1. Electron Self Energy Diagram.

$$\mathcal{L}_{QED} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - ieA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$
$$= \mathcal{L}_0 - ieA_\mu\bar{\psi}\gamma^\mu\psi$$
$$= \underset{e\to 0}{\mathcal{L}_0} - \mathcal{L}_{int}$$

① For a free theory.

The two point function (propagator) of free electrons:

$$\int d^4x\, e^{ip\cdot x} \langle 0|T\psi(x)\bar{\psi}(0)|0\rangle = \frac{i(\not{p}+m_0)}{p^2-m_0^2+i\epsilon}$$

m_0: free electron mass, bare mass (NOT measurable)

$\mathcal{L}_0$ → means no any interaction terms → no photons

→ electrons are bare, or naked.

(think about (7.33))

② For an interacting theory   lowest energy state.

$$\int d^4x\, e^{ip\cdot x} \langle \Omega|T\psi(x)\bar{\psi}(0)|\Omega\rangle = \frac{iZ_2(\not{p}+m)}{p^2-m^2+i\epsilon}$$

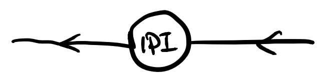
m: dressed electron mass (Measurable)

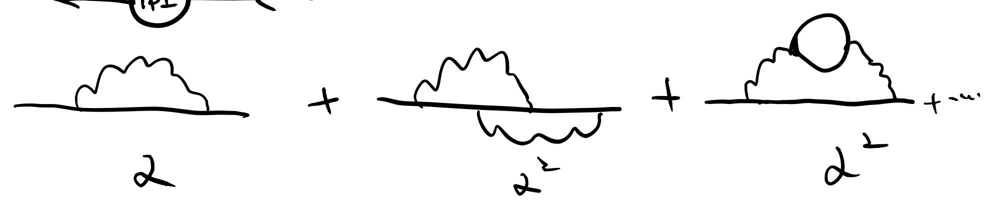
electron phyical mass

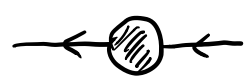
Z_2: the probability for the quantum field to create or annihilate an exact one electron eigenstate of the full Hamiltonian $\langle\Omega|\psi(0)|p,s\rangle = \sqrt{Z_2}\, u^s(p)$

One-particle irreducible (1PI) diagram: diagrams that cannot be split into two by removing a single line. Peskin P219.

$i\Sigma(p)$: defined as the sum of all 1PI diagrams with two external fermion lines:

$-i\Sigma(p)$ = 

= 

$\int d^4x\, e^{ip\cdot x}\langle\Omega|T\psi(x)\bar{\psi}(0)|\Omega\rangle$ = 

= 

$\Rightarrow$

$$\frac{iZ_2}{\not p - m} = \frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}\boxed{(-i\Sigma)\frac{i}{\not p - m_0}} + \frac{i}{\not p - m_0}\boxed{(-i\Sigma)\frac{i}{\not p - m_0}}\boxed{(-i\Sigma)\frac{i}{\not p - m_0}}$$

$$+ \cdots\cdots$$

$$= \sum_{n=0}^{\infty}\frac{i}{\not p - m_0}\left(\frac{\Sigma}{\not p - m_0}\right)^n \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$= \frac{i}{\not p - m_0}\cdot\frac{1}{1-\frac{\Sigma}{\not p - m_0}}$$

$$= \frac{i}{\not p - m_0 - \Sigma(p)}$$

Left hand side: the propagator we expect to have in the interacting theory

Right hand side: is based on the perturbative expansion of the free theory together with the electron and photon interaction term.

① How to determine m?

The physical dressed electron mass m is determined by the location of the pole at $\not{p} = m$

$$\Rightarrow \quad m - m_0 - \Sigma(\not{p})\Big|_{\not{p}=m} = 0$$

② How to determine Z_2?

By linearly expanding $\Sigma(\not{p})$ at $\not{p} = m$

$$\Sigma(\not{p}) = \Sigma(\not{p}=m) + (\not{p}-m)\frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m} + \cdots\cdots$$

$$\not{p} - m_0 - \Sigma(\not{p}) = \not{p} - m_0 - (m - m_0) - (\not{p}-m)\frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m}$$

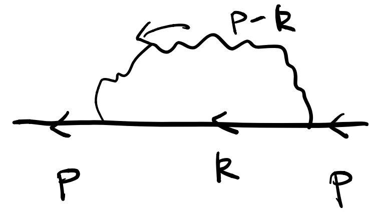
$$= (\not{p}-m) - (\not{p}-m)\frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m}$$

$$= (\not{p}-m)\left(1 - \frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m}\right)$$

$$\therefore \frac{iZ_2}{\not{p}-m} = \frac{i}{(\not{p}-m)\left(1 - \frac{d\Sigma(\not{p})}{d\not{p}}\right)} \Rightarrow Z_2^{-1} = 1 - \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}$$

$$-i\Sigma_2 = (-ie\mu^\epsilon)^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{i(\not{k}+m_0)}{k^2-m_0} \gamma_\mu \frac{-i}{(p-k)^2}$$

$$= (-ie\mu^\epsilon)^2 \int \frac{d^dk}{(2\pi)^d} \frac{(2-n)\not{k} + n m_0}{k^2-m_0} \frac{1}{(p-k)^2}$$



① using Feynman parameter, $\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2}$

$B = k^2 - m_0$, $A = (p-k)^2$

$$\Rightarrow xA + (1-x)B = x(p-k)^2 + (1-x)(k^2-m_0)$$
$$= x[p^2+k^2-2p\cdot k] + (1-x)(k^2-m_0^2)$$

complete the square and define a shifted momentum.

$$= \underbrace{k^2 - 2x\, p\cdot k} + xp^2 - (1-x)m_0^2$$
$$= (k-xp)^2 - x^2p^2 + xp^2 - (1-x)m_0^2$$
$$= (k-xp)^2 - x(x-1)p^2 - (1-x)m_0^2$$
$$= l^2 - \Delta \quad \text{where } \Delta = x(x-1)p^2 + (1-x)m_0^2$$
$$l = k - xp$$

$$= \int_0^1 dx\, (-ie\mu^\epsilon)^2 \int \frac{d^dl}{(2\pi)^d} \frac{(2-n)\not{l} + (2-d)x\not{p} + dm_0}{[l^2-\Delta]^2}$$

dropping the term linear in l

$$= \int_0^1 dx\, (-ie\mu^\epsilon)^2 \int \frac{d^dl}{(2\pi)^d} \frac{(2-d)x\not{p} + dm_0}{[l^2-\Delta]^2}$$

using: Any Minkowski space integral can be converted to a integral in the Euclidean space, eg.

$$\int \frac{d^dl}{(2\pi)^d} \frac{(l^2)^n}{(l^2-\Delta)^m} = \frac{i(-1)^n}{(-1)^m} \int \frac{d^dl_E}{(2\pi)^d} \frac{(l_E^2)^n}{(l_E^2+\Delta)^m}$$

Now, it seems like : $n=0$ $m=2$

$$= \int_0^1 dx\, (-ie\mu^\epsilon)^2 [(2-d)x\not{p} + dm_0]\, i \int \frac{d^dl_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^2}$$

Then: Euclidean:

$$\int \frac{d^dl_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^m} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m-\frac{d}{2}}$$

And. it seems like = $m=2$ $d=4-2\epsilon$

$$= \int_0^1 dx \, (-ie\mu^\epsilon)^2 \left[(2-d)x\not{p} + d m_0\right] \cdot \frac{1}{(4\pi)^{2-\epsilon}} \cdot \frac{\Gamma(\epsilon)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^\epsilon$$

where $\Delta = x(x-1)p^2 + (1-x)m_0^2$

$$= \int_0^1 dx \frac{\alpha}{2\pi} \left[(2-\epsilon)m_0 - (1-\epsilon)x\not{p}\right] \cdot \left[\frac{(1-x)m_0^2 - x(1-x)p^2}{4\pi\mu^2}\right]^{-\epsilon} \Gamma(\epsilon)$$

- For generic p^2 and m_0,

$$a^{-\varepsilon} = e^{-\varepsilon \ln a} \simeq 1 - \varepsilon \ln a$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E$$

$$\Sigma_2 \simeq \frac{\alpha}{2\pi}\int_0^1 dx \left[(2-\epsilon)m_0 - (1-\epsilon)x\not{p}\right]\left[\frac{1}{\epsilon} - \gamma_E\right]\left[1 - \epsilon \ln \frac{\Delta^2}{4\pi\mu^2}\right]$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx \left[2m_0 - x\not{p}\right]\frac{1}{\epsilon} \quad + \text{finite terms.}$$

only interested in the divergent term.

$$= \frac{\alpha}{4\pi}\frac{1}{\epsilon}(-\not{p} + 4m_0)$$

$$Z_2^{-1} = 1 - \left.\frac{d\Sigma(\not{p})}{d\not{p}}\right|_{\not{p}=m_0} = 1 + \frac{\alpha}{4\pi}\frac{1}{\epsilon} \Rightarrow Z_2 = 1 - \frac{\alpha}{4\pi}\frac{1}{\epsilon}$$

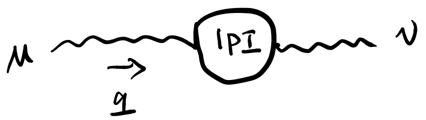
$$m - m_0 - \Sigma(\not{p})\Big|_{\not{p}=m_0} = 0 \Rightarrow m - m_0 = \frac{3\alpha}{4\pi}\frac{m_0}{\epsilon}$$

Electron mass renormalization: $\Rightarrow m = m_0 + \frac{3\alpha}{4\pi}\frac{m_0}{\epsilon}$

At α order, if require m to be finite, m_0 has to be divergent to cancel the divergence.

2. Photon Self Energy diagram, Vaccum Polarization Diagram.

Firstly, 1PI diagrams:



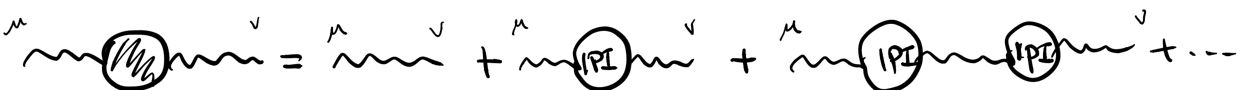
$\equiv i\Pi^{\mu\nu}(q)$

Here consider the tensor structure of $\Pi_2^{\mu\nu}$: The only possible tensor that can appear in $\Pi_2^{\mu\nu}$ is $g^{\mu\nu}$ and $q^\mu q^\nu$. However, the Ward identity tells us that $q_\mu \Pi^{\mu\nu}(q) = 0$. This implies that $\Pi^{\mu\nu}(q)$ is proportional to $(q^2 g^{\mu\nu} - q^\mu q^\nu)$

$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$

$$\boxed{\begin{aligned}&(q^2 g^{\mu\nu} - q^\mu q^\nu) q_\nu \\ &= q^2 q^\mu - q^\mu q^2 \\ &= 0\end{aligned}}$$

The full photon propagator



To α order

$$\frac{-iZ_3 g_{\mu\nu}}{q^2} = \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} i(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi_2(q^2)\frac{-ig_{\sigma\nu}}{q^2} + \cdots\cdots$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-i}{q^2}(q^2 g_{\rho\sigma} - q_\rho q_\sigma)\Pi_2(q^2)\frac{1}{q^2} + \cdots\cdots$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-i}{q^2}(q^2 g_{\mu\sigma} - \cancel{q_\mu q_\nu})\Pi_2(q^2)\frac{1}{q^2} + \cdots\cdots$$

According to the Ward identity, we need to remove all the terms which proportioned to q_μ or q_ν

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\nu}}{q^2}\Pi_2(q^2) + \cdots$$

$$= \frac{-ig_{\mu\nu}}{q^2[1 - \Pi_2(q^2)]}.$$

$$\boxed{Z_3 = \frac{1}{1-\Pi_2(q^2)} \quad \text{and define the physical charge} \quad e = \sqrt{Z_3}\, e_0}$$

$$\frac{-iZ_3 \; g_{\mu\nu}}{q^2} = \frac{-i \quad g_{\mu\nu}}{q^2[1-\Pi_2(q^2)]}$$

At $q^2 = 0 \Rightarrow Z_3 = \frac{1}{[1-\Pi_2(0)]}$

Notice that this result dose not generate a mass for photon, as compared to the electron mass renormalization, therefore photons always remain massless.

$$\frac{-iZ_3 e_0^2 \; g_{\mu\nu}}{q^2} \rightarrow \frac{-ie^2 g_{\mu\nu}}{q^2}$$

e_0: bare charge $\quad e$: physical charge $\quad e = \sqrt{Z_3}\, e_0$

$\sqrt{Z_3}\, e_0$ can be measured in experiments.

$$\frac{e_0^2}{4\pi} \rightarrow \frac{e^2}{4\pi}$$

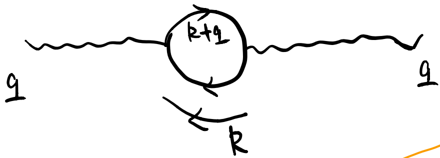
$$\alpha_0 \rightarrow \alpha_{eff} = \frac{e_0^2}{4\pi} \cdot \frac{1}{1-\Pi(q^2)}$$

$$= \frac{e^2}{4\pi} \cdot \frac{1}{[1-\Pi(q^2)]\, Z_3}$$

$$= \alpha \cdot \frac{1}{[1-(\Pi_2(q^2)-\Pi_2(0))]}$$

??!?!!

$Z_3 = \frac{1}{1-\Pi_2(0)} \approx 1+\Pi_2(0)$



see Peskin P120 Eq. (4.120)

$$i\Pi_2^{\mu\nu} = (-ie\mu^\epsilon)^2(-1)\int \frac{d^dk}{(2\pi)^d}\,\mathrm{Tr}\left[\gamma^\mu \frac{i}{\not k - m}\gamma^\nu \frac{i}{\not k+\not q - m}\right]$$

$$= (-ie\mu^\epsilon)^2(-1)\int \frac{d^dk}{(2\pi)^d}\,\mathrm{Tr}\left[\gamma^\mu \frac{i(\not k+m)}{(k^2-m^2)}\gamma^\nu\frac{i(\not k+\not q+m)}{(k+q)^2-m^2}\right]$$

For Dirac-Matrices in d-dim

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}]$$

$$\mathrm{Tr}[\gamma^\mu \not k \gamma^\nu (\not k+\not q)] = k_\rho (k+q)_\sigma\,\mathrm{Tr}[\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma]$$

$$= k_\rho(k+q)_\sigma \cdot 4[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu}]$$

$$= 4[k^\mu(k+q)^\nu - g^{\mu\nu}\,k\cdot(k+q) + k^\nu(k+q)^\mu]$$

$$m^2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4m^2 g^{\mu\nu}$$

$$= (-ie\mu^\epsilon)^2 \cdot 4\int\frac{d^dk}{(2\pi)^d}\cdot\frac{[k^\mu(k+q)^\nu - g^{\mu\nu}\,k\cdot(k+q) + k^\nu(k+q)^\mu + m^2 g^{\mu\nu}]}{(k^2-m^2)[(k+q)^2-m^2]}$$

Using Feynman parameter

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{[xA+(1-x)B]^2}$$

define $A = (k+q)^2$, $B = k^2 - m^2$

$$xA + (1-x)B = x[k^2+q^2+2k\cdot q - m^2] + (1-x)[k^2-m^2]$$

$$= k^2 + 2xk\cdot q + xq^2 - m^2$$

$$= (k+xq)^2 - x^2q^2 + xq^2 - m^2$$

$$= (k+xq)^2 - x(x-1)q^2 - m^2$$

$$= l^2 - \Delta \quad \text{where } l = k+xq \quad \cancel{\Delta = x(1-x)q^2 - m^2}$$

$$\Delta = x(x-1)q^2 + m^2$$

$$= (-ie\mu^{\epsilon})^2 \cdot 4\int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \cdot \frac{[k^\mu (k+q)^\nu - g^{\mu\nu} k\cdot(k+q) + k^\nu (k+q)^\mu + m^2 g^{\mu\nu}]}{[l^2 \cancel{+} - \Delta]^2}$$

$$= (-ie\mu^{\epsilon})^2 \cdot 4\int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \cdot \frac{[\qquad\qquad]}{[l^2 \cancel{+} - \Delta]^2}$$

$$2k^\mu k^\nu - g^{\mu\nu} k^2 + k^\mu q^\nu + k^\nu q^\mu - g^{\mu\nu} k\cdot q + m^2 g^{\mu\nu}$$

① ② ③ ④ ⑤ ⑥

① $2(k+xq)^\mu (k+xq)^\nu - 2x k^\mu q^\nu - 2x k^\nu q^\mu - 2x^2 q^\mu q^\nu$

$$= 2l^\mu l^\nu - 2x(k+xq)^\mu q^\nu + 2x^2 q^\mu q^\nu - 2x(k+xq)^\nu q^\mu + 2x^2 q^\mu q^\nu - 2x^2 q^\mu q^\nu$$

$$= 2l^\mu l^\nu \cancel{- 2x\, l^\mu q^\nu - 2x\, l^\nu q^\mu} + 2x^2 q^\mu q^\nu$$

② $-g^{\mu\nu} k^2$

$$= -g^{\mu\nu}(k+xq)^2 + 2g^{\mu\nu} x\, k\cdot q + g^{\mu\nu} x^2 q^2$$

$$= -g^{\mu\nu} l^2 + 2g^{\mu\nu} x (k+xq)\cdot q - 2x^2 g^{\mu\nu} q^2 + x^2 g^{\mu\nu} q^2$$

$$= -g^{\mu\nu} l^2 + 2\cancel{x\, g^{\mu\nu}\, l\cdot q} - x^2 g^{\mu\nu} q^2$$

③ $k^\mu q^\nu$

$$= (k+xq)^\mu q^\nu - x q^\mu q^\nu$$

$$= \cancel{l^\mu q^\nu} - x q^\mu q^\nu$$

④ $k^\nu q^\mu$

$$= (k+xq)^\nu q^\mu - x q^\nu q^\mu$$

$$= \cancel{l^\nu q^\mu} - x q^\mu q^\nu$$

⑤ $-g^{\mu\nu} k\cdot q$

$$= -g^{\mu\nu}(k+xq)\cdot q + x g^{\mu\nu} q^2$$

$$= \cancel{-g^{\mu\nu}\, l\cdot q} + x g^{\mu\nu} q^2$$

⑥ $m^2 g^{\mu\nu}$

Several useful integral identities:

$$k^\mu k^\nu = \frac{g_{\mu\nu}}{d} k^2$$

$$\Rightarrow\ = 2l^\mu l^\nu - g^{\mu\nu} l^2 + (2x^2 - 2x) q^\mu q^\nu - x^2 g^{\mu\nu} q^2 + x g^{\mu\nu} q^2 + m^2 g^{\mu\nu}$$

$$= \left(\frac{2}{d} g^{\mu\nu} - g^{\mu\nu}\right) l^2 + (2x^2 - 2x) q^\mu q^\nu - (x^2 - x) g^{\mu\nu} q^2 + m^2 g^{\mu\nu}$$

$$\rightarrow\ = (-ie\mu^{\epsilon})^2 \cdot 4\int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \cdot \frac{[(\frac{2}{d} - 1) g^{\mu\nu} l^2 + (2x^2 - 2x) q^\mu q^\nu - (x^2 - x) g^{\mu\nu} q^2 + m^2 g^{\mu\nu}]}{[l^2 \cancel{+} - \Delta]^2}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^n}{(l^2 - \Delta)^m} = \frac{i(-1)^n}{(-1)^m} \int \frac{d^d l_E}{(2\pi)^d} \frac{(l_E^2)}{(l_E^2 + \Delta)^m}$$

&

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^m} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(m - \frac{d}{2})}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m - \frac{d}{2}}$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{(l_E^2)}{(l_E^2 + \Delta)^m} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(m - \frac{d}{2} - 1)}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m - \frac{d}{2} - 1}$$

① $n=1$, $m=2$

$$\left(\frac{2}{d}-1\right)g^{\mu\nu}(-i)\int\frac{d^d l_E}{(2\pi)^d}\frac{(l_E^2)}{(l_E^2+\Delta)^m}$$

$$=(-i)\,g^{\mu\nu}\left(\frac{2}{d}-1\right)\cdot\frac{1}{(4\pi)^{d/2}}\cdot\frac{d}{2}\,\frac{\Gamma(2-\frac{d}{2}-1)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}-1}$$

$\Gamma(2)=1$

$$=(-i)g^{\mu\nu}\frac{1}{(4\pi)^{d/2}}\cdot\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\cdot\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}$$

$$=(-i)g^{\mu\nu}\frac{1}{(4\pi)^{d/2}}\cdot\Gamma\left(2-\frac{2}{d}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\cdot\Delta$$

$$=ig^{\mu\nu}\cdot\frac{1}{(4\pi)^{d/2}}\cdot\Gamma\left(2-\frac{2}{d}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\cdot(-\Delta)$$ where $\Delta=x(1-x)q^2-m^2$

② $n=0$, $m=2$

$$\left[(2x^2-2x)q^\mu q^\nu-(x^2-x)g^{\mu\nu}q^2+m^2g^{\mu\nu}\right]\cdot i\cdot\int\frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2}$$

$$=\left[(2x^2-2x)q^\mu q^\nu-(x^2-x)g^{\mu\nu}q^2+m^2g^{\mu\nu}\right]\cdot i\cdot\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\cdot\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$=i\cdot\frac{1}{(4\pi)^{d/2}}\left[(2x^2-2x)q^\mu q^\nu-(x^2-x)g^{\mu\nu}q^2+m^2g^{\mu\nu}\right]\cdot\Gamma\left(2-\frac{d}{2}\right)\cdot\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$-\Delta g^{\mu\nu}+(2x^2-2x)q^\mu q^\nu-(x^2-x)g^{\mu\nu}q^2+m^2g^{\mu\nu}$$

$$=-\left[\cancel{x(1-x)q^2-m^2}\right]g^{\mu\nu}+(2x^2-2x)q^\mu q^\nu-(x^2-x)g^{\mu\nu}q^2+m^2g^{\mu\nu}$$

$x(x-1)q^2+m^2$

$$=\cancel{-2x(x-1)\,(q^2g^{\mu\nu}-g^{\mu}q^{\nu})}$$

$$=2x(1-x)(q^2g^{\mu\nu}-q^\mu q^\nu)$$

$\frac{d}{2}=\frac{4-2\epsilon}{2}=2-\epsilon$

$2-\frac{d}{2}=\epsilon$

$$\Rightarrow=(-ie\mu^\epsilon)^2\cdot 4\int_0^1 i\cdot\frac{1}{(4\pi)^{d/2}}\cdot 2x(1-x)(q^2g^{\mu\nu}-q^\mu q^\nu)\,\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$$i\Pi_2^{\mu\nu}=-i\frac{2\alpha}{\pi}\Gamma(\epsilon)(q^2g^{\mu\nu}-q^\mu q^\nu)\int_0^1dx\,x(1-x)\left[\frac{m^2-x(1-x)q^2}{4\pi\mu^2}\right]^{-\epsilon}$$

We can write $\Pi_2^{\mu\nu}=(q^2g^{\mu\nu}-q^\mu q^\nu)\Pi_2(q^2)$

$$\Pi_2(q^2)=-\frac{2\alpha}{\pi}\Gamma(\epsilon)\int_0^1dx\,x(1-x)\left[\frac{m^2-x(1-x)q^2}{4\pi\mu^2}\right]^{-\epsilon}$$

$$\Gamma(t)=\int_0^\infty dx\,x^{t-1}e^{-x}$$

(1) At $q^2=0$

$$\Pi_2(0) = -\frac{2\alpha}{\pi}\Gamma(\epsilon)\int_0^1 dx\,(1-x)\left[\frac{m^2}{4\pi\mu^2}\right]^{-\epsilon}$$

$$\frac{1}{2}x^2-\frac{1}{3}x^3\Big|_0^1 \qquad \frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

$$= -\frac{2\alpha}{\pi}\Gamma(\epsilon)\cdot\frac{1}{6}\cdot\left[\frac{m^2}{4\pi\mu^2}\right]^{-\epsilon}$$

$$= -\frac{\alpha}{3\pi}\left[\frac{1}{\epsilon}-\gamma_E\right]\cdot\left[1-\epsilon\cdot\ln\frac{m^2}{4\pi\mu^2}\right]$$

$$= -\frac{\alpha}{3\pi}\left[\frac{1}{\epsilon}-\gamma_E-\ln\frac{m^2}{4\pi\mu^2}+\epsilon\gamma_E\ln\frac{m^2}{4\pi\mu^2}\right]$$

$$= -\frac{\alpha}{3\pi}\frac{1}{\epsilon}+\text{finite terms.} \qquad Z_3=\frac{1}{1-\Pi_2(0)} \Rightarrow Z_3\approx 1+\Pi_2(0)$$

Consider $m^2-x(1-x)q^2$ term :

$$(a+b)^2=(a-b)^2+4ab\geq 4ab$$
$$(a+b)^2\geq 4ab \qquad \sqrt{ab}\leq\frac{a+b}{2}$$

$x(1-x)\leq\frac{1}{4}$

if $m^2>\frac{1}{4}q^2$ or $-\infty<q^2<4m^2$ then $m^2-x(1-x)q^2>0$

if $m^2<\frac{1}{4}q^2$ or $4m^2<q^2<+\infty$ then $m^2-x(1-x)q^2<0$

(2) When $-\infty<q^2<4m^2$, $\Pi_2(q^2)$ is real.

$$\log_a\left(\frac{M}{N}\right)=\log_a M-\log_a N$$

$$\Pi_2(q^2)=\frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\left[\frac{1}{\epsilon}-\gamma_E-\ln\frac{\Delta^2}{4\pi\mu^2}\right]$$

$$= \frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\ln\frac{\Delta^2}{m^2} \qquad \Delta=m^2-x(1-x)q^2$$

when $-q^2\gg m^2$

$$= \frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\left[\ln\left(\frac{-q^2}{m^2}\right)+\ln\left[x(1-x)\right]\right]$$

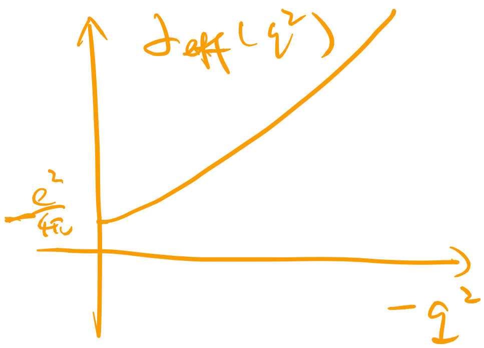
$$= \frac{2\alpha}{\pi}\left[\frac{1}{6}\ln\left[\frac{-q^2}{m^2}\right]-\frac{5}{18}\right]$$

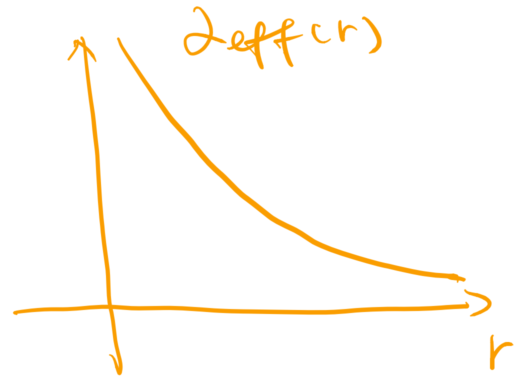
$$= \frac{\alpha}{3\pi}\left[\ln\left[\frac{-q^2}{m^2}\right]-\frac{5}{3}\right]$$

$$x(1-x) = -x^2+x = -(x^2-x) = -\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right] = -\left(x-\frac{1}{2}\right)^2+\frac{1}{4}$$

$$\alpha_{eff}=\frac{\alpha}{1-\frac{\alpha}{3\pi}\ln\frac{-q^2}{Am^2}} \qquad A=e^{\frac{5}{3}}$$

$$\frac{Z_3 e_0^2}{1-\left[\Pi_2(q^2)-\Pi_2(0)\right]} =$$





$r \sim \frac{1}{|q|}$ lifetime of photon

(3) $+\infty > q^2 > 4m^2$

$\Pi_2(q^2)$ has an imaginary part which modifies electrical potential.

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-mr}}{(mr)^{3/2}} + \cdots\right)$$

$r \downarrow \quad V(r) \uparrow$

$r \uparrow \quad V(r) \downarrow$

small r → large effective charge

large r → small effective charge

Vaccum polarization.

See Peskin P255