Advanced QFT Discussion Notes

Zhang Yuan-Yuan, Xing Wen-Jing, Wang Xiao-Dan, Chen Lin, Wang Lei, Liu Yu-Fei,...

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Contents

1 Timetable

We invite every participant to talk about one topic every Wednesday from 7pm to 9pm, the note should be uploaded by speaker before Monday 5pm for others to have a look. The book we use are reference [2], [1], [3].

The preliminary timetable is as follows:

Week 1(Yuanyuan Zhang): ϕ^4 renormalization (Chap 15 of [2], Chap 10.2 of [1], as so on...)

Week 2(Wen Jing Xing): QED renormalization I : vacuum polarization, anomalous magnetic moment (Chap 16, 17 of [2])

Week 3(Shi Yu): QED renormalization II: mass renormalization, renormalized perturbation theory (Chap 18, 19 of [2])

Week 4(): Infrared Divergence and Renormalizability, some cover on non-renormalizable theory (Chap 20, 21, 22 of [2])

Week 5(Feng Lei Liu) : QCD beta function calculation (Chap 16.5 of [1])

Week 6(): The renormalization group (Chap 23 of [2],Chap 12 of [1])

Week 7 (Xiaodan Wang): Dimensional transmutation- Gross Neveu Model

Week 8: Non-Abelian gauge theory and Faddeev-Popov method, BRST symmetry (Chap 16.1-16.4)

Week 9: Spontaneous Symmetry Breaking I: linear sigma model

Week 10: Spontaneous Symmetry Breaking I: non-linear sigma model

Week 11 (Xiaodan Wang): Spontaneous broken gauge theories: Chiral anomaly

...

2 Notes

2.1 ϕ^4 renormalization (Yuanyuan Zhang)

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0 - \frac{1}{2}m_0^2\varphi_0^2 - \frac{\lambda_0}{4!}\varphi_0^4 \tag{1}$$

Renormalization : absorb divergences into redefinition of coupling constant λ_R, mass m_R and field ϕ_R.

2.1.1 Simplier Version

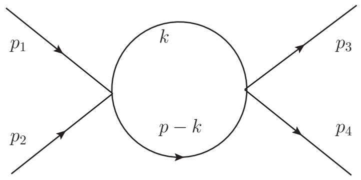
bare/on-shell perturbative theory
Massless real scalar field

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{\lambda}{4!}\phi^4 = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!}\phi^4 \tag{2}$$

Matrix element (quantity to renormalize), renormalize coupling constant λ. Leading order matrix element:

$i\mathcal{M}_1 =$  $= -i\lambda.$

The s-channel loop correction (ignore t-channel and u-channel) to matrix element



$$i\mathcal{M}_2=\frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p-k)^2}\quad\left(p=p_1+p_2, s=p^2\right) \tag{3}$$

trick to get answer, take the derivative with respect to s

$$\frac{\partial}{\partial s}\mathcal{M}_2(s)=\frac{p^\mu}{2s}\frac{\partial}{\partial p^\mu}\mathcal{M}_2(s)=\frac{i\lambda^2}{2s}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{\left(p^2-p\cdot k\right)}{(p-k)^4}=-\frac{\lambda^2}{32\pi^2}\frac{1}{s}\rightarrow\mathcal{M}_2=-\frac{\lambda^2}{16\pi^2}\ln s+c(=\frac{\lambda^2}{16\pi^2}\ln\Lambda^2) \tag{4}$$

define renormalized coupling λ_R as the value of the matrix element at a particular $s=s_0$

$$\lambda_R\equiv-\mathcal{M}\left(s_0\right)=\lambda+\frac{\lambda^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2}+\cdots(\text{Renormalization Condition!!!}) \tag{5}$$

Expand λ in terms of λ_R

$$\lambda_R=\left(\lambda_R+a\lambda_R^2+\cdots\right)+\frac{\left(\lambda_R+a\lambda_R^2+\cdots\right)^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2}+\cdots \tag{6}$$

we get

$$\lambda=\lambda_R-\frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2}+\cdots \tag{7}$$

matrix element at another scale

$$\begin{aligned}\mathcal{M}(s)&=-\lambda-\frac{\lambda^2}{32\pi^2}\ln\frac{s}{\Lambda^2}\\&=-\lambda_R-\frac{\lambda_R^2}{32\pi^2}\ln\frac{s}{s_0}+\cdots\end{aligned} \tag{8}$$

for any s that is finite order-by-order in perturbation theory. By the way, the logarithmic behavior is a characteristic of loop effects – tree-level graphs only give you rational polynomials in momenta and couplings, never logarithms.

renormalized perturbative theory write Lagrangian in terms of λ_R and counter term in the beginning

$$\mathcal{L}=-\frac{1}{2}\phi\square\phi-\frac{\lambda_R}{4!}\phi^4-\frac{\delta_\lambda}{4!}\phi^4 \tag{9}$$

working to order λ_R^2, the amplitude

$$\mathcal{M}(s)=-\lambda_R-\delta_\lambda-\frac{\lambda_R^2}{32\pi^2}\ln\frac{s}{\Lambda^2}+\mathcal{O}\left(\lambda_R^4\right) \tag{10}$$

choose $\delta_\lambda=-\frac{\lambda_R^2}{32\pi^2}\ln\frac{s_0}{\Lambda^2}$ makes

$$\begin{aligned}\mathcal{M}\left(s_0\right)&=-\lambda_R\\\mathcal{M}(s)&=-\lambda_R+\frac{\lambda_R^2}{32\pi^2}\ln\frac{s}{s_0}\end{aligned} \tag{11}$$

2.1.2 Complete Version

Write the Lagrangian in terms of renormalized parameter and counter term,

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}\left(\partial_\mu\phi_R\right)^2-\frac{1}{2}m^2\phi_R^2-\frac{\lambda_R\mu^\epsilon}{4!}\phi_R^4\\&+\frac{1}{2}\delta_\phi\left(\partial_\mu\phi_R\right)^2-\frac{1}{2}\delta_m\phi_R^2-\frac{\delta_\lambda}{4!}\phi_R^4\end{aligned} \tag{12}$$

The relation of vertex function and correlation function:

$$\begin{aligned}G_2^{-1}(p)&=\Gamma_2(p)\\G_4(p_1\cdots p_4)&\equiv\frac{\delta\left(\sum_i p_i\right)}{\prod_i\left(p_i^2+m^2\right)}\Gamma_4\left(p_1\cdots p_4\right)+O\left(\lambda^4\right)\end{aligned} \tag{13}$$

Renormalization condition

$$\Gamma_2|_{p=0} = m_R^2, \frac{d\Gamma_2}{dp^2}|_{p=0} = 1$$
$$\Gamma_4|_{p=0} = -\lambda_R \tag{14}$$

To get the correlation for massive scalar ϕ^4 field, we should first learn about functional methods. The scalar field Lagrangian in Euclidean space and the action

$$\begin{aligned}\mathcal{L} &= -\left[\frac{1}{2}\sum_i (\partial_i\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda\mu^\epsilon}{4!}\phi^4\right] - [\frac{1}{2}\delta_\phi \sum_i (\partial_i\phi)^2 + \frac{1}{2}\delta_m\phi^2 + \frac{\delta_\lambda}{4!}\phi^4] \\ S &= \int d^4x\mathcal{L}\end{aligned} \tag{15}$$

here $i = 0, 1, 2, 3$. The Fourier transform of scalar field

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\varphi_p = \int_p e^{ipx}\varphi_p \quad ; \quad \varphi_p = \int d^4x e^{-ipx}\phi(x) \tag{16}$$

Partition function

$$\begin{aligned}Z &= \int D\phi e^S = \int D\varphi e^{S_0+S_I} \\ S_0 &= -\frac{1}{2}\int d^4x \left[\sum_i (\partial_i\phi)^2 + m^2\phi^2\right] = -\frac{1}{2}\int_p \varphi_{-p}\left(p^2+m^2\right)\varphi_p \\ S_I &= -\frac{\lambda}{4!}\int d^4x\phi^4 = -\frac{\lambda}{4!}\int_{p_1\cdots p_4} \delta\left(p_1+p_2+p_3+p_4\right)\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\varphi_{p_4}\end{aligned} \tag{17}$$

Consider the case of weak coupling, $\lambda \ll 1$

$$\begin{aligned}Z &= \int D\varphi e^{S_0+S_I} = \int D\varphi e^{S_0}[1 + S_I + \frac{1}{2}S_I^2 + O\left(\lambda^3\right)] = Z_0(1 + Z_1 + Z_2 + \cdots) \\ Z_1 &= \frac{1}{Z_0}\int D\varphi e^{S_0} S_I \\ &= \frac{1}{Z_0}\int D\varphi e^{S_0}\left(-\frac{\lambda}{4!}\int_{p_1\cdots p_4}\delta\left(p_1+p_2+p_3+p_4\right)\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\varphi_{p_4}\right) \\ &= -\frac{\lambda}{4!}\int_{p_1\cdots p_4}\delta\left(p_1+p_2+p_3+p_4\right) E_0[\varphi_{p_1}\varphi_{p_2}\varphi_{p_3}\varphi_{p_4}]\end{aligned} \tag{18}$$

The calculation of correlation function use Wick's theorem. We first discuss Gaussian path integration of gaussian random variable x_i.

$$\begin{aligned}Z[J] &= \frac{1}{Z_0}\int Dx \exp\left\{-\frac{1}{2}\sum_{mn} x_m M_{mn} x_n + \sum_n x_n J_n\right\} \\ &= \exp\left(\frac{1}{2}\sum_{nm} J_n M^{-1}_{nm} J_m\right) = e^{W[J]} \\ S_0 &= -\frac{1}{2}\sum_{mn} x_m M_{mn} x_n \quad ; \quad Z_0 = \int Dx \exp\{S_0\} \\ E_0\left[x_i x_j\right] &\equiv \overbrace{x_i x_j} = \frac{1}{Z_0}\int Dx x_i x_j e^{S_0} = \frac{\partial^2}{\partial J_i \partial J_j} Z[J]\Big|_{J=0} \\ &= M^{-1}_{ij} \equiv D_{ij} : \text{ contraction} \\ E_0\left[x_1x_2x_3x_4\right] &= \frac{1}{Z}\int Dx e^{S_0} x_1x_2x_3x_4 = \frac{\partial^4 e^{W[J]}}{\partial J_1 \cdots J_4} \\ &= D_{42}D_{31} + D_{41}D_{32} + D_{43}D_{21}\end{aligned} \tag{19}$$

Wick theorem: generalization of the result for Gaussian random variables x_i to quantum fields ϕ

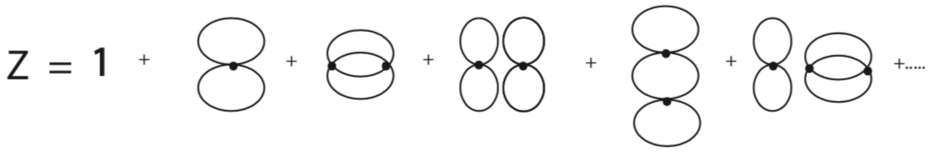
$$
\begin{aligned}
S_0 &= -\frac{1}{2}\int_p \varphi_{-p}\left(p^2+m^2\right)\varphi_p \quad ; \quad Z_0=\int D\varphi e^{S_0} \\
Z[J] &= e^{W[J]} = \exp\left\{\frac{1}{2}\int d^4x d^4y J(x) D(x-y) J(y)\right\} \\
&= \exp\left\{\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} J_{-p}\frac{1}{p^2+m^2} J_p\right\} \quad ; \quad J(x)=\int \frac{d^4p}{(2\pi)^4} e^{ipx} J_p \\
E_0[\phi(x)\phi(y)] &= D(x-y) = \int_p e^{ip(x-y)} D_p \quad : \quad D_p=\frac{1}{p^2+m^2} \\
&= \int_{pp'} e^{ipx+ip'y} D_p \delta(p+p') \\
E_0\left[\varphi_p\varphi_{p'}\right] &= \frac{\delta(p+p')}{p^2+m^2} \\
E_0\left[\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\right] &= D\left(x_1-x_2\right) D\left(x_3-x_4\right)+D\left(x_1-x_3\right) D\left(x_2-x_4\right) \\
&\quad + D\left(x_1-x_4\right) D\left(x_2-x_3\right)
\end{aligned}
\tag{20}
$$

We get our correlation from partition function, we renormalize partition function first. Calculate Z, we get Z_1 first.

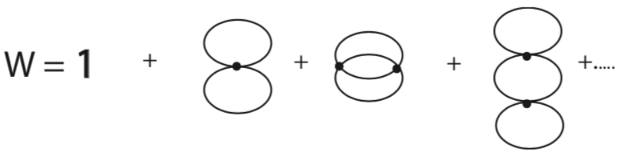
$$
\begin{aligned}
Z_1 &= -\frac{\lambda\cdot 3}{4!}\int_{p_1\cdots p_4}\delta^4\left(\sum_i p_i\right)\frac{\delta\left(p_1+p_2\right)}{p_1^2+m^2}\frac{\delta\left(p_3+p_4\right)}{p_3^2+m^2} \\
&= -\frac{\lambda}{8}\delta^4(0)\left[\int_p \frac{1}{p^2+m^2}\right]^2
\end{aligned}
\tag{21}
$$

(a) (b)

first contract two field $\varphi(p_1), \varphi(p_2)$, then contract the other two $\varphi(p_3), \varphi(p_4)$.



All contribution to Z



All contribution to W

Taking the logarithmic of Z removes all the disconnected diagrams $Z=e^W$, $W[J]$ called generating functional. The exact n-point connected correlation functions, the green function, derived from $W[J]$

$$
\begin{aligned}
G\left(x_1\cdots x_N\right) \equiv E\left[\phi\left(x_1\right)\cdots\phi\left(x_N\right)\right]_c &= \left.\frac{\delta^N W[J]}{\delta J\left(x_1\right)\cdots\delta J\left(x_N\right)}\right|_{J=0} \\
W[J] &= \sum_{N=1}^{\infty}\frac{1}{N!}\int dx_1\cdots dx_N G\left(x_1\cdots x_N\right) J\left(x_1\right)\cdots J\left(x_N\right)
\end{aligned}
\tag{22}
$$

Notice difference between E_0 and E

$$
E_0[O]=\int D\varphi O e^{S_0+J\varphi} \quad ; \quad E[O]=\int D\varphi O e^{S_0+S_I+J\varphi}
\tag{23}
$$

$$
\begin{aligned}
G\left(x_1,x_2\right) &= E\left[\phi\left(x_1\right)\phi\left(x_2\right)\right]_c = \left.\frac{\delta^2 W[J]}{\delta J\left(x_1\right)\delta J\left(x_2\right)}\right|_{J=0} \\
&= \frac{\delta^2}{\delta J\left(x_1\right)\delta J\left(x_2\right)}\left\{\ln\left(\int D\phi e^{S+J\phi}\right)\right\}\Bigg|_{J=0} = \frac{\delta}{\delta J\left(x_1\right)}\left\{\frac{1}{Z(J)}\int D\phi\phi\left(x_2\right)e^{S+J\phi}\right\} \\
&= \frac{1}{Z}\int D\phi\phi\left(x_1\right)\phi\left(x_2\right)e^{S} - \left\{\frac{1}{Z}\int D\phi\phi\left(x_1\right)e^{S}\right\}\left\{\frac{1}{Z}\int D\phi\phi\left(x_2\right)e^{S}\right\} \\
&= E\left[\phi\left(x_1\right)\phi\left(x_2\right)\right] - E\left[\phi\left(x_1\right)\right]E\left[\phi\left(x_2\right)\right] \quad (E[\varphi(x)]=0) \\
&= E\left[\phi\left(x_1\right)\phi\left(x_2\right)\right]
\end{aligned} \tag{24}
$$

We can calculate two-point correlation function in momentum space

$$
\begin{aligned}
G_2\left(p_1,p_2\right) &= \frac{1}{Z}\int D\varphi e^{S_0}\varphi\left(p_1\right)\varphi\left(p_2\right)\left[1+S_I+\frac{1}{2!}S_I^2+\cdots\right] \\
&= G_2^0+G_2^1+G_2^2+\cdots
\end{aligned} \tag{25}
$$

The calculation of different order contribution

$$
\begin{aligned}
G_2^0\left(p_1,p_2\right) &= \frac{\delta\left(p_1+p_2\right)}{(1+\delta_\phi)p_1^2+(m^2+\delta_m)} \\
G_2^1\left(p_1,p_2\right) &= E_0\left[\varphi\left(p_1\right)\varphi\left(p_2\right)S_I\right]_c \\
&= -\frac{\lambda}{4!}\int D\varphi\int_{k_1\cdots k_4} e^{S_0}\left[\varphi\left(p_1\right)\varphi\left(p_2\right)\delta\left(\sum_i k_i\right)\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\right]_c \\
&= -12\cdot\frac{\lambda}{2}\int_{k_1\cdots k_4}\frac{\delta\left(p_1+k_1\right)\delta\left(p_2+k_2\right)\delta\left(k_3+k_4\right)}{\left(p_1^2+m^2\right)\left(p_2^2+m^2\right)\left(k_3^2+m^2\right)} \\
&= -\frac{\lambda}{2}\frac{\delta\left(p_1+p_2\right)}{\left(p_1^2+m^2\right)^2}\int_k\frac{1}{\left(k^2+m^2\right)} \\
&= -\frac{\lambda}{2}\frac{\delta\left(p_1+p_2\right)}{\left[(1+\delta_\phi)p_1^2+(m^2+\delta_m)\right]^2}+J_1
\end{aligned} \tag{26}
$$

To one loop,

$$
\begin{aligned}
G_2\left(p_1,p_2\right) =& \delta\left(p_1+p_2\right)\left[\frac{1}{\left(p_1^2+m^2\right)} - \frac{\lambda\mu^\epsilon}{2}\frac{J_1}{\left(p_1^2+m^2\right)^2}\right]+O\left(\lambda^2\right) \\
\simeq& \frac{\delta\left(p_1+p_2\right)}{(1+\delta_\phi)p_1^2+(m^2+\delta_m)+\frac{\lambda\mu^\epsilon}{2}J_1}+O\left(\lambda^2\right)
\end{aligned} \tag{27}
$$

divergence in integration J_1, by dimensional regularization

$$
\begin{aligned}
J_1(\epsilon) &= \int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+m^2\right)} = m^2\frac{1}{m^\epsilon}\frac{(4\pi)^{\epsilon/2}}{(4\pi)^2}\Gamma\left(\frac{\epsilon}{2}-1\right) \\
&= \frac{m^2}{16\pi^2}\left(\frac{4\pi}{m^2}\right)^{\epsilon/2}\left[-\frac{2}{\epsilon}-\varphi(2)+O(\epsilon)\right] \quad ;\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon}+\varphi(n+1)\right]+O(\epsilon) \\
&= -\frac{m^2}{8\pi^2 m^\epsilon}\frac{1}{\epsilon} - \frac{m^2}{16\pi^2}\left[\varphi(2)+\ln 4\pi\right]+O(1) \quad ;\varphi(x) = \frac{d\ln\Gamma(x)}{dx}
\end{aligned} \tag{28}
$$

From renormalization condition, to cancel the divergence and constant, $\overline{MS}$ scheme gives

$$
\delta_m = -\frac{\lambda_R\mu^\epsilon}{2}J_1 = \frac{\lambda_R m_R^2}{16\pi^2}\left(\frac{\mu}{m_R}\right)^\epsilon\frac{1}{\epsilon}+\frac{\lambda_R\mu^\epsilon m^2}{32\pi^2}\left[\varphi(2)+\ln 4\pi\right] \quad \overline{MS} \quad ; \quad \delta_\phi = 0 \quad (1-\text{loop}) \tag{29}
$$

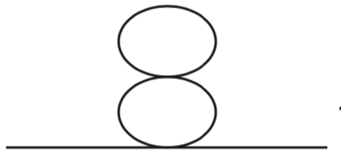


Figure 1: I_3

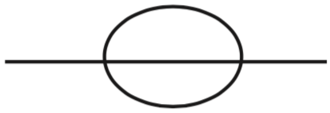


Figure 2: I_4

To two-loop, we have

$$
\begin{aligned}
G_2^2 =& E\left[\varphi\left(p_1\right)\varphi\left(p_2\right)\frac{1}{2!}S_I^2\right]_c \\
=&\frac{\lambda^2}{(4!)^2 2!}\int\delta\left(\sum_i k_i\right)\delta\left(\sum_i k_i'\right)E\left[\varphi\left(p_1\right)\varphi\left(p_2\right)\varphi\left(k_1\right)\cdots\varphi\left(k_n\right)\varphi\left(k_1'\right)\cdots\varphi\left(k_n'\right)\right] \\
=&\frac{\delta\left(p+p'\right)}{\left(p^2+m^2\right)^2}\left[I_3+I_4\right] \\
I_3 =& \frac{\lambda^2\mu^{2\epsilon}}{4}\int\frac{d^d k_1}{(2\pi)^d}\frac{d^d k_2}{\left(k_1^2+m^2\right)\left(k_2^2+m^2\right)^2} = \frac{\lambda^2\mu^{2\epsilon}}{4}J_1 J_2 \quad ; \quad J_2 = \int\frac{d^d k}{(2\pi)^d\left(k^2+m^2\right)^2} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{m^2}\right)^{2-d/2} \\
=& \lambda^2\frac{m^2}{4(16\pi^2)^2}\frac{(4\pi)^\epsilon}{m^{2\epsilon}}(\frac{\epsilon}{2}-1)\Gamma^2(\frac{\epsilon}{2}-1) = -\lambda^2\mu^{2\epsilon}\frac{m^2}{4(16\pi^2)^2}\frac{(4\pi)^\epsilon}{m^{2\epsilon}}\left[\frac{4}{\epsilon^2}+\frac{4\varphi(2)-2}{\epsilon}\right]+O(1) \\
I_4 =& \frac{\lambda^2\mu^{2\epsilon}}{6}\int\frac{d^d k_1 d^d k_2 d^d k_3 (2\pi)^d\delta\left(k_1+k_2+k_3+p\right)}{(2\pi)^{3d}\left(k_1^2+m^2\right)\left(k_2^2+m^2\right)\left(k_3^2+m^2\right)} \\
=& \frac{\lambda^2\mu^{2\epsilon}}{6(2\pi)^{2d}}\int\frac{d^d k_1 d^d k_2}{\left(k_1^2+m^2\right)\left(k_2^2+m^2\right)\left(\left(k_1+k_2-p\right)^2+m^2\right)} \\
=& -\frac{\lambda^2\mu^{2\epsilon}}{6\left(16\pi^2\right)^2}\left\{\frac{6m^2}{\epsilon^2}+\frac{6m^2}{\epsilon}\left[\frac{3}{2}+\varphi(1)+\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right]+\frac{p^2}{2\epsilon}\right\}
\end{aligned} \tag{30}
$$

The symmetry factor for I_3 ($2\times 2=4$)and) and I_4 ($2\times 3=6$) comes from 1) interchange of end/start point of same line. 2) interchange of lines. 3)equivalence of vertices.

all the Feynman diagrams that contribute to the propagator $G_2(p)$ to $O(\lambda^2)$

$$
G_2(p) = \frac{1}{p^2+m^2}+\frac{1}{\left(p^2+m^2\right)^2}\{\frac{\lambda_R m_R^2}{16\pi^2}(\frac{\mu}{m_R})^\epsilon\frac{1}{\epsilon}+\frac{\lambda_R\mu^\epsilon m^2}{32\pi^2}[\varphi(2)+\ln 4\pi]\} - \frac{1}{\left(p^2+m^2\right)^2}\frac{\lambda^2\mu^{2\epsilon}}{6(16\pi)^2\epsilon}p^2+O\left(\lambda^2/\epsilon^2,p^4\right) \tag{31}
$$

two point vertex function

$$
\Gamma_2(p) = G_2^{-1}(p) = (1+\delta_\phi)p^2+(m^2+\delta_m)-\{\frac{\lambda_R m_R^2}{16\pi^2}(\frac{\mu}{m_R})^\epsilon\frac{1}{\epsilon}+\frac{\lambda_R\mu^\epsilon m^2}{32\pi^2}[\varphi(2)+\ln 4\pi]\}+\frac{\lambda^2\mu^{2\epsilon}}{6\left(16\pi^2\right)^2\epsilon}p^2+O\left(\lambda^2/\epsilon^2,p^4\right) \tag{32}
$$

From renormalization condition, to cancel the divergence and constant, $\overline{MS}$ scheme

$$
\delta_m = \frac{\lambda_R m_R^2}{16\pi^2}(\frac{\mu}{m_R})^\epsilon\frac{1}{\epsilon}+\frac{\lambda_R\mu^\epsilon m^2}{32\pi^2}[\varphi(2)+\ln 4\pi] \quad ; \quad \delta_\phi = \frac{\lambda_R^2\mu^{2\epsilon}}{6\left(16\pi^2\right)^2\epsilon} \qquad (2-\text{loop}) \tag{33}
$$

Four-point function

$$
\begin{aligned}
G_4(p_1,\cdots,p_4) &= \frac{1}{Z}\int D\varphi e^{S_0}\left[\varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)\left(1+S_I+\frac{1}{2!}S_I^2+\cdots\right)\right]_c \\
&= \frac{1}{Z}\int D\varphi e^{S_0}\left[\varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)\left(S_I+\frac{1}{2!}S_I^2+\cdots\right)\right]_c \\
&= \frac{\delta\left(\sum_i p_i\right)}{\prod_i\left(p_i^2+m^2\right)}\left[\Gamma_4^0+\Gamma_4^1+O\left(\lambda^3\right)\right]
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\Gamma_4^0 &= -\lambda\mu^\epsilon-\delta_\lambda \\
\Gamma_4^1 &= I_2(p_1+p_2)+I_2(p_2+p_3)+I_2(p_3+p_4) \\
I_2(p_1+p_2) &= \frac{1}{2!}\left(-\frac{\lambda\mu^\epsilon}{4!}\right)^2 C_2\int_k\frac{1}{(k^2+m^2)\left((k-p_1-p_2)^2+m^2\right)} \\
I_2(p) &= \frac{\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\int_0^1 dx\left[\frac{2}{\epsilon}+\varphi(1)\right]\left[1-\frac{\epsilon}{2}\ln\left(\frac{m^2+p^2x(1-x)}{4\pi\mu^2}\right)\right] \\
&= \frac{\lambda^2\mu^{2\epsilon}}{32\pi^2}\left(\frac{2}{\epsilon}+\varphi(1)\right)-\frac{\mu^{2\epsilon}\lambda^2}{32\pi^2}\int_0^1 dx\ln\frac{m^2+p^2x(1-x)}{4\pi\mu^2} \\
&= \frac{\lambda^2\mu^{2\epsilon}}{32\pi^2}\left[\frac{2}{\epsilon}+\varphi(1)+\ln(4\pi)\right]-\frac{\mu^{2\epsilon}\lambda^2}{32\pi^2}\int_0^1 dx\ln\frac{m^2+p^2x(1-x)}{\mu^2}
\end{aligned} \tag{35}
$$

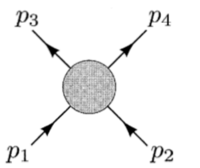
To cancel the divergence and constant, $\overline{MS}$ scheme

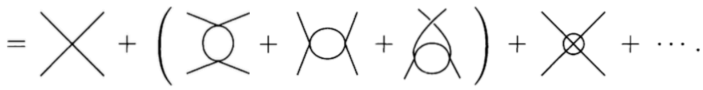
$$
\delta_\lambda = -\frac{3\lambda^2\mu^{2\epsilon}}{32\pi^2}\left[\frac{2}{\epsilon}+\varphi(1)+\ln(4\pi)\right] \tag{36}
$$

For $d=4$, the four-point function do not depend on renormalization scale

$$
\begin{aligned}
0 = \mu\frac{DG_4}{D\mu} &= -\mu\frac{\partial}{\partial\mu}\lambda+\frac{3\lambda^2}{16\pi^2} \\
\beta(\lambda) = \mu\frac{\partial}{\partial\mu}\lambda &= \frac{3\lambda^2}{16\pi^2}
\end{aligned} \tag{37}
$$

Chap 10.2 of [1] Peskin

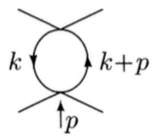
$i\mathcal{M}(p_1p_2\to p_3p_4) =$ 



$$
i\mathcal{M} = -i\lambda+(-i\lambda)^2[iV(s)+iV(t)+iV(u)]-i\delta_\lambda
$$

renormalization condition : $iM(s=4m^2,t=0,u=0)=-i\lambda$

$$
\delta_\lambda = -\lambda^2\left[V\left(4m^2\right)+2V(0)\right] \tag{38}
$$



$$
\begin{aligned}
&= \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(k+p)^2-m^2} \\
&\equiv (-i\lambda)^2\cdot iV(p^2).
\end{aligned}
$$

$$
\begin{aligned}
V(p^2) =& \frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\frac{1}{(k+p)^2-m^2} \\
& \frac{1}{AB} = \int_0^1 dx \frac{1}{[A+(B-A)x]^2} \\
=& \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-m^2+xp^2+2xk\cdot p]^2} \\
=& \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2+2xk\cdot p+x^2p^2-m^2+xp^2-x^2p^2]^2} \\
=& \frac{i}{2}\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{1}{[l^2+x(1-x)p^2-m^2]^2} \\
=& -\frac{1}{2}\int_0^1 dx \int \frac{d^dl_E}{(2\pi)^d}\frac{1}{[-l_E^2+x(1-x)p^2-m^2]^2} \quad (l^0 = il_E^0) \\
& \int \frac{d^dl_E}{(2\pi)^d}\frac{1}{\left(l_E^2+\Delta\right)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \\
=& -\frac{1}{2}\int_0^1 dx \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)}\left(\frac{1}{m^2-x(1-x)p^2}\right)^{2-\frac{d}{2}} \\
& \Gamma\left(2-\frac{d}{2}\right) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \Gamma(n) = (n-1)! \\
=& -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right)\left[\frac{4\pi}{m^2-x(1-x)p^2}\right]^{\epsilon/2} \\
& d = 4-\epsilon, a^{\frac{\epsilon}{2}} = e^{\frac{\epsilon}{2}\ln a} = 1 + \frac{\epsilon}{2}\ln a + \mathcal{O}(\epsilon) \\
& d \to 4, \epsilon \to 0 \\
& \approx -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon}-\gamma+\log(4\pi)-\log\left[m^2-x(1-x)p^2\right]\right)
\end{aligned} \tag{39}
$$

Similarily,

$$
\begin{aligned}
\delta_\lambda &= -\lambda^2\left[V\left(4m^2\right)+2V(0)\right] \\
&\approx \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon}-3\gamma+3\log(4\pi)-\log\left[m^2-x(1-x)4m^2\right]-2\log\left[m^2\right]\right)
\end{aligned} \tag{40}
$$

Then

$$
\begin{aligned}
iM =& -i\lambda+(-i\lambda)^2[iV(s)+iV(t)+iV(u)]-i\delta_\lambda \\
=& -i\lambda-i\lambda^2[V(s)+V(t)+V(u)]-i\delta_\lambda \\
=& -i\lambda-i\frac{\lambda^2}{32}\int_0^1 dx\left[\log\left(\frac{m^2-x(1-x)s}{m^2-x(1-x)4m^2}\right)+\log\left(\frac{m^2-x(1-x)t}{m^2}\right)+\log\left(\frac{m^2-x(1-x)u}{m^2}\right)\right]
\end{aligned} \tag{41}
$$

finite scattering amplitude

Reference

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