Graphic Method for Arbitrary n-body Phase Space

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Sci. Bull., in print [DOI: 10.1016/j.scib.2020.10.009].

Overview

Final State Phase Space

Phase Space - the space formed by all possible states of a system

Classical mechanics: Position-momentum Space.

Quantum mechanics: Hilbert Space.

Phase space is an essential part in calculating of physical process in quantum theory:

(1) **Physical Observables** calculation of transition probability

$$P_{i\to\{f\}} = \sum_{\{f\}} \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} \tag{1}$$

(2) **Unitarity** calculation of the imaginary part of a physical amplitude

$$2\mathrm{Im}\langle f|T|i\rangle = \sum_{\{\psi\}} \frac{\langle f|T|\psi\rangle\langle\psi|T^\dagger|i\rangle}{\langle\psi|\psi\rangle} \tag{2}$$

Final State Phase Space

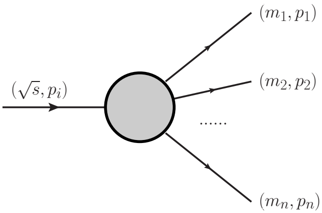


Figure: Process with center-of-mass-system energy $\sqrt{s}$ and n outgoing particles.

For scattering process, the final states are labeled by momentum(continuous part) :

$$P_{i\to\{f\}} = \sum_{\{f\}} \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} \propto \int \mathrm{d}\Phi_n(\sqrt{s}; m_1, ..., m_n)|\mathcal{M}_{fi}|^2, \tag{3}$$

where the n-**body final state phase space element** is defined as follows

$$\mathrm{d}\Phi_n(\sqrt{s}; m_1, ..., m_n) := \prod_{j=1}^{n} \frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3 2p_j^0} \delta^4(p_i - p_f). \tag{4}$$

Graphic Representation of Phase Space Decomposition

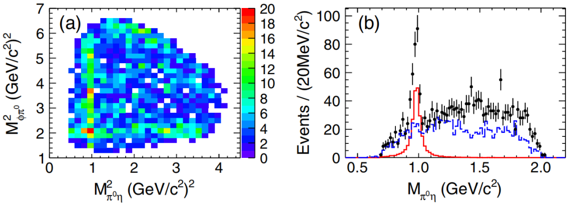


Figure: BESIII's observation of $a_0(980)^0 - f_0(980)$ mixing [Ablikim et al., 2018]: Dalitz plot for $J/\psi \to \eta\pi^0\phi$ (a) and mass projections on $m_{\pi^0\eta}$ (b).

In high energy physics, one cares about whether there is a non-trivial structure (peak, dip, cusp and so on) in the invariant mass distribution.

A key problem of the phase space calculation

how to get the formula of the phase space element expressed in terms of any given invariant masses for an n-body system?

Graphic Representation of Phase Space Decomposition

Phase Space Decomposition: recursive relation

$$\mathrm{d}\Phi_n(m; m_1, \ldots, m_n) = \mathrm{d}\Phi_k(m; m_1, \ldots, m_{(k)}) \times (2\pi)^3 \mathrm{d}m_{(k)}^2 \times \mathrm{d}\Phi_{n-k+1}(m_{(k)}; m_k, \ldots, m_n). \tag{5}$$

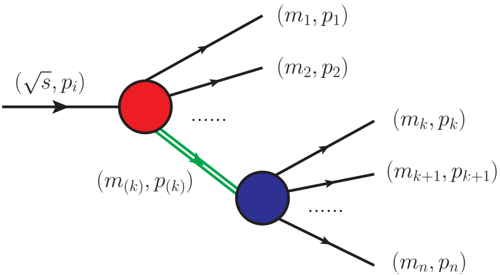


Figure: Graphic representation of the recursive relation.

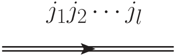
Graphic Representation of Phase Space Decomposition

Graphic method [Jing et al., 2020] is a way to calculate the phase space formula intuitively, just as what **Feynman diagrams** do in calculating scattering amplitudes.

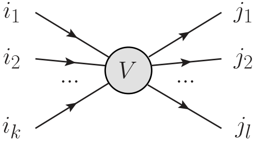
Elements and corresponding rules of graphic method:

(1) 

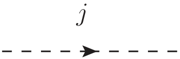
$1;$

(2) 

$(2\pi)^3 \mathrm{d}m^2_{j_1 j_2 \cdots j_l};$

(3) 

$\mathrm{d}\Phi_l(m_{i_1 \cdots i_k}; m_{j_1}, \ldots, m_{j_l});$

(4) 

$$\frac{(2\pi)^3}{\mathrm{d}^4 p_j \delta(p_j^2 - m_j^2)\theta(p_j^0)}.$$

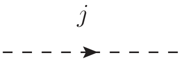
Graphic Representation of Phase Space Decomposition

Drawing rules

(1) A single line can be internal or external, and a double line can only be internal.

(2) There is one and only one route of double lines between any two vertices.

(3) If there are duplicate single lines for the same particle in the whole diagram, only one can be kept, and the rest are represented by dashed single lines:



$$\frac{(2\pi)^3}{\mathrm{d}^4 p_j \delta(p_j^2 - m_j^2)\theta(p_j^0)}.$$

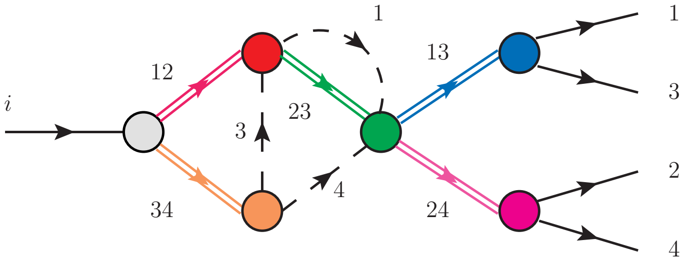
***This is to cancel out one copy of the phase space element of the particle which is double-counted in the integral measures.

(4) Invariant masses for all double lines in the whole diagram must be independent.

Graphic Representation of Phase Space Decomposition

From the drawing rules, one can find the following topological rules:

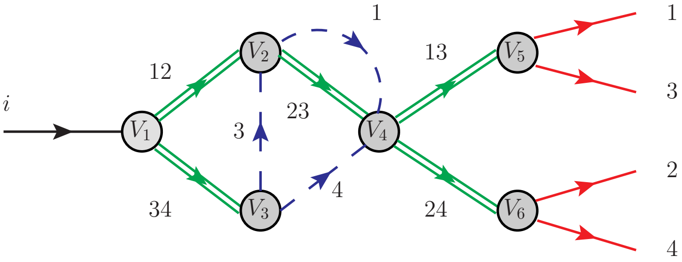
(1) the number of vertices v and that of double lines d are related as $v = d + 1$;



Graphic Representation of Phase Space Decomposition

From the drawing rules, one can find the following topological rules:

(2) the number of final state particles n, the number of double lines d, the number of internal single (solid and dashed) lines l and the number of outgoing lines for each vertex v_j are related as $n + d + l = \sum_{j=1}^{d+1} v_j$;

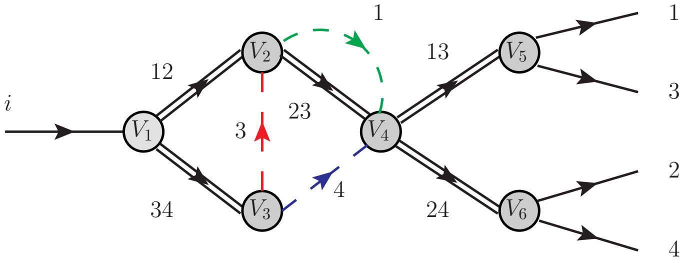


Graphic Representation of Phase Space Decomposition

From the drawing rules, one can find the following topological rules:

(3) the number of dashed lines equals the number of internal single lines l;

(4) using Euler's formula: $v - (l + d) + L = 1$, where L is the number of loops in a diagram, one finds $L = l$.



Graphic Representation of Phase Space Decomposition

Def: A diagram with $v_j = 2$ is called a **complete-expansion diagram (CED)**; otherwise, it is called an **incomplete-expansion diagram (ICED)**.

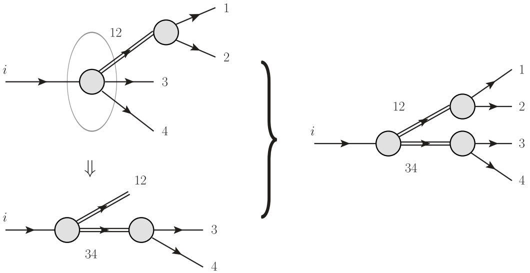
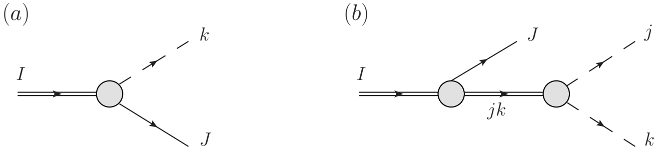


Figure: An **ICED** can be further expanded into a **CED**.

Combine **CED** and 2-**body phase space element**, an arbitrary n-body phase space can be easily expressed in integrations over any allowed invariant masses with the involved momenta being in any reference frame.

Composite corresponding rules: vertex with one(a) or two(b) dashed single lines.



case (a): $\quad (2\pi)^{-3}\ \delta\left[(p_I - p_k)^2 - m_J^2\right]$

case (b): $\quad (2\pi)^{-3}\ \mathrm{d}m_{jk}^2\ \delta\left[(p_I - p_j - p_k)^2 - m_J^2\right]\ \delta\left[(p_j + p_k)^2 - m_{jk}^2\right]$

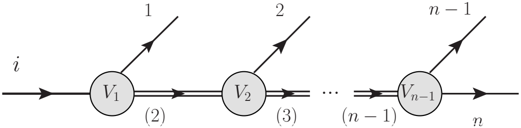
$\Rightarrow$ more dashed lines will contribute more δ-functions.

Graphic method vs **Feynman diagram**:

each loop will contributes a **on-shell momentum δ-function** vs **loop momentum integral**!

Some composite rules and examples of 4-body PSF

Example I : chain tree diagram of n-body phase space.



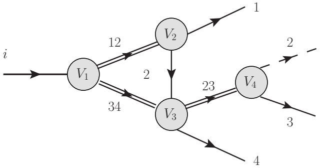
V_1 : $\mathrm{d}\Phi_2(m; m_1, m_{(2)})$, V_2 : $\mathrm{d}\Phi_2(m_{(2)}; m_2, m_{(3)})$, ..., V_{n-1} : $\mathrm{d}\Phi_2(m_{(n-1)}; m_{n-1}, m_n)$,
"(2)" : $(2\pi)^3 \mathrm{d}m^2_{(2)}$, "(3)" : $(2\pi)^3 \mathrm{d}m^2_{(3)}$, ..., "$(n-1)$" : $(2\pi)^3 \mathrm{d}m^2_{(n-1)}$,

$$\mathrm{d}\Phi_n(m; m_1, ..., m_n) = \frac{1}{2^n (2\pi)^{3n} m} |\mathbf{p}_1| \cdots |\mathbf{p}_{n-1}| \mathrm{d}\Omega_1 \cdots \mathrm{d}\Omega_{n-1} \mathrm{d}m_{(2)} \cdots \mathrm{d}m_{(n-1)}, \quad (6)$$

where $(|\mathbf{p}_i|, \Omega_i)$ is the three-momentum of the final-state particle i in the c.m. frame of the $(i, i+1, \ldots, n)$ particle system.

A much more lengthy derivation can be found in the appendix of Ref. [Jing et al., 2019].

Example II : a 1-loop diagram of 4-body PSF.



$$V_1: \ \mathrm{d}\Phi_2(m; m_{12}, m_{34}), \ V_2: \ \mathrm{d}\Phi_2(m_{12}; m_1, m_2), \ V_3: \ \mathrm{d}\Phi_2(m_{234}; m_4, m_{23}),$$

$$V_4: \ \mathrm{d}\Phi_2(m_{23}; m_2, m_3), \text{"12"}: (2\pi)^3 \mathrm{d}m_{12}^2, \ \text{"34"}: (2\pi)^3 \mathrm{d}m_{34}^2,$$

$$\text{"23"}: \ (2\pi)^3 \mathrm{d}m_{23}^2, \ \text{dashed single line "2"}: \ (2\pi)^3 \left[\mathrm{d}^4 p_2 \delta(p_2^2 - m_2^2)\theta(p_2^0)\right]^{-1}.$$

$$\mathrm{d}\Phi_4(m; m_1, m_2, m_3, m_4) = \mathrm{d}m_{12}^2 \mathrm{d}m_{34}^2 \mathrm{d}m_{23}^2 \mathrm{d}\Omega'_{12} \mathrm{d}\Omega''_1 \mathrm{d}\phi_4^* \\ \times \frac{|\mathbf{p}'_{12}||\mathbf{p}''_1|}{(2\pi)^{12} 2^7 m m_{12} m_{234} |\mathbf{p}_2^*|} \ \theta_{[-1,1]} \left(\frac{m_{23}^2 + m_2^2 - m_3^2 - 2p_{23}^{*0} p_2^{*0}}{2|\mathbf{p}_2^*||\mathbf{p}_4^*|} \right), \qquad (7)$$

where $(|\mathbf{p}'_{12}|, \Omega'_{12})$ is the three-momentum of the final-state (1,2) particle system in the c.m. frame of the initial state, $(|\mathbf{p}''_1|, \Omega''_1)$ is the three-momentum of particle 1 in the c.m. frame of particles 1 and 2, and the quantities labelled by a "*" are defined in the c.m. frame of the (2,3,4) particle system in the final state.

Summary

(1) Phase space integration is important to connect theoretical calculation with experimental observation in quantum theory; it also enters into the calculation of the imaginary part of a physical amplitude through unitarity.

(2) The phase space element can be decomposed into many phase space elements intuitively by graphic method.

(3) Graphic method is applicable in the case of general D-dimensional spacetime, which is useful when considering dimensional regularization (e.g., in the soft-gluon resummation in Ref. [Forte and Ridolfi, 2003]) and quantum field theory in arbitrary dimensions.

THANK YOU FOR YOUR ATTENTION !

Ablikim, M. et al. (2018).
Observation of $a_0^0(980)$-$f_0(980)$ Mixing.
Phys. Rev. Lett., 121(2):022001.

Forte, S. and Ridolfi, G. (2003).
Renormalization group approach to soft gluon resummation.
Nucl. Phys. B, 650:229–270.

Jing, H.-J., Sakai, S., Guo, F.-K., and Zou, B.-S. (2019).
Triangle singularities in $J/\psi \to \eta\pi^0\phi$ and $\pi^0\pi^0\phi$.
Phys. Rev. D, 100(11):114010.

Jing, H.-J., Shen, C.-W., and Guo, F.-K. (2020).
Graphic Method for Arbitrary n-body Phase Space.
Sci. Bull., in print: arXiv: 2005.01942.

Back up

2-body phase space element in 4-dimensional spacetime

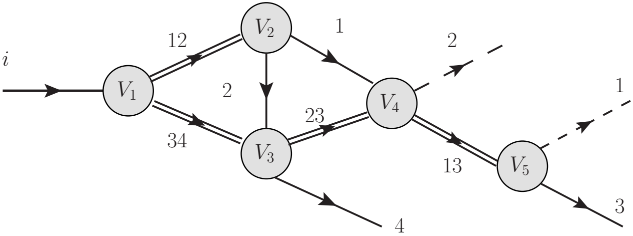
Integral measure	Integrand	$\beta=0$	$0<\beta<\beta_1^*$	$\beta\geq\beta_1^*$
$\mathrm{d}\cos\theta_1\mathrm{d}\phi_1$	$\frac{\lvert\mathbf{p}_1\rvert^2}{4\lvert(p^0\lvert\mathbf{p}_1\rvert-p_1^0\lvert\mathbf{p}\rvert\cos\theta_{01})\rvert}$	1	1	1,2
$\mathrm{d}\lvert\mathbf{p}_1\rvert\mathrm{d}\phi_1$	$\frac{\lvert\mathbf{p}_1\rvert}{4p_1^0\lvert\mathbf{p}\rvert\lvert\partial\cos\theta_{01}/\partial\cos\theta_1\rvert}$	$\aleph_1$	1,2	1,2
$\mathrm{d}p_1^0\mathrm{d}\phi_1$	$\frac{1}{4\lvert\mathbf{p}\rvert\lvert\partial\cos\theta_{01}/\partial\cos\theta_1\rvert}$	$\aleph_1$	1,2	1,2
$\mathrm{d}\lvert\mathbf{p}_1\rvert\mathrm{d}\cos\theta_1$	$\frac{\lvert\mathbf{p}_1\rvert}{4p_1^0\lvert\mathbf{p}\rvert\lvert\partial\cos\theta_{01}/\partial\phi_1\rvert}$	1,$\aleph_1$	1,2,$\aleph_1$	1,2,$\aleph_1$
$\mathrm{d}p_1^0\mathrm{d}\cos\theta_1$	$\frac{1}{4\lvert\mathbf{p}\rvert\lvert\partial\cos\theta_{01}/\partial\phi_1\rvert}$	1,$\aleph_1$	1,2,$\aleph_1$	1,2,$\aleph_1$

TABLE I. Integrand of the 2-body phase space element in spherical coordinate system with different integral variables. A factor $(2\pi)^{-6}$ has been omitted in each integrand. The definitions of p, p_1 and θ_{01} are the momentum of the initial state, the momentum of particle 1 in the final state and the relative angle between $\mathbf{p}$ and $\mathbf{p}_1$, respectively. The last three columns give the possible numbers of solutions of the physical on-shell equations of particles 1 and 2 in the final state when the corresponding two integral variables are fixed in any reference frame, where β and β_1^* are the velocity of the initial state and that of particle 1 in the rest frame of the initial state, respectively. $\aleph_1$ is the second transfinite number.

For more details, see supplemental materials of Ref. [Jing et al., 2020].

More examples of 4-body phase space element

Example III : a 2-loop diagram of 4-body PSF.



V_1 : $\mathrm{d}\Phi_2(m; m_{12}, m_{34})$, V_2 : $\mathrm{d}\Phi_2(m_{12}; m_1, m_2)$, V_3 : $\mathrm{d}\Phi_2(m_{234}; m_4, m_{23})$,

V_4 : $\mathrm{d}\Phi_2(m_{123}; m_2, m_{13})$, V_5 : $\mathrm{d}\Phi_2(m_{13}; m_1, m_3)$

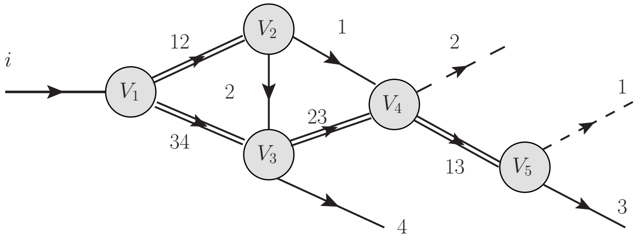
"12" : $(2\pi)^3 \mathrm{d}m_{12}^2$, "34" : $(2\pi)^3 \mathrm{d}m_{34}^2$, "23" : $(2\pi)^3 \mathrm{d}m_{23}^2$, "13" : $(2\pi)^3 \mathrm{d}m_{13}^2$,

dashed single line "1" : $(2\pi)^3 \left[\mathrm{d}^4 p_1 \delta(p_1^2 - m_1^2)\theta(p_1^0)\right]^{-1}$.

dashed single line "2" : $(2\pi)^3 \left[\mathrm{d}^4 p_2 \delta(p_2^2 - m_2^2)\theta(p_2^0)\right]^{-1}$.

More examples of 4-body phase space element

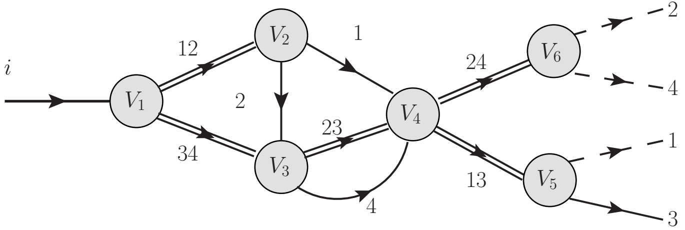
Example III : a 2-loop diagram of 4-body PSF.



$$
\begin{aligned}
\mathrm{d}\Phi_4(m;m_1,m_2,m_3,m_4) =& \,\mathrm{d}\Phi_2(m;m_{12},m_{34})(2\pi)^3\mathrm{d}m_{12}^2\mathrm{d}\Phi_2(m_{12};m_1,m_2) \\
& \times (2\pi)^3\mathrm{d}m_{34}^2(2\pi)^3\mathrm{d}m_{23}^2\mathrm{d}\Phi_2(m_{234};m_4,m_{23})(2\pi)^3\mathrm{d}m_{13}^2 \\
& \times (2\pi)^{-6}\delta\left[(p_{23}-p_2)^2-m_3^2\right]\delta\left[(p_1+p_{23}-p_2)^2-m_{13}^2\right] \\
=& \frac{|\mathbf{p}'_{12}||\mathbf{p}''_1||\mathbf{p}^*_{23}|}{(2\pi)^{12}2^7 m m_{12} m_{234}}\mathrm{d}m_{12}^2\mathrm{d}m_{34}^2\mathrm{d}m_{23}^2\mathrm{d}m_{13}^2\mathrm{d}\Omega'_{12}\mathrm{d}\Omega''_1 \\
& \times \left|\frac{\partial(p_1\cdot p_{23}, p_2\cdot p_{23})}{\partial(\cos\theta^*_{23},\phi^*_{23})}\right|^{-1}\theta_{[-1,1]}\left(\cos\tilde{\theta}^*_{23}\right)\theta_{[0,\pi]}\left(\tilde{\phi}^*_{23}\right).
\end{aligned}
$$

More examples of 4-body phase space element

Example IV : a 3-loop diagram of 4-body PSF.



V_1 : $\mathrm{d}\Phi_2(m;m_{12},m_{34})$, V_2 : $\mathrm{d}\Phi_2(m_{12};m_1,m_2)$, V_3 : $\mathrm{d}\Phi_2(m_{234};m_4,m_{23})$,
V_4 : $\mathrm{d}\Phi_2(m;m_{13},m_{24})$, V_5 : $\mathrm{d}\Phi_2(m_{13};m_1,m_3)$, V_6 : $\mathrm{d}\Phi_2(m_{24};m_2,m_4)$
"12" : $(2\pi)^3\mathrm{d}m_{12}^2$, "34" : $(2\pi)^3\mathrm{d}m_{34}^2$, "23" : $(2\pi)^3\mathrm{d}m_{23}^2$,
"13" : $(2\pi)^3\mathrm{d}m_{13}^2$, "24" : $(2\pi)^3\mathrm{d}m_{24}^2$,

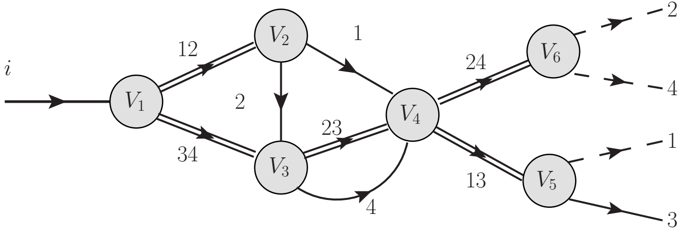
dashed single line "1" : $(2\pi)^3\left[\mathrm{d}^4p_1\delta(p_1^2-m_1^2)\theta(p_1^0)\right]^{-1}$.

dashed single line "2" : $(2\pi)^3\left[\mathrm{d}^4p_2\delta(p_2^2-m_2^2)\theta(p_2^0)\right]^{-1}$.

dashed single line "4" : $(2\pi)^3\left[\mathrm{d}^4p_4\delta(p_4^2-m_4^2)\theta(p_4^0)\right]^{-1}$.

More examples of 4-body phase space element

Example IV : a 3-loop diagram of 4-body PSF.



$$\begin{aligned}
\mathrm{d}\Phi_4(m;m_1,m_2,m_3,m_4) =& \,\mathrm{d}\Phi_2(m;m_{12},m_{34})(2\pi)^3\mathrm{d}m_{12}^2\mathrm{d}\Phi_2(m_{12};m_1,m_2) \\
& \times (2\pi)^3\mathrm{d}m_{34}^2(2\pi)^3\mathrm{d}m_{23}^2\mathrm{d}\Phi_2(m_{234};m_4,m_{23})(2\pi)^3\mathrm{d}m_{13}^2 \\
& \times (2\pi)^3\mathrm{d}m_{24}^2(2\pi)^{-3}\delta\left[(p_2+p_4)^2-m_{24}^2\right] \\
& \times (2\pi)^{-6}\delta\left[(p_{23}-p_2)^2-m_3^2\right]\delta\left[(p_1+p_{23}-p_2)^2-m_{13}^2\right] \\
=& \frac{|\mathbf{p}'_{12}||\mathbf{p}''_1||\mathbf{p}^*_{23}|}{(2\pi)^{12}2^8 m m_{12} m_{234}}\mathrm{d}m_{12}^2\mathrm{d}m_{34}^2\mathrm{d}m_{23}^2\mathrm{d}m_{13}^2\mathrm{d}m_{24}^2\mathrm{d}\Omega'_{12}\mathrm{d}\phi''_1
\end{aligned}$$

$$\times \left|\frac{\partial(p_1\cdot p_{23}, p_2\cdot p_{23}, p_2\cdot p_4)}{\partial(\cos\theta^*_{23},\phi^*_{23},\cos\theta''_1)}\right|^{-1}\theta_{[-1,1]}\left(\cos\tilde{\theta}^*_{23}\right)\theta_{[0,\pi]}\left(\tilde{\phi}^*_{23}\right)\theta_{[-1,1]}\left(\cos\tilde{\theta}''_1\right).$$

Graphic method in general spacetime dimensions

The n-body phase space element in D dimensions(here one time dimension is considered) is

$$\mathrm{d}\Phi_n(m; m_1, \ldots, m_n) = \delta^D(p - \sum_{i=1}^{n} p_i) \prod_{j=1}^{n} \frac{\mathrm{d}^{D-1}\mathbf{p}_j}{(2\pi)^{D-1} 2p_j^0}. \tag{8}$$

The drawing rules and topological rules of the D-dimensional case are exactly the same as the 4-dimensional case, and one only needs the following modifications:

(1) $$(2\pi)^3 \mathrm{d}m^2_{j_1 j_2 \cdots j_l} \to (2\pi)^{D-1} \mathrm{d}m^2_{j_1 j_2 \cdots j_l};$$

(2) $$\frac{(2\pi)^3}{\mathrm{d}^4 p_j \delta(p_j^2 - m_j^2)\theta(p_j^0)} \to \frac{(2\pi)^{D-1}}{\mathrm{d}^D p_j \delta(p_j^2 - m_j^2)\theta(p_j^0)}.$$

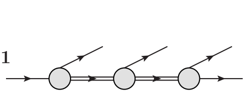
For more details, see supplemental materials of Ref. [Jing et al., 2020].

4-body Complete-expansion Diagrams

Here we give all directed acyclic graphs of 4-body phase space complete-expansion diagrams.

2 tree diagrams:

1



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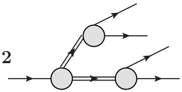
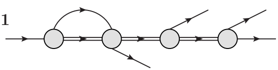
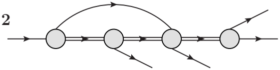


5 1-loop diagrams:

1



2



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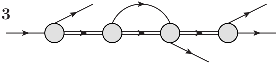

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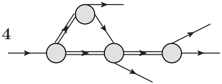

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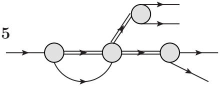
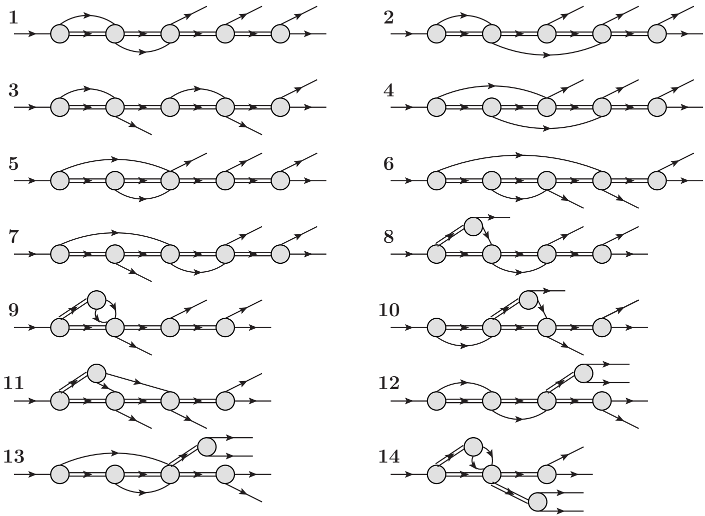
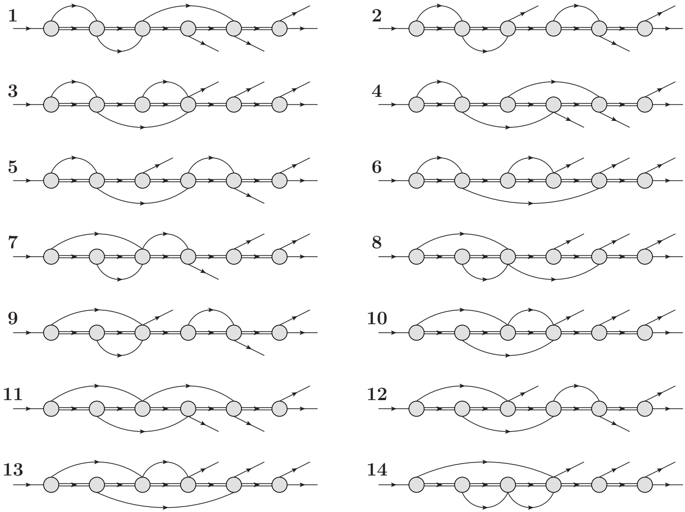


4-body Complete-expansion Diagrams

14 2-loop diagrams:

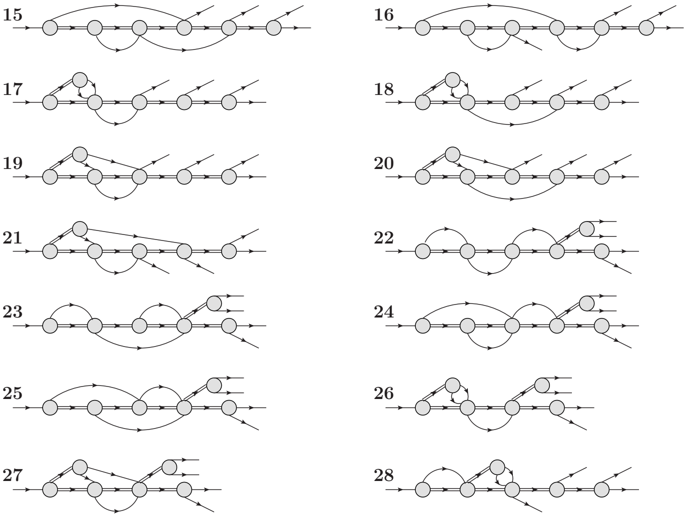


34 3-loop diagrams:



4-body Complete-expansion Diagrams

34 3-loop diagrams:



4-body Complete-expansion Diagrams

34 3-loop diagrams:

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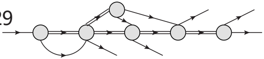

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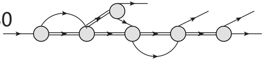

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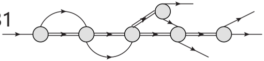

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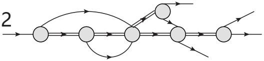

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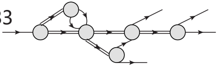

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