac{\lambda_R^2}{2(4\pi)^2}(\frac{2}{\epsilon}+\varphi(1))\int_0^1 dx\Big\{(\frac{4\pi\mu^2}{f(s)})^{\epsilon/2}+(\frac{4\pi\mu^2}{f(u)})^{\epsilon/2}+(\frac{4\pi\mu^2}{f(t)})^{\epsilon/2}\Big\}$$

where

$$f(s) = m_R^2 + sx(1-x)\ ;\ s=(p_1+p_2)^2\ ;\ u=(p_1+p_3)^2\ ;\ t=(p_1+p_4)^2$$

To cancel the $1/\epsilon$ term, we need to add the following term to the counter-term to the renormalized Lagrangian

$$\mathcal{L}_R - \frac{1}{4!}\frac{3\lambda_R}{16\pi^2}\frac{\lambda_R\mu^\epsilon}{\epsilon}\varphi_R^4$$

We hence obtain the renormalized four-point vertex function given by

$$\mu^{-\epsilon}\Gamma_4^R(p_1,\cdots,p_4) = -\lambda_R - \frac{3\lambda_R^2}{16\pi^2\epsilon} + \frac{\lambda_R^2}{2(4\pi)^2}(\frac{2}{\epsilon}+\varphi(1))\int_0^1 dx\Big\{(\frac{4\pi\mu^2}{f(s)})^{\epsilon/2}+(\frac{4\pi\mu^2}{f(u)})^{\epsilon/2}+(\frac{4\pi\mu^2}{f(t)})^{\epsilon/2}\Big\}$$

Taking the limit of $\epsilon \to 0$ yields the result obtained earlier in Eq. 10.6.6 and given by

$$\Gamma_4^R(s,u,t) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\int_0^1 dx\Big\{\ln(\frac{f(s)}{c\mu^2})+\ln(\frac{f(u)}{c\mu^2})+\ln(\frac{f(t)}{c\mu^2})\Big\}$$

where

$$c = 4\pi e^{\varphi(1)}$$

In summary, in the scheme of minimal subtraction to cancel the divergent term, one makes the addition of a counter-term to $\mathcal{L}_R$, which yields the bare Lagrangian $\mathcal{L}_B$; to the leading order, we have

$$\mathcal{L}_B \simeq \mathcal{L}_R - \frac{1}{2}\frac{\lambda_R m_R^2}{16\pi^2\epsilon}\varphi_R^2 + \frac{1}{2}\frac{\lambda_R^2}{6(16\pi^2)^2\epsilon}(\partial_\mu\varphi)^2 - \frac{1}{4!}\frac{3\lambda_R}{16\pi^2}\frac{\lambda_R\mu^\epsilon}{\epsilon}\varphi_R^4 \quad (10.10.4)$$

The definition of the renormalization constants, from Eqs. 10.10.1 and 10.10.4, yield the following results

$$Z_\varphi - 1 = -\frac{\lambda_R^2}{6(16\pi^2)^2}\frac{1}{\epsilon} + O(\lambda_R^3)$$

$$Z_m^2 Z_\varphi - 1 = \frac{\lambda_R}{16\pi^2}\frac{1}{\epsilon} + O(\lambda_R^2)$$

$$Z_\lambda Z_\varphi^2 - 1 = \frac{3\lambda_R}{16\pi^2}\frac{1}{\epsilon} + O(\lambda_R^2)$$

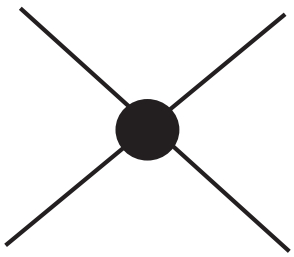

Figure 10.3 The vertex counter-term in $\mathcal{L}_{CT}$ is a new term in the Lagrangian

To lowest order, the solution given above agrees, as expected with the result obtained earlier using bare perturbation theory as given in Eq. 10.10.2.

Note that in renormalized perturbation theory, the Feynman perturbation expansion is carried completely in terms of the renormalized quantities. One first computes the Feynman diagram for a given order using the Lagrangian $\mathcal{L}_R$. One then imposes the renormalization conditions for that given order. One then adjusts, or renormalizes, the renormalization constants $Z_\varphi, Z_m, Z_\lambda$ to subtract and cancel all the divergent terms requires for fulfilling the renormalization conditions.

The remarkable and enigmatic result for a renormalizable theory is that the procedure of canceling the divergent terms in a Feynman diagram using the renormalization counter-terms renders finite all Feynman diagrams to arbitrarily high order. A deeper understanding of this rather baffling and unexpected result was given by the modern interpretation of renormalization due primarily to the pioneering work of Wilson, discussed in Section 11.10.

## 10.11 Momentum cut-off regularization

For completeness, a regularization using momentum cut-off is discussed. The one-loop results are analyzed as these are sufficient to illustrate the main features of this method.

All Feynman diagrams are evaluated in $d = 4$. All momenta in the loops of Feynman diagrams are restricted to have a magnitude less than $\Lambda$.

For the one-loop renormalization of mass we need to be evaluate

$$J_1 = \int_\Lambda \frac{d^4k}{(2\pi)^d} \frac{1}{(k^2 + m^2)} = \frac{\lambda}{16\pi^2}[\Lambda^2 - m^2 \ln(1 + \Lambda^2/m^2)]$$

using the result from Noteworthy 9.2. The divergent terms for $J_1$ are

$$J_1 = \frac{\lambda}{16\pi^2}[\Lambda^2 - m^2 \ln(\Lambda^2/m^2) + m^2 O(m^2/\Lambda^2)]$$

We analyze only mass renormalization; field renormalization is similar but needs a two-loop diagram. The one-loop calculation of the propagator is due to the tadpole diagram and, from Eq. 10.4.1, yields

$$\begin{aligned} G_2(p) &= \frac{1}{p^2 + m_B^2} - \frac{\lambda_B}{4!} \cdot 12 \cdot \frac{1}{(p^2 + m_B^2)^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{p^2 + m_B^2} + O(\lambda_B^2) \\ &= \frac{1}{p^2 + m_{\text{eff}}^2} \end{aligned}$$

where $m_{\text{eff}}$, similar to Eq. 10.5.4 is given by

$$m_{\text{phy}}^2 = m_B^2 + \frac{\lambda_B}{2} J_1 \tag{10.11.1}$$

$$= m_B^2 + \frac{\lambda_B}{32\pi^2}[\Lambda^2 - m_B^2 \ln(\Lambda^2/m_B^2) + O(m^2/\Lambda^2)] \tag{10.11.2}$$

Similar to minimal subtraction, the bare mass is chosen to cancel the divergences that appear for $m_{\text{eff}}$, and yields

$$m_B^2 = m_R^2 - \frac{\lambda_R}{32\pi^2}[\Lambda^2 - m_R^2 \ln(\Lambda^2/\mu^2)] \tag{10.11.3}$$

Note that, unlike dimensional regularization, in the momentum cut-off scheme, the bare mass has both additive and multiplicative renormalization. Eq. 10.11.3 yields $Z_m$ as follows

$$m_B^2 = Z_m^2 m_R^2 \;\; ; \;\; Z_m^2 = 1 - \frac{\lambda_R}{32\pi^2}\frac{\Lambda^2}{m_R^2} + \frac{\lambda_R}{32\pi^2}\ln(\Lambda^2/\mu^2) \tag{10.11.4}$$

Unlike the renormalization constants in dimensional regularization that are independent of mass, momentum cut-off regularization yields mass dependent constants. This is one of major advantages of dimensional regularization since one can take the limit for massless theories, something that is not so straight forward for the case of momentum cut-off regularization.

From Eqs. 10.11.2 and 10.11.3, the effective $m_{\text{eff}}$ is given by

$$m_{\text{eff}}^2 = m_R^2\{1 - \frac{\lambda_R}{16\pi^2}\ln(\frac{\mu}{m_R})\} + O(\lambda_R^2)$$

and we have have recovered the result obtained in Eq. 10.5.4 using dimensional regularization.

For the four point vertex function we need to evaluate

$$K_2(p^2) = \int \frac{d^4k}{(2\pi)^4(k^2+m^2)((k+p)^2+m^2)}$$

From Eq. 9.10.3

$$\begin{aligned} K_2(p^2) &= \int_0^1 dx \int \frac{d^4\tilde{k}}{(2\pi)^4[\tilde{k}^2+m^2+p^2x(1-x)]^2} \\ &= \int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+f^2)} \quad ; \quad f^2 = m^2+p^2x(1-x) \end{aligned}$$

Furthermore

$$\begin{aligned} \frac{1}{(2\pi)^4}\int \frac{d^4k}{(k^2+f^2)^2} &= \frac{1}{(2\pi)^4}\int_0^\Lambda dk k^3 d\Omega_3 \frac{1}{(k^2+f^2)^2} \\ &= \frac{1}{(2\pi)^4}\frac{2\pi^2}{2}\int_0^{\Lambda^2} dz \frac{z}{(z+f^2)^2} \\ &= \frac{1}{16\pi^2}\int_{f^2}^{\Lambda^2+f^2} du\Big(\frac{u-f^2}{u^2}\Big) = \frac{1}{16\pi^2}\Big[\ln(u)+\frac{f^2}{u}\Big]_{f^2}^{\Lambda^2+f^2} \\ &= \frac{1}{8\pi^2}\Big[\ln(\frac{\Lambda}{f})-\frac{1}{2}\Big]+O(1/\Lambda^2) \end{aligned}$$

and hence

$$K_2(p^2) = \frac{1}{8\pi^2}\int_0^1 dx\Big[\ln(\frac{\Lambda}{f})-\frac{1}{2}\Big]+O(1/\Lambda^2)$$

Recall from Eq. 10.6.1, and as shown in Figure 10.1

$$\begin{aligned} \Gamma_4^B(p_1,\cdots,p_4) &= -\lambda_B + \frac{1}{2}\lambda_B^2\{K_2(p_1+p_2)+K_2(p_2+p_3)+K_2(p_3+p_4)\} \\ &= -\lambda_B + \frac{\lambda_B^2}{16\pi^2}\int_0^1 dx\Big[\ln(\frac{\Lambda}{f(s)})+\ln(\frac{\Lambda}{f(u)})+\ln(\frac{\Lambda}{f(t)})-\frac{3}{2}\Big] \end{aligned} \qquad (10.11.5)$$

with the variables $s, u, t$ being defined as follows

$$s = (p_1+p_2)^2 \; ; \; u = (p_1+p_3)^2 \; ; \; t = (p_1+p_4)^2$$

Similar in spirit to the scheme of minimal subtraction, choose the renormalized coupling constant to exactly cancel the divergence due to $\Lambda$; for arbitrary scale $\mu$

$$\lambda_B = \lambda_R(\mu) + \frac{3\lambda_R^2(\mu)}{16\pi^2}\ln(\frac{\Lambda}{\mu}) \qquad (10.11.6)$$

Hence, from Eqs. 10.11.5 and 10.11.6, the renormalized four point vertex function is given by

$$\Gamma_4^R(p_1,\cdots,p_4) = -\lambda_R + \frac{\lambda_R^2}{16\pi^2}\int_0^1 dx\Big[\ln(\frac{\mu}{f(s)}) + \ln(\frac{\mu}{f(u)}) + \ln(\frac{\mu}{f(t)}) - \frac{3}{2}\Big]$$

where, since $m_B$ and $m_R$ are equal to leading order in $\lambda_R$

$$f(s) = m_R^2 + sx(1-x)$$

Hence, we have recovered the result for the renormalized $\Gamma_4^R(p_1,\cdots,p_4)$ that was obtained earlier in Eq. 10.6.6 using dimensional regularization. The results differ in the definition of $\mu$, which is to be expected since we are using different schemes to define $\mu$. This difference yields equivalent renormalized coupling constants that are related by a renormalization group transformation.

The bare and renormalized Lagrangians, to one-loop, are given as follows

$$\begin{aligned}\mathcal{L}_B &= -\frac{1}{2}(\partial_\mu\varphi_B)^2 - \frac{1}{2}m_B^2\varphi_B^2 - \frac{\lambda_B}{4!}\varphi_B^4 \\ &= -\frac{1}{2}(\partial_\mu\varphi_R)^2 - \frac{1}{2}m_R^2\varphi_R^2 - \frac{\lambda_R}{4!}\varphi_R^4 + \mathcal{L}_{c.t.}\end{aligned}$$

where the counter-term Lagrangian is

$$\mathcal{L}_{c.t.} - \frac{1}{2}(\partial_\mu\varphi_R)^2 + \frac{1}{2}\frac{\lambda_R}{32\pi^2}[\Lambda^2 - m_R^2\ln(\Lambda^2/\mu^2)]\varphi_R^2 - \frac{1}{4!}\frac{\lambda_R^2}{16\pi^2}\ln(\frac{\Lambda}{\mu})\varphi_R^4$$

The counter-term has an additive mass renormalization term that is independent of $m_R^2$, namely $\lambda_R\Lambda^2/(32\pi^2)$, and which does not appear in dimensional regularization.

## 10.12 Renormalization: Background Field method

The background field method is an efficient procedure for renormalizing the mass and coupling constant of a quantum field. This method is particularly suited to more complicated cases such as super symmetric Yang-Mills quantum fields.

Consider the bare Lagrangian of $\phi^4$ theory, namely

$$\mathcal{L}_B = -\frac{1}{2}(\partial_\mu\varphi_B)^2 - \frac{1}{2}m_B^2\varphi_B^2 - \frac{\lambda_B}{4!}\varphi_B^4 \;\;;\;\; S(\phi) = \int d^dx\mathcal{L}_B$$

One shifts the quantum field variable $\phi$ by $\phi_c$ such that

$$\phi_B \to \phi + \phi_c$$

The field $\phi_c$ is a determinate function of spacetime and solves the classical field equation, namely that

$$\frac{\delta S[\phi_c]}{\delta \phi} = 0 = -\partial_\mu \partial_\mu \phi_c + m_B^2 \phi_c + \frac{\lambda_B}{3!}\phi_c^3$$

The path integral yields

$$Z = \int D\phi e^{S(\phi+\phi_c)} = \int D\phi e^{S_0(\phi)+S(\phi_c)+S_I(\phi,\phi_c)}$$

where

$$S_0 = \int d^d x \mathcal{L}_0(\phi) = -\frac{1}{2}\int [(\partial_\mu \phi)^2 + m_B^2 \phi^2]$$

and

$$S_I(\phi,\phi_c) = \int d^d x \mathcal{L}_I(\phi,\phi_c) = -\frac{\lambda_B}{4}\int d^d x \phi_c^2(x)\phi^2(x) + O(\phi^3)$$

There is no term linear in $\phi$ since $\phi_c$ is the solution of the classical field equation. The Lagrangian $\mathcal{L}_I$ has the vertex $\phi_c^2\phi^2$ and is shown in Figure 10.4.

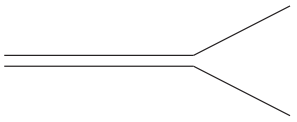

Figure 10.4 Interaction term $S_I \simeq \phi_c^2\phi^2$: the double line corresponds to $\phi_c^2$ and the two separate lines correspond to $\phi^2$.

The partition function yield the following

$$\begin{aligned} Z &= e^{S(\phi_c)} \int D\phi e^{S_0(\phi)}[1 + S_I + \frac{1}{2!}S_I^2 + o(\lambda_B^3)] \\ &= e^{S(\phi_c)+\Delta S(\phi_c)} \end{aligned}$$

The lowest order $\lambda_B$ terms yields

$$\begin{aligned} \Delta S_1 &= -\frac{\lambda_B}{4}\int \phi_c^2(x) \int \frac{d^d k}{(2\pi)^d (k^2 + m_B^2)} \\ &= -\frac{\lambda_B}{4} J_1 \int \phi_c^2(x) \end{aligned}$$

where, using result from Noteworthy 9.2

$$J_1 = \int \frac{d^d k}{(2\pi)^d (k^2 + m_B^2)} = \frac{m_B^2}{16\pi^2}(\frac{4\pi}{m_B^2})^{\epsilon/2}[-\frac{2}{\epsilon} + \varphi(2)] + O(\epsilon)$$

The term $O(\lambda_B^2)$ yields the following

$$\Delta S_2 = \frac{1}{2!}\frac{1}{16}\lambda_B^2 E[\int d^dx \phi_c^2(x)\phi^2(x) \int d^dx' \phi_c^2(x')\phi^2(x')]$$

and is represented by two Feynman diagrams shown in Figure 10.5.

Figure 10.5 The two diagrams at second order; the first is disconnected and the second is connected.

The first (disconnected) diagram yields

$$\Delta S_2(1) = \frac{1}{32}\lambda_B^2 (J_1)^2 [\int d^dx \phi_c^2(x)]^2 = \frac{(\Delta S_1)^2}{2!}$$

The second (connected) diagram gives the following

$$\Delta S_2(2) = \frac{\lambda_B^2}{16}\int_{p_1,p_2,p_3}\int_{k_1,k_2,k_3} \xi(k_1)\xi(p_1) \times \delta(\sum_{i=1}^{3} k_i)\delta(\sum_{i=1}^{3} p_i)\frac{\delta(p_2+k_2)}{p_2^2+m_B^2}\frac{\delta(p_3+k_3)}{p_3^2+m_B^2}$$

where

$$\xi(k) \equiv \int d^dx e^{ikx}\phi_c^2(x)$$

Performing the contractions yields, to leading order, the following

$$\Delta S_2(2) = \frac{\lambda_B^2}{16}\int \xi(k)\xi(-k)J_2 = \frac{\lambda_B^2}{16}J_2 \int d^dx \phi_c^4(x)$$

where, from Eq. 9.8.2

$$J_2 = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m_B^2)^2} = \frac{1}{16\pi^2}(\frac{4\pi}{m^2})^{\epsilon/2}[\frac{2}{\epsilon}+\varphi(3)] + O(\epsilon)$$

Hence

$$Z = e^{S(\phi_c)}[1 + \Delta S_1 + \frac{(\Delta S_1)^2}{2!} + \Delta S_2(2)]$$

and yields,to $O(\lambda_B^2)$ the following background action

$$S_{\text{eff}} = S(\phi_c) + \Delta S_1 + \Delta S_2(2)$$

The effective background Lagrangian is *defined* to be finite and is given by

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}(\partial_\mu \phi_c)^2 - \frac{1}{2}m_R^2\phi_c^2 - \frac{\lambda_R \mu^\epsilon}{4!}\phi_c^4$$

Note in the background field method, as expected, the field renormalization constant $Z_\varphi$ only appears at $O(\lambda^2)$.

The renormalized coupling constant is given by

$$\begin{aligned}
&\lambda_R\mu^\epsilon = \lambda_B - \frac{3}{2}J_2\lambda_B^2 = \lambda_B - \frac{3\lambda_B^2}{32\pi^2}(\frac{4\pi}{m_B^2})^{\epsilon/2}[\frac{2}{\epsilon}+\varphi(3)] \\
&= \lambda_B(1-\frac{3\lambda_B\mu^{-\epsilon}}{16\pi^2\epsilon}) - \frac{3\lambda_B^2\mu^{-\epsilon}}{16\pi^2}\ln(\frac{b\mu}{m_B}) \quad ; \quad b = 2\sqrt{\pi}e^{\varphi(3)/2} \qquad (10.12.1) \\
&\Rightarrow \lambda_B = \lambda_R\mu^\epsilon(1+\frac{3\lambda_R}{16\pi^2\epsilon}) + \mu^\epsilon\frac{3\lambda_R^2}{16\pi^2}\ln(\frac{b\mu}{m_R})
\end{aligned}$$

The renormalized mass is

$$\begin{aligned}
&m_R^2 = m_B^2 + \frac{1}{2}\lambda_B J_1 = m_B^2\Big(1+\frac{\lambda_B}{32\pi^2}(\frac{4\pi}{m_B^2})^{\epsilon/2}[-\frac{2}{\epsilon}+\varphi(2)]\Big) \\
&= m_B^2\Big(1-\frac{\lambda_R}{16\pi^2\epsilon}\Big) - m_B^2\frac{\lambda_R}{16\pi^2}\ln(\frac{\tilde{b}\mu}{m_B}) \quad ; \quad \tilde{b} = 2\sqrt{\pi}e^{-\varphi(2)/2} \qquad (10.12.2)
\end{aligned}$$

The renormalized mass and coupling constant have a $\mu$ dependent term that is absent in the result obtained earlier using the minimal scheme for both bare and renormalized perturbation theory.

The difference is because the background field method and minimal subtraction scheme have different momentum scales at which the renormalized quantities are defined. It is shown later in Section 11.5 that these two schemes yield the same renormalized theory, with the renormalized quantities of the two schemes being related by a finite renormalization group transformation.

The background effective Lagrangian is defined to be completely finite to all orders in perturbation theory and hence have provides an independent renormalization scheme. All the quantum fluctuations are integrated out and the smoothly varying classical field contains all the short distance effect of the quantum field. Since the bare quantum field Lagrangian has quantum fluctuations for all length scales, the long distance effective Lagrangian's mass and coupling constant are the renormalized parameters.

## 10.13 Renormalizability to all orders

For the $\phi^4$ theory, two divergent diagrams were analyzed. The question naturally arises whether there are more divergent diagrams and whether these appear at higher orders. A high order Feynman diagram is shown in Figure 10.6. For the theory to be renormalizable, Feynman diagrams of arbitrarily high order all have to be finite.

For every Feynman diagram, define the superficial degree of divergence to be $D$. This is a measure of how the Feynman integral diverges as the regularization is removed.

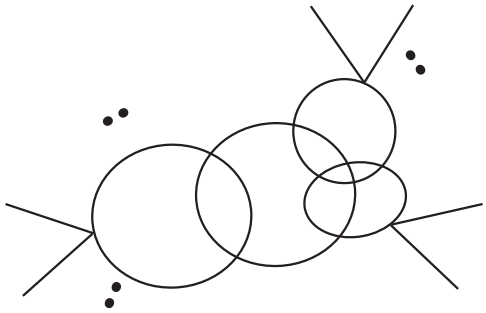

Figure 10.6 A $O(\lambda^{11})$ order Feynman diagram for a $\phi^4$ scalar field theory.

For a momentum cut off $\Lambda$, the norm of the integrations are restricted so that $k_1^2 + \cdots + k_d^2 < \Lambda^2$. Note that all the components of the momentum are restricted only for the Euclidean theory, since for Miknowski space the norm is not positive definite. However, it is sufficient to show that the quantum field theory is renormalizable in Euclidean spacetime for it to be renormalizable in Minkowski spacetime.

In general, a Feynman diverges as $\Lambda^D$. Note for $D \geq 0$ the graph is divergent, with $D = 0$ indicating a logarithmic divergence. For $D < 0$ , the graph is finite and convergent.

A Feynman diagram in general is defined by the following.

- $E$ number of external lines; an external line connects to only one vertex
- $n$ number of vertices
- $I$ number of internal lines; the two ends of an internal line are each connected to a vertex, which can be the *same* vertex as in the case of a bubble diagram, or can be connected to different vertices.
- $L$ number of loops; a loop is an independent $d$-dimensional integration

Let the spacetime dimension be $d$ and consider the general interaction terms $\phi^r$, with $r \geq 3$. Each loop corresponds to a momentum integration over $d$ dimensions and gives a divergence of $\Lambda^d$; each internal line $I$ corresponds to a propagator that contributes a $\Lambda^{-2}$ to the divergence.

Hence for $L$ loops with $I$ internal lines, $D$ is given by

$$D = dL - 2I \tag{10.13.1}$$

For a bubble Feynman diagram, given in Figure 9.9(b), $L = 1, I = 1$

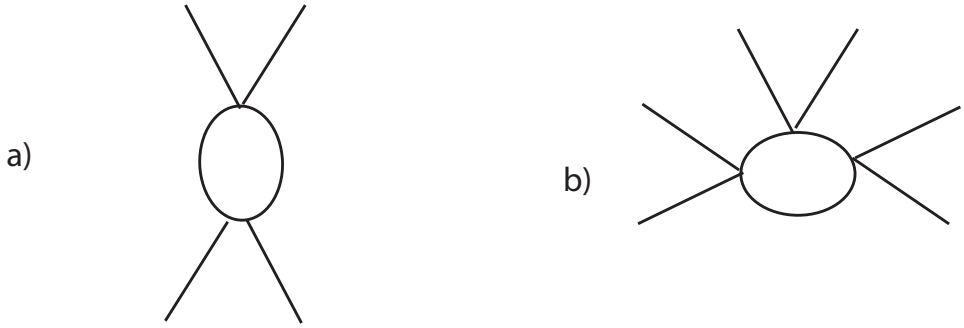


Figure 10.7 Feynman diagrams for a $\phi^4$ scalar field. a) A Feynman diagram with four external legs. b) A Feynman diagram with six external legs.

and hence $D = d - 2$, yielding a quadratic divergence $\Lambda^2$ in $d = 4$. The Feynman diagram in Figure 10.7(a) has $D = d - 2 \times 2 = d - 4$, which has $\ln \Lambda$ divergence for $d = 4$. On the other hand, diagram in Figure 10.7(b) has $D = d - 6$, which converges for $d = 4$.

The main criterion for classifying a divergent Feynman diagram needs to be the number of external lines $E$ that the diagram has – as this tells us how many Feynman diagrams are divergent for the theory and need to be rendered finite by renormalization.

We re-write the degree of divergence $D$ in terms of the number of external lines $E$. For $n$ vertices, an interaction of $\phi^r$ gives $rn$ legs that sum to $2I + E$ since each internal line connects two vertices. Hence

$$rn = E + 2I$$

The number of loops are determined by $n$ and $I$. Each vertex in the Feynman diagram has one momentum conserving delta-function constraining the momentum integrations, less one overall momentum conservation delta function for the entire Feynman diagram. Hence

$$L = I - (n - 1) = I - n + 1 \tag{10.13.2}$$

For example, in Figure 10.6, $n = 11, L = 9$ and $I = 19$; hence we have $L - 1 = 8 = I - n$, as expected.

The superficial degree of divergence, from Eqs. 10.13.1 and 10.13.2, is given by

$$D = d(I - n + 1) - 2I = (d - 2)I - d(n - 1)$$

For 1PI diagrams, discussed later in Section 12.5, all internal lines occur

within a loop and one has the condition

$$rn = E + 2I$$

and which yields

$$D = (d-2)(\frac{r}{2}n - \frac{E}{2}) - d(n-1)$$

Hence

$$D = d - \frac{d-2}{2}E + (\frac{d-2}{2}r - d)n \quad ; \quad n = 1, 2 \cdots \infty \qquad (10.13.3)$$

We have the following cases.

- $(d-2)r/2 - d = 0$ and hence $D = d - (d-2)E/2$. There are only a finite number external lines $E$ for which the superficial degree of divergent is positive. For sufficiently large $E$, we have $D < 0$.

  Since the bare Lagrangian has only a finite number of coupling constants, a renormalizable theory can only have divergent diagrams for a finite number of external legs $E$. The superficial degree of divergence is independent of the number of vertices $n$, and which implies that for each $E$ for which $D > 0$, there can be *infinitely* many divergent diagrams, with larger and larger $n$. Two examples of the infinite collection of divergent diagrams are given in Figure 10.8 for $\phi^4$ theory.

  The theory is renormalizable since the infinitely many divergent terms can all be absorbed by redefining a finite number of bare coupling constants, which correspond to the finite number of $E$ for which $D > 0$.

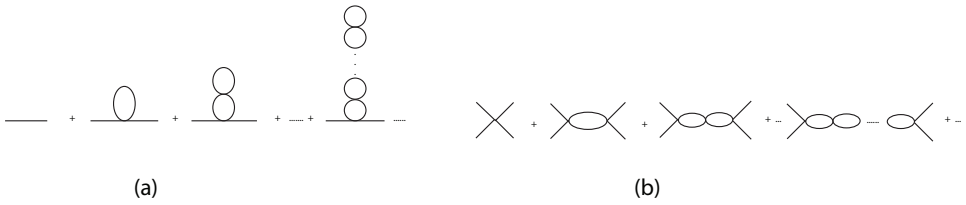


Figure 10.8 Divergent Feynman diagrams for a $\phi^4$ scalar field. a) Bubble diagrams with two external legs. b) Divergent diagram with four external legs.

- $(d-2)r/2 - d < 0$: only a finite number of Feynman diagrams are divergent since for large enough $n$, the superficial degree of divergence $D < 0$; the rest of the higher order Feynman diagrams are all convergent, and the theory is super-renormalizable

- $(d-2)r/2 - d > 0$: there are infinite number of divergent Feynman diagrams; theory is non-renormalizable. One has infinitely many divergent diagrams since the degree of divergence $D$ keeps increasing with increasing $n$; in particular, for sufficiently large $n$ there will be a divergent Feynman diagram for any number of external lines $E$. Hence, we expect that for this case the theory cannot be rendered finite by any perturbative procedure and is termed as non-renormalizable.

For $\phi^4$ theory in $d = 4$, for $r = 4$ we have $(d-2)r/2 - d = 0$ and $D = 4 - E$. Hence, there are only two divergent diagrams, namely for $E = 2$ and 4. There are infinitely many divergent diagrams for each case. For the case $E = 2$, some of the infinitely many divergent diagrams can be seen from the tower of bubble diagrams that occur for the propagator and shown in Figure xxx; all the divergences can be absorbed by re-defining the bare mass $m_B$ in terms of the renormalized mass $m_R$ and divergent counter-terms.

In general, from Eq. 10.13.3 for $d = 4$ , we have

$$D = 4 - E + n(r-4) \tag{10.13.4}$$

From equation above, $\phi^4$ is renormalizable in $d = 4$ but $\phi^6$ theory is non-renormalizable.

For a $\phi^r$ theory in $d = 2$ , from Eq. 10.13.3 , we have

$$D = -d(n-1)$$

which is independent of $r$ and hence all polynomial interactions are renormalizable. This allows for non-polynomial potentials, such as an exponential of the field $\phi$, to be renormalizable in $d = 2$ .

### *10.13.1 $\phi^3$ theory: Super-renormalizable in $d = 4$*

For $\phi^3$ theory in $d = 4$, from Eq. 10.13.4 we have

$$D = 4 - E - n \;\; ; \;\; r = 3$$

In $d = 4$, the following are only three Feynman diagrams that are divergent for $\phi^3$ theory.

- $n = 1 \;\Rightarrow\; D = 3 - E$: For $E = 1$, there is a single one-loop tadpole diagrams shown in Figure 10.9(a). There is no $E = 2$ diagram and $E = 3$ gives a tree diagram (no loops) that is finite. For $E > 3$, we have $D < 0$ and hence convergent.
- $n = 2 \;\Rightarrow\; D = 2 - E$: For $E = 2$ there is a single self-energy one-loop diagram that yields mass renormalization shown in Figure 10.9(c).

- $n = 3 \Rightarrow D = 1 - E$: For $E = 1$, there is a single two-loop tadpole divergent Feynman diagram shown in Figure 10.9(b).
- All the diagrams higher $n$ for all $E$ with $n > 3$, are convergent.

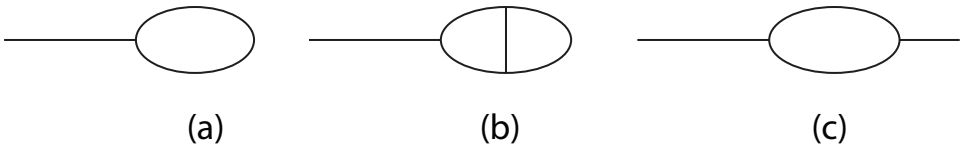


Figure 10.9 Feynman diagrams for a $\phi^3$ scalar field. a) A one-loop tadpole diagram (one external leg). b) A two-loop tadpole diagram (one external leg). c) A one-loop Feynman diagram with two external legs.

- The three divergent terms can be rendered finite by introducing a linear term in the Lagrangian $\delta h\phi$, with $\delta h$ canceling the one-loop and two-loop divergent terms coming from the tadpole diagrams given in Figures 10.9(a) and (b). A mass renormalization $\delta m^2$ is added to the mass term in the Lagrangian to yield $(m^2+\delta m^2)\phi^2$ and cancels the one-loop divergence coming from the self-energy diagram given in Figure 10.9(c).

Hence, in $d = 4$, $\phi^3$ theory is super renormalizable.

### 10.14 Superficial degree of divergence: Dimensional analysis

The degree of divergence $D < 0$ does not mean that a graph is necessarily convergent and this is the reason that it is called the superficial degree of divergence. Consider for example $E = 6$ for $\phi^4$ theory we have $D = -2$. A graph such as given in Figure 10.10(a) is convergent But graph in Figures 10.10(b) and (c) are divergent since they contain divergent subdiagrams.

However, intuitively it is clear that if one renormalizes the primitively divergent graphs, then graphs such as those given in Figures 10.10(b) and (c) no longer have divergent subgraphs, and should be convergent.

*Weinberg's theorem* states that if the $D$ of a graph is negative as well as the $D$ for all its subgraphs are negative, then the Feynman diagram converges.

Consider the Lagrangian in $d$ dimensions namely

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - g\phi^r$$

Since $\int d^dx\mathcal{L}$ is dimensionless we have

$$\phi : a^d\frac{1}{a^2}a^{2d_\phi} = 1 \Rightarrow d_\phi = \frac{1}{2}(2-d)$$

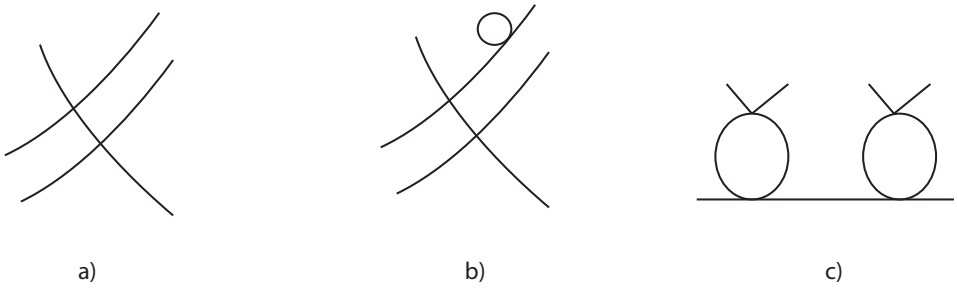


Figure 10.10 Feynman diagrams for a $\phi^4$ scalar field. (a) Finite Feynman diagram. (b) Divergent Feynman diagram. (c) Divergent Feynman diagram.

$$g : a^{-d_g} a^{r d_\phi} a^d = 1 \Rightarrow d_g = -d - r d_\phi$$

or

$$d_g = \frac{d-2}{2} r - d$$

Hence one can write $D$, from Eq. 10.13.3, as follows

$$D = d - (\frac{d}{2} - 1)E - n d_g$$

Each vertex has a coupling constant $g$; for a $n$th order Feynman diagram, the dimensionless contribution to, for example, the generating functional $W$ is given by $g^n I_n$, where $I_n$ is the Feynman diagram that is given by integral over all the loop integrations. The dimension of $g \sim \mu^{d_g}$.

- For $d_g > 0$, the coupling constant $g^n$ yields a dimensional factor of $\mu^{n|d_g|}$ and hence, to make the product $g^n I_n$ dimensionless, the Feynman diagram must go as $\Lambda^{-n|dg|}$: the higher the order of the diagram, the more convergent it is. Hence the theory is super-renormalizable as there are only at best a finite number of graphs that are divergent.
- For $d_g < 0$, the dimensional term $\mu^{-|d_g|}$ coming from the coupling constant must be compensated by a divergent Feynman diagram going as $\Lambda^{|dg|}$. The higher the order, the more divergent is the Feynman diagram, and the theory is non-renormalizable.
- For $d_g = 0$, there are infinitely many divergent diagrams but these appear only in a finite number with a maximum given number of external legs. All the divergent diagrams can be rendered finite by renormalizing a finite number of parameters in the action that correspondence to these external legs. Hence, the theory is renormalizable for $d_g = 0$.

In conclusion, for renormalizable theories in $d$ dimensions, $d_g = 0$ and the coupling constant $g$ is *dimensionless*.

## 10.15 Summary

It is one of the mysteries of renormalization that additive infinities, such as $m + \delta m$ and $\lambda + \delta\lambda$, can be removed by the multiplicative redefinitions of the original (bare) Lagrangian. The other mystery is how is it possible that by redefining the bare Lagrangian all the higher correlation functions are rendered finite – and yield a completely finite theory to all orders in perturbation theory.

A partial answer is given in the next Chapter on the renormalization group. The existence of a 'ultra-violet fixed point' for the case of high energy physics and of an 'infra-violet fixed point' in the case of phase transitions – these concepts are defined in the next Chapter 11 – is the fundamental reason that a theory is renormalizable. What one obtains in perturbation theory is the result of existence of this fixed point, order by order.

# 11

# The Renormalization Group

The perturbation expansion of $\phi^4$ nonlinear scale quantum field yields divergent results. An analysis of bare perturbation theory shows that a redefinition of the bare parameters, using the scheme of multiplicative renormalization, removes all the divergences. Furthermore, it was demonstrated that both, the bare and renormalized perturbation expansions, yield the same results.

The momentum scale $\mu$ that appears in dimensional regularization is a general feature of renormalization. In every scheme, a momentum or length scale must be chosen for defining the renormalized coupling constant and masses in terms of the bare ones. The renormalization group reveals the physical significance of the apparently arbitrary scale $\mu$. The renormalization group is a mathematical formalism for studying the behavior of a system under a change of scale, which in our case means changing $\mu$. In particular, it determines how the coupling constants change with a change of scale, with the renormalization group transformation relating the coupling constants at two different scales.[1]

A transparent analysis of renormalization is provided by bare perturbation theory using dimensional regularization. The renormalized theory is defined entirely in terms of the bare theory, with the renormalized theory being defined in terms of the bare theory by the procedure of multiplicative redefinition. The Callan-Symanzik equation analyzes the change in the correlation functions holding the bare coupling constants and masses fixed – and is a realization of the renormalization group. Wilson's approach to the renormalization group, based on integrating out the high momentum degrees

[1] The renormalization group technically speaking is a semi-group, with the transformation only increasing the scale; the renormalization group transformation is closed under multiplication, but does *not* have an inverse, as is the case for semi-group.

of freedom, is more general and includes the Callan-Symanzik equation as a special case.

From Eq. 10.1.2, the renormalization constants $Z$ depend only on $\lambda_R$ and $\epsilon$; since we are going to hold $\lambda_B$ fixed, we solve for $\lambda_R$ in terms of $\lambda_B$, that is

$$\lambda_R = \lambda_R(\mu^{-\epsilon}\lambda_B, \epsilon)$$

Using the fact that the constants $Z$ in dimensional regularization are independent of mass, we have

$$\lambda_B = Z_\lambda(\lambda_R, \epsilon)\lambda_R\mu^\epsilon \tag{11.0.1}$$

$$m_B = Z_m(\lambda_R, \epsilon)m_R \tag{11.0.2}$$

$$\phi_B = Z_\phi^{1/2}(\lambda_R, \epsilon)\phi_R \tag{11.0.3}$$

where $\epsilon = 4-d$. Recall that $\lambda_B, m_B, \epsilon$ and $\phi_B$ are all *independent* of the scale parameter $\mu$. It is for this reason that the renormalized coupling constant $\lambda_R(\mu)$ is of central importance in determining how the behavior changes with the change of scale $\mu$.

## 11.1 Callan-Symanzik equation

The Callan-Symanzik equation is an example of the renormalization group equation as it describes how the correlation functions of the quantum field theory change as one changes the scale at which the renormalized coupling constants and masses are defined.

Consider the bare $n$-point vacuum expectation value

$$G_n^B(p_1, \cdots p_n) = \langle\Omega|T\{\phi_B(p_1)\cdots\phi_B(p_n)\}|\Omega\rangle = G_n^B(\lambda_B, m_B, \epsilon)$$

The renormalized Greens function are defined by

$$G_n^B(\lambda_B, m_B, \epsilon) = Z_\phi^{n/2}(\mu, \lambda_B, \epsilon)G_n^R(\lambda_R, m_R, \epsilon)$$

and we have

$$G^B(\lambda_B, m_B, \epsilon) = Z_\phi^{n/2}G_n^R(\lambda_R, m_R, \mu)$$

Since the bare theory is independent of $\mu$, we obtain

$$\mu\frac{d}{d\mu}G^B(\lambda_B, \epsilon) = 0$$

Above condition yields the following Callan-Symanzik equation

$$0 = \frac{d}{d\mu}(Z_\phi^{n/2}G_n^R) = \frac{n}{2}Z_\phi^{n/2}\frac{1}{Z}\frac{dZ}{d\mu}G_n^R + Z_\phi^{n/2}\frac{d}{d\mu}G_n^R$$

or

$$\mu\frac{d}{d\mu}G_n^R + \frac{n}{2}\mu\frac{d\ln Z}{d\mu}G_n^R = 0$$

Hence we obtain

$$(\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda_R}{\partial\mu}\frac{\partial}{\partial\lambda_R} + \mu\frac{\partial m_R}{\partial\mu}\frac{\partial}{\partial m_R} + \frac{n}{2}\mu\frac{\partial\ln Z}{\partial\mu})\Big|_{\lambda_B,m_B,\epsilon} G_n^R = 0$$

or

$$(\mu\frac{\partial}{\partial\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + \gamma_m(\lambda_R)\frac{\partial}{\partial m_R} + \frac{n}{2}\gamma_\phi(\lambda_R))\Big|_{\lambda_B,m_B,\epsilon} G_n^R = 0 \quad (11.1.1)$$

Note the important point that it is necessary to keep $\lambda_B, m_B, \epsilon$ constant as is required by the definition of the bare Lagrangian, which is independent of the renormalized quantities.

In the scheme of bare perturbation theory, the renormalization functions are evaluated by the variation of the renormalized theory as a function of $\mu$, while keeping the bare theory, namely $\lambda_B, m_B, \epsilon$ fixed. This yields

$$\beta(\lambda_R) = \mu\frac{\partial\lambda_R}{\partial\mu}\Big|_{\lambda_B,m_B,\epsilon} \; ; \; \gamma_m(\lambda_R) = \frac{1}{m_R}\mu\frac{\partial m_R}{\partial\mu}\Big|_{\lambda_B,m_B,\epsilon}$$
$$\gamma_\phi(\lambda_R) = \frac{1}{2}\mu\frac{\partial\ln Z}{\partial\mu}\Big|_{\lambda_B,m_B,\epsilon}$$

The field renormalization constant $Z_\phi$ is given in Eq. 10.4.5 and yields

$$Z_\phi = 1 - \frac{1}{\epsilon}\frac{1}{12}\frac{\lambda_B^2\mu^{-2\epsilon}}{(4\pi)^4} + O(\epsilon, \lambda_B^3)$$

Hence

$$\gamma_\phi = \frac{1}{2}\mu\frac{\partial\ln Z_\phi}{\partial\mu}\Big|_{\lambda_B,m_B,\epsilon} = \frac{1}{12}\frac{\lambda_R^2}{(4\pi)^4} + O(\lambda_R^3)$$

The finite renormalized mass $m_R$, from Eq. 10.5.2, is given by

$$m_R = (1 - \frac{\lambda_B\mu^{-\epsilon}}{32\pi^2\epsilon})m_B + O(\lambda_R^2)$$

The anomalous mass dimension is given by

$$\gamma_m = \frac{1}{m_R}\mu\frac{\partial m_R}{\partial\mu}\Big|_{\lambda_B,m_B,\epsilon} = \frac{1}{32\pi^2}\lambda_R \quad (11.1.2)$$

Anticipating the result for the $\beta$ function obtained in Eq. 11.1.4, we have

$$\beta = \frac{3}{16\pi^2}\lambda_R^2 \; ; \; \gamma_m = \frac{1}{32\pi^2}\lambda_R \; ; \; \gamma_\phi = \frac{1}{12}\frac{\lambda_R^2}{(4\pi)^4} \quad (11.1.3)$$

### 11.1.1 Running coupling constant

The behaviour of the coupling constant as one changes the scale $\mu$ is a fundamental property of a quantum field theory and determines the $\beta$-function. Recall the bare theory given by $\mathcal{L}_B$ is defined independent of $\mu$ and this induces a dependence of the renormalized quantities on the scale $\mu$. Recall the beta function is defined by

$$\beta = \mu \frac{\partial \lambda_R}{\partial \mu}\Big|_{\lambda_B, m_B, \epsilon}$$

For the $\phi^4$ theory, the relation of the bare and renormalized coupling constant is given by Eq. 10.6.4

$$\lambda_B = \Big(1 + \frac{3\lambda_R}{16\pi^2\epsilon}\Big)\lambda_R \mu^\epsilon + O\big(\lambda_R^3\big)$$

Since the bare coupling constant is independent of the scale $\mu$, we have

$$\mu \frac{\partial \lambda_B}{\partial \mu} = 0$$

and it then follows that

$$0 = \mu \frac{\partial \lambda_R}{\partial \mu}\mu^\epsilon(1 + \frac{3\lambda_R}{16\pi^2\epsilon}) + \epsilon\lambda_R\mu^\epsilon(1 + \frac{3\lambda_R}{16\pi^2\epsilon}) + \lambda_R\mu^\epsilon \frac{3}{16\pi^2\epsilon}\mu\frac{\partial\lambda_R}{\partial\mu}$$

or

$$\begin{aligned} \beta = \mu\frac{\partial\lambda_R}{\partial\mu} &= -\frac{\epsilon\lambda_R(1+\frac{3\lambda_R}{16\pi^2\epsilon})}{(1+\frac{6\lambda_R}{16\pi^2\epsilon})} = -\epsilon\lambda_R(1 - \frac{3\lambda_R}{16\pi^2\epsilon}) \\ \Rightarrow \beta &= -\epsilon\lambda_R + \frac{3\lambda_R}{16\pi^2} \end{aligned} \tag{11.1.4}$$

Hence, for $\epsilon \to 0$, the beta function is given by

$$\beta(\lambda_R) = \frac{3\lambda_R^2}{16\pi^2} + O(\lambda_R^3)$$

The two-loop result is

$$\beta(\lambda_R) = \frac{3\lambda_R^2}{16\pi^2} - \frac{17\lambda_R^3}{3(4\pi)^4} + O(\lambda_R^4)$$

The renormalized coupling constant $\lambda_R = \lambda_R(\mu)$ is a *running* coupling constant that 'runs' with the scale $\mu$; it has a dependence on $\mu$ to precisely off-set the variation of the renormalized theory so that the bare theory is independent of the arbitrary scale $\mu$.

## 11.2 Callan-Symanzik equation: anomalous dimensions

Consider the renormalized $n$-point function

$$G_n(p, \lambda_R, m_R, \epsilon, \mu) \equiv G_n^R(p_1 \cdots p_n, \lambda_R, m_R, \epsilon, \mu)$$

which is related to the bare $n$-point function by the following

$$G_B(p, \lambda_B, m_B, \epsilon) = Z_\phi^{n/2} G_n(p, \lambda_R, m_R, \epsilon, \mu)$$

To obtain the variation of $G_n$ under a change of scale $\mu$, let

$$\mu(t) = e^t \mu_0 \quad \Rightarrow \quad \mu \frac{d}{d\mu} = \mu \frac{dt}{d\mu} \frac{d}{dt} = \frac{d}{dt}$$

The Callan- Symanzik equation is given by

$$0 = \frac{d}{dt}\{G_B(p, \lambda_B, m_B, \epsilon)\} = \frac{d}{dt}\{Z_\phi^{n/2} G_n(p, \lambda_R, m_R, \epsilon, \mu)\}$$

and which yields

$$[\mu \frac{d}{d\mu} + \frac{n}{2}\gamma_\phi(t)] G_n(p, e^t \mu_0, \lambda_R(t)) = 0$$

Hence

$$[\frac{d}{dt} + \frac{n}{2}\gamma_\phi(t)] G_n(t) = 0 \quad \Rightarrow \quad \frac{d}{dt}\Big(e^{\frac{n}{2}\int_0^t dt' \gamma_\phi(t)(t')} G_n(t)\Big) = 0$$

Integrating above equation yields

$$G_n(p, \mu_0, \lambda_R(0)) = e^{n\Gamma(t)} G_n(p, e^t \mu_0, \lambda_R(t)) \tag{11.2.1}$$

where

$$\Gamma(t) = \frac{1}{2}\int_0^t dt' \gamma_\phi(t')$$

Changing variables from $\mu$ to $\lambda_R$ yields

$$dt = \frac{d\mu}{\mu} = \frac{d\lambda_R}{\beta(\lambda_R)} \quad \Rightarrow \quad \Gamma = \frac{1}{2}\int_{\lambda_R(0)}^{\lambda_R(t)} d\lambda \frac{\gamma_\phi(\lambda)}{\beta(\lambda)}$$

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Noteworthy 11.1: Dimension of connected Greens function

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The 'engineering dimension' of $G_n(x_1, \cdots, x_n) \equiv G_n(x)$ is determined by the dimension of the scalar field $\phi$ – given by $d_\phi = (2-d)/2$. From its definition

$$[G_n(x)] = \mu^{-n d_\phi} = \mu^{n(d/2-1)}$$

The momentum space Greens function is given by

$$G_n(p_1 \cdots p_n) = \int dx_1 \cdots dx_n e^{i\sum_{i=1}^{N} p_i x_i} G_n(x_1 \cdots x_n)$$

Hence

$$\begin{aligned}[G_n(p_1 \cdots p_n)] &\equiv [G_n(p)] = \mu^{-nd}\mu^{n(d/2-1)} = \mu^{-n(d/2+1)} \\ &= \mu^{-d_n} \quad ; \quad d_n = n(d/2+1) \end{aligned} \tag{11.2.2}$$

Consider the example of the two point function given by

$$G_2(p_1, p_2) = \delta(p_1 + p_2) G_2(p_1)$$

with

$$G_2(p) = \int d^4x e^{-ipx} E[\phi_R(x)\phi_R(0)] = \int d^4x e^{-ipx} G_2(x)$$

The two point function for the free field is given by

$$G_2^0(p_1, p_2) = \frac{\delta(p_1 + p_2)}{p^2}$$

Hence, for the free field

$$G_2^0(e^t p_1, e^t p_2) = e^{-td-2t} \frac{\delta(p_1 + p_2)}{p^2}$$

and we recover the engineering dimension $d_n$ given in Eq. 11.2.2 with $n = 2$.

In general, for a free field the scaling properties of a Greens function is determined by its engineering dimension, whereas for a renormalizable quantum field theory, the scaling properties will be determined by the interactions.

---

Rescaling $p$ to $e^t p$ in Eq. 11.2.1 yields

$$G_n(e^t p, \mu_0, \lambda_R(0)) = e^{n\Gamma(t)} G_n(e^t p, e^t \mu_0, \lambda_R(t)) \tag{11.2.3}$$

For simplicity, suppose all the $p_i$'s are much larger the renormalized mass $m_R$, so that we can ignore all the masses and in effect, set $m_R = 0$.

Note $e^t \mu_0$ has the scale of momentum and behaves under scaling like a momentum variable. The scaling of all the momenta, from Eq. 11.2.2, yields

$$G_n(e^t p, e^t \mu_0, \lambda_R(t)) = e^{-td_n} G_n(p, \mu_0, \lambda_R(t)) \tag{11.2.4}$$

The coupling constant $\lambda_R(t)$ remains unchanged above since it is dimensionless and does not scale as a momentum.

Hence, from Eqs. 11.2.3 and 11.2.4, we obtain

$$G_n(e^t p, \mu_0, \lambda_R(0)) = e^{(-tn(d/2+1)+n\Gamma(t))} G_n(p, \mu_0, \lambda_R(t)) \qquad (11.2.5)$$

The engineering dimension of the Greens function $G_n$, namely $d_n$, has been modified due to the renormalization arising from the higher momentum degrees of freedom and the Greens function has an anomalous dimension given by $d_n - n\Gamma$. From Eq. 11.2.5, if follows that the high momentum $e^t p$ asymptotic behavior of $G_n$ is determined by its anomalous dimension and by value of $\lambda_R(t)$ at the high momentum.

## 11.3 The Beta function

From Eq. 11.1.3, the $\beta$-function for $\phi^4$ theory in $d = 4$ spacetime dimensions is given by

$$\beta = \frac{d\lambda_R}{d\ln\mu} = \frac{3\lambda_R^2}{16\pi^2}$$

Integrating the $\beta$ function yields $\lambda_R(\mu)$, which is given by

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 - \frac{3\lambda_R(\mu_0)}{16\pi^2}\ln(\mu/\mu_0)} \qquad (11.3.1)$$

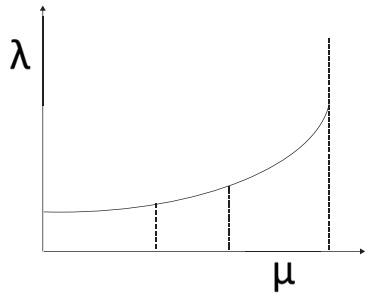


Figure 11.1 The running coupling constant $\lambda_R(\mu)$.

Figures 11.1 shows that $\lambda_R(\mu)$ increases as $\mu$ increases; the approximation breaks down for $\mu_m$ given by

$$1 - \frac{3\lambda_0}{16\pi^2}\ln(\mu_m/\mu_0) = 1 \quad \Rightarrow \quad \mu_m = e^{\frac{16\pi^2}{3\lambda_0}}\mu_0$$

The expression for the running coupling constant $\lambda_R(\mu)$ is valid only for $\mu < \mu_m$. The value of the coupling constant $\lambda_R(\mu)$ at $\mu$ in terms of the

theory at $\mu_0$ is free from any divergence. It is only when $\mu \to \infty$ that the ultra-violet divergences appear.

As expressed in the Callan-Symanzik equation, the value of $\mu$ is arbitrary. In particular, a change in $\mu_0$ can be compensated by a corresponding change in the value of $\lambda_R(\mu_0)$ and yields

$$\mu_0 \frac{\partial \lambda_R}{\partial \mu_0}\Big|_\mu = 0$$

Hence, there has to be a constraint on the dependence of $\lambda_R(\mu_0)$ on $\mu_0$ such that

$$\lambda_R(\mu) = f\{\alpha \ln(\mu/\mu_0) + h(\lambda_R(\mu_0))\} \tag{11.3.2}$$

For $\mu = \mu_0$

$$\lambda_R(\mu_0) = f(h(\lambda_R(\mu_0))) \quad \Rightarrow \quad f(h(x)) = x \quad \Rightarrow \quad f = h^{-1} \tag{11.3.3}$$

For the $\phi^4$ theory, from Eq. 11.5.1

$$\lambda_R(\mu) = \frac{1}{\lambda^{-1}(\mu_0) - \frac{3}{16\pi^2}\ln(\mu/\mu_0)} = f\Big\{h\big(\lambda(\mu_0)\big) - \frac{3}{16\pi^2}\ln(\mu/\mu_0)\Big\}$$

where

$$f(x) = \frac{1}{x} \;\; ; \;\; h(x) = \frac{1}{x}$$

Hence Eq. 11.3.3 is a non-perturbative constraint on the running coupling constraint $\lambda_R(\mu)$ and leads to improvements in perturbation theory.

Note the expression

$$\lambda(\mu) = \lambda(\mu_0)\{1 + \frac{3\lambda(\mu_0)}{16\pi^2}\ln(\mu/\mu_0)\} + O(\lambda_R^3)$$

does *not* have the form in Eq. 11.3.2. It is only the Equation 11.5.1, obtained by integrating the $\beta$- function, that has the correct functional form, as required by Equation 11.3.2.

Equation 11.3.2 yields the scale $s$ defined by

$$\lambda_R(\mu) = f(\ln(\mu/s))$$

with

$$s = \mu_0 e^{-h(\lambda_R(\mu_0))}$$

The appearance of the arbitrary scale $s$ breaks scale invariance; in other words, the necessity of renormalizing a quantum field theory leads to the breaking of scale invariance.

The fundamental reason that the renormalization group improves perturbation is because it replaces the bare coupling constant $\lambda_B$ by the renormalized running coupling constant $\lambda_R(\mu)$. As long as $\lambda_R(\mu) << 1$, the theory has an improved perturbation expansion due to the fact that the leading logarithms have all been summed up in the expression of $\lambda_R(\mu)$.

## 11.4 Physical mass and coupling constant

The derivation of the beta function $\beta$ and of $\gamma_m$ that determines how the renormalized coupling constant $\lambda_R$ and mass $m_R$ depend on the scale $\mu$, respectively, was based on the formal properties of the Callan-Symanzik equation; the same result is derived from a more intuitive point of view.

Recall from Eq. 10.5.4, the physically measured mass is given in terms of the renormalized coupling constant and scale parameter $\mu$ by the following

$$m_{\text{phy}} = m_R\{1 - \frac{\lambda_R}{32\pi^2}\ln(\frac{\mu}{m_R})\} + O(\lambda_R^2)$$

The choice of $m_R$ for setting the measure for $\mu$ is arbitrary, and we replace it with $E$ for greater generality and obtain

$$m_{\text{phy}} = m_R\{1 - \frac{\lambda_R}{32\pi^2}\ln(\frac{\mu}{E})\} + O(\lambda_R^2)$$

The $\beta$ function, as given in Eq. 11.1.3, is

$$\beta = \mu\frac{\partial\lambda_R}{\partial\mu} = \frac{3\lambda_R^2}{16\pi^2}$$

The physically measured mass and coupling constant are independent of the arbitrary scale $\mu$, which requires making both $\lambda_R$ and $m_R$ to be functions of $\mu$. The dependence of $\lambda_R(\mu)$ and $m_R(\mu)$ on $\mu$ is fixed so that $m_{\text{phy}}^2, \Gamma_4^R(s,u,t)$ are independent of the arbitrary scale $\mu$.

The renormalized vertex function is given by Eq. 10.6.7

$$\Gamma_4^R(s,u,t) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\Big\{\ln(\frac{s}{\tilde{c}\mu^2}) + \ln(\frac{u}{\tilde{c}\mu^2}) + \ln(\frac{t}{\tilde{c}\mu^2})\Big\} \qquad (11.4.1)$$

The renormalized $\Gamma_4^R$ is independent of $\mu$, as indeed it must be since the

scale $\mu$ is arbitrary; to see this note that

$$\mu\frac{d\Gamma_4^R(s,u,t)}{d\mu} = \mu\frac{\partial\lambda_R}{\partial\mu}\frac{\partial\Gamma_4^R(s,u,t)}{\partial\lambda_R} + \mu\frac{\partial\Gamma_4^R(s,u,t)}{\partial\mu}$$
$$= -\beta + \frac{\lambda_R^2}{32\pi^2}\cdot 2\cdot 3 + O(\lambda_R^3) = 0 + O(\lambda_R^3)$$

Similarly, for the physical mass, since $\beta = O(\lambda_R^2)$, we have

$$\mu\frac{dm_{\rm phy}}{d\mu} = \mu\frac{\partial m_R}{\partial\mu}\{1 - \frac{\lambda_R}{32\pi^2}\ln(\frac{\mu}{E})\} - m_R\frac{\lambda_R}{32\pi^2} + O(\lambda_R^2)$$

Since

$$\gamma_m = \frac{\mu}{m_R}\frac{\partial m_R}{\partial\mu} = \frac{\lambda_R}{32\pi^2}$$

we obtain the expected result that

$$\mu\frac{dm_{\rm phy}}{d\mu} = 0 + O(\lambda_R^2)$$

We could have obtained the results for $\beta$ and $\gamma_m$ by demanding that physical quantities like the observed mass and four point vertex function be independent of the scale $\mu$. We would then have arrived at the results we obtained using the Callan-Symanzik equation and given in Eq. 11.1.3.

## 11.5 Renormalization group: mass and coupling constant

The renormalization group relates the renormalized coupling constants and masses at different scales. A reflection of this is found in the various schemes for renormalizing a quantum field, and this aspect is studied using the one-loop results for $\varphi^4$ theory.

The result obtained for the $\beta$-function in Eq. 11.5.1 is valid to $O(\lambda_R^2)$; this yields

$$\lambda_R(\mu) = \lambda_R(\mu_0)\Big(1 + \frac{3\lambda_R(\mu_0)}{16\pi^2}\ln(\frac{\mu}{\mu_0})\Big) \quad ; \quad \mu > \mu_0 \tag{11.5.1}$$

As can be seen from Eq. 11.5.1, the description of the quantum field at the scale $\mu_0$ (given by $\lambda_R(\mu_0)$) in terms of the behavior of the quantum field at higher momentum scale $\mu$ (given by $\lambda_R(\mu)$) is completely finite since the ratio of these two scales given by $\mu/\mu_0$ is finite. The earlier result, obtained in Eq. 10.7.2, relating the renormalized coupling constants at two different momentum scales is equivalent to Eq. 11.5.1.

All the divergences of quantum field theory are due to describing the correlation functions at a finite scale $\mu_0$ in terms of the bare quantum field

that is defined at the cut-off scale $\Lambda$ – and which needs to be taken to infinity to restore full Lorentz invariance. This feature of the quantum field can also be seen in Eq. 11.5.1: if we set $\mu = \Lambda$ we recover the relation of the bare to the renormalized coupling constant given in Eq. 10.11.6; the transformation from $\lambda_R(\mu)$ to $\lambda_R(\mu_0)$ becomes singular at $\Lambda \to \infty$.

To illustrate the workings of the renormalization group, consider the renormalization procedure using the background field method and the scheme of minimal subtraction.

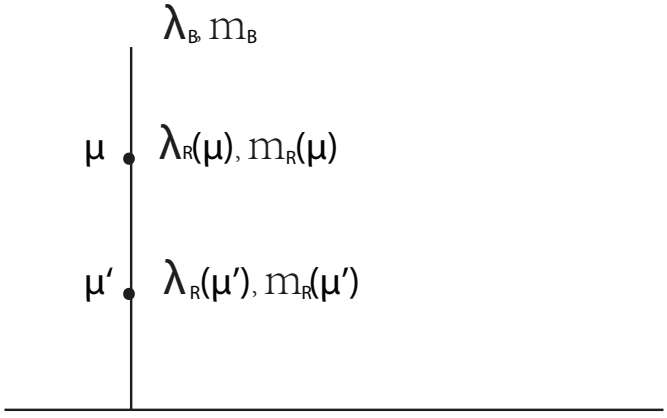


Figure 11.2 The vertex counter-term in $\mathcal{L}_{CT}$ is a new term in the Lagrangian

The renormalized coupling constant in the background field method, from Eq. 10.12.1, is given by

$$\lambda_R^{(1)}\mu^\epsilon = \lambda_B(1 - \frac{3\lambda_B\mu^{-\epsilon}}{16\pi^2\epsilon}) - \frac{3\lambda_B^2\mu^{-\epsilon}}{16\pi^2}\ln(\frac{b\mu}{m_B}) \quad ; \quad b = 2\sqrt{\pi}e^{\varphi(3)/2} \tag{11.5.2}$$

The scale at which $\lambda_R^{(1)}$ is not necessarily $\mu$ since there are many ways of defining the scale, as discussed in Section 10.2; for generality, let $\mu'$ be the scale for $\lambda_R^{(1)} = \lambda_R^{(1)}(\mu')$. From Eq. 10.6.4, the minimal subtraction scheme is defined at momentum scale $\mu$ and yields the following renormalized coupling constant

$$\lambda_R^{(2)}(\mu)\mu^\epsilon = \lambda_B(1 - \frac{3\lambda_B\mu^{-\epsilon}}{16\pi^2\epsilon}) \tag{11.5.3}$$

Subtracting Eq. 11.5.2 from Eq. 11.5.3 yields, using the notation

$$\lambda_R^{(2)}(\mu) = \lambda_R(\mu) \quad ; \quad \lambda_R^{(1)}(\mu') = \lambda_R(\mu')$$

yields the following

$$\lambda_R(\mu) - \lambda_R(\mu') = \frac{3\lambda_R^2(\mu')}{16\pi^2} \ln(\frac{b\mu}{m_B}) \tag{11.5.4}$$

Hence, we see from Eqs. 11.5.4 and 11.5.1, that

$$\mu' = m_R/b$$

The running coupling constants for the background field method and minimal subtraction scheme are defined at two different scales – and are related by the renormalization group transformation since they both are representations of the same underlying quantum field.

A similar analysis can be done for the two methods for obtaining the renormalized mass. Since

$$\gamma_m = \frac{\mu}{m_R}\frac{\partial m_R}{\partial \mu} = \frac{\partial \ln m_R}{\partial \ln \mu} = \frac{\lambda_R}{32\pi^2}$$

we have, using Eq. 11.5.1

$$\ln\left(\frac{m_R(\mu)}{m_R(\mu')}\right) = \frac{1}{32\pi^2}\int_{\mu'}^{\mu} d\ln\mu\ \lambda_R(\mu) = \frac{\lambda_R(\mu')}{32\pi^2}\ln(\frac{\mu}{\mu'}) \tag{11.5.5}$$

Hence, to lowest order

$$m_R^2(\mu) = m_R^2(\mu')\left(1 + \frac{\lambda_R(\mu')}{16\pi^2}\ln(\frac{\mu}{\mu'})\right) \tag{11.5.6}$$

Similar to the notation used for the renormalized coupling constant, let us denote the two masses by a scales denoted by $\mu'$ and $\mu$. The two renormalized masses are the following

$$m_R^2(\mu') : \text{ background field } \ ; \ \ m_R^2(\mu) : \text{ minimal subtraction}$$

The background field method has the following renormalized mass that is given by Eq. 10.12.2

$$m_R^2(\mu') = m_B^2\Big(1 - \frac{\lambda_R}{16\pi^2\epsilon}\Big) - m_B^2\frac{\lambda_R}{16\pi^2}\ln(\frac{\tilde{b}\mu}{m_B}) \ \ ; \ \ \tilde{b} = 2\sqrt{\pi}e^{-\varphi(2)/2} \tag{11.5.7}$$

The minimal subtraction scheme yields the renormalized given by Eq. 10.5.1

$$m_R^2(\mu) = m_B^2\Big(1 - \frac{\lambda_R}{16\pi^2\epsilon}\Big) + O(\lambda_B^2) \tag{11.5.8}$$

Hence, subtracting Eq. 11.5.7 from Eq. 11.5.8 yields, to lowest order

$$m_R^2(\mu) - m_R^2(\mu') = m_R^2\frac{\lambda_R}{16\pi^2}\ln(\frac{\tilde{b}\mu}{m_R})$$

Hence, from Eq. 11.5.6 and above, we obtain the following

$$\mu' = m_R/\tilde{b}$$

As was the case for the renormalized coupling constant, the renormalized mass obtained by the background field method is related to the one obtained using minimal subtraction by the renormalization group transformation.

## 11.6 The Wilson-Fisher fixed point

The $\beta$ function for $\phi^4$ theory in $d = 4 - \epsilon$ dimension, from Eq. 11.1.4, is given by

$$\beta = \mu\frac{\partial\lambda}{\partial\mu} = -\epsilon\lambda + a\lambda^2 \quad ; \quad a = \frac{3}{16\pi^2} \tag{11.6.1}$$

where $\lambda$ is the renormalized coupling constant. In critical phenomenon, the statistical phenomenon has $\epsilon = 1$ since the system being studied has $d = 3$, and sometimes the system has $d = 2, 1$ dimensions. Hence, the behavior for $\epsilon \neq 0$ is of physical interest.

The fixed point is given by $\lambda_*$ such that

$$\beta(\lambda_*) = 0 \quad \Rightarrow \quad \lambda_* = \epsilon/a$$

Let $t = \ln\mu$; then

$$\int_{\lambda_0}^{\lambda}\frac{d\lambda}{\lambda(\lambda-\lambda_*)} = a\int_{t_0}^{t}dt \quad \Rightarrow \quad \frac{1}{\lambda_*}\int_{\lambda_0}^{\lambda}d\lambda(\frac{1}{\lambda-\lambda_*} - \frac{1}{\lambda}) = a(t_0 - t)$$

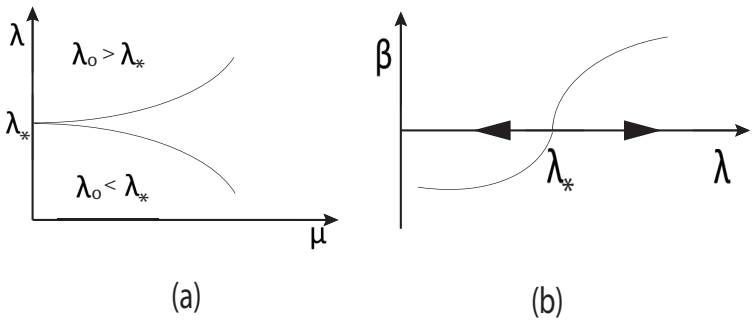


Figure 11.3 The running coupling constant and the Infra-red (IR) Wilson-Fisher Fixed point (IR) stable fixed points. (a) IR Running coupling constant. (b) IR Beta function.

which yields

$$\lambda(t) = \frac{\lambda_*}{1 - (1 - \frac{\lambda_*}{\lambda_0})e^{\epsilon(t-t_0)}} \quad ; \quad \lambda(t_0) = \lambda_0 \tag{11.6.2}$$

There are different solutions, shown in Figure 11.3, depending on the initial starting value of $\lambda_0$.

$$I.\ \ \lambda_0 > \lambda_* : 1 - \frac{\lambda_*}{\lambda_0} = c^2 > 0 \quad \Rightarrow \quad \lambda \to \infty \ \ as \ \ e^{\epsilon(t-t_0)} \to \frac{1}{c^2}$$

$$II.\ \ \lambda_0 < \lambda_* : \ 1 - \frac{\lambda_*}{\lambda_0} = -d^2 > 0 \quad \Rightarrow \quad \lambda = \frac{\lambda_*}{1 + d^2 e^{\epsilon(t-t_0)}} \to 0$$

$$III.\ \ \lambda_0 = \lambda_* \ \text{ for all } \ t$$

Since all the running coupling constants flow *away* from $\lambda_*$, the fixed point $\lambda_*$ is infra-red stable. For the $\phi^4$ theory for $\epsilon \neq 0$, the the running coupling constant and beta function for IR- fixed point is given in Figure 11.3 (a) and (b).

### *11.6.1 Ultra-violet fixed point*

Consider the beta function given by The $\beta$ function in $d = 4 - \epsilon$ dimension is given by

$$\beta = \mu \frac{\partial g}{\partial \mu} = \epsilon g - a g^2 \tag{11.6.3}$$

and $g$ is the renormalized coupling constant. A beta function similar to this case occurs for the Yang-Mills gauge fields of Quantum Chromodynamics, with the difference being that the beta function has the following behavior

$$\beta = \mu \frac{\partial g}{\partial \mu} = \epsilon g - a g^3 \tag{11.6.4}$$

with $a = 11 N_c/(48\pi^2)$ for $SU(\mathcal{N}_c)$, from Peskin and Schroeder (1995). We analyze the beta function given in Eq. 11.6.3 since it is simpler and has the same qualitative behavior as Yang-Mills gauge fields.

Note that Eq. 11.6.3 is the same as Eq. 11.6.1, except that the signs of both $\epsilon$ and $a$ are reversed. Hence, for both these beta functions, the fixed point is given by $g^*$

$$\beta(g_*) = 0 \quad \Rightarrow \quad g_* = \epsilon/a$$

The solution of the running coupling is the same as Eq. 11.6.2 except for the

crucial difference that the sign of $\epsilon$ in Eq. 11.6.2. Hence, for $t = \ln(\mu/\mu_0)$, we have

$$g(t) = \frac{g_*}{1 - (1 - \frac{g_*}{g_0})e^{-\epsilon(t-t_0)}}$$

All initial couplings flow to $g_*$ as $t \to \infty$, as shown in Figure 11.4(a). This is quite the opposite of the Wilson-Fisher fixed point for which all couplings flow to infinity as one increases the momentum scale. In fact, as discussed earlier, once the coupling constant becomes large, one cannot use the results of the one-loop beta function as perturbation theory is no longer valid.

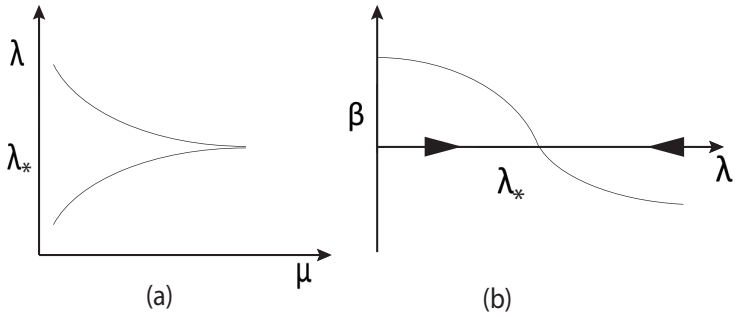


Figure 11.4 Ultra-violet (UV) stable fixed points. (a) UV Running coupling constant. (b) UV Beta function.

For case of $\epsilon = 0$ we have

$$\begin{aligned} &\beta = \mu\frac{\partial g}{\partial \mu} = -ag^2 \\ &\Rightarrow \quad g(t) = \frac{1}{\frac{1}{g_0} + a(t - t_0)} = \frac{1}{\frac{1}{g_0} + a\ln(\mu/\mu_0)} \end{aligned} \tag{11.6.5}$$

Note that Eq. 11.6.5 has the general form of a running coupling constant as given in Eq. 11.3.2.

Eq. 11.6.5 is the famous result of asymptotic freedom that shows that, since the sign yields $a > 0$, the coupling becomes weaker as the scale of momentum $\mu$ is increased. An asymptotically free theory is the only quantum field theory that can be defined all the way to infinite momentum since the weak coupling behavior allows one to analyze the infinite momentum properties of the theory. The UV free theory is also called asymptotically free, with the most famous free asymptotically free theory being the Yang-Mills gauge field.

### 11.7 Beta function and fixed-points for $d = 4$

In general, the fixed points of the beta function are defined by

$$\beta(\lambda_*) = 0$$

The flow of the running coupling constants for various fixed points are determined by the sign of

$$\beta' \equiv \frac{\partial \beta(\lambda_*)}{\partial \lambda}$$

The running coupling $\lambda_R(\mu)$ defines the effective coupling at momentum $\mu$, where $\mu \in [0, \infty]$. There are various behavior for $\lambda_R(\mu)$ that correspond to different $\beta$- function; are shown in Figures below.

The theory in Figure 11.4 (a) and (b) shows the behavior of an UV(ultra-violet) stable fixed point, with all initial coupling flowing towards the value of $\lambda_*$. The $\beta$- function is given by Figure Figure 11.4 (b). The arrows on the $\lambda$ axis indicate the flow of $\lambda_R(\mu)$ under the renormalization group and is the result of

$$\beta(\lambda_*) = 0 \;\; ; \;\; \beta'(\lambda_*) < 0 \quad : \text{UV stable fixed point}$$

For the beta-function given in Eq. 11.6.3

$$\beta'(\lambda_*) = -\epsilon < 0$$

and theory has an ultra-violet stable fixed point.

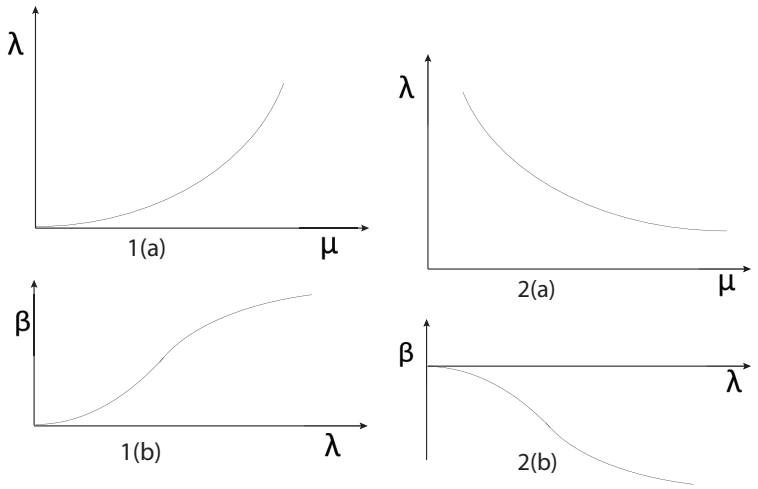


Figure 11.5 Beta functions for $\epsilon = 0$ that yields $d = 4$. Figures 1(a) and 1(b) show the behavior of an infra-red stable theory and Figures 2(a) and 2(b) show the behavior of an asymptotically free theory.

An IR(infra-red) stable theory is one for which the all initial values of the running coupling constant flow away from the fixed point value $\lambda_*$, as shown in Figure 11.4 (a) and (b).

$$\beta(\lambda_*) = 0, \quad ; \quad \beta'(\lambda_*) > 0 \quad : \text{IR stable fixed point}$$

The Wilson-Fisher fixed point yields

$$\beta'(\lambda_*) = \epsilon > 0$$

and theory has an infra-red stable fixed point.

The beta-function for the infra-red and ultra-violet stable fixed points, for $\epsilon = 0$, are given in 1 and 2 in Figures 11.5 respectively. An IR-free fixed point $\beta(\lambda) \simeq \lambda^2 > 0$ is a theory, given by 1(a) and 1(b) in Figure 11.5, which have a finite value at large $\mu$ and, as $\mu \to 0$, the running coupling constant flows to the value of zero at $\mu = 0$. In contrast, for an UV-free fixed point, also called an asymptotically free theory and given by 2(a) and 2(b) in 11.5, for $\mu \simeq \infty$ , $\beta(\lambda) \simeq -\lambda^2 < 0$, and the running coupling constant increases indefinitely as $\mu \to 0$.

## 11.8 Fixed point and scaling

Recall from Eq. 11.2.5, under the scaling of the momentum, the Greens function has the following behavior

$$G_n(e^t p, \mu_0, \lambda_R(0)) = e^{-tn(d/2+1)+n\Gamma(t)} G_n(p, \mu_0, \lambda_R(t))$$

where $\mu = e^t \mu_0$.

Consider the two point function; Eq. 11.2.5 yields

$$G_2(e^t p_1, e^t p_2, \mu_0, \lambda_R(0)) = e^{-td-2t+2\Gamma(t)} G_2(p_1, p_2, \mu_0, \lambda_R(t)) \quad (11.8.1)$$

The two point function is given by

$$G_2(p_1, p_2) = \delta(p_1 + p_2) G_2(p_1) \quad (11.8.2)$$

Hence, from Eq. 11.8.1 and Eq. 11.8.2

$$G_2(e^t p, \mu_0, \lambda_R(0)) = e^{-2t+2\Gamma(t)} G_2(p, \mu_0, \lambda_R(t)) \quad (11.8.3)$$

As it stands, Eq. 11.8.3 cannot be explicitly solved due to the presence of $\lambda_R(t)$ on the right hand side. However, for large $t$ the existence of an UV fixed point allows for a solution.

The high momentum behavior of the theory is determined by the limit

of $t \to \infty$. Suppose the theory has an UV fixed point; the running coupling constant will converge to $\lambda*$, its value at the fixed point, and hence

$$t \to \infty \quad \Rightarrow \quad \lambda_R(t) \to \lambda_*$$

Let value of $\gamma_\phi$ at the fixed point be denoted by $\gamma_{\phi*} = \gamma_\phi(\lambda_*)$. We then have

$$\Gamma(t) = \frac{1}{2}\int_0^t dt' \gamma(\lambda_R(t')) \approx \frac{1}{2} t \gamma_\phi(\lambda_*) = \frac{1}{2} t \gamma_{\phi*}$$

One can obtain the scaling properties of the Greens function directly from the Callan-Symanzik equation. From Eq. 11.8.3

$$\lim_{t\to\infty} G_2(e^t p, \mu_0, \lambda_R(0)) = e^{-2t + t\gamma_{\phi*}} G_2(p, \mu_0, \lambda_*) \tag{11.8.4}$$

From Eq. 11.8.4, we see that the correlation is a pure scaling function of the momentum and hence we obtain

$$G_2(p) = \frac{C(\lambda_*)}{p^2} (\frac{p^2}{\mu_0^2})^{\gamma_{\phi*}} \tag{11.8.5}$$

where $C(\gamma_{\phi*})$ is a constant. The result that we have obtained is based on the explicit solution of Callan-Symanzik equation. The expression $(\frac{p^2}{\mu_0^2})^{\gamma_{\phi*}}$ is given by

$$(\frac{p^2}{\mu_0^2})^{\gamma_{\phi*}} = 1 + \gamma_{\phi*} \ln(\frac{p^2}{\mu_0^2}) + \frac{1}{2}(\gamma_{\phi*})^2 \ln^2(\frac{p^2}{\mu_0^2}) + .... \tag{11.8.6}$$

Each logarithm term is given by a Feynman diagram and the Callan-Symanzik equation sums up an infinite set of Feynman diagrams to yields the scaling result.

For the Wilson-Fisher fixed point, we have

$$\lambda_* = \epsilon \cdot \frac{16\pi^2}{3} \quad \Rightarrow \quad \gamma_{\phi*} = \gamma_\phi(\lambda_*) = \frac{1}{12}\frac{\lambda_*^2}{(4\pi)^4} = \frac{\epsilon^2}{108}$$

The parameter $\gamma_{\phi*}$ is a physical quantity that can be measured. In phase transitions $\gamma_{\phi*}$ yields critical exponents for the correlation function and in high energy experiments it determines the scaling properties of scattering cross-sections with large momentum transfer.

## 11.9 Infra-red fixed points and Phase transitions

In the Landau-Ginzburg-Wilson approach to phase transitions, the long distance properties of a critical system are modeled by degrees of freedom defined on a continuous space. The short distance structure of the system,

for example the lattice spacing of a solid, is unimportant for its critical properties. The long distance properties of the system have many universal properties – for example exact spherical symmetry that is absent for the lattice. The long range order of the system is realized by the system having an infinite correlation length $\xi$.

The order parameter of the phase transitions is encoded in the field $\phi$; the statistical fluctuations of the classical statistical mechanical system are mathematically equivalent to considering $\phi$ to be a quantum field. The Landau-Ginzburg-Wilson Lagrangian, similar to the discussion in Section 3.5, is given by the nonlinear Lagrangian defined in $d$-dimensional Euclidean space

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_B \partial^\mu \phi_B - \frac{1}{2} m_B^2 \phi_B^2 - \frac{1}{4!}\lambda_B \phi_B^4$$

The parameter $m_B^2$ in the Lagrangian yields a well defined and convergent quantum field theory for both positive and negative values.

Consider a system undergoing a second order phase transition at temperature $T_c$; in the Landau-Ginzburg-Wilson phenomenological approach, $m_B^2$ has the following dependence on temperature $T$

$$m_B^2 = \mu_0^2 (\frac{T}{T_c} - 1) \tag{11.9.1}$$

where $\mu_0$ carries the dimension of momentum.

The 'mass' of the Landau-Ginzburg-Wilson Lagrangian changes its sign as the system goes from the disordered to ordered phase since

$$\begin{array}{rcl} m_B^2 & > 0 & : \quad T > T_c \\ m_B^2 & < 0 & : \quad T < T_c \end{array} \tag{11.9.2}$$

For $T > T_c$, the system has $E[\phi] = 0$ and is said to be in the disordered phase. For $T < T_c$, the system has $E[\phi] > 0$ and undergoes a phase transition, with the potential developing a double-well, as shown in Figure 12.5; the system spontaneously chooses one of the minima as its ground state, and in doing so breaks the symmetry of $\phi \to -\phi$.

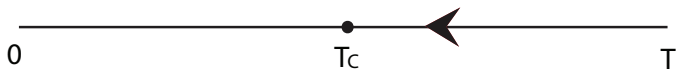


Figure 11.6 Approaching the critical temperature $T_c$ from the high temperature disordered phase, with $T > T_c$.

We examine the system as it approaches $T_c$ from the disordered phase,

namely for temperatures $T > T_c$, as shown in Figure 11.6. The bare Lagrangian describes the short distance behavior of the system; the large distance behavior of the system is given by renormalizing the system and examining its behavior for momentum scale $\mu \to 0$.

As shown by Wilson, at the phase transition, namely for $T = T_c$, the system is scale invariant and is described by the quantum field being at the *infra-red fixed point*. The flow of the coupling constant as one renormalizes the system to decreasing valuef of $\mu$ is shown in Figure 11.7. Hence, to study the critical properties of the system, we study the long distance behavior of the system near the Wilson-Fisher infra-red fixed point. In particular critical indices, which describe the universal characteristics of a second order phase transition, are determined by the infra-red fixed point.

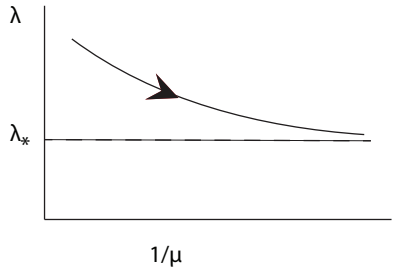


Figure 11.7 The Wilson-Fisher fixed point for phase transitions.

Thermodynamic systems have $d = 3$ and hence $\epsilon = 1$; although one considers the limit of $\epsilon \to 0$ for studying the renormalization of quantum fields, in the study of phase transitions, the system has a well defined expansion for $\epsilon = 1$.

### *11.9.1 Critical exponents*

The correlation function of the system near criticality ($T \to T_c$) is

$$G(x) = E[\phi(0)\phi(x)] = \frac{1}{|x|^{d-2+\eta}} \exp\{-|x|/\xi\} \tag{11.9.3}$$

where the correlation length near criticality is given by

$$\xi(T) = \frac{\xi_0}{(T/T_c - 1)^\nu} \to \infty \quad \text{as} \quad T \to T_c \tag{11.9.4}$$

The *critical exponents* $\eta, \nu$ are dimensionless numbers that are experimentaill mesured. The same crtical exponents describe phase transitions for a large class of critical systems, and is a reflection of *Universality* of phase

transition, which states that vastly different systems having the same symmetries and dimensionality have the same critical exponents. The universality arises due to the fact that the fixed point Lagrangian is independent of many of the details of the starting bare Lagrangian.

From Eq. 11.8.5, the correlation function at the fixed point is given by

$$G_2(p) = \frac{C(\lambda_*)}{p^2}(\frac{p^2}{\mu_0^2})^{\gamma_{\phi*}}$$

where $C(\gamma_{\phi*})$ is a constant. For the Wilson-Fisher infra-red fixed point, the exponent $\gamma_{\phi*}$ is given by Eq. 11.8.6

$$\gamma_{\phi*} = \frac{\epsilon^2}{108}$$

Fourier transforming $G_2(p)$ yields ($K$ is a constant)

$$G_2(x) = \int_{-\infty}^{+\infty} \frac{d^d p}{(2\pi)^d} G_2(p) = K \frac{1}{|x|^{d-2+2\gamma_{\phi*}}}$$

and hence the critical exponent $\eta$ is given by

$$\eta = 2\gamma_{\phi*} = \frac{\epsilon^2}{54} + O(\epsilon^3)$$

The experimental value of $\eta$ for a wide range of phase transitions is in the range of 0 - 0.1 and hence the crude estimate that we have obtained of $\eta \approx 0.02$ is consistent with experiments. For the $O(N)$ symmetric $\phi^4$ theory discussed in Section 10.8, it can be shown that

$$\eta = \frac{N+2}{2(N+8)^2}\epsilon^2 + O(\epsilon^3)$$

The critical exponent $\nu$ is obtained by studying mass renormalization. The anomalous mass dimension, from Eq. 11.1.2 is given by

$$\gamma_m = \frac{1}{m_R}\mu\frac{\partial m_R}{\partial \mu} \tag{11.9.5}$$

At the fixed point, $\gamma_m = \gamma_{m*}$ and is independent of $\mu$. Hence, integrating Eq. 11.9.5 at the fixed point yields

$$m_R^2(\mu_0) = \left(\frac{\mu_0}{\mu}\right)^{2\gamma_{m*}} m_R^2(\mu) \quad ; \quad \mu_0 > \mu$$

For the Wilson-Fisher fixed point, we have

$$\gamma_{m*} = \frac{1}{32\pi^2}\lambda_* = \frac{\epsilon}{6} + O(\epsilon^2) \quad ; \quad \lambda_* = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2)$$

The bare theory is given by the value of $\mu_0 >> 1$; for statistical mechanical systems, the large momentum scale $\mu_0$ can be taken to be equal to the inverse of the lattice spacing of a solid, which is about $10^{-9}$m. Holding $\mu_0$ fixed and large, we need to decrease $\mu$ so that we find the description of the large distance behavior of the system in terms of the bare Lagrangian, since it is the bare Lagrangian that is driven to criticality as $T \to T_c$.

The physically relevant scale for a system going critical is its correlation length, and hence we take

$$\mu_0 = \frac{1}{\xi_0} \quad ; \quad \mu = \frac{1}{\xi} \tag{11.9.6}$$

Since mass has dimension of $\mu$, we define the dimensionless ratio

$$\frac{m_R^2(\mu_0)}{\mu_0^2} = \left(\frac{\mu_0}{\mu}\right)^{2\gamma_{m*}-2} \frac{m_R^2(\mu)}{\mu^2} \tag{11.9.7}$$

We choose the scale $\mu$ such that it equals the renormalized mass and hence

$$m_R^2(\mu) = \mu^2$$

From Eq. 11.9.1

$$\frac{m_R^2(\mu_0)}{\mu_0^2} = \frac{m_B^2}{\mu_0^2} = \frac{T}{T_c} - 1$$

Hence Eqs. 11.9.6 and 11.9.7 yield

$$\frac{T}{T_c} - 1 = \left(\frac{\xi}{\xi_0}\right)^{2\gamma_{m*}-2} \quad \Rightarrow \quad \xi = \left(\frac{T_c}{T - T_c}\right)^{\nu} \xi_0$$

and we obtain the critical exponent

$$\nu = \frac{1}{2 - 2\gamma_{m*}}$$

For the Wilson-Fisher fixed point the exponent is given by

$$\nu = \frac{1}{2 - 2\gamma_{m*}} = \frac{3}{6 - \epsilon} = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)$$

which is significantly different from the free field result of $\nu = 1/2$. The experimental value of $\nu$ for $d = 3$ – three dimensional phase transitions – is in the range of 0.6 - 0.7 and hence the crude estimate that we have obtained of $\nu \approx 0.6$ is surprisingly accurate. From the value of $\gamma_m$ and $\beta$ obtained for the $O(N)$ symmetric $\phi^4$ theory in Section 10.8, it can be shown that

$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)}\epsilon + O(\epsilon^2)$$

## 11.10 Wilson Renormalization Group

Consider an arbitrary action $S[\phi]$ of a scalar field $\phi$. Let the theory have an ultra-violet momentum cut-off $\Lambda$, and the action is denoted by $S[\phi;\Lambda] = S_0$. The field $\phi(x)$ has the following expansion in terms of $\phi_p$ , the momentum degrees of freedom

$$\phi(x) = \int_{|p|<\Lambda} \frac{d^d p}{(2\pi)^d} \phi_p \equiv \int_{p,\Lambda} \phi_p \;\; ; \;\; |p| < \Lambda$$

The quantum field theory is defined by the path integral

$$Z = \int D\phi e^{S_0} = \prod_{p,0<|p|<\Lambda} \int d\phi_p e^{S_0}$$

Choose a new momentum $\Lambda' = e^{-t}\Lambda < \Lambda$; the fast and slow degrees of freedom are defined in the following manner

$$\phi(x) = \int_{p,\Lambda} \phi_p = \int_{p,|p|<\Lambda'} \phi^s_p + \int_{p,\Lambda'<|p|<\Lambda} \phi^f_p = \phi_f + \phi_s$$

The fast degrees of freedom form a shell in momentum space at the cut-off, and the slow variables are the rest of the degrees of freedom.

The renormalized action, having a maximum momentum $\Lambda'$, is defined by the following Wilson renormalization group transformation

$$Z = \int D\phi_s D\phi_f e^{S_0[\phi_s+\phi_f]} = \prod_{p,0<|p|<\Lambda'} \int d\phi_p e^{S'[\phi_s]}$$

where

$$e^{S'[\phi_s]} = \int D\phi_f e^{S_0[\phi_s+\phi_f]} = \prod_{p,\Lambda'<|p|<\Lambda} \int d\phi_p e^{S_0[\phi_s+\phi_f]}$$

The action $S'[\phi_s]$ can be computed using Feynman diagrams, or numerically, depending on the nature of the problem. Note the computation of $S'[\phi_s]$ is similar to the calculation of the background field action discussed in Section 10.12; the nonlinear terms in the action $S_0$ lead to the renormalization of the coupling constants and also yield new terms in the action $S'[\phi_s]$ not present in the starting action $S_0$.

A rescaling of the momentum of the slow degrees of freedom $\phi_s(p)$ by defining $q = e^t p$ that is required to restore the range of momentum of $\phi_s$ back to $\Lambda$. A *second* rescaling is that of the slow degrees of freedom $\phi_s$ and is necessary to restore the scale of fluctuation of the quantum field $\phi_s$,

since otherwise one will not be able to obtain a nontrivial result. The two rescalings are done in the following manner

$$\phi^s_p = \zeta \hat{\phi}_{e^t p} = \zeta \hat{\phi}_q \;\; ; \;\; q = e^t p \;\; ; \;\; 0 < |q| < \Lambda \;\; ; \;\; 0 < |p| < e^{-t}\Lambda$$

The renormalized action $S_t[\hat{\phi}]$ at scale $\mu = e^{-t}\Lambda$ is given by

$$S_t[\hat{\phi}] = S'[\zeta \phi_s]$$

The field renormalization $\zeta$ can be fixed similar to perturbative renormalization by requiring the kinetic term for $\hat{\phi}$ in $S_t[\hat{\phi}]$ have the form

$$-\frac{1}{2}\int_{q,|q|<\Lambda} q^2 |\hat{\phi}_q|^2$$

The partition function, upto a constant, is given by

$$Z = \int D\hat{\phi} e^{S_t[\hat{\phi}]} = \prod_{p,0<|p|<\Lambda} \int d\hat{\phi}_p e^{S_t[\hat{\phi}]}$$

The actions $S_t$ and $S_0$ describe the *same* system; the only difference is that the bare action $S_0$ can be used for computing correlation functions with momentum upto $\Lambda$ whereas the action $S_t$ can be used for computing correlation functions up to a maximum momentum of $e^{-t}\Lambda$. In other words, by integrating out the high momentum degrees of freedom we have not lost any information: the renormalized theory can be used for obtaining the correlation functions and other properties of the original bare theory with a maximum momentum of $|p| < e^{-t}\Lambda$ using the action $S_t = S_t[\hat{\phi}]$.

An *infinitesimal* change in the renormalized action is obtained by setting $t' = t + \epsilon$ and integrating out the degrees of freedom in the momentum shell $[e^{-t}\Lambda, e^{-(t+\epsilon)}\Lambda]$. This yields the renormalized action $S_{t+\epsilon}$ from $S_t$. We hence obtain the *renormalization group transformation* $\mathcal{R}$ given by

$$\frac{\partial S_t}{\partial t} = \mathcal{R}(S_t) \quad : \text{Renormalization Group transformation}$$

The Callan-Zymanzik equation determines the change of the renormalized correlation functions as one changes the momentum scale $\mu = e^{-t}\Lambda$ at which the renormalized theory is defined. The renormalization group transformation $\partial S_t/\partial t$ is the generalization of the Callan-Zymanzik equation and is defined directly in terms of the action; the change in the action as one changes $\mu$ can, in general, change the functional form of the action.

Since there is a change of scale in going from $S_0$ to $S_t$, any physical quantity computed by the action $S_t$ must be rescaled. In particular, let the correlation at $t = 0$ be denoted by $\xi_0$ and let the correlation length computed

using action $S_t$ be denoted by $\xi_t$. The momentum scale has been changed by the relation $p = e^{-t}q$ in going from $S_0$ to $S_t$. Hence, distances for $S_t$ must scale as the inverse of momentum and we have the relation

$$\xi_0 = e^t \xi_t$$

## 11.11 Fixed points and renormalized trajectory

The fixed points of the renormalization group play a central role in defining a renormalized quantum field as well as in the theory of phase transitions. The role and existence of a fixed point action has a transparent and intuitive formulation is Wilson's approach to renormalization.

Consider the space $\mathcal{A}$ of all possible actions $S$; one can represent any action by an infinite dimensional functional Taylor's expansion in terms of the derivatives of the fields. The coefficient functions of this expansion can be considered as the 'co-ordinates' of the space of all possible actions. Hence, action space is infinite dimensional. The *renormalization trajectory* refers to the trajectory in action space $\mathcal{A}$ followed by $S_t$ for $t \in [0, \infty]$ .

The fixed point action $S_*$ is invariant under the renormalization group transformation $\mathcal{R}$ and is given by

$$\mathcal{R}(S_*) = 0 \quad : \text{Fixed point action}$$

Note the remarkable fact that the fixed point action $S_*$ is entirely determined by the renormalization group transformation $\mathcal{R}$ and is independent of the cut-off action $S[\Lambda]$ that one may start from. This is the explanation of the universality of phase transition discussed earlier in the context of the critical indices that describe a phase transition.

The fixed point action explains why a phase transition is scale invariant. Since $\mathcal{R}(S_*) = 0$, we have

$$\frac{\partial S_t}{\partial t} = 0 \quad \Rightarrow \quad S_t = S_* \quad : \text{Scale invariant}$$

The renormalization group transformation changes the scale of the action and $S_t$ describes the behavior of the Lagrangian at the scale $e^{-t}\Lambda$. Since the action is *invariant* under a scale transformation the fixed point action describes a phenomenon that is s*cale invariant.*

For the fixed point action $S_*$, the correlation function is given by

$$\xi_* = e^t \xi_*$$

Hence, at the fixed point, the action's correlation function is either 0 or $\infty$

$$\xi_* = 0, \infty$$

Ignoring the trivial case of $\xi_* = 0$ that corresponds to a system for which all the degrees of freedom are decoupled, we find that the fixed point action $S_*$ describes a critical system with infinite correlation

$$\xi_* = \infty \quad : \quad \text{critical system}$$


Figure 11.8 (a) The renormalized trajectory flows away from the UV fixed point action. (b) The renormalized trajectory flows into the IR fixed point action.

The renormalization trajectory flows *away* from the UV (ultra-violet) fixed point at one integrates out the high momentum degrees of freedom and in doing so one increases $t$; the renormalization trajectory is shown in Figure 11.8(a). On the other hand, the renormalization trajectory flows *into* the IR (infra-red) fixed as one probes the large distance behavior of and is shown in Figure 11.8(b).

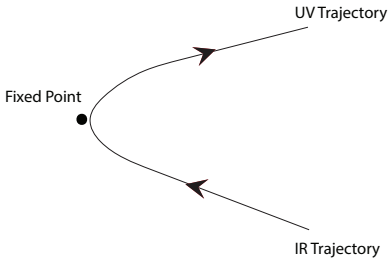


Figure 11.9 The infra-red renormalized trajectory approaches a fixed point action and the ultra-violet trajectory recedes from it.

As one can see from the description of the renormalization flows and fixed points, a fixed point is neither intrinsically UV or IR: rather, its behavior is determined by the renormalization trajectory. As shown in Figure 11.9, when the renormalization trajectory flows towards a fixed point, the fixed point appears to be an IR fixed point, and the same fixed point appears at an UV fixed point when the renormalization trajectory flows away from it.

The Wilson formulation of renormalization allows us to isolate the specific degree of freedom that one is interested in, and then go on to calculate the effect of all the other degrees of freedom on the case in question. In the case of high energy physics, one is interested in the behavior of the system at high energies and one can then study the degrees of freedom near the UV fixed point. In contrast, for critical systems, it is the long distance degrees of freedom that are of importance and one is led to study the system near the IR fixed point.

Noteworthy 11.2: Quantum fields, phase transitions and universality

The fixed points of the renormalization group describe the renormalizability of quantum fields as well as provide a description and explanation of second order phase transitions. This at first sight appears rather unexpected and counter-intuitive since the renormalizability of quantum fields arises from its short-distance properties. In contrast, second order phase transitions are characterized by the large distance behavior of the thermodynamic system, with the correlation length at the phase transition point becoming infinite.

These two results are a consequence of the properties and interpretation of the fixed points. In case of quantum fields, the fixed points describe the ultra-violet properties of the system. The existence of the UV fixed point is necessary for the theory to have a well defined behavior as one indefinitely increases the momentum scale of at which the quantum field is being probed.

In contrast, for thermodynamic systems undergoing phase transitions, their short distance properties are unimportant and it is the *long distance properties* of the system that are described by infra-red stable fixed points. As one probes the system at large and larger distances, the renormalized theory approaches the infra-red fixed point and exhibits the required infinite correlation length.

Critical systems exhibit universality; this remarkable result is due to the fact that many different systems can flow into the same fixed point when they are critical. This example exhibits the key features that differentiates a universal from the particular. A particular system has its own specific characteristics for all length scales; when it goes critical, only its infinitely long distance behavior is relevant, and it shares this long distance behavior with a large class of systems.

# 12

# The Effective Action

## 12.1 Introduction

The classical action is the foundation of classical mechanics and the minimum in the variation of the classical action determines the classical field equation. The classical action also determines the equilibrium configuration for the classical field, which is fixed by the minimum value of the classical action. For constant solutions of the field equations, the classical action is equal, up to a volume factor, to the potential energy of the field.

Hence, in classical physics, the symmetry breaking for a system is determined by the potential of the classical field. In Sections 3.5 and 3.6, the symmetry breaking for classical fields were studied for nonlinear fields. In Section 3.7, the Higgs mechanism, which is the reflection of the interplay of symmetry breaking with gauge symmetry, was discussed.

All the discussions in Chapter 2 on symmetry breaking were classical and one needs to answer the question: how does one study the breaking of symmetries for a quantum field? Is there a generalization of the classical action that incorporates all the quantum effects and yields a criterion, similar to the minimization of the potential energy for the classical field, for symmetry breaking for quantum fields. The answer is given by the *effective action* $\Gamma$, and which in turn yields the *effective potential*.

The effective action $\Gamma$ is the generating functional of all the one particle irreducible vertices (defined below) similar to $W[J]$ being the generator functional of all the connected correlation functions. To evaluate $\Gamma$ one needs to renormalize the theory and express $\Gamma$ in terms of the renormalized mass and coupling constant.

The effective potential provides the criterion for symmetry breaking for a quantum field. For example, consider a theory with its Lagrangian being symmetric under the transformation $\phi \to -\phi$. Suppose the minimum of

the effective potential is given by $\bar{\phi}$; it will be shown to be the vacuum expectation of the quantum field $\phi$, namely

$$\bar{\phi} = \langle \Omega | \phi | \Omega \rangle = E[\phi]$$

If $\bar{\phi} \neq 0$, then the symmetry is said to be spontaneously broken.

## 12.2 The effective action Γ

The generator functional $W[J]$ of all the connected correlation functions of a quantum field is given in Eq. 9.5.1 as follows

$$Z[J] = e^{W[J]} = \int D\phi e^{S + \int J\phi}$$

where $J = J(x)$ is a classical external current.

Define $\phi_c = \phi_c(x)$ to be the expectation value of the quantum field in presence of the external source $J(x)$

$$\phi_c(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z} \int D\phi e^{S + \int j\phi} \phi(x) \tag{12.2.1}$$

The effective action is introduced to replace the role of $J(x)$ by the vacuum expectation value $\phi_c(x)$. Define the **effective action** by the following

$$\Gamma[\phi_c] = W[J] - \int d^d x J(x) \phi_c(x) \tag{12.2.2}$$

Note by its very definition, the effective action $\Gamma[\phi_c]$ is *independent* of $J(x)$, since Eq. 12.2.1 follows from Eq. 12.2.2 only if $\delta\Gamma/\delta J = 0$. What this means is that we need to consider $J$ to be a function of $\phi_c$ in the expression for $\Gamma$ and hence $J$ is not an independent source term.

For $W[J]$ the external source $J(x)$ is an independent classical source and $\phi_c(x)$ is a function of $J(x)$. In contrast, in $\Gamma[\phi_c]$, the independent classical source is $\phi_c(x)$. In fact Eq. 12.2.2 is a Legendre transformation, taking a functional $W[J]$ of $J$ to $\Gamma[\phi_c]$, which is a functional of $\phi_c$.

As expected of a Legendre transformation one has from Eq. 12.2.2, we have

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = \int d^d y \frac{\delta W[J]}{\delta J(y)} \frac{\delta J[y]}{\delta \phi_c(x)} - \int d^d y \frac{\delta J[y]}{\delta \phi_c(x)} \phi_c(y) - J(x)$$

or, from Eq. 12.2.1

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = -J(x) \tag{12.2.3}$$

Eqs. 12.2.1 and 12.2.3 are dual to each other,

To obtain $\Gamma[\phi_c]$ one first solves Eq. 12.2.1 to obtain $J = J(\phi_c, x)$, considered as a functional of $\phi_c$. One then replaces $J$ by $J(\phi_c, x)$ in Eq. 12.2.2 to obtain $\Gamma[\phi_c]$.

## 12.3 Classical action and $\Gamma$

To concretely illustrate the transformation from $W[J]$ to $\Gamma[\phi_c]$ consider the free scalar field

$$S = -\frac{1}{2}\int d^dx((\vec{\nabla}\phi)^2 + m^2\phi^2)$$

where $W[J]$ is given by

$$W[J] = \ln(\int D\phi e^{S+\int J\phi}/Z) = \frac{1}{2}\int d^dx d^dy J(x)D(x-y)J(y)$$

The propagator is given by

$$D(x) = \int \frac{d^dp}{2\pi)^d}\frac{e^{ipx}}{p^2+m^2}$$

From Eq. 12.2.1, the classical function $\phi_c(x)$ is given by

$$\phi_c(x) = \frac{\delta W[J]}{\delta J(x)} = \int d^dy D(x-y)J(y)$$

Inverting above equation yields

$$J(x) = \int d^dy D^{-1}(x-y)\phi_c(y)$$

and for the free field, the effective action $\Gamma[\phi_c]$ is given by

$$\begin{aligned}\Gamma[\phi_c] &= \frac{1}{2}\int_{x,y} J(x)D(x-y)J(y) - \int J(x)\phi_c(x) \\ &= \frac{1}{2}\int \phi_c D^{-1}DD^{-1}\phi_c - \int \phi_c D^{-1}\phi_c = -\frac{1}{2}\int \phi_c D^{-1}\phi_c\end{aligned}$$

Hence

$$\Gamma[\phi_c] = -\frac{1}{2}\int d^dx((\vec{\nabla}\phi_c)^2 + m^2\phi_c^2)$$

We have obtained that for the free field, $\Gamma[\phi_c]$ is equal to the action $S[\phi_c]$.

We now show that the result for the free field is valid to leading order in $\hbar$ for a nonlinear classical action as well.

Restoring $\hbar$, the definition of $W[J]$ is given by

$$Z[j] = e^{W[J]/\hbar} = \frac{1}{Z}\int D\phi e^{(S+\int J\phi)/\hbar}$$

The classical field equation is given by

$$\frac{\delta S[\phi_{cl}]}{\delta\phi(x)} + J(x) = 0$$

To leading order in $\hbar$, $W[J]$ is given by the stationary point of the integrand of the functional integral and yields

$$Z[j] = \exp\{(S[\phi_{cl}] + \int J\phi_{cl})/\hbar + O(1)\}$$

Hence

$$W[J] \approx W_{cl}[J] = S[\phi_{cl}] + \int J\phi_{cl}(x) + O(\hbar)$$

and we obtain

$$\phi_c(x) = \frac{\delta W[J]}{\delta J(x)} = \phi_{cl}(x) + O(\hbar)$$

In other words, to leading order in $\hbar$, the vacuum expectation value of the quantum field, given by $\phi_c$ is equal to the classical field $\phi_{cl}$. Eq. 12.2.2 yields

$$\Gamma[\phi_c] = \Gamma[\phi_{cl}] + O(\hbar) = \Big\{S[\phi_{cl}] + \int J\phi_{cl}(x)\Big\} - \int J\phi_{cl}(x)$$

or

$$\Gamma[\phi_c] = S[\phi_{cl}] + O(\hbar) \tag{12.3.1}$$

Hence, in the classical limit of $\hbar \to 0$, the effective action $\Gamma[\phi_c]$ is equal to the classical action.

## 12.4 Semi-classical expansion of $\Gamma[\phi_c]$

Consider a scalar field given by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_i\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) + J\phi$$

The classical solution $\phi_{cl}$ is given by

$$(-\partial^2 + m^2)\phi_{cl} + V'(\phi_{cl}) = J$$

Expanding the action about $\phi_{cl}$ by the change of field variables

$$\tilde{\phi} = \phi_{cl} + \phi$$

yields

$$S[\phi] + \int J\phi = S[\phi_{cl}] + \int J\phi_{cl} + \tilde{S}$$

where

$$\tilde{S} = -\frac{1}{2}\int((\partial\tilde{\phi})^2 + m^2\tilde{\phi}^2) - \frac{1}{2}\int V''(\tilde{\phi})\tilde{\phi}^2 + O(\tilde{\phi}^3)$$

To leading order in $\hbar$

$$e^{W[J]/\hbar} \approx e^{W_{cl}[J]/\hbar}\int D\tilde{\phi}e^{\tilde{S}} = e^{W_{cl}[J]/\hbar}\frac{const}{\sqrt{det(-\partial^2 + m^2 + V''(\tilde{\phi}))}}$$

Recall $detM = e^{Tr\ln M}$ Hence, up to irrelevant constants

$$W[J] = W_{cl}[J] - \frac{\hbar}{2}Tr\ln(1 - \frac{V''(\tilde{\phi})}{-\partial^2 + m^2}) + O(\hbar^2)$$

where

$$W_{cl}[J] = S_{cl}[\phi_{cl}] + \int J\phi_{cl}$$

To evaluate $\Gamma[\phi_c]$ note that

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = \phi_{cl} + O(\hbar^2)$$

and this yields

$$\Gamma[\phi_c] = S_{cl}[\phi_c] + \frac{\hbar}{2}Tr\ln(1 + \frac{V''(\tilde{\phi})}{-\partial^2 + m^2}) + O(\hbar^2)$$

As expected to lowest order $\Gamma[\phi_c]$ is the classical action. The first order correlation in $\hbar$ requires summing over all the one loop diagrams and the loop-expansion is discussed in Section 12.8.

## 12.5 The connected vertex functions

In the calculation of the correlators, it was seen that the propagators for the external lines could be removed for studying for example the theory's renormalization, as in Figure 9.10 and discussed in Section 9.7.

The Feynman diagram shown in Figure 12.1(a), called one particle reducible, can be completely split up into three distinct disconnected subdiagrams by cutting two single lines (which recall stands for the propagator). There is no independent integration along the line being cut.

The cutting of the single lines, and indicated by an arrow in Figure 12.1, leads to subdiagrams diagrams, shown in Figure 12.1(b), that all cannot be split further into two subdiagrams by cutting a line and are called *one particle irreducible* diagrams, and denoted by 1PI. For 1PI diagrams, all internal lines occur only within a loop.

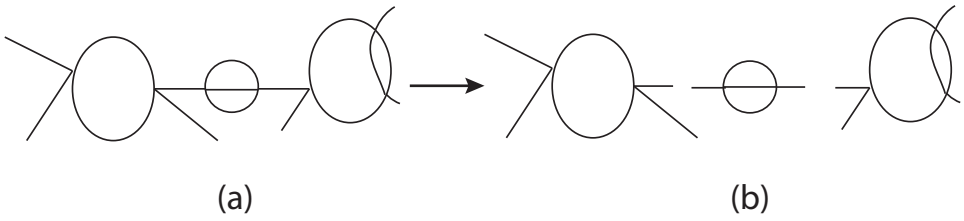


Figure 12.1 a) One particle reducible Feynman diagram. b) All the subdiagram on the right hand side are 1PI (one particle irreducible).

It is intuitively expected that the 1PI subdiagrams contain all the essential divergences, and that each of the 1PI diagrams can be renormalized separately. The one particle reducible diagrams can then be renormalized using the 1PI renormalized diagrams.

To formalize the framework such that the 1PI Feynman diagrams can be isolated, we analyze the effective action $\Gamma[\phi_c]$. In general $\Gamma[\phi_c]$ has the following Taylors expansion

$$\Gamma[\phi_c] = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n \Gamma_n(x_1 \cdots x_n) \phi_c(x_1) \cdots \phi_c(x_n) \quad (12.5.1)$$

The function $\Gamma_n(x_1 \cdots x_n)$ are the connected vertex functions; these are the 1PI (one particle irreducible) Feynman diagrams with the external legs removed, called amputated. In our discussion below, we will derive explicit expressions for $\Gamma_n$ in terms of the connected Greens functions $G_n$ and will show that the amputated Greens functions yield $\Gamma_n$; however, we will not prove that the $\Gamma_n$ are given by the 1PI Feynman diagrams as the proof is not required in any of our discussions.

For connected Greens functions $G_n$ from Eq. 9.5.2 we have

$$W[J] = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n G_n(x_1 \cdots x_n) J(x_1) \cdots J(x_n)$$

Since $\phi$ is a bosonic quantum field, $\Gamma_n(x_1 \cdots x_n)$ and $G_n(x_1 \cdots x_n)$ are completely symmetric functions of the spacetime coordinates $x_1 \cdots x_n$.

Recall

$$\frac{\delta \Gamma}{\delta \phi_c(x)} = -J(x) \quad \Rightarrow \quad \frac{\delta^2 \Gamma}{\delta J(x) \delta \phi_c(x)} = -\delta(x-y)$$

From equation above and Eq. 9.5.2, we have

$$-\delta(x-y) = \int d^d z \frac{\delta \phi_c(z)}{\delta J(x)} \frac{\delta^2 \Gamma}{\delta \phi_c(z) \delta \phi_c(x)} = \int d^d z \frac{\delta^2 W[J]}{\delta J(y) \delta J(z)} \frac{\delta^2 \Gamma}{\delta \phi_c(z) \delta \phi_c(x)}$$

$$= \int d^d z G_2(y-z)\Gamma_2(z-x)$$

Hence, in compact matrix notation

$$G_2\Gamma_2 = -1 \quad \Rightarrow \quad \Gamma_2 = -G_2^{(-1)} \tag{12.5.2}$$

Furthermore, Eqs. 12.2.3 and 12.5.1 yield

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = -J(x) \quad \Rightarrow \quad \frac{\delta J}{\delta\phi_c} = -\frac{\delta^2\Gamma}{\delta\phi_c\delta\phi_c} = -\Gamma_2 \tag{12.5.3}$$

From Eq. 12.2.1

$$\phi_c(x) \equiv \frac{\delta W[J]}{\delta J(x)} \quad \Rightarrow \quad \frac{\delta\phi_c(x)}{\delta J(y)} = \frac{\delta^2 W[J]}{\delta J(y)J(x)} = G_2(x,y)$$

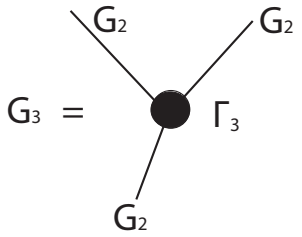


Figure 12.2 Feynman diagram representation of $G_3 = G_2G_2G_2\Gamma_3$.

The chain rule for $\delta/\delta J(x)$ yields

$$\frac{\delta}{\delta J(x)} = \int d^d y \frac{\delta\phi_c(y)}{\delta J(x)}\frac{\delta}{\delta\phi_c(y)} = \int d^d y G_2(x,y)\frac{\delta}{\delta\phi_c(y)}$$

In simplified notation

$$\frac{\delta}{\delta J} = G_2\frac{\delta}{\delta\phi_c}$$

Applying $\delta/\delta J(x)$ to Eq. 12.5.2 yields, in abbreviated notation

$$0 = \frac{\delta}{\delta J}\left\{\frac{\delta^2 W}{\delta J\delta J}\frac{\delta^2\Gamma}{\delta\phi_c\delta\phi_c}\right\} = \frac{\delta^3 W}{\delta J\delta J\delta J}\frac{\delta^2\Gamma}{\delta\phi_c\delta\phi_c} + G_2G_2\frac{\delta^3\Gamma}{\delta\phi_c\delta\phi_c\delta\phi_c}$$

$$= G_3\Gamma_2 + G_2G_2\Gamma_3$$

Using $\Gamma_3\Gamma_2 = \Gamma_2\Gamma_3$, since both $\Gamma_3$ and $\Gamma_2$ are completely symmetric functions of the spacetime coordinates, yields the following final result

$$G_3 = G_2G_2G_2\Gamma_3 \quad \Rightarrow \quad \Gamma_3 = -\Gamma_2\Gamma_2\Gamma_2G_3 \tag{12.5.4}$$

Eq. 12.5.4 is represented graphically in Figure 12.2 and its explicit expression is given by

$$G_3(x_1, x_2, x_3) = \int dy_1 dy_2 dy_3 G_2(x_1, y_1) G_2(x_2, y_2) G_2(x_3, y_3) \Gamma^{(3)}(y_1, y_2, y_3)$$

Inverting $G_2$ in Eq. 12.5.4 we obtain

$$\Gamma_3 = -\Gamma_2 \Gamma_2 \Gamma_2 G_3$$

which, written explicitly, yields

$$\Gamma_3(x_1, x_2, x_3) = -\int dy_1 dy_2 dy_3 \Gamma_2(x_1, y_1) \Gamma_2(x_2, y_2) \Gamma_2(x_3, y_3) G_3(y_1, y_2, y_3)$$

The equation above shows that $\Gamma_2$ cuts the external legs of $G_3$ and yields $\Gamma_3$, and is shown in Figure 12.2.

In effect $\Gamma^{(3)}$ is obtained by amputating the external legs of $G_3$ using the exact propagator $G_2$. It can be shown that $\Gamma^{(3)}$ itself is connected vertex function.

### *12.5.1 Four point vertex*

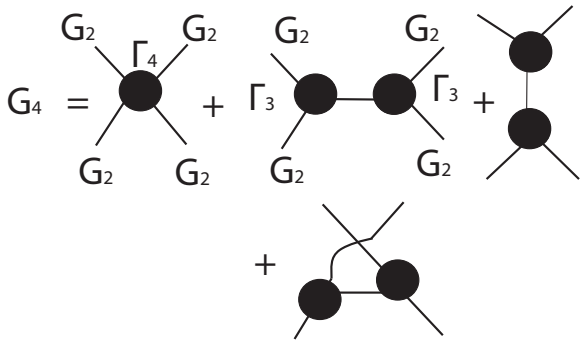


Figure 12.3 The four point vertex function $\Gamma_4$ in terms of the connected correlation functions and $\Gamma_3$.

Recall $G_3 = G_2 G_2 G_2 \Gamma_3$, where $G_2$ is the exact 2-point correlation function. Employing a procedure similar to the one used for obtaining $\Gamma_3$, the four point connected vertex function is given by the following

$$\begin{aligned} G_4 = \frac{\delta G_3}{\delta J} &= \frac{\delta}{\delta J}\{G_2 G_2 G_2 \Gamma_3\} \qquad (12.5.5) \\ &= G_2 G_2 G_2 G_2 \Gamma_4 + G_3 G_2 G_2 \Gamma_3 + G_2 G_3 G_2 \Gamma_3 + G_2 G_2 G_3 \Gamma_3 \\ &= G_2 G_2 G_2 G_2 \Gamma_4 + G_2 G_2 \Gamma_3 G_2 G_2 \Gamma_3 + \text{symmetrization} \qquad (12.5.6) \end{aligned}$$

Figure 12.3 gives the Feynman diagrams for Eq. 12.5.5 to all orders in perturbation theory, and was earlier represented to lowest order in perturbation theory, to $O(\lambda)$, in Figure 9.11. The definition for $\Gamma_4$ is given in Eq. 9.7.1 to lowest order in $\lambda$, with $\Gamma_4 \approx \Gamma_4^0 + \Gamma_4^1$, and with the lowest order propagators $G_2 \approx 1/(p^2 + m^2)$ being factored out, as in Eq. 12.5.5.

In general, for $\Gamma_n$, an equation similar to Eq. 12.5.5 can be obtained in terms of the connected correlation functions $G_n$ and lower order vertex functions.

## 12.6 Loop Expansion for the Effective Action

Loop expansion is the evaluation of the effective action in powers of $\hbar$. Consider the partition function given by

$$Z = \int D\phi \exp\{\frac{1}{\hbar} S\}$$

For Feynman perturbation expansion, the action can consist of many fields and, in general, has a quadratic part and an interaction vertex given by the following break up

$$S = S_0 + S_I$$

and which yields

$$Z = \int D\phi \exp\{\frac{1}{\hbar} S_0\} \sum_{n=0}^{\infty} \frac{1}{n!\hbar^n} S_I^n$$

Note that $Z$ contains only the amputated Greens functions. For the $n$-th order term in the perturbation expansion, we have the following:

- Each propagator contributes a factor of $\hbar$; each internal line $I$ corresponds to a propagator and hence yields a factor of $\hbar^I$
- Each vertex contributes a factor of $1/\hbar$, and for $n$ vertices yields a factor of $\hbar^{-n}$
- Hence, the $n$-th order diagram has a pre-factor of $\hbar^{I-n}$

From Eq. 10.13.2, we have $I - n = L - 1$, where $L$ is the number of loops. The prefactor is given by

$$\hbar^{I-n} = \hbar^{L-1}$$

Hence, an expansion in the number of loops is an expansion of the effective action about the quadratic action, in a power series in $\hbar$. An expansion aout the classical solution also yields a loop expansion about the classical solution.

For a single coupling constant, the loop is equivalent to an expansion in powers of the coupling constant. Consider the $\phi^4$ theory with the action given by

$$S = -\int d^4x\{\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\}$$

Rescale the field $\phi$ to $\phi/\sqrt{\lambda}$; then the action is

$$S = -\frac{1}{\lambda}\int d^4x\{\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\phi^4\}$$

and we see that $\lambda$ plays the role of $\hbar$ since the effective expansion parameter for the loop expansion becomes $\lambda\hbar$. This is the reason that for $\phi^4$ theory, the concept of the loop expansion and a perturbation expansion in the coupling constant have been used inter-changeably.

For a Lagrangian consisting of many quantum fields, the loop expansion takes into account the contribution of all the coupling constants and organizes the perturbation about the classical solution that is different from a perturbation expansion in the coupling constants.

### *12.6.1 Dimensional analysis*

The engineering dimension of the vertex function $\Gamma_n$ is evaluated. The Fourier transform of the connected correlation function yields

$$G_n(p_1\cdots p_n) = \int dx_1\cdots dx_n e^{i\sum_{i=1}^n p_i x_i} G_n(x_1\cdots x_n)$$

Using translation invariance of the theory implies that

$$G_2(x,y) = G_2(x-y)$$

For constant $z$, translation invariance yields

$$\Gamma_n(x_1\cdots x_n) = \Gamma_n(x_1+z\cdots x_n+z)$$

$$= \int dp_1\cdots dp_n e^{i\sum_{i=1}^n p_i x_i}\Gamma_n(p_1\cdots p_n)\delta(\sum_{i=1}^n p_i)$$

In general, as seen from Eq. 12.5.5, $\Gamma_n$ has an expansion in terms of $G_n$ and lower order vertex functions. For dimensional analysis, we need to identify only the dimensional aspect of the definition of $\Gamma_n$, and it is sufficient to

keep the leading term. Hence, indicating dimensions by [..], we have

$$[G_n(p_1 \cdots p_n)] = \Big[G_2(p_1) \cdots G_2(p_n)\Gamma_n(p_1 \cdots p_n)\delta(\sum_{i=1}^{n} p_i)\Big] \quad (12.6.1)$$

From Eq. 11.2.2, the dimension of the momentum Greens function is

$$[G_n(p_1 \cdots p_n)] = \mu^{-n(d/2+1)}$$

For the propagator $n = 2$ and hence

$$[G_2(p_1, p_2)] = \mu^{-(d+2)}$$

Defining $G_2(p)$ by

$$G_2(p, p') = \delta^d(p + p')G_2(p)$$

yields the engineering dimension

$$[G_2(p)] = \mu^d[G_2(p, p')] = \mu^{-2}$$

The engineering dimension of $\Gamma_n(p_1 \cdots p_n)$ is given, from Eqs. 11.2.2 and 12.6.1, by the following

$$[G_n(p_1 \cdots p_n)] = [G_2(p)]^n \mu^{-d}[\Gamma_n(p_1 \cdots p_n)]$$

or

$$[\Gamma_n(p_1 \cdots p_n)] = \mu^{-n(d/2+1)}\mu^{2n}\mu^d = \mu^{d-n(d/2-1)}$$

Hence for $d = 4$, the engineering dimension is given by

$$[\Gamma_n(p_1 \cdots p_n)] = \mu^{4-n}$$

The vertex function $\Gamma_n$ develops an anomalous dimension, similar to the case discussed in Section 11.8, due to the effects of renormalization.

## 12.7 The effective potential $V_{\text{eff}}$

The effective action has a physical interpretation in terms of the energy of the system. Consider the special case of a constant current $J$ and a field $\phi_c$ that depends only on space. In terms of the Hamiltonian $H$, the definition of the generating functional is given by

$$e^{W[J]} = \lim_{\tau \to \infty} tr\Big(e^{-\tau H + \tau J \int d^3x \phi_c}\Big)$$

Let $|\Omega_J\rangle$ be the vacuum state given by

$$\Big(H - J \int d^3x \phi_c\Big)|\Omega_J\rangle = -\frac{W[J]}{\tau}|\Omega_J\rangle$$

Since $\tau \to \infty$, only the energy of the vacuum state is required; hence

$$\lim_{\tau\to\infty} tr\Big(e^{-\tau H+\tau J\int d^3x\phi_c}\Big) \simeq \lim_{\tau\to\infty}\langle\Omega_J|e^{-\tau H+\tau J\int d^3x\phi_c}|\Omega_J\rangle = e^{W[J]}$$

The result above shows that for constant $J$, the effective action $W[J]$ is equal to *minus* of the energy of the vacuum $|\Omega_J\rangle$, divided by $\tau$.

The $J$ dependent vacuum energy per unit volume $E$ and $\phi_c$ are given by

$$E = \langle\Omega_J|H|\Omega_J\rangle = E(J) \;\; ; \;\; \phi_c = \langle\Omega_J|\phi(x)|\Omega_J\rangle = \phi_c(J) \;\; : \;\; \text{constant}$$

Since $J$ and $\phi_c$ are spacetime constants, the integral over spacetime factorizes and the spacetime volume of the system is an overall multiplicative constant. Hence

$$W[J] = \mathcal{V}(-E + J\phi_c) \;\; ; \;\; \mathcal{V} = \tau\int d^3x$$

and which yields the effective action

$$\Gamma = W[J] - J\int d^4x\phi_c = -E\mathcal{V} \tag{12.7.1}$$

The defining equation for $J$ yields

$$\frac{\partial E(J)}{\partial\phi_c} = J = J(\phi_c)$$

The vacuum expectation value of the quantum field for the true vacuum, namely $\bar{\phi}$, is obtained when $J = 0$; hence

$$\bar{\phi} = \langle\Omega_J|\phi(x)|\Omega_J\rangle\big|_{J=0} \quad\Rightarrow\quad \lim_{J\to 0}\phi_c \to \bar{\phi}$$

Conversely, we have the limit

$$\lim_{\phi_c\to\bar{\phi}} J(\phi_c) \to J(\bar{\phi}) = 0$$

We conclude that the true ground state has a minimum energy for $J = 0$

$$\frac{\partial E(J(\phi_c))}{\partial\phi_c}\Big|_{\phi_c=\bar{\phi}} = J(\bar{\phi}) = 0 \quad\Rightarrow\quad \frac{\partial E(\bar{\phi})}{\partial\phi_c} = 0$$

In summary, the minimization of the effective action $\Gamma$ yields the quantum ground state, with the correct value for $\bar{\phi}$, and which has all the quantum contributions taken into account. Hence the effective action yields the appropriate criterion to be used for defining the quantum corrected ground state.

The effective action $\Gamma$ has an expansion in terms of the powers of the derivatives of the field $\phi_c$

$$\Gamma[\phi_c] = \int d^d x[-V_{\rm eff}(\phi_c) + f(\phi_c)(\partial_\mu \phi_c)^2 + g(\phi_c)(\partial_\mu \partial_\nu \phi_c)^2 + ...] \quad (12.7.2)$$

The effective potential $V_{\rm eff}(\phi_c)$ is the value of $\Gamma[\phi_c]$ for $\phi_c = B$, where $B$ is a constant. From the gradient expansion for $\Gamma$ given in Eq. 12.7.2, we see from Eq. 12.7.1 that for $\phi_c = B$

$$V_{\rm eff}(B) = -\Gamma[B]/\mathcal{V} = E$$

Hence the effective potential has the correct sign in Eq. 12.7.2, being equal to energy for quantum field configurations that are constant. As a function of $\phi_c$, the minimum value of $V_{\rm eff}$ is the minimum value of $E$ – which as discussed above yields the true quantum corrected ground state. The definition and sign of $V_{\rm eff}$ in Eq. 12.7.2 has been chosen to make the effective potential a quantum corrected generalization of the concept of the potential in classical mechanics.

From the discussion above, the value of $B$ for which $V_{\rm eff}(B)$ is a minimum is equal to the vacuum expectation value of the quantum field $\phi$ in the true quantum corrected vacuum state and hence determines whether the field spontaneously breaks a symmetry of the Lagrangian.

The ultra-violet divergences of the quantum field also appear in the effective action and hence in the effective potential as well. In general, for field renormalization one needs to set the coefficient of the kinetic term in the action, namely $(\partial_\mu \phi_c)^2$ to unity. Since we are considering only a constant value for $B$ there is no kinetic term in $V_{\rm eff}(B)$, and one has to refer to the effective action for field renormalization. The derivation of the effective potential is being done to one-loop of $\phi^4$ theory, and to one-loop there is no field renormalization. Hence, we will not address this any further.

There are a variety of conditions that one can impose to define the renormalized mass and coupling constant, depending of the nature of the calculation. In general, one needs to impose two conditions to renormalize the effective potential, and which are taken to follow the scheme given in Eq. 10.2.1; the quadratic term of the effective potential defines the renormalized mass and the quartic term defines the renormalized coupling constant. Hence we have

$$\frac{d^2 V_{eff}}{dB^2}\Big|_{B=M} = m_R^2 \;\; ; \;\; \frac{d^4 V_{eff}}{dB^4}\Big|_{B=M} = \lambda_R(M) = \lambda_R \quad (12.7.3)$$

where $M$ is an arbitrary mass scale analogous to the scale $\mu$, which appears in Eq. 10.2.1.

## 12.8 One loop effective action for $\phi^4$ theory

The semi-classical expansion directly computes the effective action by doing a loop expansion about the classical field $\phi_{cl}$ that satisfies the Euler-Lagrange equation. The background field method is another procedure for computing the effective action and is similar to the background field method used for renormalizing the theory, as discussed in Section 10.12.

The one-loop effective action is computed for the $\phi^4$ theory and the result is used to ascertain if there is spontaneous breaking of symmetry for the quantum field due to the quantum corrections to the classical theory. The background field method is used as it provides efficient procedure for computing the one-loop effective potential.

Consider the action $S[\phi]$ that yields the generating function

$$Z(J) = e^{W(J)} = \int D\varphi e^{S+\int J\varphi}$$

The classical function of spacetime $\phi_c$ is a functional of $J$, and is the expectation value of the quantum field.

$$\frac{\delta W[J]}{\delta J(x)} = \phi_c(x)$$

The effective action is given by the following definition

$$\Gamma[\phi_c] = W[J] - \int d^dx J(x)\phi_c(x) \quad \Rightarrow \quad \frac{\delta\Gamma[\phi_c]}{\delta\phi_c](x)} = -J(x)$$

Hence

$$e^{\Gamma[\phi_c]} = \frac{1}{Z}\int D\phi e^{S[\phi]+\int d^dx(\phi-\phi_c)\frac{\delta\Gamma[\phi_c]}{\delta\phi_c}}$$

Shifting $\phi \to \phi + \phi_c$ yields

$$e^{\Gamma[\phi_c]} = \frac{1}{Z}\int D\phi e^{S[\phi+\phi_c]+\int d^dx\phi\frac{\delta\Gamma[\phi_c]}{\delta\phi_c}}$$

Consider the $\phi_c^0$ to be the solution of

$$\frac{\delta\Gamma[\phi_c^0]}{\delta\phi_c} = 0$$

Hence

$$e^{\Gamma[\phi_c^0]} = \frac{1}{Z}\int D\phi e^{S[\phi+\phi_c^0]}$$

To leading order $\Gamma[\phi_c]$ is given by the classical action and, from Eq. 12.3.1

$$\phi_c^0 = \phi_{cl} + O(\hbar)$$

The action $S[\phi]$ is considered to be the bare action and the *renormalized* effective action $\Gamma_R[\phi_c^0]$ is given by

$$\Gamma_R[\phi_c^0] = \ln\left(\frac{1}{Z}\int D\phi \exp\left\{S[\phi+\phi_c^0] + \text{counter terms}\right\}\right)$$

Consider the $\phi^4$ theory; let the background field $\phi_c^0$ be devoted by $B$

$$\phi_c^0 \equiv B = \text{constant}$$

The $\phi^4$ Lagrangian is taken to be written in terms of the bare quantities, namely $\phi_B, \lambda_B,\ m_B$, and is given by

$$\mathcal{L}_B[\phi] = -\frac{1}{2}\partial_\mu\phi_B\partial_\mu\phi_B - \frac{1}{2}m_B^2\phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4$$

Dropping the subscript on $\mathcal{L}_B$ and $\phi_B$ yields

$$\mathcal{L}[\phi+B] = \mathcal{L}_0[B] - \frac{1}{2}(\partial\phi)^2 - \frac{m_B^2}{2}\phi^2 - \frac{1}{4}\lambda_B B^2\phi^2 = \mathcal{L}_0[B] + \mathcal{L}[B,\phi]$$

As given in Eq. 12.3.1, to leading order in $\hbar$, $B$ is equal to the classical field $B = \phi_{cl}$; hence the expansion does not have any linear terms in the quantum field $\phi$. The expansion about constant $B$ yields

$$\mathcal{L}_0[B] = -\frac{m_B^2}{2}B^2 - \frac{1}{4}\lambda_B B^4$$

and

$$\mathcal{L}[B,\phi] = -\frac{1}{2}(\partial\phi)^2 - \frac{m_B^2}{2}\phi^2 - \frac{1}{4}\lambda_B B^2\phi^2$$

The effective action is given by

$$\Gamma = \Gamma_0 + \Gamma_1 + O(\hbar^2)$$

and

$$V_{\text{eff}}(B) = V_0(B) + V_1(B)$$

The lowest order contribution is given by

$$\Gamma_0 = \int d^dx\mathcal{L}_0[B] = -\mathcal{V}V_0(B)$$

Hence, to lowest order, the effective potential is

$$V_0(B) = \frac{1}{2}m_B B^2 + \frac{\lambda_B}{4}B^4$$

The $O(\hbar$ contribution to the effective potential is given by $\Gamma_1$ given by

$$e^{\Gamma_1} = \int D\phi \exp\{S\} \quad ; \quad S = \int d^dx\mathcal{L}[B,\phi]$$

where

$$S = -\frac{1}{2}\int d^d x[(\partial\phi)^2 + m_B^2\phi^2 + \frac{\lambda_B B^2}{2}\phi^2]$$

The path integral for $\Gamma_1$, with $B =$ constant, gives the one loop correlation to $V_{\text{eff}}$.

To develop a perturbation expansion for $\Gamma_1$, consider the term $(1/2)\lambda_B B^2\phi^2$ to be an interaction vertex and expand it into an infinite power series. Using the free action

$$S_0 = -\frac{1}{2}\int d^d x[(\partial\phi)^2 + m_B^2\phi^2]$$

an expansion of the path integral in powers of the vertex yields

$$e^{\Gamma_1} = \int D\phi e^{S_0}[\sum_{n=0}^{\infty}(-\frac{\lambda_B B^2}{4})^n\phi^{2n}] \tag{12.8.1}$$

The path integral for $\Gamma_1$ given in Eq. 12.8.1 consists of all the one loop diagrams with vertices given by $\lambda_B B^2$ as shown in Figure 12.4.

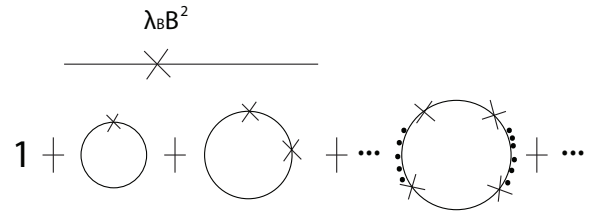


Figure 12.4 The vertex for the one loop diagram together with all the diagrams that contribute to the one-loop effective action.

One can in fact perform the path integral and sum up the infinite series of one-loop diagrams. Instead of summing up all the one-loop diagrams, one can instead perform the path integral using the fact that $B$ is a constant. Differentiating $\Gamma_1$ with respect to $B$ yields the following differential equation

$$\frac{d\Gamma_1}{dB} = \frac{1}{Z}\int D\phi e^S(-\frac{1}{2}\int d^d x\lambda_B B\phi^2) \;\; ; \;\; Z = \int D\phi e^S \tag{12.8.2}$$

Hence, $d\Gamma_1/dB$ is given by the propagator with an effective mass

$$m_B^2 + \frac{\lambda_B B^2}{2} = m_e^2$$

From Eq. 12.8.2 we obtain

$$\frac{d\Gamma_1}{dB} = -\frac{1}{2}\lambda_B B\int d^d x E[\phi^2(x)] = -\frac{1}{2}\lambda_B B\int d^d x\int\frac{d^d p}{(2\pi)^d}\frac{1}{(p^2 + m_B^2 + \frac{\lambda_B B^2}{2})}$$

The spacetime volume $\int d^dx = \mathcal{V}$ factors out in $\Gamma_1$ and we have

$$V_1(B) = -\Gamma_1/\mathcal{V}$$

The $V_1(B)$ component of the effective potential satisfies

$$\frac{dV_1}{dB} = +\frac{1}{2}\lambda_B B \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p^2 + m_B^2 + \frac{\lambda_B B^2}{2})}$$

Dimensional regularization, from Eq. 9.8.6, yields

$$\frac{dV_1}{dB} = -\frac{\lambda_B m_e^2}{32\pi^2} B[\frac{2}{\epsilon} + \varphi(2) - \ln(\frac{m_e^2}{4\pi\mu^2})]$$

An arbitrary scale $\mu$ has been introduced to carry out dimensional regularization and which, in particular, renders the logarithm dimensionally consistent.

The zero of $V_{\text{eff}}(B)$ is set by $V_{\text{eff}}(0) = 0$. Let $\xi = m_e^2$ and $d\xi = \lambda_B B dB$; integrating above expression yields

$$V_1(B) = \int_0^B dB' \frac{dV_1}{dB'} \tag{12.8.3}$$

$$= -\frac{1}{32\pi^2}\int_a^b d\xi\xi(\frac{2}{\epsilon} + \varphi(2)) + \frac{1}{32\pi^2}\int_a^b d\xi\xi \ln(\frac{\xi}{4\pi\mu^2}) \tag{12.8.4}$$

where

$$a = m_B^2 \quad ; \quad b = m_B^2 + \frac{\lambda_B B^2}{2}$$

Performing the integrals yields[1]

$$V_1 = -\frac{1}{64\pi^2}(\frac{2}{\epsilon} + \varphi(2))(\lambda_B m_B^2 B^2 + \frac{\lambda_B^2 B^4}{4}) - \frac{1}{128\pi^2}[\lambda_B m_B^2 B^2 + \frac{\lambda_B^2 B^4}{4}]$$
$$+\frac{1}{64\pi^2}\Big[(m_B^2 + \frac{\lambda_B B^2}{2})^2 \ln\big(\frac{m_B^2 + \frac{\lambda_B B^2}{2}}{4\pi\mu^2}\big) - m_B^4 \ln\big(\frac{m_B^2}{4\pi\mu^2}\big)\Big] \tag{12.8.5}$$

### *12.8.1 Renormalizing $V_{\text{eff}}$*

To renormalize $V_{\text{eff}}$, the bare Lagrangian is written in terms of the renormalized Lagrangian and counter term as following

$$\mathcal{L}_B = \mathcal{L}_R + \Delta\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(m_R^2 + \delta m^2)(\phi + B)^2 - \frac{1}{4!}(\lambda_R + \delta\lambda)(\phi + B)^4$$

[1] Note $\int dx x \ln(x) = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2$.

or

$$\mathcal{L}_B = -\frac{1}{2}(m_R^2+\delta m^2)B^2-\frac{1}{4!}(\lambda_R+\delta\lambda)B^4-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m_R^2(\phi)^2-\frac{\lambda_R}{4!}(\phi)^2B^2+O(\hbar^2)$$

The counter terms are $O(\lambda)$ and hence all higher terms involving $\delta\lambda,\ \delta m$ are of $O(\lambda^2)$. The only change in $V_{\text{eff}}$ is for the $V_0$, and for $V_1$, one simply replaces $m_B, \lambda_B$ by $m_R, \lambda_R$ as required. Hence we obtain

$$V_{\text{eff}}(B, m_R, \lambda_R) = V_0(m_R^2 + \delta m^2, \lambda_R + \delta\lambda) + V_1(B, m_R, \lambda_R) \quad (12.8.6)$$

where

$$V_0(m_R^2 + \delta m^2, \lambda_R + \delta\lambda) = \frac{1}{2}(m_R^2 + \delta m^2)B^2 + \frac{1}{4!}(\lambda_R + \delta\lambda)B^4$$

The renormalization of $V_{\text{eff}}$ is done by the imposing two conditions to determine the renormalized mass and coupling constant, as given in Eq. 12.7.3. For the massive $\phi^4$ theory, the renormalization point is chosen to be $M = 0$. Hence, the following conditions are imposed for determining the renormalized mass and coupling constant

$$\frac{d^2V_{\text{eff}}}{dB^2}\Big|_{B=0} = m_R^2 \ \ ; \ \ \frac{d^4V_{\text{eff}}}{dB^4}\Big|_{B=0} = \lambda_R$$

To implement the renormalization condition, we expand $V_1$ as a polynomial in $B$ to $O(B^4)$; note

$$\begin{aligned}
&(m_B^2 + \frac{\lambda_B B^2}{2})^2 \ln\big(\frac{m_B^2 + \frac{\lambda_B B^2}{2}}{4\pi\mu^2}\big) - m_B^4 \ln\big(\frac{m_B^2}{4\pi\mu^2}\big) \\
&= (\lambda_B m_B^2 B^2 + \frac{1}{4}\lambda^2 B^4) \ln\big(\frac{m_B^2}{4\pi\mu^2}\big) + (m_B^2 + \frac{\lambda_B B^2}{2})^2 \ln(1 + \frac{\lambda_B B^2}{2m_B^2}) \\
&\approx (\lambda_B m_B^2 B^2 + \frac{1}{4}\lambda_B^2 B^4) \ln\big(\frac{m_B^2}{4\pi\mu^2}\big) + \frac{1}{2}\lambda_B m_B^2 B^2 + \frac{3}{8}\lambda_B^2 B^4 + O(B^6) \quad (12.8.7)
\end{aligned}$$

Hence, to $O(B^4)$, Eqs. 12.8.5 and 12.8.7 yield

$$\begin{aligned}
V_1 \approx &-\frac{1}{64\pi^2}(\frac{2}{\epsilon} + \varphi(2))(\lambda_B m_B^2 B^2 + \frac{\lambda_B^2 B^4}{4}) + \frac{1}{256\pi^2}\lambda_B^2 B^4 \\
&+ \frac{1}{64\pi^2}(\lambda_B m_B^2 B^2 + \frac{1}{4}\lambda_B^2 B^4) \ln\big(\frac{m_B^2}{4\pi\mu^2}\big) \quad (12.8.8)
\end{aligned}$$

Note that the $B^2$ term has simplified, with two terms canceling out in the expansion of $V_1$ given in Eq. 12.8.8 above.

Imposing the condition for fixing the renormalized mass

$$\frac{d^2V_{\text{eff}}}{dB^2}\Big|_{B=0} = m_R^2$$

yields

$$m_R^2 + \delta m^2 - \frac{\lambda_R m_R^2}{32\pi^2}(\frac{2}{\epsilon} + \varphi(2)) + \frac{\lambda_R m_R^2}{32\pi^2}\ln\frac{m_R^2}{4\pi\mu^2} = m_R^2$$

or

$$\delta m^2 = \frac{\lambda_R m_R^2}{32\pi^2}(\frac{2}{\epsilon} + \varphi(2) - \ln\frac{m_R^2}{4\pi\mu^2})$$

The renormalized coupling constant is determined by

$$\frac{d^4 V_{\text{eff}}}{dB^4}\Big|_{B=0} = \lambda_R$$

and yields

$$\lambda_R = \frac{3\lambda_R^2}{32\pi^2} + \lambda_R + \delta\lambda - \frac{3\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} + \varphi(2) - \ln\frac{m_R^2}{4\pi\mu^2})$$

or

$$\delta\lambda = \frac{3\lambda_R^2}{32\pi^2}(\frac{2}{\epsilon} + \varphi(2) - \ln\frac{m_R^2}{4\pi\mu^2}) - \frac{3\lambda_R^2}{32\pi^2}$$

The renormalized effective potential is given by

$$V_{\text{eff}}(B, m_R, \lambda_R) = V_0 + V_1 = \frac{m_R B^2}{2} + \frac{\lambda_R B^4}{4!} - \frac{1}{4!}\frac{3\lambda_R^2 B^4}{32\pi^2}$$

$$-\frac{1}{128\pi^2}(\lambda_R m_R^2 B^2 + \frac{1}{4}\lambda_R^2 B^4) + \frac{1}{64\pi^2}(m_R^2 + \frac{\lambda_R B^2}{2})^2\ln(1 + \frac{\lambda_R B^2}{2m_R^2})$$

and hence

$$V_{\text{eff}}(B, m_R, \lambda_R) = V_0 + V_1 = \frac{m_R B^2}{2} + \frac{\lambda_R B^4}{4!} - \frac{\lambda_R m_R^2 B^2}{128\pi^2}$$

$$-\frac{3\lambda_R^2 B^4}{512\pi^2} + \frac{1}{64\pi^2}(m_R^2 + \frac{\lambda_R B^2}{2})^2\ln(1 + \frac{\lambda_R B^2}{2m_R^2})$$

All the divergent terms in $V_{\text{eff}}$ due to $m_B, \lambda_B$ cancel, leaving a finite expression for $V_{\text{eff}}$in terms of $m_R, \lambda_R$. The arbitrary mass scale $M$ does not appear; this because it is set to zero by the condition defining the renormalized parameters being imposed for $B = 0$.

The effective potential does not have a finite limit as $m_R \to 0$ due to the logarithmic term. This apparent divergence appears due to the definition of $m_B, \lambda_R$ at $B = 0$. For the massless case, the renormalized mass and coupling constant are defined using a non-zero value for the scale $M$.

## 12.9 Effective potential: massless $\phi^4$ theory

To renormalize the massless $\phi^4$ theory, we start with the massive theory and define the renormalized coupling constant at a scale $M$ by the following conditions.

- The renormalized mass is zero, that is $m_R^2 = 0$
- Mass renormalization is zero, that is $\delta m^2 = 0$
- The renormalized coupling constant is defined at a non-zero scale $M$

$$\frac{d^4 V_{\text{eff}}}{dB^4}\Big|_{B=M} = \lambda_R(M) = \lambda_M$$

Note that $m_R^2 = 0$ and $\delta m^2 = 0$ are two separate conditions.

The renormalized potential, using the result given in Eq. 12.8.6, yields

$$\begin{aligned} V_{\text{eff}}(B) = {} & \frac{m_R^2 B^2}{64\pi^2} \ln \frac{m_R^2 + \frac{\lambda B^2}{2}}{m_R^2} + \frac{B^2}{2}\Big[m_R^2 + \delta m^2 \\ & - \frac{3\lambda_M m_R^2}{32\pi^2}\Big(\frac{2}{\epsilon} + \varphi(2)\Big) + \frac{3\lambda_M m_R^2}{32\pi^2} \ln \frac{m_R^2 + \frac{\lambda B^2}{2}}{4\pi\mu^2} - \frac{3\lambda_M m_R^2}{64\pi^2}\Big] \\ & + \frac{B^4}{4!}\Big[\lambda_M + \delta\lambda - \frac{3\lambda_M^2}{32\pi^2}\Big(\frac{2}{\epsilon} + \varphi(2)\Big) + \frac{3\lambda_M^2}{32\pi^2} \ln \frac{m_R^2 + \frac{\lambda B^2}{2}}{4\pi\mu^2} - \frac{3\lambda_M^2}{64\pi^2}\Big] \end{aligned}$$

The massless limit is taken by setting $m_R^2 \to 0$ and imposing no mass renormalization to one loop namely $\delta m^2 = 0$. All the mass terms go to zero and makes the $B^2$-term absent from $V_{\text{eff}}(B)$. This result is expected since a $B^2$ term in the effective action is a reflection of the effective action having a mass term.

The massless effective action is given by

$$V_{\text{eff}}(B) = \frac{B^4}{4!}\Big[\lambda_M + \delta\lambda - \frac{3\lambda_M^2}{32\pi^2}\Big(\frac{2}{\epsilon} + \varphi(2)\Big) + \frac{3\lambda_M^2}{32\pi^2} \ln \frac{\lambda B^2}{8\pi\mu^2} - \frac{3\lambda_M^2}{64\pi^2}\Big]$$

The renormalized coupling constant is defined by

$$\lambda_M = \frac{d^4 V_{\text{eff}}}{dB^4}\Big|_{B=M}$$

Using the identity

$$\frac{d^4}{dB^4}\{B^4 \ln \frac{\lambda_M B^2}{8\pi\mu^2}\} = 24 \ln \frac{\lambda_M B^2}{8\pi\mu^2} + 100$$

yields

$$\lambda_M = \Big[\lambda_M + \delta\lambda - \frac{3\lambda_M^2}{32\pi^2}\Big(\frac{2}{\epsilon} + \varphi(2)\Big) - \frac{3\lambda_M^2}{64\pi^2}\Big] + \frac{3\lambda_M^2}{32\pi^2}\{\ln \frac{\lambda_M M^2}{8\pi\mu^2} + \frac{100}{4!}\}$$

The coupling constant counter-term is given by

$$\delta\lambda = \frac{\lambda_M^2}{32\pi^2}[3(\frac{2}{\epsilon} + \varphi(2)) - 3\ln\frac{\lambda_M M^2}{8\pi\mu^2} - 11]$$

Substituting $\delta\lambda$ yields

$$V_{\text{eff}} = \frac{B^4}{4!}\Big[\lambda_M - \frac{\lambda_M^2}{32\pi^2}(3\ln\frac{\lambda_M^2}{8\pi\mu^2} - 11) + \frac{3\lambda_M^2}{32\pi^2}\ln\frac{\lambda_M B^2}{8\pi\mu^2} - \frac{3\lambda_M M^2}{64\pi^2}\Big]$$

Hence, in the final result the arbitrary scale $\mu$ cancels out and the one loop massless effective action is given by

$$V_{\text{eff}} = \frac{\lambda_M B^4}{4!} + \frac{\lambda_M^2 B^4}{256\pi^2}\Big(\ln(\frac{B^2}{M^2}) - \frac{25}{6}\Big)$$

### *12.9.1 Renormalization group invariance*

The effective potential depends on an arbitrary scale $M$. All physical quantities must be independent of $M$ and hence as is expected from renormalization theory, $\lambda_M$ must have a dependence on $M$ to exactly cancel the dependence of $V_{\text{eff}}$ on $M$. Consider

$$M\frac{\partial V_{\text{eff}}}{\partial M} = \beta\Big[\frac{B^4}{4!} + \frac{\lambda_M B^4}{128\pi^2}\big(\ln(\frac{B^2}{M^2}) - \frac{25}{6}\big)\Big] - \frac{\lambda_M^2 B^4}{128\pi^2} \qquad (12.9.1)$$

where the beta-function is defined by

$$\beta = M\frac{\partial\lambda_M}{\partial M}$$

To obtain the beta function from $V_{\text{eff}}$, consider another scale $M'$ to define $\lambda'_M$, namely

$$\lambda'_M = \frac{d^4 V_{\text{eff}}}{dB^4}\Big|_{B=M'} = \lambda_M + \frac{3\lambda_M^2}{32\pi^2}\ln(\frac{M'^2}{M^2})$$

The infinitesimal change

$$M' = M + dM$$

yields

$$d\lambda_M = \lambda'_M - \lambda_M \quad \Rightarrow \quad d\lambda_M = \frac{3\lambda_M^2}{16\pi^2}\frac{dM}{M}$$

The beta function is given by

$$\beta = M\frac{\partial\lambda_M}{\partial M} = \frac{3\lambda_M^2}{16\pi^2}$$

and which was obtained in Eq. 11.1.4 using renormalization theory.

Hence, from Eq. 12.9.1, since $\beta$ is of $O(\lambda_M^2)$

$$M\frac{\partial V_{\text{eff}}}{\partial M} = \beta\frac{B^4}{4!} - \frac{\lambda_M^2 B^4}{128\pi^2} + O(\lambda_M^3) = 0$$

Hence $V_{\text{eff}}$ is independent of the scale $M$.

## 12.10 Spontaneous symmetry Breaking

The definition of the effective action yields

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = -J(x)$$

For a Lagrangian that is translation invariant, one has that

$$\bar{\phi} = \frac{1}{Z}\int D\phi e^S \phi(x) \quad : \text{constant}$$

Recall in the presence of the external current $J(x)$, the expectation value is defined by

$$\phi_c(x,J) = \frac{1}{Z}\int D\phi e^{S+\int J(x)\phi(x)}$$

where the dependence of $\phi_c(x,J)$ on $J$ has been explicitly indicated.

In terms of the external current, we have

$$\bar{\phi} = \frac{\delta W[J]}{\delta J}\Big|_{J=0} = \lim_{J\to 0}\phi_c(x,J)$$

Recall that for the effective action, $J$ is a function of $\phi_c$ namely $J(x) = J(x,\phi_c)$ hence for $\phi_c \to \bar{\phi}$ we have

$$\lim_{\phi_c\to\bar{\phi}} J(x,\phi_c) = 0$$

Hence

$$\frac{\delta\Gamma}{\delta\phi_c(x)}\Big|_{\phi_c=\bar{\phi}} = -J(\phi_c)\Big|_{\phi_c=\bar{\phi}} = 0 \tag{12.10.1}$$

For a constant classical field B, the effective action is up to a constant volume factor equal to the effective potential

$$\Gamma/V = -V_{\text{eff}}(B)$$

and, from Eq. 12.10.1, this yields

$$\frac{\partial V_{\text{eff}}(B)}{\partial B}\Big|_{B=\bar{\phi}} = 0$$

In other words the quantum corrected value of $\bar{\phi}$ is given by the extremization of $V_{\text{eff}}(B)$. We conclude that for a system with parity symmetry, a non-zero $\bar{\phi}$ signifies the spontaneous symmetry breaking.

For the massless $\phi^4$ theory

$$V_{\text{eff}}(B, M) = \frac{\lambda_M B^2}{2} + \frac{3\lambda_M^2 B^4}{256\pi^2}[\ln(\frac{B^2}{M^2}) - \frac{25}{6}]$$

The potential for the one loop massless theory is shown in Figure 12.5, and can be seen to have a double well structure, with two minimas that are non-zero.

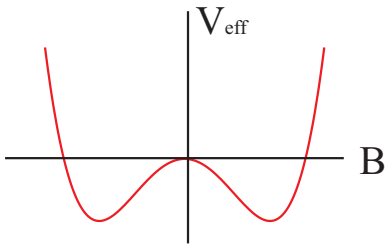


Figure 12.5 The potential $V_{\text{eff}}(B)$ for the one loop $\phi^4$ massless theory.

The classical theory has the symmetry of $\phi \to -\phi$ and hence $\bar{\phi}$ is classically equal to zero. The quantum corrected value of $\bar{\phi}$ , to one loop $O(\hbar)$ is given by

$$\frac{\partial V_{\text{eff}}(B, M)}{\partial B}\Big|_{B=\bar{\phi}} = 0$$

and yields

$$\frac{\lambda_M B^3}{3!} + \frac{\lambda_M^2 B^3}{64\pi^2}\Big[\ln(\frac{\bar{\phi}^2}{M^2}) - \frac{25}{6}\Big] + \frac{\lambda_M^2 B^3}{128\pi^2} = 0$$

One obtains the non-zero solution

$$\frac{\lambda_M}{6} + \frac{\lambda_M^2}{64\pi^2}\ln(\frac{\bar{\phi}^2}{M^2}) - \frac{11\lambda_M^2}{192\pi^2} = 0 \qquad (12.10.2)$$

Solving Eq. 12.10.2 for $\bar{\phi}$ yields

$$\bar{\phi} = \pm M \exp(\frac{11}{6}) \exp(-\frac{16\pi^2}{3\lambda_M})$$

Note the result for $\bar{\phi}$ has a non-perturbative dependence on $\lambda_M$, since $\bar{\phi}$ has

an essential singularity at $\lambda_M = 0$. The non-perturbative result is obtained by summing the infinite set of one-loop diagrams.

The second derivative of the effective potential at $B = \bar{\phi}$, using Eq. 12.10.2, is given by

$$\begin{aligned}\frac{d^2V_{\text{eff}}(\bar{\phi}, M)}{dB^2} &= \frac{\lambda_M\bar{\phi}^2}{2!} + \frac{\lambda_M^2\bar{\phi}^2}{64\pi^2}\ln(\frac{\bar{\phi}^2}{M^2}) - \frac{54}{64\pi^2}\lambda_M^2\bar{\phi}^2 \\ &= \frac{\lambda_M^2\bar{\phi}^2}{32\pi^2} > 0\end{aligned}$$

The effective potential has a minimum at $B = \bar{\phi}$ and hence, to one loop, $\bar{\phi}$ minimizes the effective potential. The value of the effective potential is

$$V_{\text{eff}}(\bar{\phi}, M) = -\frac{\lambda_M^2\bar{\phi}^4}{512\pi^2} < 0$$

The result obtained for $\bar{\phi}$ looks non-physical since it directly depends on the arbitrary scale $M$. To verify that in fact the result is non-physical, we check if $\bar{\phi}$ is invariant under a change of scale [2]

$$\frac{d\ln\bar{\phi}}{d\ln M} = \frac{16\pi^2\beta}{3\lambda_M^2} + 1 = 2$$

hence

$$\frac{d\bar{\phi}}{dM} \neq 0$$

and the result is *not* independent of the renormalization scale $M$. It can be shown that the computation of $\bar{\phi}$ using perturbation theory to one loop is not consistent since contributions from higher order loops are equally important.

A more intuitive reason that $V_{\text{eff}}$ does not yield a consistent result is because the $O(\lambda_M)$ and $O(\lambda_M^2)$ terms in Eq. 12.10.2, being of different orders, cannot compensate for the change of scale coming from $\ln(\bar{\phi}^2/M^2)$ is one varies $M$ holding $\bar{\phi}$ fixed.

One way of making $\bar{\phi}$ renormalization group invariant is to introduce two independent coupling constants, by coupling the complex field scalar field to the $U(1)$ gauge field. This theory leads to a consistent spontaneous symmetry breaking – due to the vacuum state being changed by quantum corrections – since the two coupling constants are sufficient to compensate for the variation of $M$ in $\bar{\phi}$.

[2] The other branch yields $d\bar{\phi}/dM = 0$, but both branches need to be renormalization group invariant.

## 12.11 Conclusions

The exact value of the effective action $\Gamma[\phi_c]$, correct to all orders in $\hbar$, contains all the quantum correction to $S[\phi_{cl}]$, and is the reason $\Gamma[\phi_c]$ is called the effective action.

The effective action depends on the deterministic function $\phi_c$, and is the generator functional for all the 1PI connected vertex functions, and provides an exact and complete solution of a quantum field theory. A recursive method was discussed that can generate all the higher order connected vertex functions from the fundamental equation for the propagator.

For the case of a constant $\phi_c = B$, the effective action yields the effective potential $V_{\text{eff}}$, which is the generalization of the classical concept of the potential energy. The effective potential for the $\phi^4$ theory was evaluated to one loop, for both the massive and masless theories. For the case of a constant $\phi_c = B$, the effective potential was evaluated and it was shown that to one loop, that the massless effective potential is renormalization group invariant.

The classical field is determined by $\delta S[\phi_{cl}]/\delta\phi = 0$. This equation is generalized to the quantum case with the equation $\delta\Gamma[\phi_c]/\delta\phi_c = -J$. For the case of spontaneous symmetry breaking, the function $\phi_c = \bar{\phi} \neq 0$ such that $J[\bar{\phi}] \to 0$, and we have $\delta\Gamma[\bar{\phi}]/\delta\phi_c = 0$. This is the exact quantum generalization of the classical extermal condition on the classical action $S$, with $\bar{\phi}$ being the vacuum expectation value of the quantum field to all orders in $\hbar$.

# 13

# Massless scalar QED

## 13.1 Introduction

This chapter is devoted to the study of spontaneous symmetry breaking in one loop massless scalar QED. The Lagrangian of the model, introduced for classical fields in Section 3.6, describes a massive complex scalar field interacting with itself and at the same time coupled to a the electromagnetic vector field.

We will show that, unlike the case of $\phi^4$, symmetry breaking of massless QED can be consistently described by the effective actin calculated to one loop level, and the one loop is sufficient to justify the validity of spontaneous symmetry breaking to all higher loops. The photon field is gauge invariant, and a gauge has to be chosen to study symmetry breaking.

Based on the method developed in Chapter 12, the one loop effective potential is computed in Section 13.3. In Section 13.4, the presence of symmetry breaking due to the one loop corrections is verified. The final section is spent on proving the renormalization group invariance of the theory.

## 13.2 The Lagrangian and Gauge Invariance

The Euclidean Lagrangian density of nonlinear complex scalar field interacting with potential $\lambda(\phi\phi^*)^2$, interacting with the photon field $A_\mu$ is given by

$$\mathcal{L} = -\frac{1}{2}\sum_{\mu}(D_\mu\phi)(D_\mu\phi)^* - \frac{\lambda}{4!}(\phi\phi^*)^2 - \frac{1}{4}\sum_{\mu\nu}F_{\mu\nu}^2 \qquad (13.2.1)$$

$$S = \int d^dx\mathcal{L}_E$$

where $\phi$ is the complex charged scalar field, and

$$D_\mu = \partial_\mu + ieA_\mu \tag{13.2.2}$$

is the covariant derivative of QED.

Define two real scalar field $\phi_1$ and $\phi_2$ such that

$$\phi = \phi_1 + i\phi_2 \tag{13.2.3}$$

In terms of these real fields, the Lagrangian given in Eq. 13.3.1 becomes

$$\mathcal{L} = -\Big\{\frac{1}{2}\sum_\mu [(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + (\partial_\mu\phi_2 + eA_\mu\phi_1)^2] + \frac{\lambda}{4!}(\phi_1^2+\phi_2^2)^2 + \frac{1}{4}\sum_{\mu\nu} F_{\mu\nu}^2\Big\} \tag{13.2.4}$$

Since QED obeys local gauge symmetry, the Lagrangian density 13.2.1 can be shown to be invariant under the gauge transformation.

$$A_\mu \to A_\mu + \partial_\mu\lambda \tag{13.2.5}$$

The covariant derivative gets transformed into

$$D_\mu^{'} = D_\mu + ie\lambda \tag{13.2.6}$$

The scalar field $\phi$ transforms as

$$\phi \to \exp[-ie\lambda]\phi \tag{13.2.7}$$

Note from 13.2.7 that only a complex scalar field can be gauge invariantly coupled to $A_\mu$ since under a gauge transformation the scalar field picks up a complex phase.

The gauge transformation yields

$$D_\mu^{'}\phi^{'} = \exp[-ie\lambda]D_\mu\phi \tag{13.2.8}$$

The Lagrangian in Eq. 13.2.1 gives

$$(D_\mu\phi)(D_\mu\phi)^* = (D_{\mu'}\phi')(D_{\mu'}\phi')^* \quad : \quad \text{Gauge-invariant}$$

## 13.3 The Effective Potential to One Loop

Using the background field method as in Section 12.8, we shift our fields

$$\phi \to \phi + \phi_c \qquad \text{and} \qquad A_\mu \to A_\mu + A_\mu^c \tag{13.3.1}$$

where $\phi_c$ and $A_\mu^c$ are the classical fields.

The effective potential is defined via the effective action $\Gamma_E[A^c_\mu, \phi_c]$ by

$$\exp\{\Gamma[A^c_\mu, \phi_c]\} = \int DAD\phi \exp\{S[A + A^c_\mu, \phi + \phi_c]\}$$

The effective action is gauge invariant under transformations of the background field $A^c_\mu$ in that

$$\Gamma[A^c_\mu, \phi_c] = \Gamma[A^c_\mu + \partial_\mu \lambda, \phi_c] \quad : \quad \text{gauge invariant}$$

In general, one has to choose the *background gauge* so that after gauge-fixing, the effective action is gauge invariant. However, we have a simplification for the computation of the effective action for the following reason. Similar to the case in Section 12.8, the background gauge field is constant, that is

$$A^c_\mu = \text{constant}$$

Consider a gauge transformation

$$A'^{\,c}_\mu = A^c_\mu + \partial_\mu \lambda \tag{13.3.2}$$

Since $A^c_\mu$ is a constant, we choose the following $\lambda$

$$\lambda = -\sum_\mu A^c_\mu x_\mu \quad \Rightarrow \quad \partial_\mu \lambda = -A^c_\mu \quad \Rightarrow \quad A'^{\,c}_\mu = 0$$

Hence, a constant gauge field is gauge equivalent to zero, that is $A^c_\mu = 0$, and yields

$$\Gamma[A^c_\mu, \phi_c] = \Gamma[\phi_c]$$

The fact that the effective potential does *not* depend on the background gauge field allows us to dispense with choosing a background gauge and we can choose the gauge-fixing term as discussed in Chapter 6. The most suitable gauge for the calculation is the *Landau gauge* given by

$$\delta(\partial_\mu A_\mu) \equiv \prod_x \delta(\partial_\mu A_\mu(x))$$

The effective potential, in the Landau gauge, is defined via the effective action $\Gamma[\phi_c]$ by

$$\exp\{\Gamma[\phi_c]\} = \int DAD\phi \delta(\partial_\mu A_\mu) \exp S[A, \phi + \phi_c] \tag{13.3.3}$$

In the Landau gauge, the gauge constraint appears in the path integral measure and the gauge field action simplifies to

$$-\frac{1}{4}\int d^d x \sum_{\mu\nu} F^2_{\mu\nu} = -\frac{1}{2}\int d^d x A_\mu(-\partial^2) A_\mu$$

Thus the effective potential is a function of only the classical scalar field $\phi_c$. The bare action functional for massless scalar QED is given by

$$S = -\int d^4x \frac{1}{2}\Big[(\partial_\mu\phi_1 - eA_\mu(\phi_2+\phi_{2c}))^2 + (\partial_\mu\phi_2 + eA_\mu(\phi_1+\phi_{1c}))^2 + \frac{\lambda_B}{4!}[(\phi_1+\phi_{1c})^2 + (\phi_2+\phi_{2c})^2]^2 - \frac{1}{2}A_\mu\partial^2 A_\mu\Big] \tag{13.3.4}$$

where

$$\phi_c = \phi_{1c} + i\phi_{2c} \tag{13.3.5}$$

Keeping only fields that are quadratic in the quantum fields, we have

$$\begin{aligned}&\frac{1}{2}\int d^4x[\partial_\mu\phi_1 - eA_\mu(\phi_2+\phi_{2c})]^2 \\ &= \int d^4x[(\partial_\mu\phi_1)^2 - 2eA_\mu\phi_{2c}\partial_\mu\phi_1 + e^2A^2\phi_{2c}^2] + O(\phi^2 A)\end{aligned} \tag{13.3.6}$$

with

$$A^2 = \sum_{\mu=0}^{3} A_\mu^2 \tag{13.3.7}$$

Due to the Landau gauge, we have

$$\int d^4x\phi_{2c}(\partial_\mu\phi_1)A_\mu = \text{surface term} = 0 \tag{13.3.8}$$

and Eq.13.3.6 reduces to

$$\frac{1}{2}\int d^4x[(\partial_\mu\phi_1)^2 + e^2A^2\phi_{2c}^2] \tag{13.3.9}$$

Similarly keeping only the quadratic terms in the quantum field yields

$$\frac{1}{2}\int d^4x[\partial_\mu\phi_2 + eA_\mu(\phi_1+\phi_{1c})^2] \to \frac{1}{2}\int d^4x[(\partial_\mu\phi_2)^2 + e^2A^2\phi_{1c}^2] \tag{13.3.10}$$

For the self-interaction term, expanding the quartic term to only $O(\phi^2)$ for the quantum field yields

$$\begin{aligned}&\frac{\lambda_B}{4!}[(\phi_1+\phi_{1c})^2 + (\phi_2+\phi_{2c})^2]^2 \\ \to\ &\frac{\lambda_B}{4!}[2\phi_c^2(\phi_1^2+\phi_2^2) + 4\phi_1^2\phi_{1c}^2 + 4\phi_2^2\phi_{2c}^2 + 8\phi_1\phi_2\phi_{1c}\phi_{2c}] \\ =\ &\frac{\lambda_B}{4!}\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\begin{pmatrix}2\phi_c^2+4\phi_{1c}^2 & 4\phi_{1c}\phi_{2c} \\ 4\phi_{1c}\phi_{2c} & 2\phi_c^2+4\phi_{2c}^2\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \\ =\ &\frac{\lambda_B}{4!}\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}U\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}\end{aligned} \tag{13.3.11}$$

where

$$\phi_c^2 = \phi_{1c}^2 + \phi_{2c}^2 \tag{13.3.12}$$

Since U is a symmetric matrix, it can be diagonalized by a similarity transformation. Hence we have

$$U = M \begin{pmatrix} 2\phi_c^2 & 0 \\ 0 & 6\phi_c^2 \end{pmatrix} M^{-1} \tag{13.3.13}$$

with

$$M = \frac{1}{\phi_c} \begin{pmatrix} \phi_{2c} & -\phi_{1c} \\ \phi_{1c} & \phi_{2c} \end{pmatrix} \tag{13.3.14}$$

In the diagonalized frame, Eq. 13.3.11 yields

$$\frac{\lambda_B}{4!}[2\phi_c^2\phi_1'^2 + 6\phi_c^2\phi_2'^2] \tag{13.3.15}$$

and

$$(\partial_\mu \phi_1')^2 + (\partial_\mu \phi_2')^2 = (\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 \tag{13.3.16}$$

with

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \frac{1}{\phi_c} \begin{pmatrix} \phi_{2c} & -\phi_{1c} \\ \phi_{1c} & \phi_{2c} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{13.3.17}$$

Since $M$ is orthogonal, the measure transforms trivially

$$D\phi_1 D\phi_2 = D\phi_1' D\phi_2' \tag{13.3.18}$$

The one loop connected functional is hence, dropping the primes

$$\begin{aligned} W[\phi_c] &= -\int d^4x \frac{\lambda_B}{4!}\phi_c^4 + N \ln[\int DAD\phi_1 D\phi_2 \delta(\partial_\mu A_\mu) \exp(S[\phi_c]) \\ &= -\int d^4x V_{eff}(\phi_c) \end{aligned} \tag{13.3.19}$$

Where $S[\phi_c]$ is the one loop action functional given by

$$\begin{aligned} S[\phi_c] = -\int d^4x \Big\{ &\frac{1}{2}[(\partial_\mu \phi_1)^2 + \frac{1}{6}\lambda_B \phi_c^2 \phi_1^2] + \frac{1}{2}[(\partial_\mu \phi_2)^2 + \frac{1}{2}\lambda_B \phi_c^2 \phi_2^2] \\ &+ \frac{1}{2}(-A_\mu \partial^2 A_\mu + e^2 \phi_c^2 A^2) \Big\} \end{aligned} \tag{13.3.20}$$

The summing of all one loop diagrams can be done by converting the computation of the effective potential $V_{eff}$ into a first order differential equation, as exemplified by the calculation in Section 12.8. The method

used in tSection 12.8 applies equally well to the present case. Thus from 13.3.19 and 13.3.20

$$\frac{dV_{eff}}{d\phi_c} = \frac{\lambda_B}{3!}\phi_c^3 - \frac{1}{\mathcal{V}}E\left[\frac{dS}{d\phi_c}\right] \tag{13.3.21}$$

where $\mathcal{V}$ is spacetime volume and expectation value is denoted by

$$E[Y(\phi)] = \frac{1}{Z}\int DA_\mu D\phi_1 D\phi_2 \delta(\partial_\mu A_\mu)\exp\{S[\phi_c]\}Y(\phi) \tag{13.3.22}$$

Eq. 13.3.21 is explicitly given, from Eq. 13.3.20, by the following

$$\frac{dV_{eff}}{d\phi_c} = \frac{\lambda_B}{3!}\phi_c^3 + \frac{1}{\mathcal{V}}\frac{\lambda_B}{3!}\phi_c E[\phi_1^2] + \frac{1}{\mathcal{V}}\frac{\lambda_B}{2}\phi_c E[\phi_2^2] + \frac{1}{\mathcal{V}}e^2\phi_c E[A^2] \tag{13.3.23}$$

Note that the computation of $dV_{eff}/d\phi_c$ has been reduced to the evaluation of the two point function, as was the case for the real scalar field discussed in Section 12.8.

From Eq. 13.3.20, define the effective for the scalar fields by the following

$$m_1^2 = \frac{1}{6}\lambda_B\phi_c^2 \;\;;\;\; m_2^2 = \frac{1}{2}\lambda_B\phi_c^2$$

The second term on the right hand side of 13.3.23 yields

$$\frac{1}{\mathcal{V}}\frac{\lambda_B\phi_c}{3!}E[\phi_1^2] = \frac{\lambda_B\phi_c}{6}\left(\frac{1}{2\pi}\right)^4\int\frac{d^4k}{k^2+m_1^2}$$

Recall from Eq. 9.8.7

$$\int\frac{d^dk}{(2\pi)^d(k^2+m^2)} = -\frac{m^2}{16\pi^2}[\frac{2}{\epsilon}+\varphi(2)-\ln(\frac{m^2}{4\pi\mu^2})] + O(\epsilon)$$

Hence, using dimensional regularization, we obtain

$$\begin{aligned}\frac{1}{\mathcal{V}}\frac{\lambda_B\phi_c}{3!}E[\phi_1^2] &= \lim_{\epsilon\to 0} -\frac{\lambda_B m_1^2\phi_c}{3\cdot 32\pi^2}(\frac{2}{\epsilon}+\varphi(2)-\ln\frac{m_1^2}{4\pi\mu^2}) \\ &= \lim_{\epsilon\to 0} -\frac{1}{9}\left(\frac{\lambda_B^2\phi_c^3}{64\pi^2}\right)(\frac{2}{\epsilon}+\varphi(2)-\ln\frac{\lambda_B\phi_c^2}{24\pi\mu^2})\end{aligned} \tag{13.3.24}$$

Similarly, for the $\phi_2$ field, we have

$$\begin{aligned}\frac{1}{\mathcal{V}}\frac{\lambda_B}{2}\phi_c E[\phi_2^2] &= \lim_{\epsilon\to 0} -\frac{\lambda_B m_2^2\phi_c}{32\pi^2}(\frac{2}{\epsilon}+\varphi(2)-\ln\frac{m_2^2}{4\pi\mu^2}) \\ &= \lim_{\epsilon\to 0} -\left(\frac{\lambda_B^2\phi_c^3}{64\pi^2}\right)(\frac{2}{\epsilon}+\varphi(2)-\ln\frac{\lambda_B\phi_c^2}{8\pi\mu^2})\end{aligned} \tag{13.3.25}$$

The fourth term requires the propagator of the gauge field, which is given in Eq. 6.4.9 for the Landau gauge. For the massive case we have

$$E[A_\mu(x)A_\nu(y)] = (\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2})\frac{1}{(-\partial^2)+m^2}\delta_{x-y}$$
$$\Rightarrow \sum_\mu E[A_\mu(x)A_\mu(y)] = 3\int \frac{d^4k}{(2\pi)^4(k^2+m^2)}e^{ik(x-y)}$$
$$\Rightarrow E[A^2(x)] = 3\int \frac{d^4k}{(2\pi)^4(k^2+m^2)}$$

Hence, using dimensional regularization, we obtain

$$\frac{1}{\mathcal{V}}e^2\phi_c E[A^2] = 3e^2\phi_c\left(\frac{1}{2\pi}\right)^{4-\epsilon}\int\frac{d^{4-\epsilon}k}{k^2+e^2\phi_c^2}$$
$$= \lim_{\epsilon\to 0} -\frac{3e^4\phi_c^3}{16\pi^2}(\frac{2}{\epsilon}+\varphi(2)-\ln\frac{e^2\phi_c^2}{4\pi\mu^2}) \qquad (13.3.26)$$

Writing $\lambda_B = \lambda_M + \delta\lambda$, where again $\lambda$ denoting the renormalized coupling constant, to one loop,

$$\frac{dV_{eff}}{d\phi_c} = \lim_{\epsilon\to 0}\frac{\lambda_M}{3!}\phi_c^3 + \frac{\delta\lambda}{3!}\phi_c^3 - \phi_c^3\Big[\left(\frac{10}{9}\right)(\frac{\lambda_M^2}{64\pi^2}) + \frac{3e^4}{16\pi^2}\Big]\times[\frac{2}{\epsilon}+\varphi(2)]$$
$$+\left(\frac{1}{9}\right)\frac{\lambda_M^2\phi_c^3}{64\pi^2}\ln\frac{\lambda_M\phi_c^2}{24\pi\mu^2} + \frac{\lambda_M^2\phi_c^3}{64\pi^2}\ln\frac{\lambda_M\phi_c^2}{8\pi\mu^2} + \frac{3e^4\phi_c^3}{16\pi^2}\ln\frac{e^2\phi_c^2}{4\pi\mu^2} \qquad (13.3.27)$$

Applying the renormalization prescription

$$\frac{d^4V_{eff}}{d\phi_c^4}|_{\phi_c=M} = \lambda_M \qquad (13.3.28)$$

we get[1]

$$\delta\lambda = [\frac{2}{\epsilon}+\varphi(2)]\cdot[\frac{5\lambda_M^2}{48\pi^2}+\frac{9e^4}{8\pi^2}] - \frac{\lambda_M^2}{96\pi^2}\ln\frac{\lambda_M M^2}{6(4\pi\mu^2)}$$
$$-\frac{3\lambda_M^2}{32\pi^2} - \frac{9e^4}{8\pi^2}\ln\frac{e^2M^2}{4\pi\mu^2} - \frac{55\lambda_M^2}{144\pi^2} - \frac{33e^4}{8\pi^2} \qquad (13.3.29)$$

Substituting into 13.3.27 leads to

$$\frac{dV_{eff}}{d\phi_c} = \frac{\lambda_M\phi_c^3}{3!} + \phi_c^3[\frac{5\lambda_M^2}{288\pi^2}+\frac{3e^4}{16\pi^2}]\ln\frac{\phi_c^2}{M^2} - \phi_c^3[\frac{55\lambda_M^2}{864\pi^2}+\frac{11e^4}{16\pi^2}] \qquad (13.3.30)$$

Integrating 13.3.30 with $V_{eff}(0) = 0$, we get the one loop renormalized

[1] Note $d^3(x^3\ln(x^2/a^2))/dx^3 = 6\ln(x^2/a^2)+22$.

effective potential

$$V_{eff}(\phi_c) = \frac{\lambda_M \phi_c^4}{4!} + \frac{\phi_c^4}{4}[\frac{5\lambda_M^2}{288\pi^2} + \frac{3e^4}{16\pi^2}] \ln \frac{\phi_c^2}{M^2} - \frac{125\lambda_M^2}{6915\pi^2} - \frac{25e^4}{128\pi^2}$$
$$= \frac{\lambda_M \phi_c^4}{4!} + \phi_c^4[\frac{5\lambda_M^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}] \times [\ln \frac{\phi_c^2}{M^2} - \frac{25}{6}] + O(\lambda^3, e^4, \lambda e^3) \quad (13.3.31)$$

The expression obtained above for $V_{eff}(\phi_c)$ agrees with the result obtained by Coleman and Weinberg (1973).

The method of integrating a first order differential equation for obtaining $V_{eff}(\phi_c)$, both for scalar QED and $\phi^4$ theory, is more transparent and computationally more tractable than the method used by Coleman and Weinberg (1973) – which consists of an infinite summation of all one loop Feynman diagrams.

## 13.4 Spontaneous Symmetry Breaking

The one loop effective potential is given by

$$V_{eff}[\phi_c] = \frac{\lambda_M}{4!}\Phi_c^4 + \phi_c^4(\frac{5}{1152\pi^2}\lambda_M^2 + \frac{3e^4}{64\pi^2})[\ln \frac{\phi_c^2}{M^2} - \frac{25}{6}] \quad (13.4.1)$$

The ground state is determined by the equation

$$\frac{dV_{eff}}{d\phi_c}|_{\phi_c=\bar{\phi}} = 0$$

Leading to a non-trivial solution

$$\frac{\lambda_M}{3!} + 4(\frac{5\lambda_M^2}{1152\pi^2} + \frac{3e^4}{64\pi^2})[\ln \frac{\bar{\phi}^2}{M^2} - \frac{25}{6}] + 2[\frac{5\lambda_M^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}] = 0 \quad (13.4.2)$$

Multiply 13.4.2 by $\frac{\bar{\phi}^4}{4}$, we see that

$$V_{eff}(\bar{\phi}) = -\frac{\bar{\phi}^4}{2}[\frac{5\lambda_M^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}] < 0 \quad (13.4.3)$$

From 13.4.2 the non-trivial vacuum state is hence given by

$$(\frac{\lambda_M}{6} - \frac{55\lambda_M^2}{864\pi^2} - \frac{11e^4}{16\pi^2}) + (\frac{5\lambda_M^2}{288\pi^2} + \frac{3e^4}{16\pi^2}) \ln \frac{\bar{\phi}^2}{M^2} = 0 \quad (13.4.4)$$

or

$$\bar{\phi} \sim M \exp(-\frac{c}{\lambda}) \quad (13.4.5)$$

which is again essentially a non-perturbative result due to the $1/\lambda$ singularity. As is shown by Coleman and Weinberg (1973), the one loop contribution dominates over all higher order loop corrections. The calculation shows that

the spontaneous breaking of symmetry of the theory in fact takes place and is consistently described by the one-loop result.

## 13.5 Renormalization group invariance

Changing the renormalization scale to $\phi_c = M', \lambda \to \lambda_{M'}$,

$$\frac{d^4 V_{eff}}{d\phi_c^4}|_{\phi_c = M'} = \lambda_{M'} \tag{13.5.1}$$

Using 13.3.31, we have

$$\lambda_M + [\frac{5\lambda_M^2}{48\pi^2} + \frac{9e^4}{8\pi^2}] \times [\ln \frac{M'^2}{M^2} - \frac{25}{6}] + \frac{125\lambda_M^2}{288\pi^2} + \frac{75e^4}{16\pi^2} = \lambda_{M'}$$

or

$$\lambda_{M'} = \lambda_M + [\frac{5\lambda_M^2}{48\pi^2} + \frac{9e^4}{8\pi^2}] \ln \frac{M'^2}{M^2} \tag{13.5.2}$$

Hence

$$\beta = \frac{5\lambda^2}{24\pi^2} + \frac{9e^4}{4\pi^2} + O(\lambda^3, e^4, \lambda e^3) \tag{13.5.3}$$

Note the $\beta$ function for the scalar field is the special of $N = 2$ of the earlier result for the $O(N)$ symmetric scalar field obtained in Eq. 10.8.2.

Differentiating the effective potential 13.3.31 with respect to $\ln M$ yields

$$\frac{dV_{eff}}{d\ln M} = \frac{\beta\phi_c^4}{4!} - \phi_c^4[\frac{5\lambda^2}{576\pi^2} + \frac{3e^4}{32\pi^2}] + O(\lambda^3, \lambda e^3, e^4)$$

as given by Coleman and Weinberg (1973).

Note it can be shown that for scalar QED[2]

$$\beta(e) = M\frac{de}{dM} = O(e^3)$$

and hence under a change of $M$, the change in $e$ given by $\beta(e)$ contributes $dV_{eff}/d\ln M$ only to $O(e^6)$ and can be ignored.

Substitution of 13.5.3 gives, at the one loop level

$$\frac{dV_{eff}}{d\ln M} = \frac{dV_{eff}}{dM} = 0 \tag{13.5.4}$$

which is an expected result.

[2] For photons coupled to electrons by charge $e$, the beta function is given by $\beta = e^3/(12\pi^2)$.

Also, a similar analysis on the vacuum state 13.4.4 gives, to one loop accuracy

$$\frac{\beta}{6} - [\frac{5\lambda^2}{144\pi^2} + \frac{3e^4}{8\pi^2}] + [\frac{5\lambda^2}{144\pi^2} + \frac{3e^4}{8\pi^2}]\frac{d\ln\bar{\phi}}{d\ln M} = 0$$

Inserting the value of $\beta$ leads to the crucial result

$$\frac{d\ln\bar{\phi}}{d\ln M} = \frac{d\bar{\phi}}{dM} = 0 \tag{13.5.5}$$

Thus unlike the case of massless $\phi^4$, RG invariance of $\bar{\phi}$ is preserved in massless scalar QED. This is connected with the fact that contribution to the effective potential diminish as one goes to higher orders in the present model, and hence RG invariance has to be valid for each term in the perturbative expansion.

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