Lattice calculation of the mass difference between the long- and short-lived K mesons for physical quark masses

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Outline

Elementary particles
Three types of interactions

1. Electromagnetic(QED):
   - agreements to high precision between theoretical and experimental values
   - perturbation theory
2. Strong(QCD):
   - asymptotic freedom
   - difficulties at $\sim \Lambda_{\mathrm{QCD}}$
3. Weak: least understood; good checks for new physics:
   - Unitarity of CKM matrix
   - CP violation
   - Weak decaying processes...

Illustration: Typoform

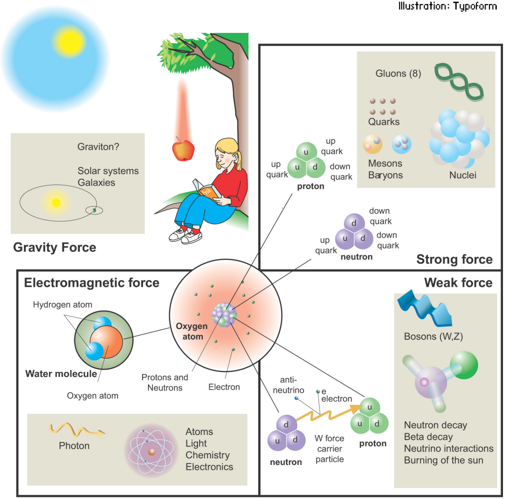


Figure: **from**
https://www.nobelprize.org/prizes/physics/2004/popular-information/

$K^0 - \overline{K^0}$ mixing and Δm_K

$K^0(S=-1)$ and $\overline{K^0}(S=+1)$, each having definite strangeness, which is conserved in the strong processes, mix through second order weak interactions.

$$i\frac{d}{dt}\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix} = (M - \frac{i}{2}\Gamma)\begin{pmatrix} K^0(t) \\ \overline{K^0}(t) \end{pmatrix}, \quad (1)$$

where the matrix M is given by:

$$M_{ij} = m_K^{(0)}\delta_{ij} + \mathcal{P}\sum_n \frac{\langle K_i^0|H_W|n\rangle\langle n|H_W|K_j^0\rangle}{m_K - E_n}, \quad (2)$$

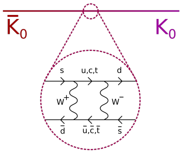


Figure: **from wikipedia**

If the small effects of CP violation are neglected, long-lived (K_L) and short-lived (K_S) are the two eigenstates:

$$K_S \approx \frac{K^0 - \overline{K^0}}{\sqrt{2}}, \quad K_L \approx \frac{K^0 + \overline{K^0}}{\sqrt{2}}. \quad (3)$$

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12}. \quad (4)$$

Different life times:

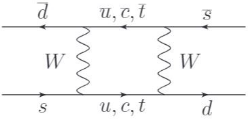
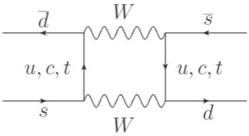
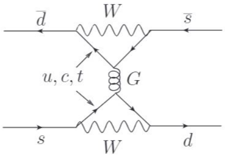
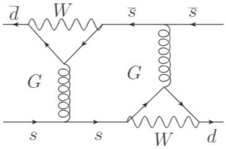
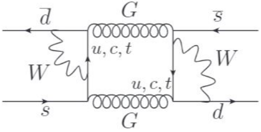
$K_S \xrightarrow[\text{even}]{\text{CP}} \pi\pi$,

$2m_\pi \approx 280\text{MeV} < m_K$

$K_L \xrightarrow[\text{odd}]{\text{CP}} \pi\pi\pi$,

$3m_\pi \approx 420\text{MeV} \lesssim m_K$

Diagrams related to Δm_K

box	QCD penguin	disconnected
 	 	

Physics motivation

Δm_K is given by:

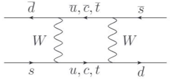
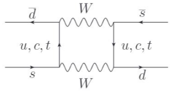
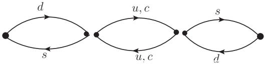
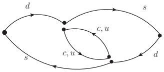
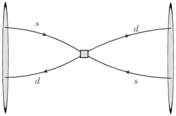
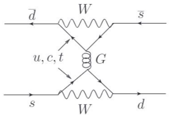
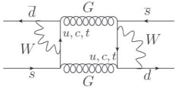
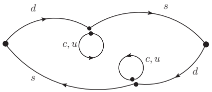
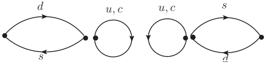
$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12} = 2\mathcal{P}\sum_n \frac{\langle \overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}. \quad (5)$$

- This quantity is:
  1. **Tiny** if compared to the K^0 mass $\sim$ 498 MeV, and precisely measured
     $$\Delta m_{K,exp} = 3.483(6) \times 10^{-12} \text{ MeV}$$
  2. **Sensitive to new physics**: FCNC via 2nd order weak interaction
- Methods to calculate Δm_K? Perturbation theory?
  Although the weak interaction itself can be treated precisely with perturbation theory, the kaon mixing process involves mesons(QCD related).
  - High-energy part: QCD perturbation theory works well
  - Low-energy part: QCD perturbative method fails, need non-perturbative calculation methods.

The operator product expansion(OPE) and Δm_K

OPE: full theory $H_W \xrightarrow[\text{heavy particles}]{\text{integrate out}} H_{\text{eff}} = \sum_j C_j(\mu) O_j(\mu)$, renormalized at scale μ

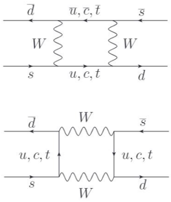
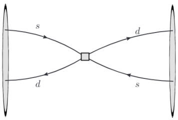
$C_j(\mu)$: short-distance, perturbative; $O_j(\mu)$: long-distance, non-perturbative

H_W^{SM}	$H_{\text{eff}}^{\Delta S=1}$	$H_{\text{eff}}^{\Delta S=2}$
 	 	
 	 	

Earlier calculations of Δm_K: charm quark is integrated out

The specific division $\mu < m_c$ in OPE where charm quark is integrated out.

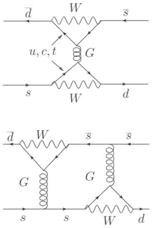
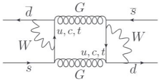
short-distance box only:

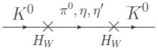
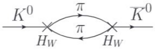
$H_W^{\rm SM} \to H_{\rm eff}^{\Delta S=2}$	
	

$$\mathcal{H}_{eff}^{\Delta S=2} = C(\mu) O_{LL}(\mu), \qquad (6)$$

$$O_{LL} = (\bar{s}d)_{V-A}(\bar{s}d)_{V-A}, \qquad (7)$$

leaving out:

QCD penguin	disconnected
	

long-distance box	
	

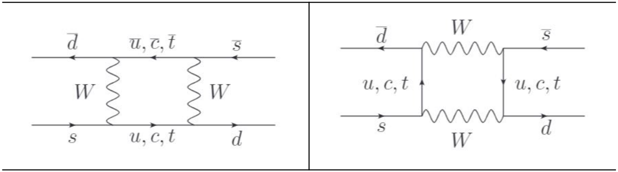
Only 36% accuracy in the next-to-next-to-leading-order(NNLO) calculation of the QCD correction factors using perturbation theory: slow convergence of the perturbative series

J. Brod and M. Gorbahn, Phys. Rev. Lett. 108, 121801 (2012)

→ **better to treat charm quark non-perturbatively on the lattice**

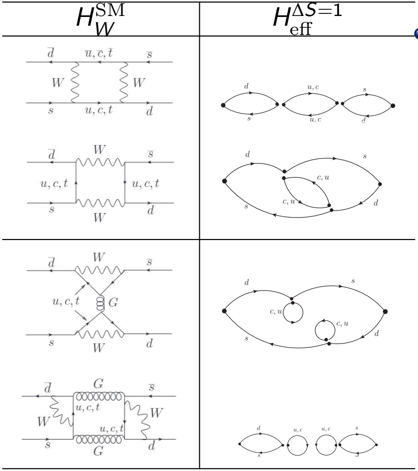
GIM mechanism and the short- and long-distance characteristics of Δm_K

GIM mechanism: flavor-changing neutral currents(FCNC) are suppressed in loop diagrams → charm quark → the CKM matrix



- Quark mixing: at each weak vertex
  → a product of CKM matrix elements $V_{qd} V^*_{q's}$, where $q, q' = u, c, t$.
- Define $\lambda_q = V_{q,d} V^*_{q,s}$, $q = u, c, t$,
  unitarity of the CKM matrix → $\lambda_u + \lambda_c + \lambda_t = 0 \rightarrow \lambda_c = -\lambda_u - \lambda_t$
- Specific diagram with GIM mechanism:
  $$\text{ⓧ}_{GIM} = \lambda_u^2 \text{ⓧ}^{(u-c)(u-c)} + \lambda_t^2 \text{ⓧ}^{(t-c)(t-c)} + 2\lambda_u \lambda_t \text{ⓧ}^{(u-c)(t-c)}$$
- For $\Delta m_K = 2\mathrm{Re} M_{12}$, the first term dominates.

Non-perturbative calculation of Δm_K using a renormalization scale above the charm quark mass



- $\mathrm{Re}\,\textcircled{X}_{GIM} \simeq \lambda_u^2\,\textcircled{X}^{(u-c)(u-c)}$
- The $\Delta S = 1$ effective weak Hamiltonian:

$$
H_{\mathrm{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \sum_{q,q'=u,c} V_{qd} V_{q's}^{*} \left(C_1 Q_1^{qq'} + C_2 Q_2^{qq'} \right) \tag{8}
$$

where the $Q_i^{qq'}{}_{i=1,2}$ are current-current opeartors, defined as:

$$
Q_1^{qq'} = (\bar{s}_i \gamma^\mu (1-\gamma^5) d_i)(\bar{q}_j \gamma^\mu (1-\gamma^5) q'_j)
$$

$$
Q_2^{qq'} = (\bar{s}_i \gamma^\mu (1-\gamma^5) d_j)(\bar{q}_j \gamma^\mu (1-\gamma^5) q'_i)
$$

Physics motivation

Δm_K is given by:

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12} = 2\mathcal{P}\sum_n \frac{\langle\overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}. \qquad (9)$$

- This quantity is:
  1. **Tiny** if compared to the K^0 mass ~ 498 MeV, and precisely measured
     $$\Delta m_{K,exp} = 3.483(6) \times 10^{-12} \text{ MeV}$$
  2. **Sensitive to new physics**: FCNC via 2nd order weak interaction
  3. Significant contribution from scale of m_c(GIM mechanism)
  4. **Difficult to compute by treating charm quark perturbatively**: strong coupling at m_c scale

  J. Brod and M. Gorbahn, Phys. Rev. Lett. 108, 121801 (2012)
- Calculate Δm_K with lattice QCD, treating charm quark non-perturbatively:
  - from first principles
  - non-perturbative, no convergence problem
  - systematic errors(finite volume corrections, finite lattice spacing effects, etc) can be controlled

Status of the calculation

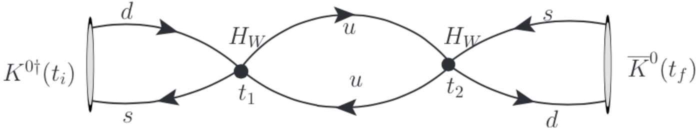
- **"Long-distance contribution at the $K_L - K_S$ mass difference"**,
  N. H. Christ, T. Izubuchi, C. T. Sachrajda, A. Soni and J. Yu
  Phys. Rev. D 88(2013), 014508
  Development of techniques and exploratory calculation on a $16^3 \times 32$ lattice with unphysical masses($m_\pi = 421MeV$) including only connected diagrams
- **"$K_L - K_S$ mass difference from Lattice QCD"**
  Z. Bai, N. H. Christ, T. Izubuchi, C. T. Sachrajda, A. Soni and J. Yu
  Phys. Rev. Lett. 113(2014), 112003
  All diagrams included on a $24^3 \times 64$ lattice with unphysical masses
- **"The $K_L - K_S$ mass difference"**
  Z. Bai, N. H. Christ, C. T. Sachrajda
  EPJ Web of Conferences 175(2018), 13017
  All diagrams included on a $64^3 \times 128$ lattice with **physical masses** on 59 configurations: $\Delta m_k = 5.5(1.7)_{stat} \times 10^{-12}$ MeV.
- In this talk, I will present the calculation of Δm_K on 152 configurations and a new analysis method employed to calculate Δm_K with better reduction of statistical error on this larger set of configurations.

- Δm_K is given by:

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathrm{Re}M_{12} = 2\mathcal{P}\sum_n \frac{\langle \overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}. \tag{10}$$

- What we can calculate are four-point correlators on the lattice:

$$G(t_1, t_2, t_i, t_f) \equiv \langle 0|T\{\overline{K^0}(t_f)H_W(t_2)H_W(t_1)K^0(t_i)\}|0\rangle \tag{11}$$



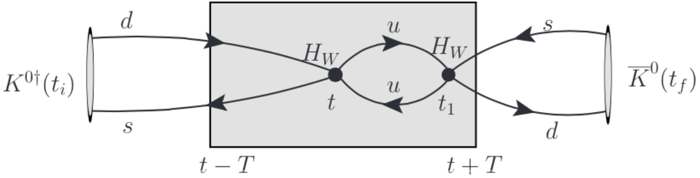
Extract Δm_K from single-integrated correlators

- The single-integrated correlator is defined as:

$$\mathcal{A}^s(t,T) \equiv \frac{1}{2!}\sum_{t_1=t-T}^{t+T} \langle 0|T\{\overline{K^0}(t_f)H_W(t_1)H_W(t)K^0(t_i)\}|0\rangle \quad (12)$$

- If we insert a complete set of intermediate states, we find:

$$\mathcal{A}^s = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}(-1+e^{(m_K-E_n)(T+1)}) \quad (13)$$



Subtraction of the light states

- Single-integration method requires subtraction of the terms from light states:

$$\mathcal{A}^s = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \overline{K^0}|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n}\{-1+e^{(m_K-E_n)(T+1)}\} \quad (14)$$

- For $|n\rangle$(in our case $|0\rangle$, $|\pi\pi\rangle$, $|\eta\rangle$, $|\pi\rangle$) with $E_n < m_K$ or $E_n \sim m_K$: the exponential terms will be significant. We can:
  - freedom of adding $c_s\bar{s}d$, $c_p\bar{s}\gamma^5 d$ operators to the weak Hamiltonian. Here we choose:

$$\langle 0|H_W - c_p\bar{s}\gamma_5 d|K^0\rangle = 0, \langle\eta|H_W - c_s\bar{s}d|\bar{K}^0\rangle = 0$$

  - subtract contributions from other states($|\pi\rangle$, $|\pi\pi\rangle$) explicitly

Operators of Δm_K calculation

- The $\Delta S = 1$ effective weak Hamiltonian:

$$H_W = \frac{G_F}{\sqrt{2}} \sum_{q,q'=u,c} V_{qd} V^*_{q's} (C_1 Q_1^{qq'} + C_2 Q_2^{qq'}) \tag{15}$$

where the ${Q_i^{qq'}}_{i=1,2}$ are current-current opeartors, defined as:

$$Q_1^{qq'} = (\bar{s}_i \gamma^\mu (1-\gamma^5) d_i)(\bar{q}_j \gamma^\mu (1-\gamma^5) q'_j)$$

$$Q_2^{qq'} = (\bar{s}_i \gamma^\mu (1-\gamma^5) d_j)(\bar{q}_j \gamma^\mu (1-\gamma^5) q'_i)$$

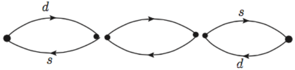
- There are four states need to subtracted: $|0\rangle$, $|\pi\pi\rangle$, $|\eta\rangle$, $|\pi\rangle$. We add $c_s \bar{s} d$, $c_p \bar{s} \gamma^5 d$ operators to weak operators to make:

$$\langle 0 | Q_i - c_{pi} \bar{s} \gamma_5 d | K^0 \rangle = 0, \langle \eta | Q_i - c_{si} \bar{s} d | K^0 \rangle = 0 \tag{16}$$

$$Q'_i = Q_i - c_{pi} \bar{s} \gamma_5 d - c_{si} \bar{s} d \tag{17}$$

Diagrams in the calculation of Δm_K

- For contractions among Q_i, there are four types of diagrams to be evaluated.



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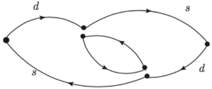

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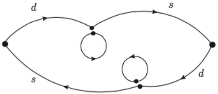

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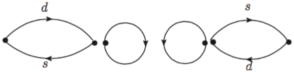
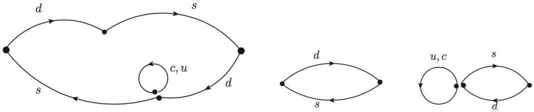


type 4

- In addition, there are "mixed" diagrams from the contractions between the $c_s \bar{s} d$ $c_p \bar{s} \gamma^5 d$ operators and Q_i operators.



Non-perturbative renormalizations

- Renormalization of lattice operator $Q_{1,2}$ and and obtain the Wilson coefficients C_i^{lat} in 3 steps:

$$C_i^{lat} = C_a^{\overline{MS}} (1 + \Delta r)_{ab}^{RI \to \overline{MS}} Z_{bi}^{lat \to RI}$$

  - Non-perturbative Renormalization: from the lattice to the RI-SMOM

$$Z^{lat \to RI} = \begin{bmatrix} 0.5642 & -0.03934 \\ -0.03934 & 0.5642 \end{bmatrix} \tag{18}$$

  - Perturbation theory: from the RI-SMOM to the $\overline{MS}$

C. Lehner, C. Sturm, Phys. Rev. D 84(2011), 014001

$$\Delta r^{RI \to \overline{MS}} = 10^{-3} \times \begin{bmatrix} -2.28 & 6.85 \\ 6.85 & -2.28 \end{bmatrix} \tag{19}$$

  - Use Wilson coefficients in the $\overline{MS}$ scheme

G. Buchalla, A.J. Buras and M.E. Lautenbacher, arXiv:hep-ph/9512380

$$C^{\overline{MS}} = 10^{-3} \times \begin{bmatrix} -0.260 & 1.118 \end{bmatrix} \tag{20}$$

Finite-volume corrections

Lattice calculations are performed with a finite space-time volume rather than an infinite volume.

- without multi-particle states: corrections $\sim e^{-mL}$
- with multi-particle states: corrections having power-law dependence.

Finite-volume corrections to Δm_K:

- the scattering among multiple particles in the finite volume: $I = 0$ two-pion state.
  "Effects of finite volume on the $K_L - K_S$ mass difference"
  N.H. Christ, X. Feng, G. Martinelli and C.T. Sachrajda, Phys.Rev.D 91 (2015) 11, 114510
- The correction $\delta(\Delta m_K)^{FV} = 2\mathrm{Re}(\delta M_{12})$.
  δM_{12} is defined as:

$$M_{12}^{\infty} = M_{12}^{V} + \delta M_{12}, \tag{21}$$

and given by:

$$\delta M_{12} = -\cot(h(m_K)) \frac{dh(E)}{dE}\bigg|_{E=m_K} \times f(m_K), \tag{22}$$

$2f(m_k)$	$h = \delta + \phi$	$\cot h$	dh/dE	$\cot h \times dh/dE$	$\delta(\Delta m_K)^{FV}$
−0.0086(25)	−0.49(6)	−1.85(27)	33.5(4)	−62(10)	−0.54(18)

Finite lattice spacing effects

Lattice calculations are performed on a discretized space with finite lattice spacing a. As $a \to 0$, we obtain the continuum limit.

- Elimination of $\mathcal{O}(a)$ finite lattice spacing errors
- Sources of $\mathcal{O}(a^2)$ finite lattice spacing errors
  - heavy charm quark, $\sim (m_c a)^2$ gives 25%
  - effects from low-energy scale $\sim \Lambda_{\mathrm{QCD}}$
- Scaling tests: perform calculations of three- and four-point quantities on two lattices with different lattice spacings. We need a coarser lattice to be compared with a finer lattice.
  - **64I(2.4 GeV)** $\leftrightarrow$ 96I(2.8 GeV) Hard to do
  - **24I(1.8 GeV)** $\leftrightarrow$ 32I(2.4 GeV) ✓

We estimate the finite lattice spacing error in our Δm_K calculation to be of order of 40%.

Lattice talk 2021

Lattice name	Action (F+G)	a^{-1} (GeV)	Lattice Volume	β	b+c	L_s	m_l	m_h	m_{res}
24I	DWF+I	1.785(5)	$24^3 \times 64 \times 16$	2.13	1.0	16	0.0050	0.0400	0.00308
32I	DWF+I	2.383(9)	$32^3 \times 64 \times 16$	2.25	1.0	16	0.0040	0.0300	0.000664

Δm_K calculation with physical quark masses

- $64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action with physical pion mass (136 MeV).

Lattice ensemble	Action (F+G)	a^{-1} (GeV)	Lattice Volume	β	b+c	L_s	m_l	m_h	m_{res}
64I	MDWF+I	2.359(7)	$64^3 \times 128 \times 12$	2.25	2.0	12	0.000678	0.02661	0.000314

- Data analysis:
  - Sample AMA correction:

data type	CG stop residual
Sloppy	$1e-4$
Exact	$1e-8$

Diagram types	sample AMA correction	# of Sloppy	# of Exact
Type-3&4	Y	116	36
Type-1&2	N	0	36

the super-jackknife method is used to estimate the statistical errors for the AMA corrected data.

  - Disconnected Type4 diagrams:
    save left- and right-pieces separately and use multiple source-sink separation for fitting.

Overview of the calculation of Δm_K

Quantities to be calculated are:

- two-point correlation functions:
  - meson masses: m_π, m_K, $m_{\pi\pi}$, m_η
  - normlization factors of meson interpolating operators: N_π, N_K, $N_{\pi\pi}$, N_η
- three-point correlation functions:
  - light state matrix elements to be subtracted: $\langle\pi|Q_i'|K^0\rangle = \langle\pi|Q_i|K^0\rangle - c_{\mathrm{s}i}\langle\pi|\bar{s}d|K^0\rangle$, and $\langle\pi\pi_{I=0}|Q_i - c_{\mathrm{p}i}\bar{s}\gamma_5 d|K^0\rangle$.
  - coefficients of the $\bar{s}d$ and $\bar{s}\gamma_5 d$ operators: $c_{\mathrm{s}i} = \frac{\langle\eta|Q_i|K^0\rangle}{\langle\eta|\bar{s}d|K^0\rangle}$, $\quad c_{\mathrm{p}i} = \frac{\langle 0|Q_i|K^0\rangle}{\langle 0|\bar{s}\gamma_5 d|K^0\rangle}$.
- four-point correlation functions:
  - unintegrated correlation functions calculated from diagrams having light state contribution subtracted: $\widetilde{G}^{\mathrm{sub}}(\delta) = \widetilde{G}(\delta) - \sum_{n\in\{n_l\}} \langle\bar{K}^0|H_W|n\rangle\langle n|H_W|K^0\rangle e^{(m_K - E_n)\delta}$
  - single-integrated correlation functions: $\widetilde{\mathcal{A}}^S(T) = \sum_{\delta=1}^{T} \widetilde{G}^{\mathrm{sub}}(\delta) + \frac{1}{2}\widetilde{G}^{\mathrm{sub}}(0) \to \Delta m_K$

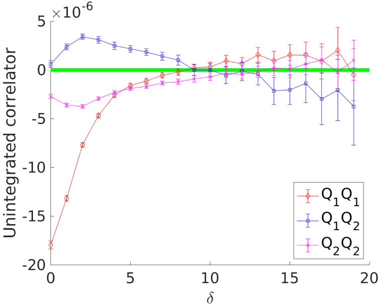
Δm_K using single-integrated correlators preliminary

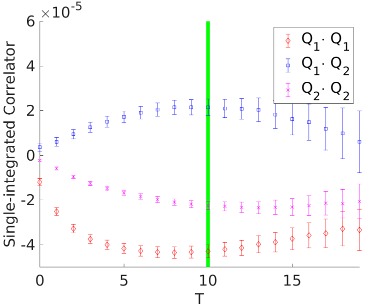
- Subtract light states from the averaged unintegrated correlator:

$$\widetilde{G}_{ij}^{\mathrm{sub}}(\delta) = \widetilde{G}_{ij}(\delta) - \sum_{n \in \{n_l\}} \langle \bar{K}^0 | Q_i' | n \rangle \langle n | Q_j' | K^0 \rangle e^{(m_K - E_n)\delta} \tag{23}$$

- Perform a single-integration over δ for the subtracted correlator between $\delta = 0$ and $\delta = T$ to obtain:

$$\widetilde{\mathcal{A}}_{ij}^{S}(T) = \sum_{\delta=1}^{T} \widetilde{G}_{ij}^{\mathrm{sub}}(\delta) + \frac{1}{2} \widetilde{G}_{ij}^{\mathrm{sub}}(0) \tag{24}$$





Sample AMA corrections preliminary

Our use of the sample AMA method reduced the computational cost of the calculation by a factor of 2.3, while the statistical error on the AMA correction will add to the total statistical error.

Analysis method	type 3&4 error from "sloppy"	type 3&4 error from correction	type 3&4 error in total
Double-integration	0.60	0.24	0.65
Single-integration	0.39	0.29	0.49

We can conclude that the AMA method does not contribute much to the error in our final answer.

Results for Δm_K preliminary

- The finite-volume correction to Δm_K is estimated to be:

$$\delta(\Delta m_K)^{FV} = -0.54(18) \times 10^{-12}\mathrm{MeV}. \tag{25}$$

- Based on the scaling tests, we estimate the finite lattice spacing error of our Δm_K calculation to be $\sim 40\%$. We choose to use the results from the single-integration method:

Analysis method	Δm_K	Δm_K(type1&2)	Δm_K(type3&4)
Double-integration	6.31(0.98)	6.71(0.48)	-0.20(0.65)
Single-integration	**6.34(0.57)**	6.24(0.24)	0.33(0.50)

After including the finite volume correction, our result for Δm_K based on 152 configurations with physical quark masses is:

$$\Delta m_K = 5.8(0.6)_{\mathrm{stat}}(2.3)_{\mathrm{sys}} \times 10^{-12}\mathrm{MeV}. \tag{26}$$

Conclusion and outlook

- Our **preliminary** result for Δm_K based on 152 configurations is:

$$\Delta m_K = 5.8(0.6)_{\text{stat}}(2.3)_{\text{sys}} \times 10^{-12}\text{MeV}, \quad (27)$$

to be compared to the experimental value:

$$(\Delta m_K)^{exp} = 3.483(6) \times 10^{-12}\text{MeV}. \quad (28)$$

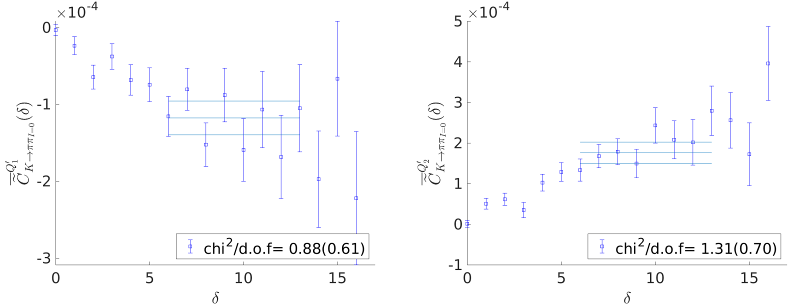
We find reasonable agreement given the large finite lattice spacing errors.

- Outlook:
Future calculations on the Summit supercomputer:
  - Δm_K: on $96^3 \times 192$ lattice with $a^{-1} = 2.8$ GeV
    - Better estimate of finite lattice spacing effect:
    64I(2.4 GeV) ↔ 96I(2.8 GeV)   continuum limit to be explored
    - Further improvement of the precision to ~ 5% level.
  - ϵ_K: with Joe Karpie, improve the accuracy of ϵ_K to sub-percent level.

Thanks for your attention!

Three-point light-state matrix elements: K to $\pi\pi$ matrix elements

$$\overline{\widetilde{C}}^{Q_i'}_{K\to\pi\pi_{I=0}}(\delta) = \left\langle \widetilde{C}^{Q_i'}_{K\to\pi\pi_{I=0}}(\Delta, t) \right\rangle_\Delta = \left\langle \frac{N_{\pi\pi_{I=0}} N_K C^{Q_i'}_{K\to\pi\pi_{I=0}}(\Delta,\delta)}{C^{2\mathrm{pt}}_{\pi\pi_{I=0}}(\Delta - t)\, C^{2\mathrm{pt}}_K(t)} \right\rangle_\Delta, \tag{58}$$



Obtaining Δm_K from single-integrated correlators with operators Q'_i and Q'_j

Separate fitting of the single-integrated correlator $\mathcal{A}^S$ with weak Hamiltonian into fitting the integrated correlator with Q'_1 and Q'_2:

$$\mathcal{A}^S_{ij}(T) = N_K^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 | Q'_i | n \rangle \langle n | Q'_j | K^0 \rangle}{m_K - E_n} \{-1 + e^{-(E_n - m_K)T}\}. \tag{59}$$

The relationship between $\mathcal{A}^S_{ij}(T)$ and $\mathcal{A}^S(T)$ is thus given by:

$$\mathcal{A}^S(T) = \frac{G_F^2}{2} \lambda_u^2 \sum_{i,j=1,2} C_i C_j \mathcal{A}^S_{ij}(T). \tag{60}$$

The value of Δm_K is then given by:

$$\Delta m_K^{\mathrm{lat}} = \frac{G_F^2}{2} \lambda_u^2 \sum_{i,j=1,2} (-2) \times C_i^{\mathrm{lat}} C_j^{\mathrm{lat}} k_{ij}. \tag{61}$$

Obtaining Δm_K from double-integrated correlators

The double-integrated correlator is defined as:

$$\mathcal{A} = \frac{1}{2} \sum_{t_2=t_a}^{t_b} \sum_{t_1=t_a}^{t_b} \langle 0| T\{\overline{K}^0(t_f) H_W(t_2) H_W(t_1) K^0(t_i)\}|0\rangle \qquad (62)$$

If we insert a complete set of intermediate states

$$\mathcal{A} = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \overline{K}^0|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n} \{-T + \frac{e^{(m_K-E_n)T} - 1}{m_K - E_n}\} \qquad (63)$$

we identify the coefficient of the term linear in the size of integration box $T = t_b - t_a + 1$ as proportional to the expression for Δm_K

$$\Delta m_K^{lat} \equiv 2 \sum_n \frac{\langle \overline{K}^0|H_W|n\rangle\langle n|H_W|K^0\rangle}{m_K - E_n} \qquad (64)$$

Finite lattice spacing effects: scaling test

- The parameters used are listed below:

Lattice	β	b+c	L_s	a^{-1}	m_l	m_h
24I	2.13	1.0	16	1.785	0.005	0.04
32I	2.25	1.0	16	2.383	0.004	0.03

Lattice	m_x	m_y	m_π	m_K	m_c's
24I	0.00667	0.0321	0.2079	0.3125	0.15:0.05:0.35
32I	0.00649	0.0249	0.1557	0.2332	$(0.15{:}0.05{:}0.35)\frac{1.785}{2.383}$

Table: Parameters related to the lattices for measurements.