# Particle physics methods for gravitational wave physics

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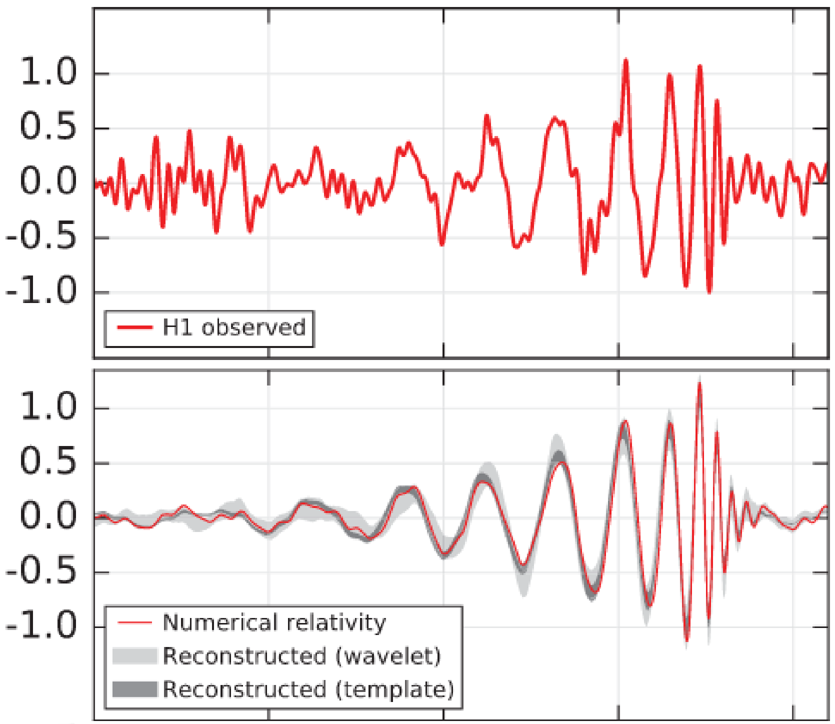
# Outline

- Introduction
- Perturbative gravity
- Classical physics from quantum amplitudes
- $\hbar$ expansion – method of regions

# Introduction

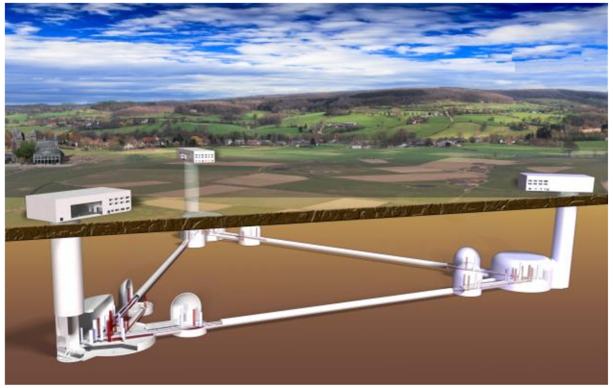
# Motivation – GW physics

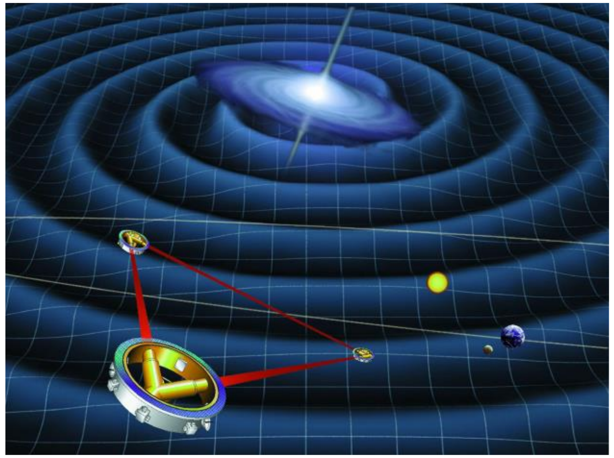
- GW discovery in 2015 by LIGO / VIRGO. Future ground-based and space-based detectors offer much higher sensitivity.

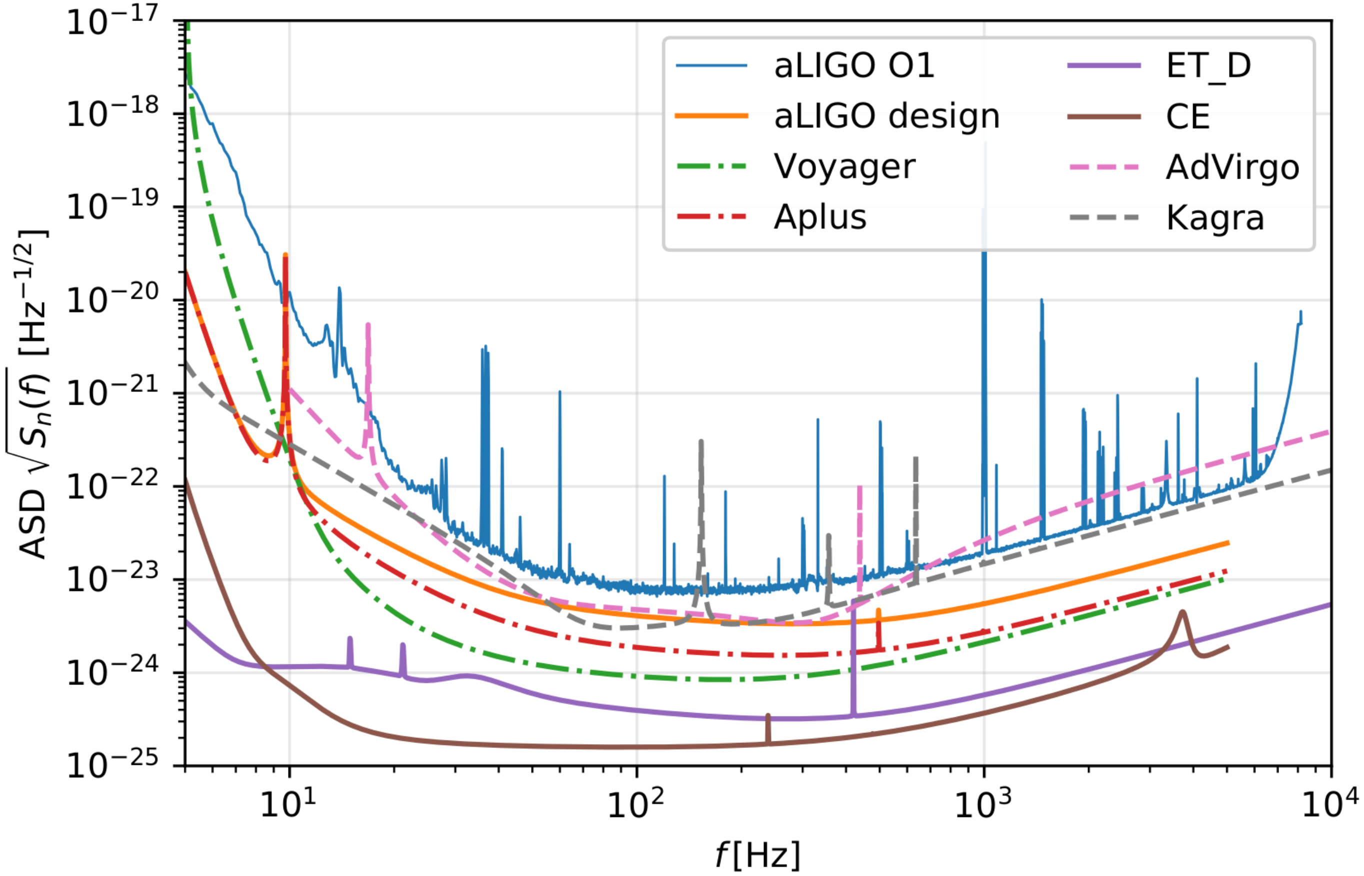


# Motivation – future detectors

- Theoretical predictions need orders of magnitude improvement!







# Post-Newtonian expansion

- Joint expansion in $GM/R$ and $v^2$, locked together by Virial's theorem.

$$m = m_A + m_B, \quad \nu = \mu/m$$
$$\mu = m_A m_B / m$$

$$\frac{H}{\mu} = \boxed{\underbrace{\frac{P^2}{2}}_{\mathcal{O}(v^2)} - \underbrace{\frac{Gm}{R}}_{\mathcal{O}(G)}} + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} + H_{3\mathrm{PN}} + H_{4\mathrm{PN}} \ldots$$

0PN, Newton

$H_{2\mathrm{PN}}$ (1980), $H_{3\mathrm{PN}}$ (2000), $H_{4\mathrm{PN}}$ (2014)

$$H_{1\mathrm{PN}} \to \mathcal{O}(v^4) + \mathcal{O}(Gv^2) + \mathcal{O}(G^2)$$

$$\boxed{+\frac{1}{c^2}\left\{-\frac{P^4}{8} + \frac{3\nu P^4}{8} + \frac{Gm}{R}\left(-\frac{P_R^2\nu}{2} - \frac{3P^2}{2} - \frac{\nu P^2}{2}\right) + \frac{G^2m^2}{2R^2}\right\}}$$

1PN, Einstein, Infeld, Hoffman, 1938

# Post-Minkowskian expansion

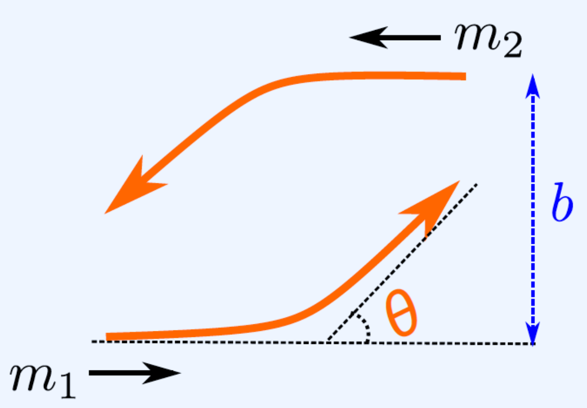
- Expansion in coupling constant $GM/R$, **exact velocity dependence.** [Bertotti, Kerr, Plebanski, Portilla, Westpfahl, Gollder, Bel, Damour, Derulle, Ibanez, Martin, Ledvinka, Schaefer, Bicak…]

- Most accurate PM scattering angle until ~2019 [Westpfahl, '85]

**Scattering angle** of two black holes, as function of $m_1, m_2, b, E_{\rm cm}$.

$$2\sin\left(\frac{\theta}{2}\right) = \frac{4G(m_1+m_2)}{b}\left(\frac{\hat{E}^4 - 2m_1^2m_2^2}{\hat{E}^4 - 4m_1^2m_2^2} + \frac{3\pi}{16}\frac{G(m_1+m_2)}{b}\frac{5\hat{E}^4 - 4m_1^2m_2^2}{\hat{E}^4 - 4m_1^2m_2^2}\right)$$

*where* $\hat{E}^2 \equiv E_{\rm cm}^2 - (m_1^2 + m_2^2), \;\; c = 1$



# New method: (multi-loop) scattering amplitudes

- In relativistic QFT, perturbative expansion in coupling constant, exactly analogous to post-Minkowskian expansion in GR.

- Effective field theory: when two black holes are at a large distance, approximated by point particles.
  - For Schwarzschild BH, massive scalar particles coupled to gravity
  - For Kerr BH, spin coupling captured by massive spin-1/2, spin-1, or higher spin particle

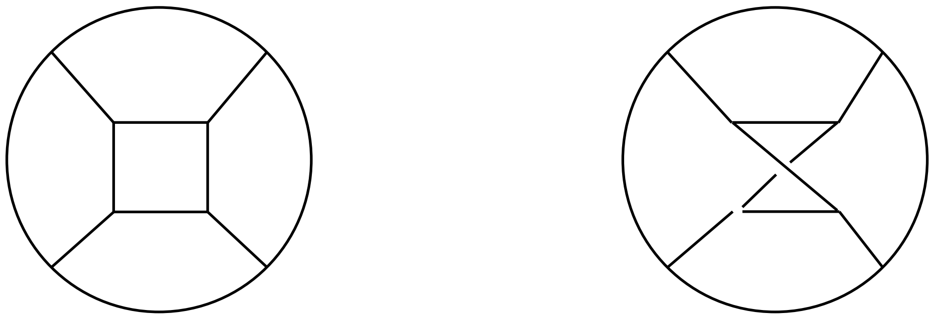
- Many advanced loop amplitude techniques developed for particle physics – used to push calculations beyond best classical results!

# Previously: multi-loop amplitudes for gravity

- Ultraviolet behavior of $N = 8$ supergravity at 5 loops.

  [Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, MZ, '18]

Vacuum integrals at high loop orders

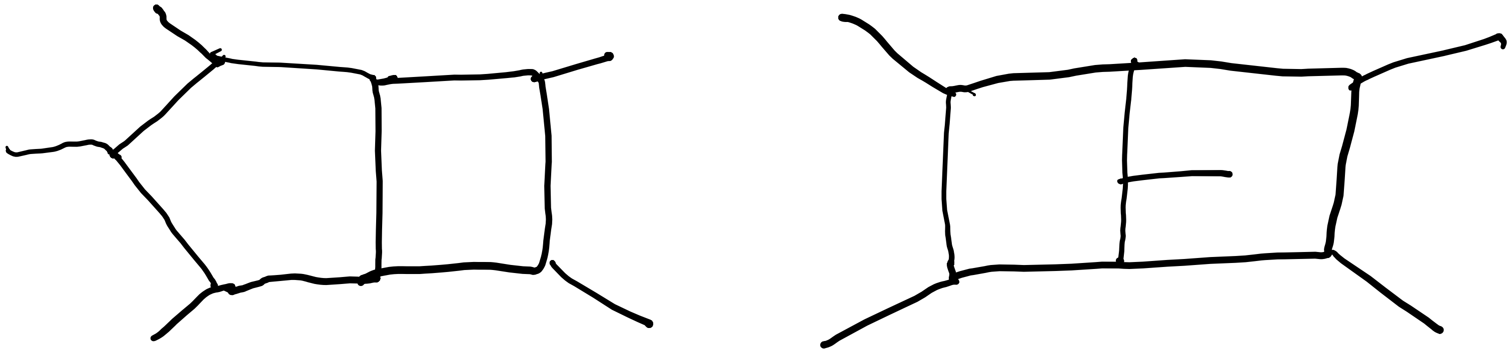


Many particle physics methods used for gravity!

- 2-loop 5-point amplitude of $N = 8$ supergravity.

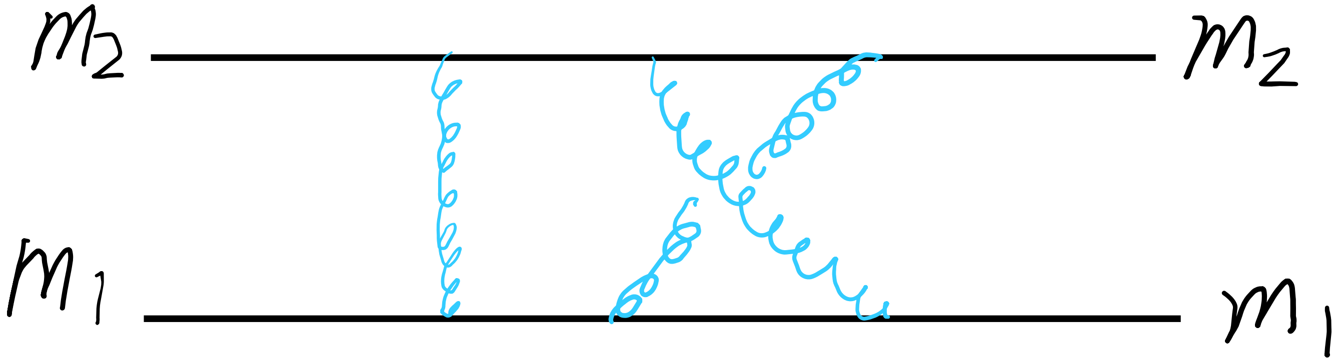
  [Abreu, Dixon, Page, Herrmann, MZ, '19; Chicherin, Gehrmann, Henn, Wasser, Zhang, '19]

Same master integrals as e.g. $pp \to 3j$ for QCD.

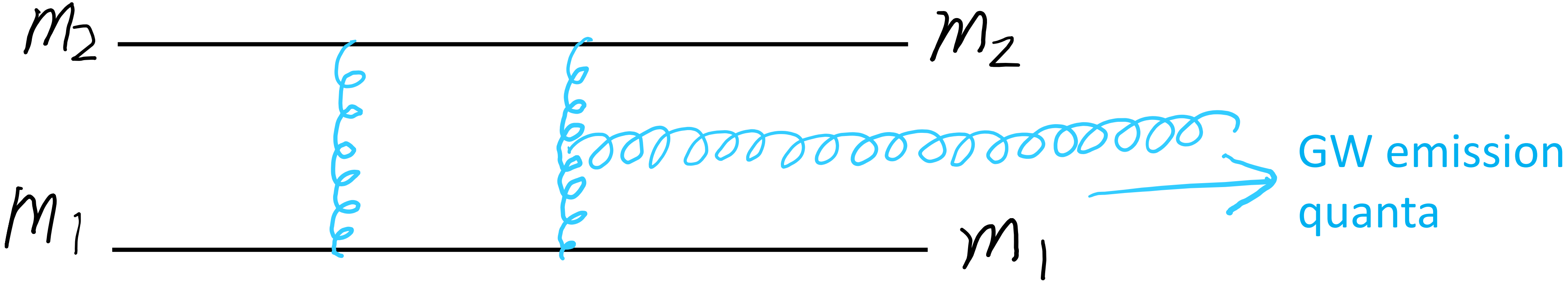


# What amplitudes do we need?

- $\Phi + \phi \to \Phi + \phi$ via graviton exchange. Conservative potential.




- $\Phi + \phi \to \Phi + \phi + h$. Graviton emission / energy loss.



# Perturbative gravity

c.f. arXiv/1702.00319

- Einstein-Hilbert Lagrangian, with additional massive scalar matter,

$$S = \int d^4x \sqrt{-g}\left[-2R + \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right)\right] + \cdots$$

Ricci curvature scalar

- Dynamic metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$, $\kappa^2 = 8\pi G$.

Minkowski

perturbation

- Integration volume $d^4x\,\sqrt{-g}, g \equiv \det(g_{\mu\nu})$.

$$ds = \sqrt{g_{\mu\nu}\,dx^\mu\,dx^\nu}, \quad V \sim (ds)^4 \sim |g_{\mu\nu}|^2\, d^4x$$
$$\sim \sqrt{-g}\, d^4x$$

# Perturbative gravity

- Einstein-Hilbert Lagrangian, with additional massive scalar matter,

$$S = \int d^4x \sqrt{-g} \left[ -2R + \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi - m^2\phi^2\right)\right] + \cdots$$

- Dynamic metric $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$, $\kappa^2 = 8\pi G$.

$$-g = -\det(g_{\mu\nu}) = \det(\eta^\mu{}_\rho\, g_{\rho\nu})$$

$$= \det(\mathbb{1} + \kappa h^\mu{}_\nu) = \exp[\mathrm{Tr}\log(\mathbb{1} + \kappa h^\mu{}_\nu)]$$

$$= \exp\left[\mathrm{Tr}\left(\kappa h^\mu{}_\nu + \frac{\kappa^2}{2} h^\mu{}_\rho h^\rho{}_\nu + \frac{\kappa^3}{3} h^\mu{}_\rho h^\rho{}_\sigma h^\sigma{}_\nu \cdots\right)\right]$$

$$= 1 + \kappa h^\mu{}_\mu + \ldots \quad \Rightarrow \sqrt{-g} = 1 + \frac{\kappa}{2} h^\mu{}_\mu$$

# Perturbative gravity

- Einstein-Hilbert Lagrangian, with additional massive scalar matter,

$$S = \int d^4x \sqrt{-g}\left[-2R + \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi - m^2\phi^2\right)\right] + \cdots$$

- $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$ .

$$= \left(g_{\mu\nu}\right)^{-1} = \left(\eta_{\mu\nu} + \kappa\, h_{\mu\nu}\right)^{-1}$$

$$= \eta^{\mu\nu} - \kappa\, h^{\mu\nu} \ \ldots.$$

Matrix

$$(A+B)^{-1} = \left[A(\mathbb{1}+A^{-1}B)\right]^{-1} = A^{-1} - A^{-1}BA^{-1} + \frac{1}{2}A^{-1}BA^{-1}BA^{-1} \ldots$$

# Perturbative gravity

- Einstein-Hilbert Lagrangian, with additional massive scalar matter,

$$S = \int d^4x \sqrt{-g} \left[ -2R + \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right)\right] + \cdots$$

- We showed $\sqrt{-g} = 1 + \frac{1}{2}\kappa\, h^\mu_\mu \ldots$ , $g^{\mu\nu} = \eta^{\mu\nu} - \kappa\, h^{\mu\nu} \ldots$

- Matter part of Lagrangian

$$\frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \frac{1}{2}\left(\partial^\mu\phi\,\partial_\mu\phi - m^2\phi^2\right)$$

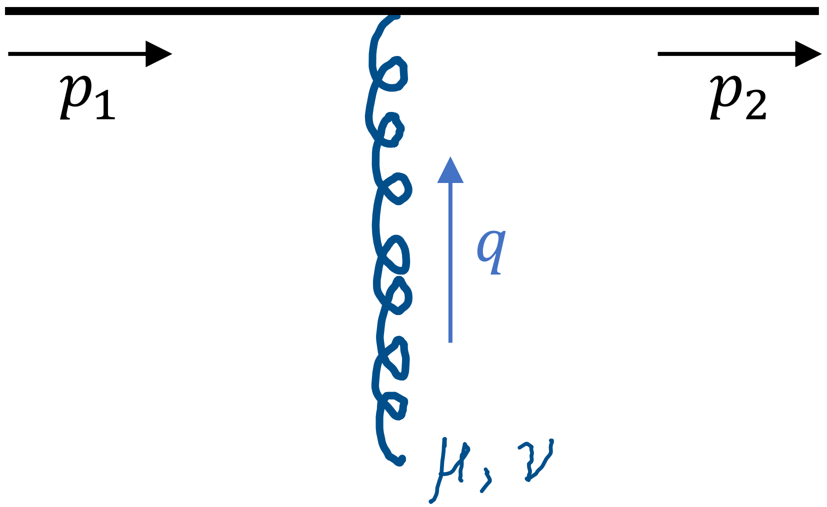
$$+\frac{1}{2}\kappa\left[\frac{1}{2}h^\mu_\mu\left(\partial^\nu\phi\,\partial_\nu\phi - m^2\phi^2\right) - h^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right] + \mathcal{O}(\kappa^2)$$

# Scalar-graviton vertex

- Matter part of Lagrangian

$$\frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\partial_\mu\phi\;\partial_\nu\phi = \frac{1}{2}\partial^\mu\phi\;\partial_\mu\phi - m^2\phi^2$$

$$+\frac{1}{2}\kappa\left[\frac{1}{2}h^\mu_\mu\left(\partial^\nu\phi\;\partial_\nu\phi - m^2\phi^2\right) - h^{\mu\nu}\partial_\mu\phi\;\partial_\nu\phi\right] + \;\mathcal{O}(\kappa^2)$$




$$= \frac{i\kappa}{2}\left[-\left(p_1^\mu\, p_2^\nu + p_1^\mu\, p_2^\nu\right) + \eta^{\mu\nu}\left(p_1\cdot p_2 - m^2\right)\right]$$

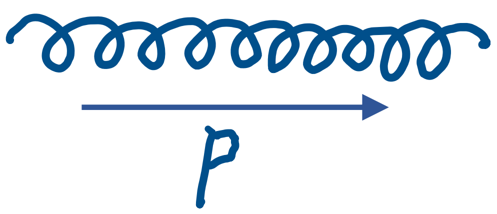
# Graviton propagtor

- Pure gravity part of Lagrangian, plus gauge-fixing term to make the quadratic terms invertible.

$$S = \int d^4x \sqrt{-g}\left[-2R + \mathcal{C}^{\nu}\mathcal{C}_{\nu}\right] + \cdots$$

$$\mathcal{C}_{\nu} = \partial_{\mu} h^{\mu}_{\nu} - \frac{1}{2}\partial_{\nu} h^{\mu}_{\mu}$$

gauge fixing vector
for de-Donder gauge

$\mu_1, \nu_1$  $\mu_2, \nu_2$



$$= \frac{i}{p^2}\left(\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2} - \frac{2}{d-2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\right)$$

# Graviton self-interactions

$$S = \int d^4x \sqrt{-g}[-2R + C^\nu C_\nu] + \cdots \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

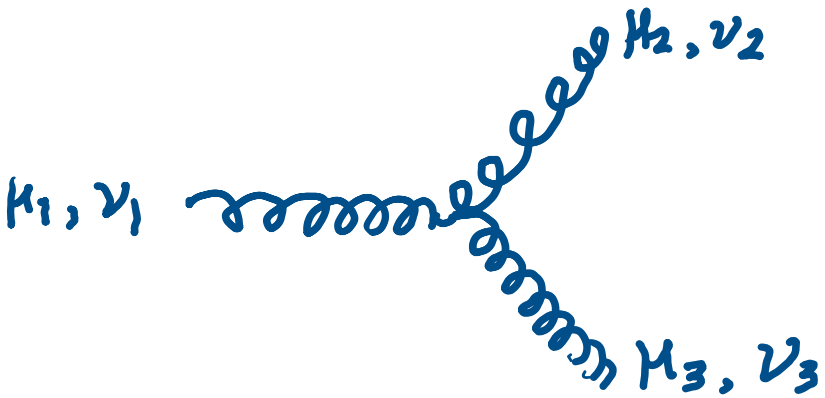
$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu} V^\rho ,$$

$$\Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \sim \partial h$$

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - \partial_\nu \Gamma_{\mu\alpha}{}^{\beta} + \Gamma_{\mu\rho}{}^{\beta} \Gamma_{\nu\alpha}{}^{\rho} - \Gamma_{\mu\rho}{}^{\beta} \Gamma_{\mu\alpha}{}^{\rho} \sim \partial^2 h + (\partial h)^2$$

$$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu} , \qquad R = g^{\nu\alpha} R_{\nu\alpha}$$

# 3-graviton vertex




$$= -\frac{1}{2}k_1^2\,\eta_{\mu_1\nu_2}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}\cdots$$

~100 terms!

Modern simplifications: double copy, generalized unitarity, nonlinear gauge fixing...

$$-\frac{1}{2}k_1.k_1\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}-\frac{1}{2}k_1.k_2\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}-\frac{1}{2}k_1.k_3\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}-\frac{1}{2}k_2.k_2\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}-\frac{1}{2}k_2.k_3\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}-\frac{1}{2}k_3.k_3\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}-3\,k_1.k_2\,\eta_{\mu_1,\nu_3}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_2}-2\,k_3.k_3\,\eta_{\mu_1,\nu_3}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_2}-3\,k_1.k_2\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}-$$

$$2\,k_3.k_3\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}+k_1.k_1\,\eta_{\mu_1,\mu_2}\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}+\frac{3}{2}k_1.k_2\,\eta_{\mu_1,\mu_2}\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}+k_1.k_3\,\eta_{\mu_1,\mu_2}\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}+k_2.k_2\,\eta_{\mu_1,\mu_2}\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}+k_2.k_3\,\eta_{\mu_1,\mu_2}\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}+k_3.k_3\,\eta_{\mu_1,\mu_2}\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}+k_1.k_1\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}+k_1.k_2\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}-2\,k_1.k_3\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}-$$

$$k_2.k_2\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}+\frac{3}{2}k_2.k_3\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}+k_3.k_3\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}+k_1.k_1\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}+k_1.k_2\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}+\frac{3}{2}k_1.k_3\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}+k_2.k_2\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}+k_2.k_3\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}+k_3.k_3\,\eta_{\mu_1,\mu_3}\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}-$$

$$2\,k_1.k_1\,\eta_{\mu_1,\mu_3}\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}-3\,k_2.k_3\,\eta_{\mu_1,\mu_3}\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}-2\,k_1.k_1\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}-3\,k_1.k_3\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}-2\,k_2.k_2\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}-3\,k_2.k_3\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}-2\,\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_1}k_1^{\mu_2}+4\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_1^{\mu_2}-2\,\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_1^{\mu_3}+4\,\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_1^{\mu_3}+$$

$$2\,\eta_{\mu_1,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_2}k_1^{\mu_3}-2\,\eta_{\mu_1,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_2}k_1^{\mu_3}+\frac{1}{2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}k_1^{\mu_1}k_1^{\nu_2}-\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_1^{\nu_2}+\eta_{\mu_1,\nu_2}\eta_{\mu_3,\nu_3}k_1^{\mu_2}k_1^{\nu_2}-2\,\eta_{\mu_1,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_2}k_1^{\nu_2}+\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}k_1^{\mu_3}k_1^{\nu_3}-2\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\nu_2}k_1^{\mu_3}k_1^{\nu_3}-\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_2}k_2^{\mu_1}+2\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_2}k_2^{\mu_1}+$$

$$2\,\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_3}k_2^{\mu_1}-2\,\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_1}k_2^{\mu_2}+4\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_2^{\mu_2}+4\,\eta_{\mu_1,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_3}k_2^{\mu_2}-2\,\eta_{\mu_1,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_3}k_2^{\mu_2}+\eta_{\mu_1,\nu_2}\eta_{\mu_3,\nu_3}k_1^{\nu_2}k_2^{\mu_2}-2\,\eta_{\mu_1,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\nu_2}k_2^{\mu_2}-2\,\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}k_2^{\mu_1}k_2^{\mu_2}+4\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_2^{\mu_2}+4\,\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_1}k_2^{\mu_3}-$$

$$2\,\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_2^{\mu_3}+2\,\eta_{\mu_1,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_2}k_2^{\mu_3}-2\,\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_3}k_1^{\nu_2}k_2^{\mu_3}+\frac{1}{2}\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}k_1^{\nu_3}k_2^{\mu_3}-\frac{3}{2}\eta_{\mu_1,\mu_2}\eta_{\nu_2,\nu_2}k_1^{\nu_3}k_2^{\mu_3}+2\,\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}k_2^{\mu_1}k_2^{\mu_3}-4\,\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_2^{\mu_3}+2\,\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_2^{\mu_3}+2\,\eta_{\mu_1,\nu_2}\eta_{\nu_2,\nu_3}k_2^{\mu_2}k_2^{\mu_3}+\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}k_1^{\mu_1}k_2^{\nu_2}-$$

$$2\,\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_2^{\nu_2}-2\,\eta_{\mu_1,\nu_3}\eta_{\mu_2,\nu_2}k_1^{\mu_3}k_2^{\nu_2}+\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}k_2^{\mu_1}k_2^{\nu_2}-2\,\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_2^{\nu_2}+\frac{1}{2}\eta_{\mu_1,\nu_2}\eta_{\mu_3,\nu_3}k_2^{\mu_2}k_2^{\nu_2}-\eta_{\mu_1,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\mu_2}k_2^{\nu_2}+\frac{1}{2}\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}k_1^{\mu_3}k_2^{\nu_3}-\frac{3}{2}\eta_{\mu_1,\mu_2}\eta_{\nu_2,\nu_2}k_1^{\mu_3}k_2^{\nu_3}+\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}k_2^{\mu_3}k_2^{\nu_3}-2\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\nu_2}k_2^{\mu_3}k_2^{\nu_3}+$$

$$2\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_2}k_3^{\mu_1}+\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_3}k_3^{\mu_1}-\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_3}k_3^{\mu_1}+\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_3}k_3^{\mu_1}-2\,\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}k_2^{\mu_2}k_3^{\mu_1}+4\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\mu_2}k_3^{\mu_1}+2\,\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}k_2^{\mu_3}k_3^{\mu_1}+\frac{1}{2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}k_2^{\nu_2}k_3^{\mu_1}-\frac{3}{2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\nu_2}k_3^{\mu_1}-2\,\eta_{\mu_2,\nu_2}\eta_{\nu_2,\mu_3}k_2^{\nu_3}k_3^{\mu_1}-$$

$$2\,\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_1}k_3^{\mu_2}+4\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_3^{\mu_2}+2\,\eta_{\mu_1,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_3}k_3^{\mu_2}+\frac{1}{2}\eta_{\mu_1,\nu_2}\eta_{\mu_3,\nu_3}k_1^{\nu_2}k_3^{\mu_2}-\frac{3}{2}\eta_{\mu_1,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\nu_2}k_3^{\mu_2}-2\,\eta_{\mu_1,\nu_2}\eta_{\nu_2,\mu_3}k_1^{\nu_3}k_3^{\mu_2}+2\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_3^{\mu_2}+\eta_{\mu_1,\nu_3}\eta_{\nu_2,\nu_2}k_2^{\mu_3}k_3^{\mu_2}-4\,\eta_{\mu_3,\nu_3}\eta_{\nu_2,\nu_2}k_3^{\mu_1}k_3^{\mu_2}+4\,\eta_{\nu_2,\mu_3}\eta_{\nu_2,\nu_3}k_3^{\mu_1}k_3^{\mu_2}+$$

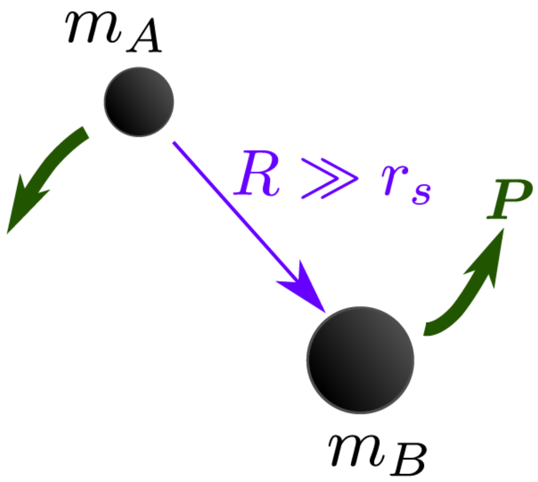
$$2\,\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}k_1^{\mu_1}k_3^{\mu_3}-2\,\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_3^{\mu_3}+2\,\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_3^{\mu_3}+2\,\eta_{\mu_1,\nu_2}\eta_{\nu_2,\nu_3}k_1^{\mu_2}k_3^{\mu_3}+\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}k_1^{\nu_3}k_3^{\mu_3}-2\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\nu_2}k_1^{\nu_3}k_3^{\mu_3}-2\,\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_3^{\mu_3}+4\,\eta_{\nu_2,\mu_2}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_3^{\mu_3}+2\,\eta_{\mu_1,\nu_3}\eta_{\nu_2,\nu_2}k_2^{\mu_2}k_3^{\mu_3}+\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}k_2^{\nu_3}k_3^{\mu_3}-$$

$$2\,\eta_{\mu_1,\mu_2}\eta_{\nu_2,\nu_2}k_2^{\nu_3}k_3^{\mu_3}+4\,\eta_{\mu_2,\nu_3}\eta_{\nu_2,\nu_2}k_3^{\mu_1}k_3^{\mu_3}-2\,\eta_{\mu_2,\nu_2}\eta_{\nu_2,\nu_3}k_3^{\mu_1}k_3^{\mu_3}+4\,\eta_{\mu_1,\nu_3}\eta_{\nu_2,\nu_2}k_3^{\mu_2}k_3^{\mu_3}-2\,\eta_{\mu_1,\nu_2}\eta_{\nu_2,\nu_3}k_3^{\mu_2}k_3^{\mu_3}+\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}k_1^{\mu_1}k_3^{\nu_2}-2\,\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_1}k_3^{\nu_2}-\frac{3}{2}\eta_{\mu_1,\nu_2}\eta_{\mu_3,\nu_3}k_1^{\mu_2}k_3^{\nu_2}-\frac{3}{2}\eta_{\mu_1,\mu_3}\eta_{\nu_2,\nu_3}k_1^{\mu_2}k_3^{\nu_2}+\frac{1}{2}\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}k_2^{\mu_1}k_3^{\nu_2}-$$

$$2\,\eta_{\mu_3,\nu_3}\eta_{\nu_2,\mu_2}k_2^{\mu_1}k_3^{\nu_2}-\frac{3}{2}\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\mu_1}k_3^{\nu_2}+\eta_{\mu_1,\nu_2}\eta_{\mu_3,\nu_3}k_2^{\mu_2}k_3^{\nu_2}-2\,\eta_{\mu_1,\mu_3}\eta_{\nu_2,\nu_3}k_2^{\mu_2}k_3^{\nu_2}+\eta_{\mu_2,\nu_2}\eta_{\mu_3,\nu_3}k_3^{\mu_1}k_3^{\nu_2}-2\,\eta_{\mu_2,\mu_3}\eta_{\nu_2,\nu_3}k_3^{\mu_1}k_3^{\nu_2}+\eta_{\mu_1,\nu_2}\eta_{\mu_3,\nu_3}k_3^{\mu_2}k_3^{\nu_2}-2\,\eta_{\mu_1,\mu_3}\eta_{\nu_2,\nu_3}k_3^{\mu_2}k_3^{\nu_2}+\frac{1}{2}\eta_{\mu_1,\nu_2}\eta_{\mu_2,\nu_2}k_3^{\mu_3}k_3^{\nu_3}-\eta_{\mu_1,\mu_2}\eta_{\nu_2,\nu_2}k_3^{\mu_3}k_3^{\nu_3}$$

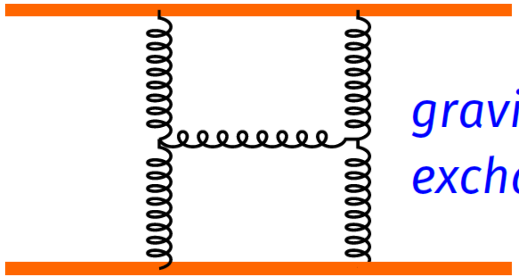
# Classical phyics from quantum amplitudes

# Point particle effective theory




point particle EFT

[Goldberger, Rothstein, '04]



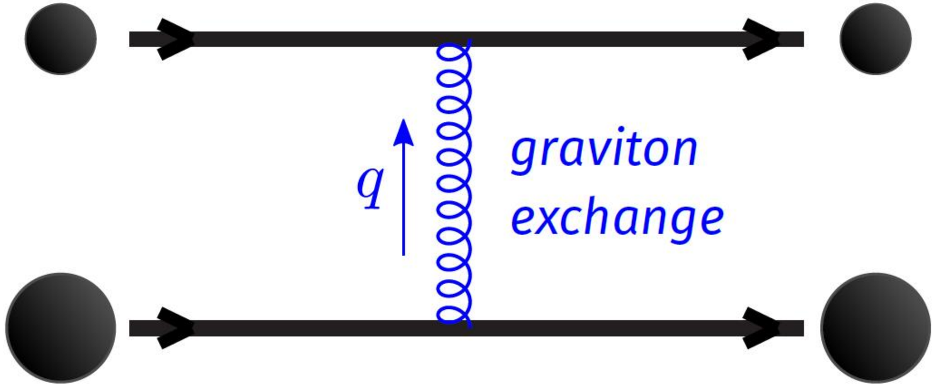

- Lagrangian: $S = S_{\text{Einstein-Hilbert}} + S_{\text{scalar}} + S_{\text{finite-size}} \longrightarrow$ Tidal deformation. Highly suppressed effect.

$$S = \int d^4x \sqrt{-g} \left[ -2R + \frac{1}{2} \left( g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - m^2 \phi^2 \right) \right] + \cdots$$

- Perturbative expansion $g_{\mu\nu} = \eta_{\mu\nu} + \kappa \, h_{\mu\nu}, \;\; \kappa^2 \equiv 8\pi G.$

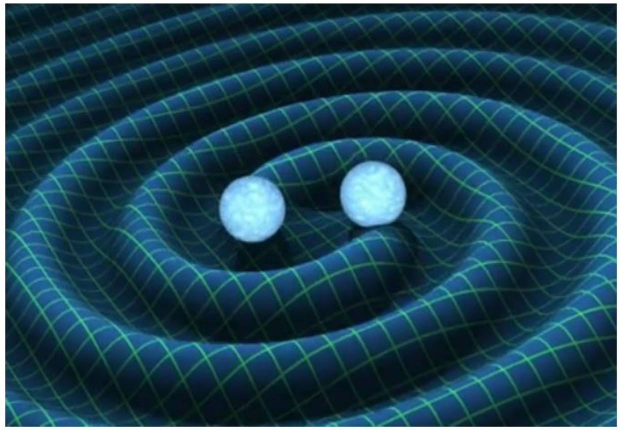
Feynman diagrams, generalized unitarity, double copy…

# Classical from quantum – it's subtle!




momentum transfer $q \sim \hbar/R \ll m_1, m_2$

$\hbar \to 0?$



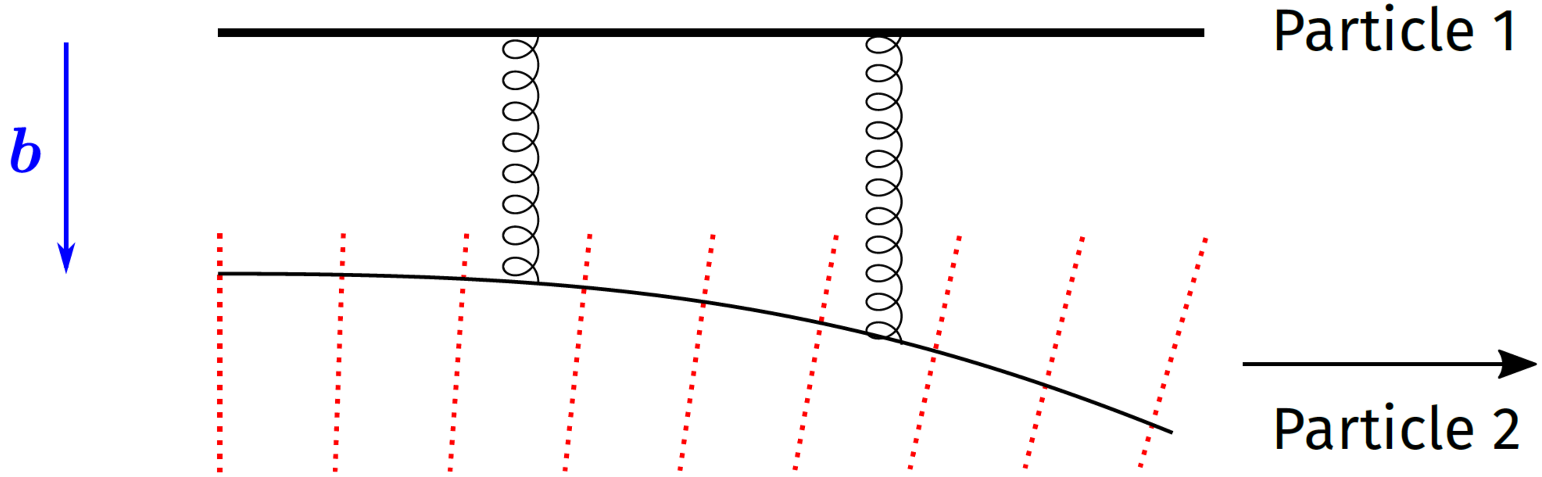
[picture: LIGO]

- Naïve $\hbar$ expansion won't work. Increasingly worse $1/\hbar$ divergences at each higher loop order, from (J)WKB approximation.

$$M \sim \exp\left(\frac{i}{\hbar}\int V(x)\, dx\right)$$

"super-classical / classically divergent terms"

# Intuition: *Huygens principle*

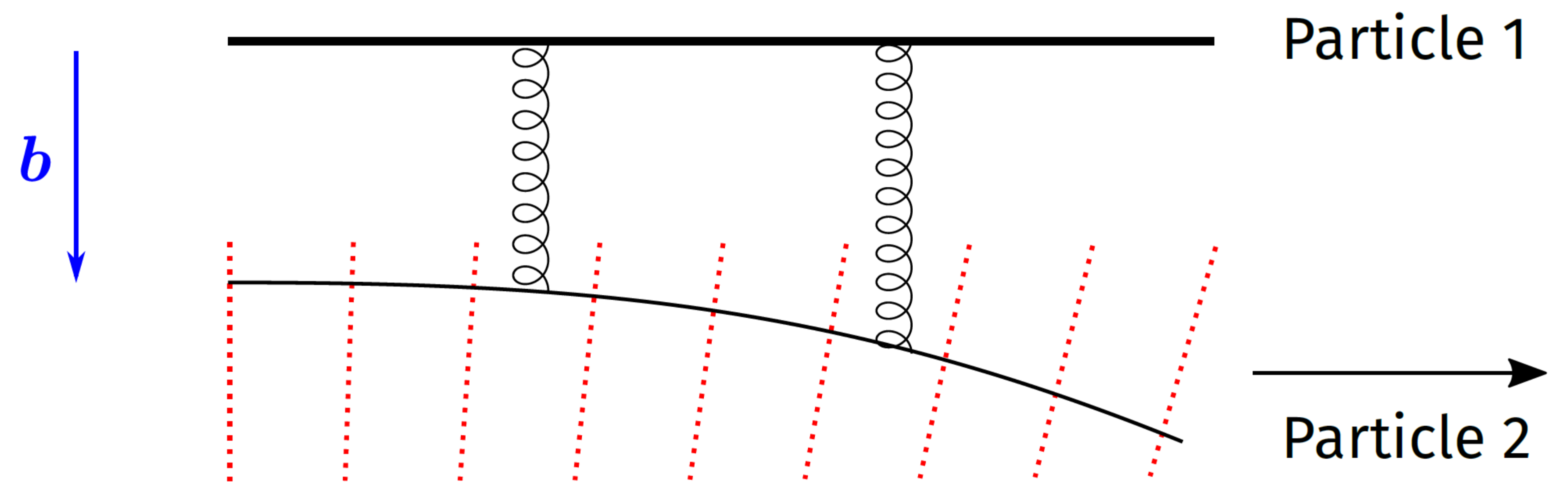



Conservative case: amplitude $\tilde{M}(b) \propto \exp\left[i\chi(b)\right]$

Scattering angle ∝ phase gradient ∝ $\partial\chi(b)/\partial b$.
Time delay ∝ $\partial\chi(b)/\partial E$.

Logarithm + differentiation cancels $1/\hbar$ divergence and infrared divergence.

# Stationary phase – "eikonal approximation"



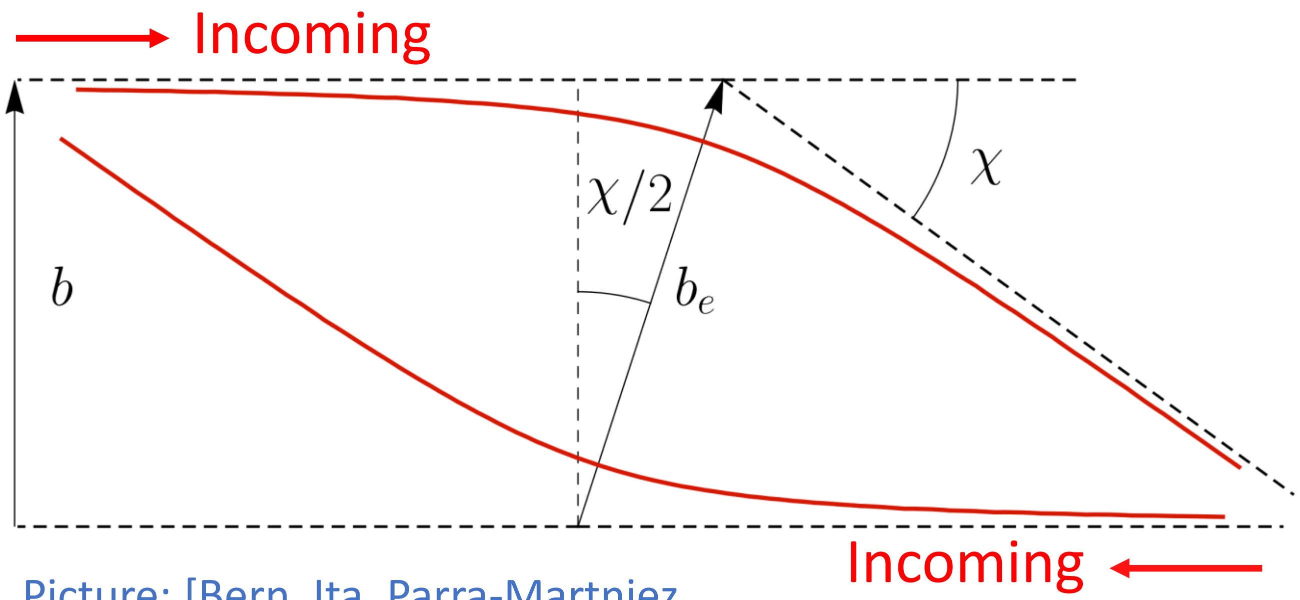

Conservative case: amplitude $\tilde{M}(b) \propto \exp\left[i\chi(b)\right]$

$$M(q) = \int d^2b\, \widetilde{M}(b)\ e^{iq\cdot b} = \int d^2b \exp[i(\chi(b) + q\cdot b)]$$

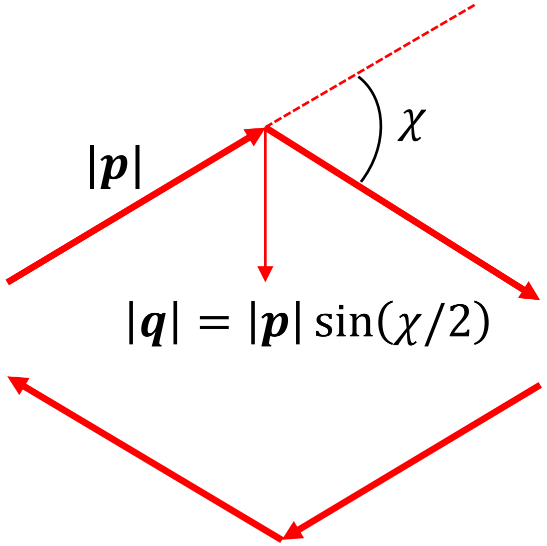
Steepest descent / stationary phase approximation:

$$|q| = \partial\chi(b)/\partial b$$

# Kinematic corrections for eikonal method




Picture: [Bern, Ita, Parra-Martniez, Ruf, arXiv:2002.02459]




$$\sin\frac{\chi}{2} = \frac{|\boldsymbol{q}|}{|\boldsymbol{p}|}, \qquad |\boldsymbol{q}| = \frac{\partial \chi}{\partial b_e}, \qquad b_e = \frac{b_\infty}{\cos(\chi/2)}$$

Needed at NNLO and beyond.

# Collider-like observable from amplitudes

[Kosower, Maybee, O'Connell (KMOC), '18]

- Interested in observables in hyperbolic black hole scattering, such as the **impulse** on a black hole (deflection angle) and **energy loss** from GW emission.

- Measurements at **fixed impact parameter**, which is not integrated over, unlike collider observables.

- Collider observables are usually quadratic in the amplitude,

$$S = 1 + i\,T, \qquad |\text{out}\rangle = S|\text{in}\rangle$$

expectation value of observable: $\langle \text{in}|S^{\dagger}\mathcal{O}\,S|\text{in}\rangle = \langle \text{in}|T^{\dagger}\mathcal{O}\,T|\text{in}\rangle$

no contribution     measurement operator

# Collider-like observable from amplitudes

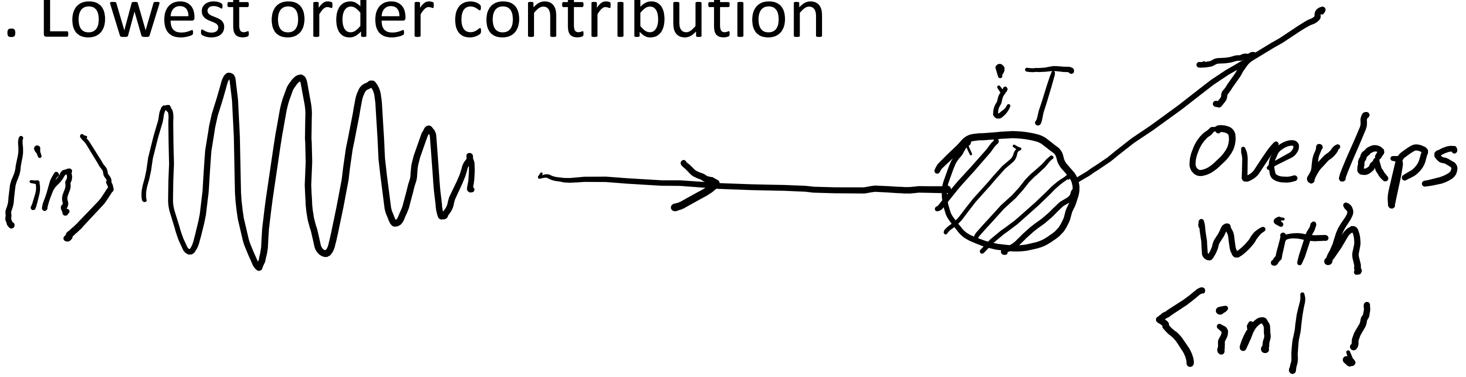
[Kosower, Maybee, O'Connell (KMOC), '18]

- Classical scattering observables can be linear in the amplitude.
- Consider fixed impact parameter, $|\text{in}\rangle \sim \int d^4p \exp(ip \cdot b_T)\psi_1(p)$

impact parameter

wave packet;
detail not important

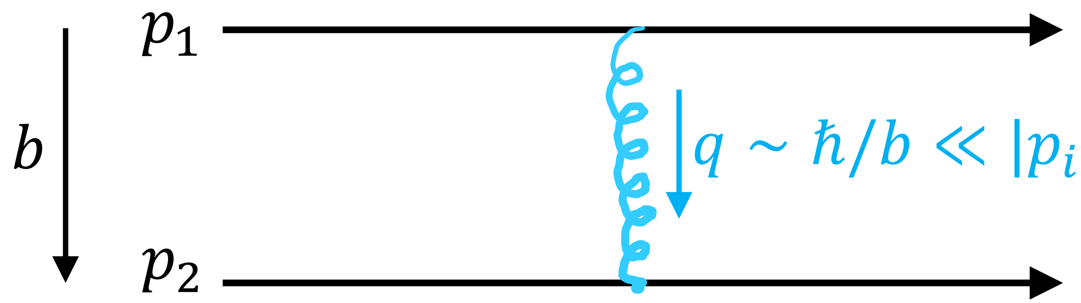
- Consider the impulse observable $\Delta p^\mu$, expectation value $\langle \text{in} | S^\dagger \mathcal{O}\, S | \text{in} \rangle$, $S = 1 + iT$. Lowest order contribution $\langle \text{in} | 1 \cdot \mathcal{O} \cdot iT | \text{in} \rangle$.



# LO impulse - Amplitude

*We'll be sloppy with constant factors throughout this example.*

Tree amplitude




$q \sim \hbar/b \ll |p_i|$, soft transverse momentum

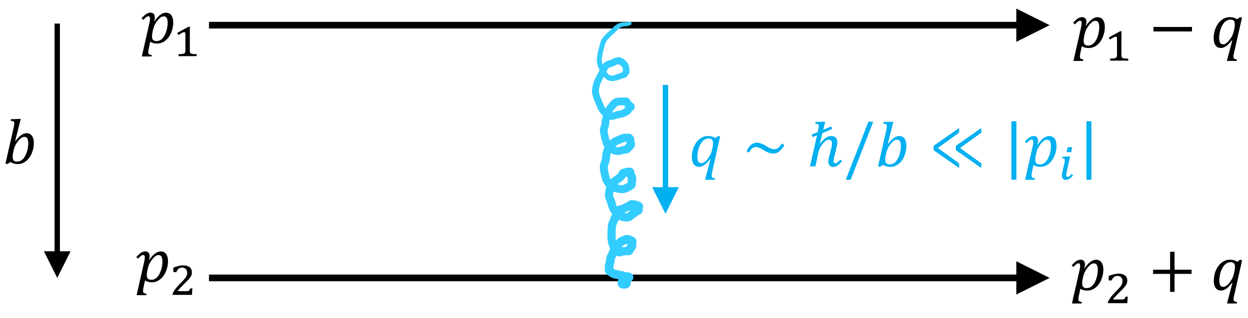
$$= \frac{-16\pi G}{q^2}\left(m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2 - \underbrace{(p_1 \cdot p_2)q^2}\right)$$

*(See later slides for derviation)*

cancels $1/q^2$ pole $\Rightarrow$ four-scalar contact interaction, no long-range classical effect.

$$\sim \frac{c}{q^2} + \text{ contract term.}$$

# LO impulse – Phase space




$\sim \dfrac{c}{q_T^2}$ + contract term.

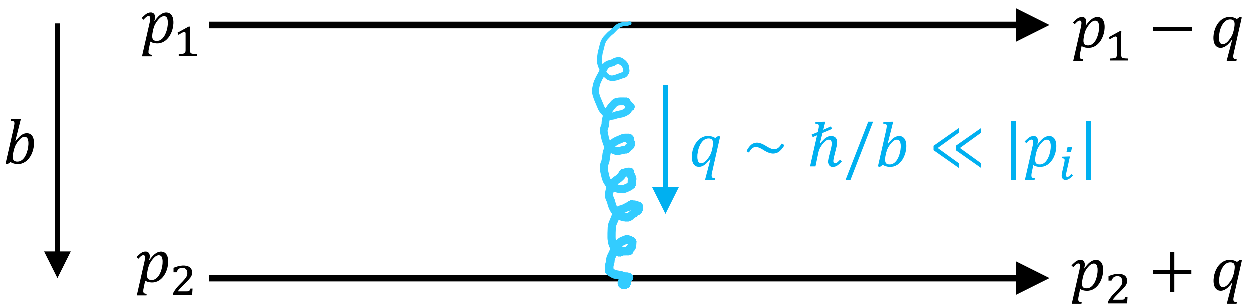
Phase space $\int d^4q\, \delta^4\left[(p_1 - q)^2 - m_1^2\right] \delta^4\left[(p_2 + q)^2 - m_2^2\right]$

$\approx\ d^4q\, \delta^4[-2p_1 \cdot q]\delta^4[2p_2 \cdot q]$

Choose frame $p_1 = m_1(1{,}0{,}0{,}0), p_2 = m_2(\sqrt{1 + v^2}, 0, 0, v)$.
Integrate $q^0$ and $q^z$ against delta functions.

$$= \frac{1}{m_1 m_2 v} d^2 q_T$$

# LO impulse - Expectation value




$\sim \frac{c}{q_T^2}$ + contract term.

$|\bar{p}_1\rangle \sim \int d^4p \exp(ip \cdot b)\psi_1(p_1)\,,$ $\qquad |\bar{p}_2\rangle \sim \int d^4p\, \psi_2(p_2)$

Transverse impact parameter

wavepacket at zero impact parameter

impulse measurement

Lowest order contribution $\langle \text{in}|1 \cdot \mathcal{O} \cdot\ iT|\text{in}\rangle \sim \left(c/q_T^2\right) q^\mu\, e^{iq_T\cdot b}$

$e^{-i(p_1-q)\cdot b}$ $\qquad e^{ip_1\cdot b}$

# LO impulse – Putting it together

$M \sim \frac{c}{q_T^2} +$ contract term.  Phase space measure $= \frac{1}{m_1 m_2 v} d^2 q_T$

impulse measurement

Lowest order contribution $\langle \text{in} | 1 \cdot \mathcal{O} \cdot \; iT | \text{in} \rangle \sim \mathcal{M}\, q^\mu\, e^{i q_T \cdot b}$

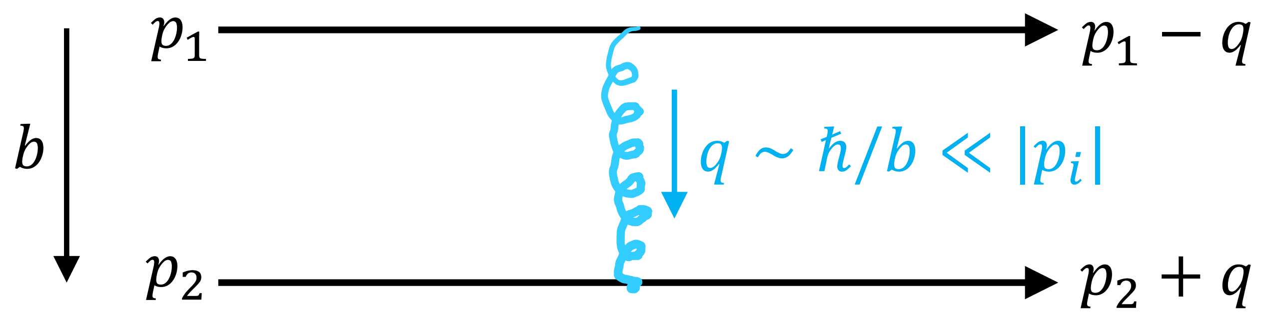
$e^{-i(p_1 - q)\cdot b}$  $e^{i p_1 \cdot b}$

Expected impulse $= \frac{1}{m_1 m_2 v} \int d^2 q_T\, M\, q^\mu\, e^{i q_T \cdot b}$

2D Fourier transform; IR divergence has no $b$ dependence, disappears after differentiation.

$$= \frac{1}{m_1 m_2 v} \int d^2 q_T\, \mathcal{M}\, q^\mu\, e^{i q_T \cdot b} = \frac{1}{m_1 m_2 v} \left( -i \frac{\partial}{\partial b^\mu} \right) \widetilde{M}(b)$$

# LO impulse – Result



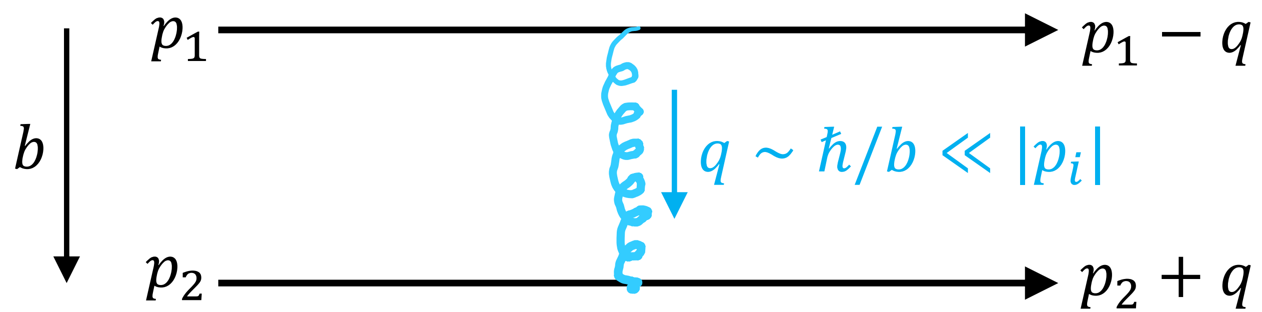

$$M \sim \frac{1}{q_T^2}\left[m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2\right] + \cdots$$

$$\text{Impulse} = \frac{1}{m_1 m_2 v}\left(-i\frac{\partial}{\partial b^\mu}\right)\widetilde{M}(b) = \frac{G m_1 m_2}{|b|}\frac{2\left[2(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]}{v}\,\hat{b}^\mu,$$

where $v = \sqrt{(p_1 \cdot p_2)^2/\left(m_1^2 m_2^2\right) - 1}$, $\hat{b}^\mu = b^\mu/|b|$.

Zero velocity limit $p_1 \cdot p_2 = m_1 m_2$ agrees with Newtonian hyperbolic orbit.

# LO impulse – compare with eikonal




$$M \sim \frac{1}{q_T^2}\left[m_1^2 m_2^2 - 2(p_1 \cdot p_2)^2\right] + \cdots$$

LO Impulse $= \dfrac{1}{m_1 m_2 v}\left(-i\dfrac{\partial}{\partial b^\mu}\right)\widetilde{M}^{(0)}(b)$

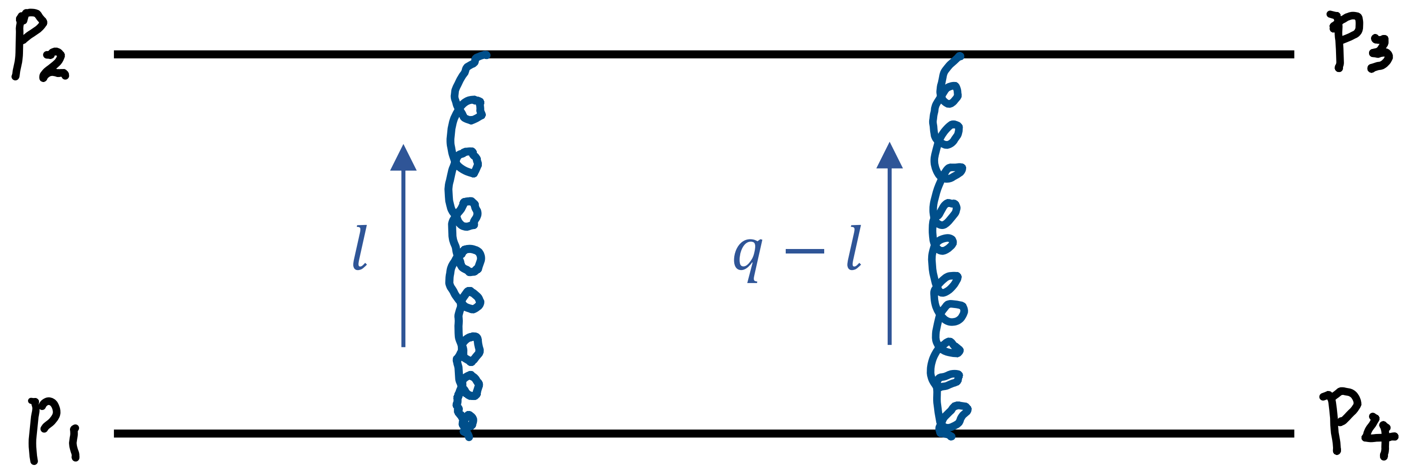
Eikonal: $\tilde{M}(b) = \exp\left[i\chi(b)\right],$ $\quad |q| = \partial\chi(b)/\partial b$

Agree if $\widetilde{M}(b) = (m_1 m_2 v)\exp\left[M^{(0)}(b)/(m_1 m_2 v) + \text{higher orders}\right]$. ✓

# $\hbar$ expansion – method of regions

# Expanding Feynman integrals

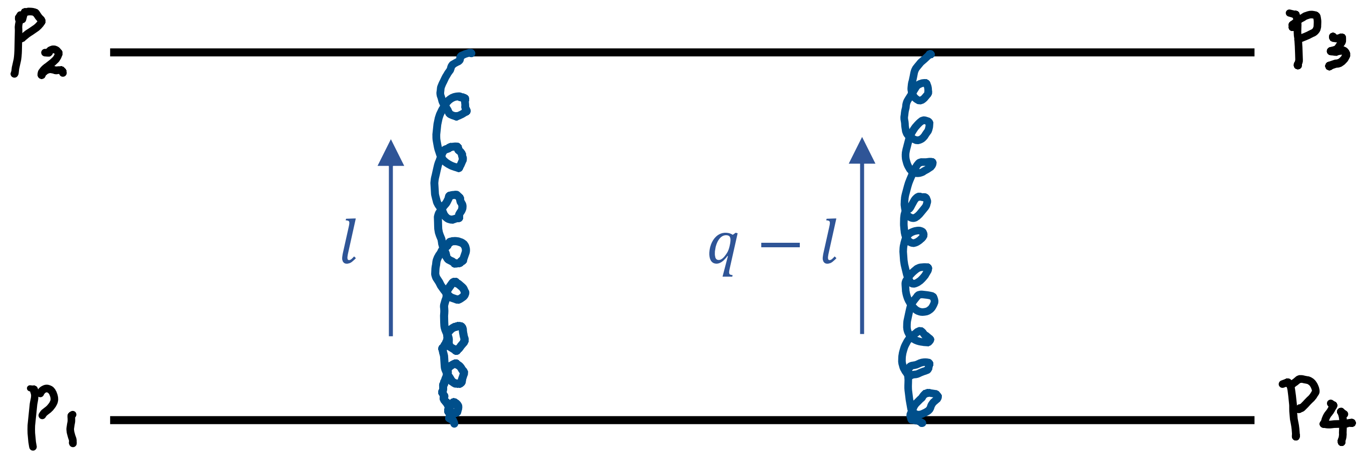
- One loop correction to $\Phi + \phi \to \Phi + \phi$




- External kinematics fixed, $q \sim \hbar/R \ll |p|$. But loop momentum $l$ is integrated over entire $\mathbb{R}^4$ - can be small or large. How do we expand?

# Method of regions

- Asymptotic series in $|q|/|p|$, to any order, is a sum of two regions:

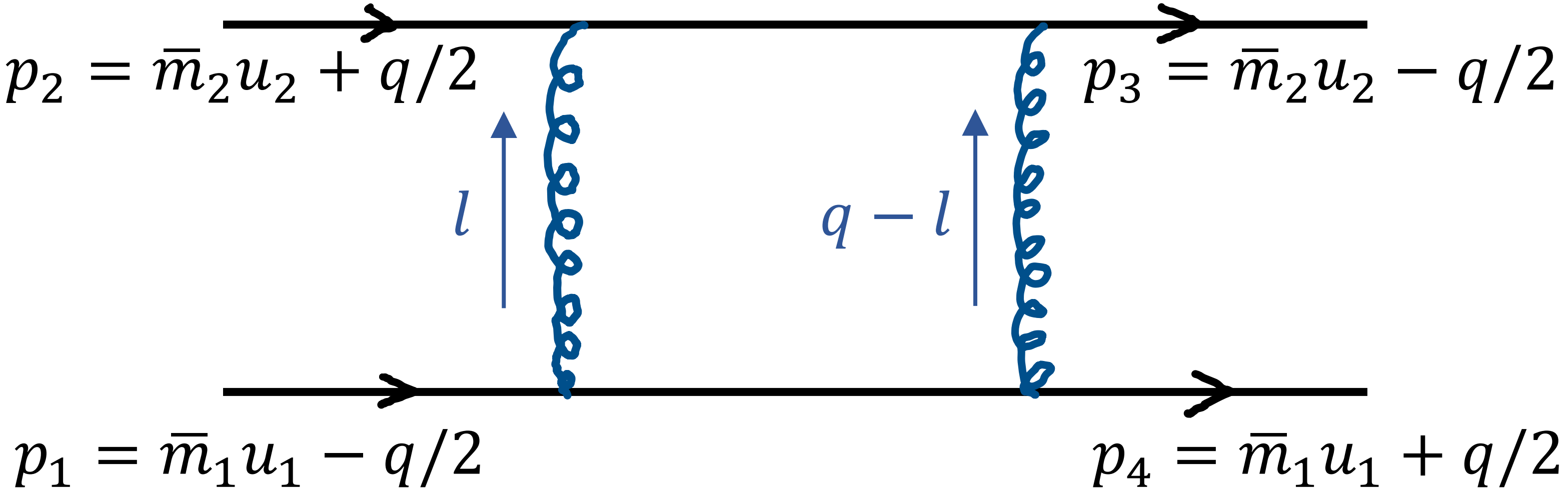



1. **Soft region,** $|l| \sim |q| \ll |p|$, expand in small $|q|/|p|$, $|l|/|p|$.
2. Hard region, $|l| \sim |p| \gg |q|$, expand in small $|q|/|p|$, $|q|/|l|$, not needed for classical physics.

# Treatment of classical soft region

[Parra-Martinez, Ruf, MZ, arXiv:2005.04236]

- Symmetric parametrization



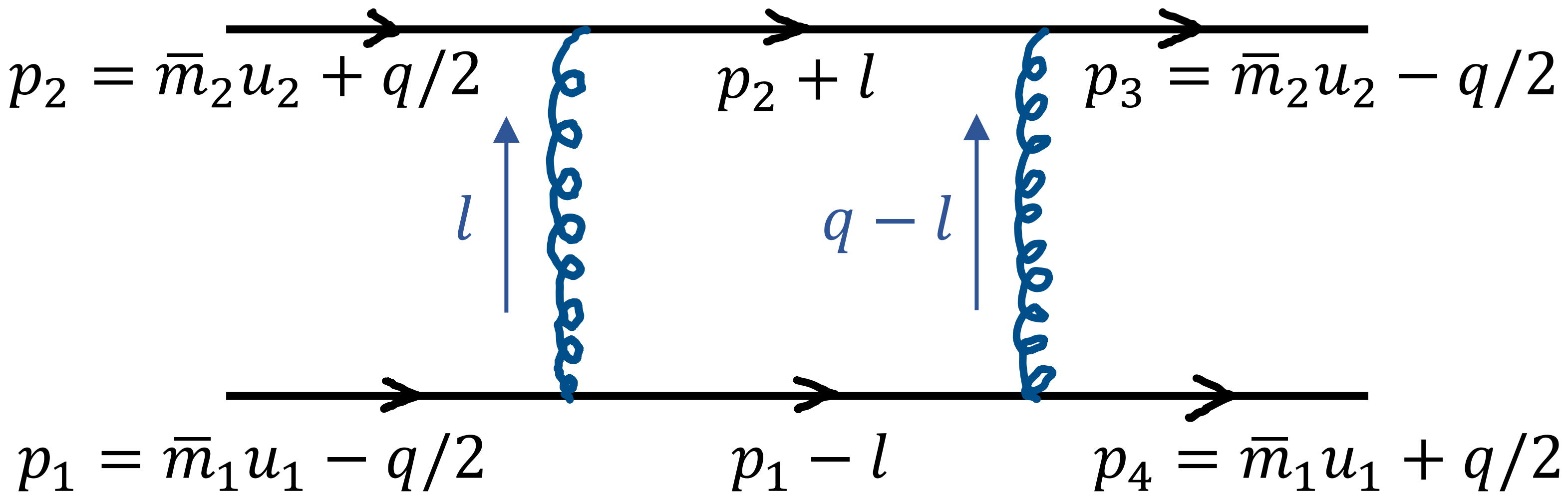

$u_1 \cdot q = u_2 \cdot q = 0$ , $u_1^2 = u_2^2 = 1$ , $q^2 = t$ ,

$u_1 \cdot u_2 = y$ ← only nontrivial variable in expanded integrals.

Conversion : $m_1^2 = p_1^2 = \bar{m}_1^2 + q^2/4$ .

# Treatment of classical soft region

[Parra-Martinez, Ruf, MZ, arXiv:2005.04236]




$|q|, |l| \ll \bar{m}_1, \bar{m}_2$

$u_1^2 = u_2^2 = 1,$

$u_1 \cdot u_2 = y$

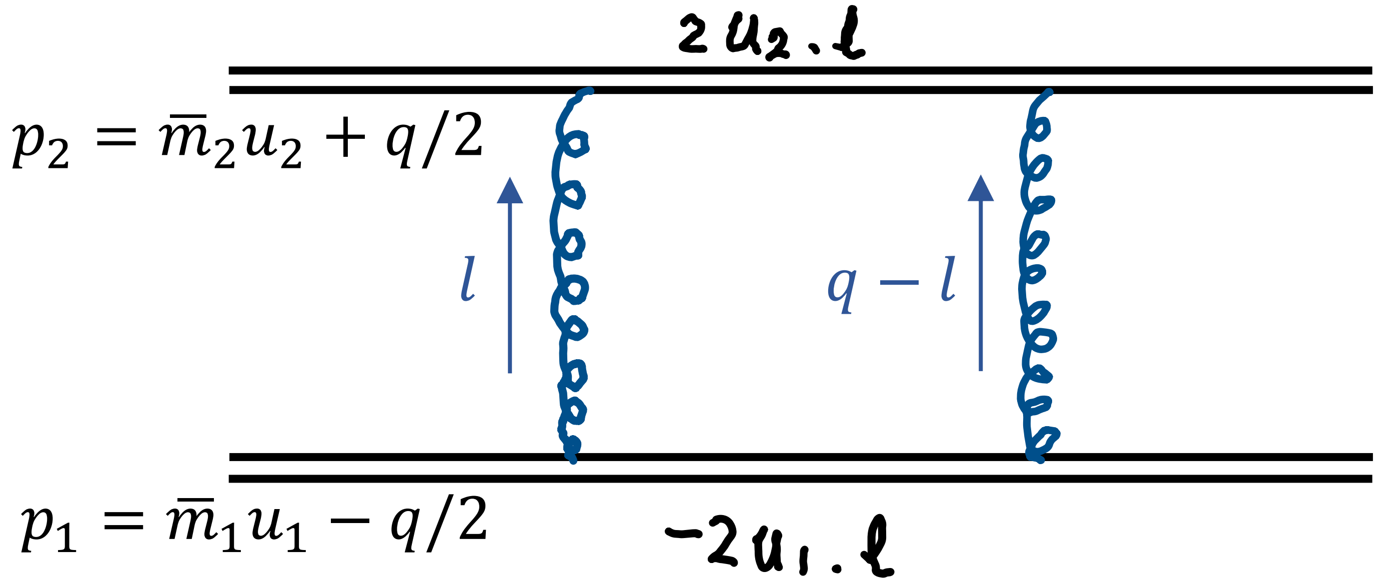
$$\frac{1}{(p_1 - l)^2 - m_1^2} = \frac{1}{-2p_1 \cdot l + l^2} = \frac{1}{-2\bar{m}_1 u_1 \cdot l + (l^2 + q \cdot l)}$$

$\to O(\lambda) \quad \to O(\lambda^2)$

$$= \frac{1}{\bar{m}_1} \frac{1}{-2u_1 \cdot l} + \frac{1}{\bar{m}_1^2} \frac{-(l^2 + q \cdot l)}{(-2u_1 \cdot l)^2} \cdots$$

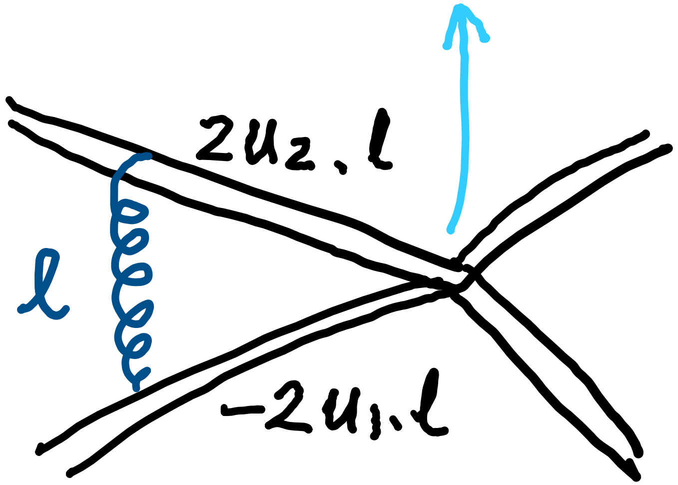
# Treatment of classical soft region

[Parra-Martinez, Ruf, MZ, arXiv:2005.04236]



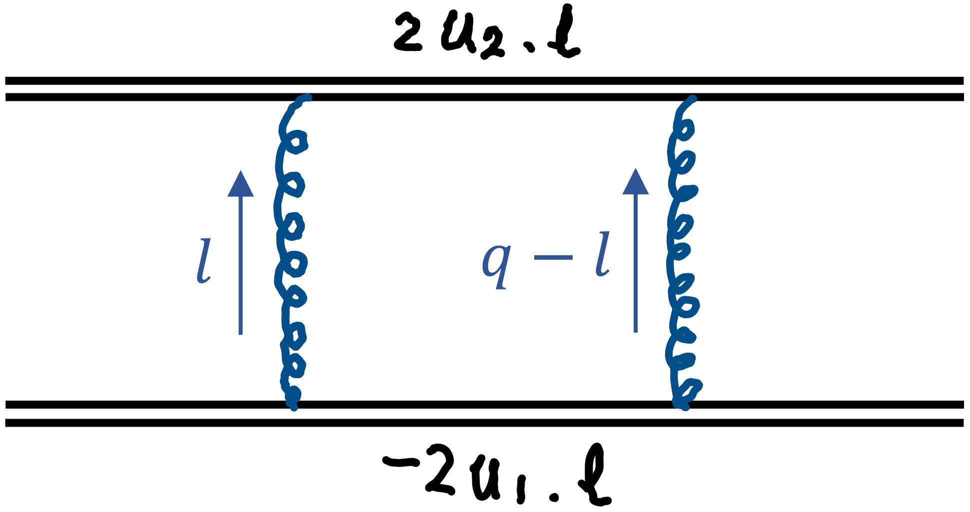

Soft expansion **linearizes** massive propagators.

Contact diagram irrelevant for long-range classical physics!




Certain subsectors are scaleless, vanish in dim. reg, for example, if we collapse $1/(q-l)^2$ propagator.

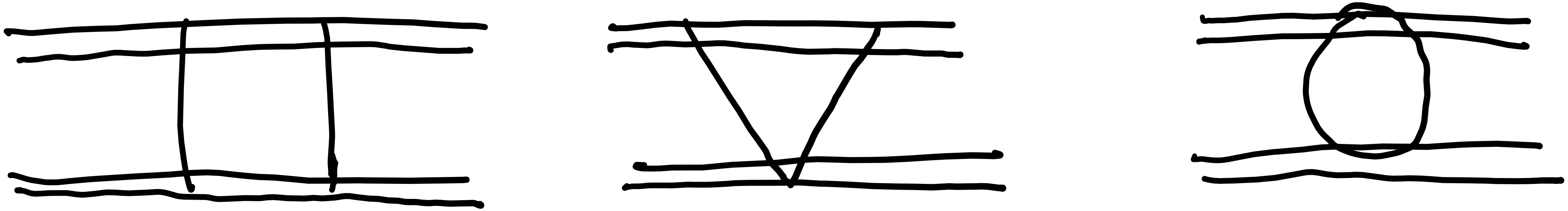
# Integration by parts (IBP)




$$\int d^d\ell \; \partial_\mu \frac{\ell^\mu (\ldots)}{\rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4}}$$

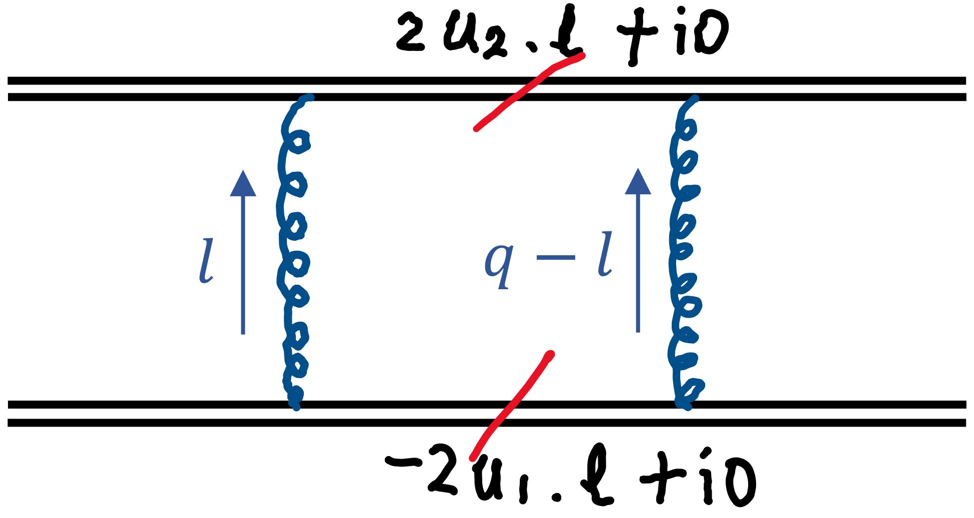
$$= 0$$

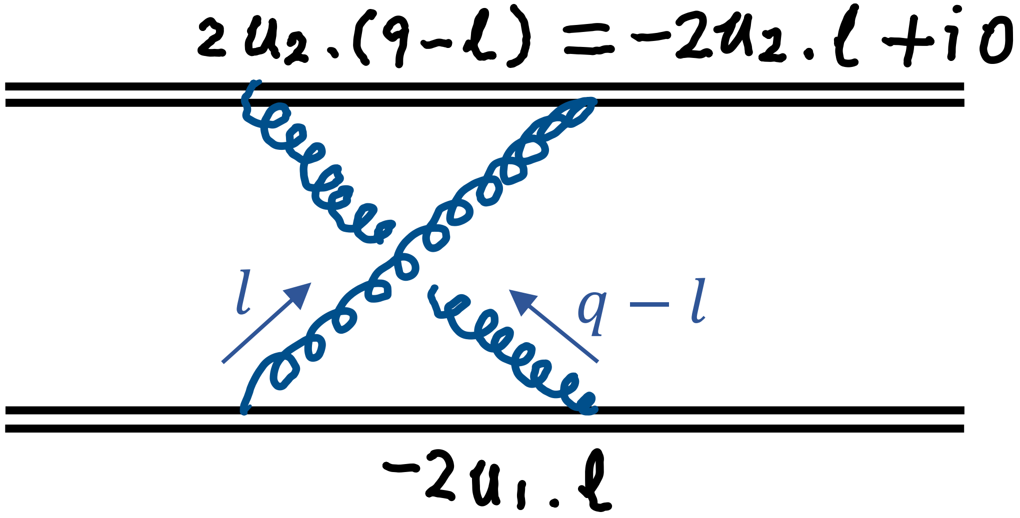
Master integrals



# Localization on matter poles

- Box and crossed box diagrams combine nontrivially into exponentiation of tree-level result.




$$\frac{1}{2u_2 \cdot l + i0} + \frac{1}{-2u_2 \cdot l + i0} = -2\pi i \, \delta(2u_2 \cdot l)$$

Similar permutation sum localizes $2\pi i \, \delta(2u_1 \cdot l)$

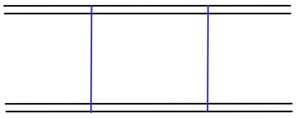
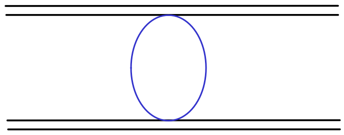
$t, z$ localized, $x, y$ remains

# Differential equations

- Derivatives of masters reduced back to linear sum of masters, by IBP

- See external slides.

Example:

$$\frac{\partial}{\partial v}\left( v \right.$$  $$\left.\right) = \frac{\partial \log(\sqrt{1+v^2}-v)}{\partial v} \times$$ 

Vanishes in potential region defined in terms of matter propagator residues

$$= -\frac{1}{\varepsilon} i\pi,$$ constant in potential region.

Explains nontrivial magic cancellation at every higher order in v, in direct expansion.

If working in soft region without truncation to potential region, RHS is a v-indep. constant,

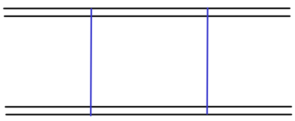
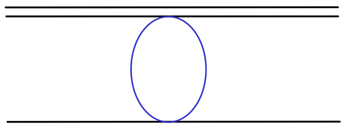
$$\Rightarrow \; = -\frac{1}{\varepsilon}\left[ i\pi + \log\left(\sqrt{1+v^2}-v\right)\right]$$

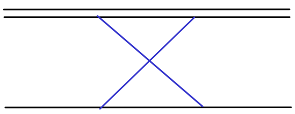
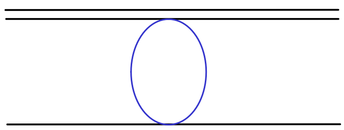
leading order in v expansion purely from potential region

$$-v + \frac{v^3}{6} - \frac{3v^5}{40} \ldots$$

Box + crossed box = const. in both potential region and full soft region. Can we see this at the level of differential equaitons?

*See also [Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21].*
*Combining in Feynman parametrization: Cristofoli, Damgaard, Di Vecchia, Heissenberg, '20]*

$$\frac{\partial}{\partial v}\left( v \right.$$  $$\left.\right) = \frac{\partial \log(\sqrt{1+v^2}-v)}{\partial v} \times$$ 

$$\frac{\partial}{\partial v}\left( v \right.$$  $$\left.\right) = (-1)\frac{\partial \log(\sqrt{1+v^2}-v)}{\partial v} \times$$ 

$$\Rightarrow \frac{\partial}{\partial v}\Big(\underbrace{v \cdot I_{box} + v \cdot I_{xbox}}_{\text{constant, due to localization on matter poles.}}\Big) = 0$$

# NEW RESULTS FOR CONSERVATIVE DYNAMICS

Newton 1687

1PN EIH 1938

2PN 1980

3PN 2000

4PN 2014

5PN: distinguish BH / NS

6PN: LISA, ET...

**Talks by Andreas Maier, Donato Bini**

$$G(1 + v^2 + v^4 + v^6 + v^8 + \ldots) \longrightarrow$$ **1PM**

$$G^2(1 + v^2 + v^4 + v^6 + v^8 + \ldots) \longrightarrow$$ **2PM** [Westpfahl 1985]

$$G^3(1 + v^2 + v^4 + v^6 + v^8 + \ldots) \longrightarrow$$ **3PM** [Bern, Cheung, Roiban, Shen, Solon, **MZ**, 2019]

$$G^4(1 + v^2 + v^4 + v^6 + v^8 + \ldots) \longrightarrow$$ **4PM (potential)** [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, 2021]

$$G^5(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$

$$G^6(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$

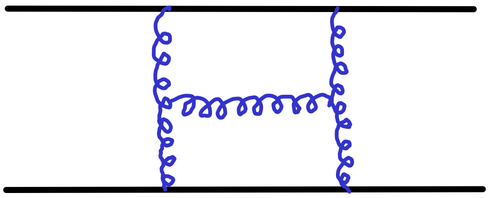
[adapted from Mikhail Solon's slide]

**Breathrough after 30+ year hiatus.**

### Phase space integrals and reverse unitarity

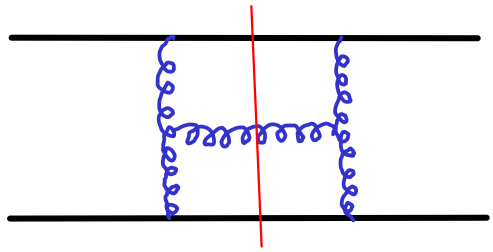
Setup: classical limit of observables from S-matrix. [Kosower, Maybee, O'Connell '18]

Virtual diagram $\mathcal{M}^{(2)}$



contributes to impulse on scattered black hole (deflection angle)

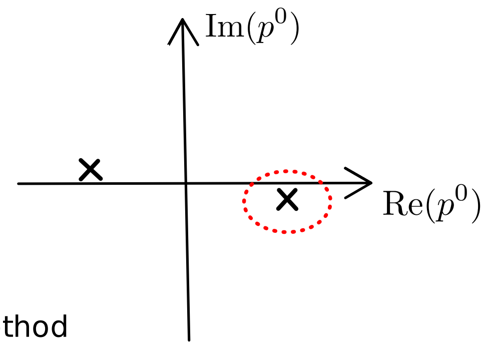
Real emission diagram $\mathcal{M}^{(1)}\mathcal{M}^{*(1)}$



contributes to impluse & energy loss

(uncut) Feynman propagator $1/(p^2 - m^2 + i0)$

cut propagtor for phase space
$2\pi\,\theta(p^0)\delta(p^2 - m^2)$
from picking up only the +ve energy residue in Feynman propagator




IBP & Differential equations unchanged!

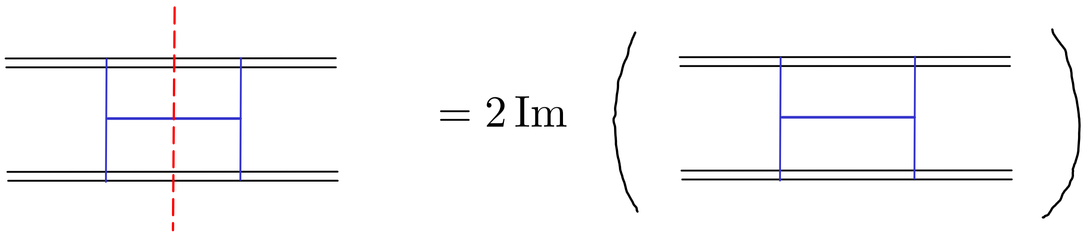
Only change boundary conditions for DEs, known as method of Reverse Unitarity.

Important in perturbative QCD for Higgs cross sections at NNLO and N3LO, and energy correlations in electron-positron collider event shapes.

First application of reverse unitarity to gravitational physics in [Herrmann, Parra Martinez, Ruf, MZ, 2101.07255 (PRL), 2104.03957].

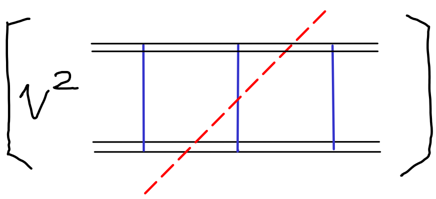
We re-used DEs in canonical basis for virtual integrals in [Parra-Martinez, Ruf, MZ, '20].

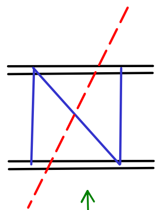
*Example use of reverse unitarity*




(only one Cutkosky cut, optical theorem enough)

(Virtual integrals computed via differential equations)

$\frac{\partial}{\partial v}$ 

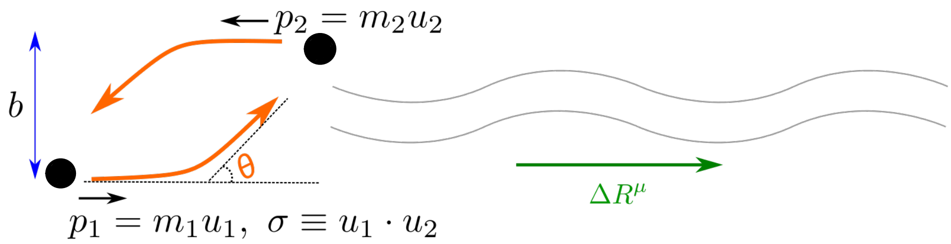

$=$ simpler integrals  etc.

More than one Cutkosky cut. Need serious use of reverse unitarity, including DEs on cut.

Known!

## Result for radaiated energy at 3rd-post-Minkowskian order

*talk by Enrico Herrmann & Michael Ruf*




$$\Delta R^\mu = \frac{G^3 m_1^2 m_2^2}{|b|^3} \frac{u_1^\mu + u_2^\mu}{\sigma + 1} \mathcal{E}(\sigma) + \mathcal{O}(G^4)\,.$$

$$\mathcal{E}(\sigma) = f_1 + f_2 \log\left(\frac{\sigma+1}{2}\right) + f_3 \frac{\sigma \operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\,,$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48\,(\sigma^2-1)^{3/2}}\,,$$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}\,, \quad f_3 = \frac{(2\sigma^2-3)\,(35\sigma^4 - 30\sigma^2 + 11)}{8(\sigma^2-1)^{3/2}}$$