QCD running coupling by Feng-Lei

QCD Lagrangian

$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{fermion} + \mathcal{L}_{gf} + \mathcal{L}_{FP}$ FP: Faddeer-Popov

since we only interested in the QCD running coupling. we just show the relevant Feynman rules in here

$$\mathcal{L}_0 = \bar{\psi}_0 i\not{D} \psi_0 + g_{s0} \bar{\psi}_0 \gamma^\mu T^a \psi_0 A^a_\mu - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

Redefine the fields and parameters

$$A^0_\mu = Z_A^{\frac{1}{2}} A_\mu \qquad \psi_0 = Z_\psi^{\frac{1}{2}} \psi \qquad g_s^0 = Z_g g_s$$

$$\mathcal{L} = \mathcal{L}_r + \delta\mathcal{L}$$

$$= Z_\psi \bar{\psi} i\not{D} \psi + g Z_g Z_\psi Z_A^{\frac{1}{2}} \bar{\psi} \gamma^\mu T^a \psi A^a_\mu - \frac{1}{4} Z_A F^a_{\mu\nu} F^{\mu\nu a} + \cdots$$

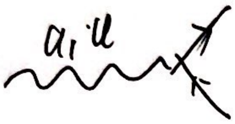
write $Z_i = 1 + \delta Z_i$ define $Z_F = Z_g Z_\psi Z_A^{\frac{1}{2}}$

$$\mathcal{L}_r = \bar{\psi} i\not{D} \psi + g \bar{\psi} \gamma^\mu T^a \psi A^a_\mu - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

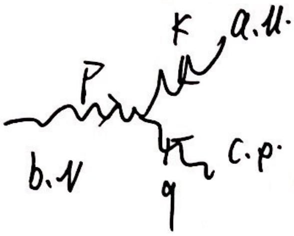
$$\delta\mathcal{L} = \delta Z_\psi \bar{\psi} i\not{D} \psi + \delta Z_F g \bar{\psi} \gamma^\mu T^a \psi A^a_\mu - \frac{1}{4} \delta Z_A F^a_{\mu\nu} F^{\mu\nu a} + \cdots$$

From §7.

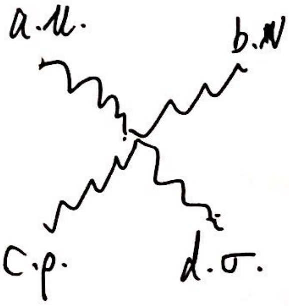
Feynman rules:



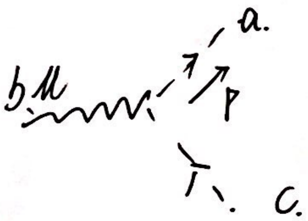
$i g_s \gamma^\mu T^a$.



$g f^{abc}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu]$

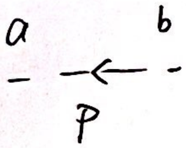


$$-ig^2[f^{ade}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})$$
$$+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})$$
$$+f^{ade}f^{bce}\cdot(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})]$$



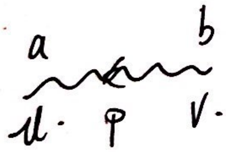
$-g f^{abc} p^\mu$.

Ghost propagator.



$$\frac{i\delta^{ab}}{p^2+i\epsilon}.$$

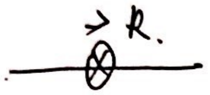
gluon propagator.



$$\frac{-i}{p^2}\delta_{ab}\left[g_{\mu\nu} - \frac{(1-\xi)p_\mu p_\nu}{p^2}\right]$$ $\xi=1$ Feynman-'t Hooft gauge.

$$= \frac{-i}{p^2}\delta_{ab}\, g_{\mu\nu}.$$

1,



$i\not{k}\delta Z_\psi$.

$i\delta Z_f g\gamma^\mu T^a$.

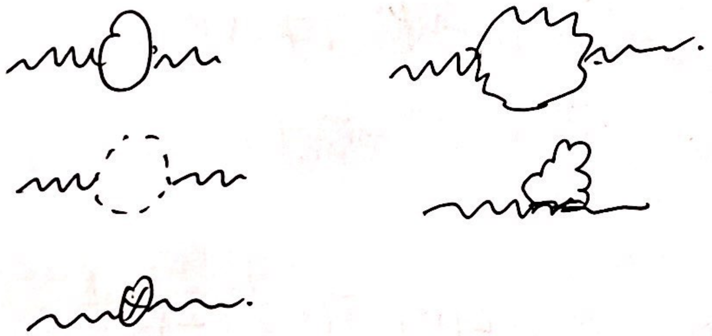
$= i(k^2 g^{\mu\nu} - k^\mu k^\nu)\delta^{ab}\delta Z_A$.

From the definition. we see

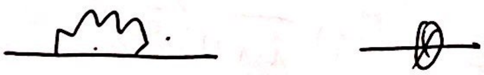
$$g_s = \frac{g_{s0}}{Z_g} = g_{s0}\,\frac{Z_\psi Z_A^{\frac{1}{2}}}{Z_F}.$$

Therefore. we should consider three class graphs which are. showed as follows.

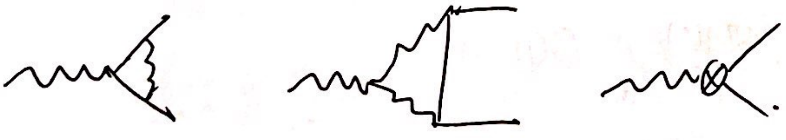
① The vaccum polarization graphes



② The self energy of fermion. $\delta Z_f \sim \delta_2$.



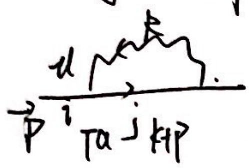
③. The vertex graphes.



note. $\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$

δZ_4 (or δ_2). peskin. 16.25.

The feynman self energy.



$$\int \frac{d^n k}{(2\pi)^n} (i g_s u^\varepsilon)^2 T^a \gamma_u \frac{i(\not p+\not k)}{(p+k)^2} \gamma^u T^a \frac{-i}{k^2}$$

$$= -(T^a T^a) g_s^2 u^{2\varepsilon} \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_u(\not p+\not k)\gamma^u}{k^2(p+k)^2}$$

$\gamma_u(\not p+\not k)\gamma^u = -(n-2)(\not p+\not k)$ $\qquad$ $\gamma^u\gamma^v\gamma_u = (2g^{uv}\gamma_u - \gamma^v\gamma^u\gamma_u)$

$= (2-n)\gamma^v$.

 $= C_2(r) g^2 (n-2) \int \frac{d^n k}{(2\pi)^n} \frac{\not p+\not k}{k^2(p+k)^2}$.

$\frac{1}{k^2(p+k)^2} = \int_0^1 dx \frac{1}{(l^2-\Delta)^2}$ with $\Delta = x(x-1)p^2$ $\quad l = k+xp$.

$$\int \frac{d^n p}{(2\pi)^n} \frac{\not p+\not k}{k^2(k+p)^2}$$

$$= \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{(1-x)\not p}{(l^2-\Delta)^2} = \int_0^1 dx\, (1-x)\not p \frac{(-1)^2 i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

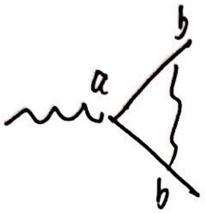
$$= \int_0^1 dx\, (1-x)\not p \frac{i}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon) \frac{1}{x(x-1)} \left(\frac{1}{q^2}\right)^\varepsilon.$$

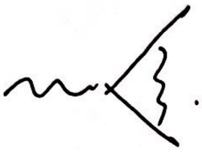
when $q=p \Rightarrow x(1-x) \approx 1$

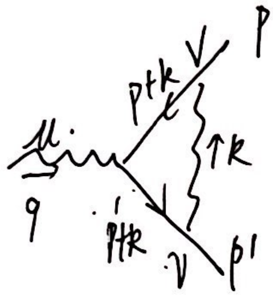
 $= \frac{i g^2}{(4\pi)^2} \left(\frac{4\pi u^2}{q^2}\right)^\varepsilon \not q\, C_2(r)\Gamma(\varepsilon)$.

for $\overline{MS}$ scheme

$$\delta Z_4 = -\frac{g_s^2}{16\pi^2} C_2(r) \left(\frac{1}{\varepsilon} - \gamma + \ln 4\pi\right)$$



$= T^a T^a T^b \otimes$ 



$$= \int \frac{d^n k}{(2\pi)^n} (i g_s \mu^\varepsilon)\gamma_\nu \frac{i(\not p' + \not k)}{(p'+k)^2} \, i g_s \mu^\varepsilon \gamma_\mu \frac{i(\not p+\not k)}{(p+k)^2} (i g_s \mu^\varepsilon)\gamma^\nu$$

$$= g_s\mu^\varepsilon g_s^2 \mu^{2\varepsilon}\int \frac{d^nk}{(2\pi)^n}\frac{\gamma_\nu(\not p'+\not k)\gamma_\mu(\not p+\not k)\gamma^\nu}{(p'+k)^2(p+k)^2k^2}$$

Since we only interested in the divergence part.

$$\int \frac{d^nk}{(2\pi)^n}\frac{\gamma_\nu(\not p'+\not k)\gamma_\mu(\not p+\not k)\gamma^\nu}{(p'+k)^2(p+k)^2k^2} = D\int \frac{d^nk}{(2\pi)^n}\frac{\gamma_\nu \not k \gamma_\mu \not k \gamma^\nu}{k^2k^2k^2}$$

$\gamma_\nu \not k \gamma_\mu \not k \gamma^\nu = -(n-2)(2k^\mu \not k - k^2\gamma^\mu)$.

$\gamma_\nu \gamma^\rho \gamma_\mu \gamma^\sigma \gamma^\nu k_\rho k_\sigma$

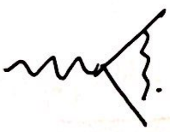
$= (2g^{\nu\rho} - \gamma^\rho\gamma^\nu)\gamma_\mu \gamma^\sigma \gamma^\nu k^\rho k^\sigma$.

$= (2\gamma_\mu \gamma^\sigma \gamma^\rho - \gamma^\rho\gamma^\nu\gamma_\mu\gamma^\sigma\gamma^\nu)k^\rho k^\sigma$

$=$

$k^\mu k^a = \frac{g^{\mu a}k^2}{n}$.

$\sim k^2\gamma^\mu$.



$$= g_s\mu^\varepsilon g_s^2\mu^{2\varepsilon}\gamma_\mu \int\frac{d^nk}{(2\pi)^n}\frac{1}{(k^2)^2}$$

$$= g_s\mu^\varepsilon g_s^2\mu^{2\varepsilon}\gamma_\mu \frac{(-1)}{(4\pi)^{\frac{d}{2}}}\, i\,\Gamma(\varepsilon)\left(\frac{1}{q^2}\right)^\varepsilon.$$

$$= g_s\mu^\varepsilon g_s^2\gamma_\mu \frac{i}{(4\pi)^2}\left(\frac{4\pi}{q^2}\right)^\varepsilon \Gamma(\varepsilon).$$

$[T^a, T^b] = i f^{abc} T^c$.

4.

$$T^bT^aT^b = T^bT^aT^b + T^b[T^a, T^b]$$

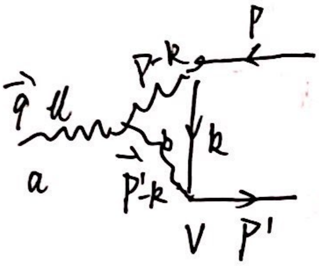
$$= C_2(r)\, T^a + T^b\, i f^{abc} T^c$$

$$= C_2(r) T^a + \frac{i}{2}\left[T^bT^c f^{abc} + T^bT^c f^{abc}\right]$$

$$= C_2(r) T^a + \frac{i}{2}[T^b, T^c] f^{abc}$$

$$= \left[C_2(r) - \frac{1}{2}C_2(G)\right] T^a$$

$$= i g_s \mu^\varepsilon \frac{g_s^2}{16\pi^2}\left[C_2(r) - \frac{1}{2}C_2(G)\right] T^a \gamma^\mu \left(\frac{4\pi\mu^2}{q^2}\right)^\varepsilon \Gamma(\varepsilon)$$



$$\int \frac{d^nk}{(2\pi)^n} (i g_s \mu^\varepsilon \gamma_\nu T^b) \frac{i}{\not k} (i g_s \mu^\varepsilon \gamma_\rho T^c) \frac{-i}{(p-k)^2}$$

$$g_s \mu^\varepsilon f^{abc}\left[g^{\mu\nu}(p'-p+p'-k)^\rho + g^{\nu\rho}(-p'+k-p+k)^\mu + g^{\rho\mu}(p-k-p'+p)^\nu\right] \frac{-i}{(p'-k)^2}$$

$$= \int \frac{d^nk}{(2\pi)^n} i g_s \mu^\varepsilon \gamma_\nu T^b \frac{i\not k}{k^2} i g_s \mu^\varepsilon \gamma_\rho T^c \frac{-i}{(p-k)^2} \frac{-i}{(p'-k)^2}$$

$$g_s \mu^\varepsilon f^{abc}\left[g^{\mu\nu}(2p'-k-p)^\rho + g^{\nu\rho}(-p'-p+2k)^\mu + g^{\rho\mu}(2p-k-p')^\nu\right]$$

$$f^{abc}T^bT^c = \frac{1}{2}\left[f^{abc}T^bT^c + f^{abc}T^bT^c\right] = \frac{1}{2}f^{abc}[T^b, T^c] = \frac{i}{2}C_2(G)\delta^{ab}T^a$$

$$= -\int \frac{d^nk}{(2\pi)^n} \cdot \frac{g_s^3 \mu^{3\varepsilon}}{2} C_2(G) T^a \cdot \frac{\gamma_\nu \not k \gamma_\rho \left[g^{\mu\nu}(2p'-k-p)^\rho + g^{\nu\rho}(-p'-p+2k)^\mu + g^{\rho\mu}(2p-k-p')^\nu\right]}{k^2 (p-k)^2 (p'-k)^2}$$

only focus on the UV part.

$$= -\int \frac{d^n k}{(2\pi)^n} \frac{g_s^3 \mu^{3\epsilon}}{2} C_2(G)\, T^a \frac{\gamma_\nu \not{k} \gamma_\rho [g^{\mu\nu} k^\rho - 2g^{\nu\rho} k^\mu + g^{\rho\mu} k^\nu]}{[k^2]^3}.$$

$$\gamma_\nu \not{k} \gamma_\rho [g^{\mu\nu} k^\rho - 2g^{\nu\rho} k^\mu + g^{\rho\mu} k^\nu] \Rightarrow 3\gamma^\mu k^2.$$

$\gamma_\nu \not{k} \gamma_\rho g^{\mu\nu} k^\rho$

$= \gamma^\mu \gamma^\sigma \gamma_\rho k^\sigma k^\rho$

$= \gamma^\mu \gamma^\sigma \gamma^\rho \frac{g_{\sigma\rho} k^2}{n}$

$= \gamma^\mu k^2.$

$-2\gamma_\nu \not{k} \gamma_\rho g^{\nu\rho} k^\mu$

$= -2\gamma^\rho \not{k} \gamma_\rho k^\mu$

$= 2(n-2)\gamma^\sigma k^\sigma k^\mu$

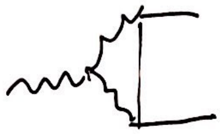
$= \frac{2(n-2)}{n} \gamma^\mu k^2$

$= \gamma^\mu k^2$

当 $n=4$

$\gamma^\mu \gamma^\nu \gamma_\mu = (2-n)\gamma^\nu$

$p^\sigma p^\mu = \frac{g^{\sigma\mu} p^2}{n}.$



$$= \frac{i g_s \mu^\epsilon}{(2\pi)^{2\epsilon}} \frac{3}{2} C_2(G)\, g_s^2 \mu^{2\epsilon} T^a \gamma^\mu \Gamma(\epsilon) \left(\frac{1}{q^2}\right)^\epsilon.$$

$$= i g_s \gamma^\mu T^a \frac{g_s^2}{16\pi^2} \frac{3}{2} C_2(G) \left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon \Gamma(\epsilon).$$

finite.

for $\overline{MS}$ scheme.

$$\delta Z_F = -\frac{g_s^2}{16\pi^2} [C_2(F) + C_2(G)] \left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right]$$

Summary

$$\delta Z_A = \frac{g_s^2}{16\pi^2}\left[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]\left(\frac{1}{\varepsilon} - \gamma + \ln 4\pi\right)$$

$$\delta Z_\psi = \frac{-g_s^2}{16\pi^2}C_2(r)\left(\frac{1}{\varepsilon} - \gamma + \ln 4\pi\right)$$

$$\delta Z_F = \frac{-g_s^2}{16\pi^2}\left[C_2(r) + C_2(G)\right]\left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi\right]$$

$$Z_g = \frac{Z_F}{Z_\psi Z_A^{\frac{1}{2}}}$$

$$= \left[1 - \frac{g_s^2}{16\pi^2}\left(C_2(r) + C_2(G)\right]\Delta'\right]\left[1 + \frac{g_s^2}{16\pi^2}C_2(r)\Delta'\right)$$
$$\left[1 - \frac{1}{2}\frac{g_s^2}{16\pi^2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]\Delta'\right].$$

$$= 1 - \frac{g_s^2}{16\pi^2}\left[C_2(r) + C_2(G) - C_2(r) + \frac{1}{2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right)\right]\Delta'$$

$$= 1 - \frac{g_s^2}{16\pi^2}\left[\frac{11}{6}C_2(G) - \frac{4}{3}n_f C(r)\right]\Delta'$$

with $\Delta' = \frac{1}{\varepsilon} - \gamma + \ln 4\pi$

T.

β function.

Note. $g_0 = \mu^{\varepsilon} g Z_g$

$a_{s0} = (\mu^2)^{\varepsilon} a_s Z_g^2$.

$$\mu^2 \frac{d a_{s0}}{d\mu^2} = 0$$

$$\mu^2 \frac{d}{d\mu^2}\left[(\mu^2)^{\varepsilon} a_s Z_g^2\right]$$

$$= \mu^2\left[\varepsilon(\mu^2)^{\varepsilon-1} a_s Z_g^2 + \mu^{2\varepsilon}\frac{da_s}{d\mu^2} Z_g^2 + (\mu^2)^{\varepsilon} a_s \frac{dZ_g^2}{d\mu^2}\right]$$

$$= (\mu^2)^{\varepsilon}\left[\varepsilon a_s Z_g^2 + \mu^2\frac{da_s}{d\mu^2} Z_g^2 + a_s\mu^2\frac{da_s}{d\mu^2}\frac{dZ_g^2}{da_s}\right]$$

$$= (\mu^2)^{\varepsilon}\left[\varepsilon a_s Z_g^2 + \beta(a_s,\varepsilon) Z_g^2 + 2 a_s \beta(a_s,\varepsilon) Z_g \frac{dZ_g}{da_s}\right]$$

$$= (\mu^2)^{\varepsilon} Z_g\left[\beta(a_s,\varepsilon) Z_g + \varepsilon a_s Z_g + 2 a_s \beta(a_s,\varepsilon)\frac{dZ_g}{da_s}\right] = 0$$

$$\left[\beta(a_s,\varepsilon) + \varepsilon a_s + 2 a_s \beta(a_s,\varepsilon)\frac{d}{da_s}\right] Z_g = 0$$

where. $\begin{cases} \beta(a_s,\varepsilon) = \mu^2 \frac{da_s}{d\mu^2}. \\ Z_g = 1 + \sum_i \frac{Z^{(i)}}{\varepsilon^i} \\ \beta(a_s,\varepsilon) = \beta(a_s) + \sum_{i=1} \beta^{(i)}(a_s)\varepsilon^i \end{cases}$ add 各阶的修正项

$$\beta(a_s) + \sum_{i=1}\beta^i(a_s)\varepsilon^i + \varepsilon a_s + 2a_s\left[\beta(a_s) + \sum \beta^i(a_s)\varepsilon^i\right]\frac{d}{da_s}\Big]\left(1 + \sum_i \frac{Z^i}{\varepsilon^i}\right) = 0$$

$\Rightarrow \begin{cases} \beta^i(a_s) = 0 \quad i > 1 \\ \beta^1(a_s) = -a_s \quad \text{一阶项} \\ \beta(a_s) = 2 a_s^2 \frac{d}{da_s} Z^{(1)} \end{cases}$

$$Z_g = 1 - \frac{g_s^2}{16\pi^2}\left[\frac{11}{6}\cdot C_2(G) - \frac{2}{3} n_f C(r)\right]\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right)$$

$$Z^{(1)} = -\frac{\alpha_s}{4\pi}\left(\frac{11}{6} C_2(G) - \frac{2}{3} n_f C(r)\right]$$

$$\beta(\alpha_s) = 2\alpha_s^2 \frac{d}{d\alpha_s} Z^{(1)}$$

$$= -\frac{\alpha_s^2}{4\pi}\left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r)\right]$$

thus. the one-loop beta function.

$$\beta(\alpha_s) = b_0 \alpha_s^2 \quad \text{with } b_0 = -\frac{1}{4\pi}\left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r)\right]$$

For SU(N) the fundamental representation.

$$C(N) = \frac{1}{2} \qquad C_2(N) = \frac{N^2-1}{2N} \qquad C_2(G) = C(G) = N$$

thus $\int_{\mu^2}^{Q^2} d\ln\mu^2 = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta}$ from. $\beta = \mu^2 \frac{d\alpha_s}{d\mu^2}$.

$$\Rightarrow \ln\frac{Q^2}{\mu^2} = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{\beta} = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{d\alpha_s}{b_0 \alpha_s^2}.$$

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 - b_0 \alpha_s(\mu^2) \ln\frac{Q^2}{\mu^2}}$$

$$= \frac{\alpha_s(\mu^2)}{1 + \frac{1}{4\pi}(11 - \frac{2}{3} n_f)\alpha_s(\mu^2) \ln\frac{Q^2}{\mu^2}}.$$ peskin. 17.12.

leading order. running α_s.