Lattice calculation of the muon $g-2$

Luchang Jin

University of Connecticut / RIKEN BNL Research Center

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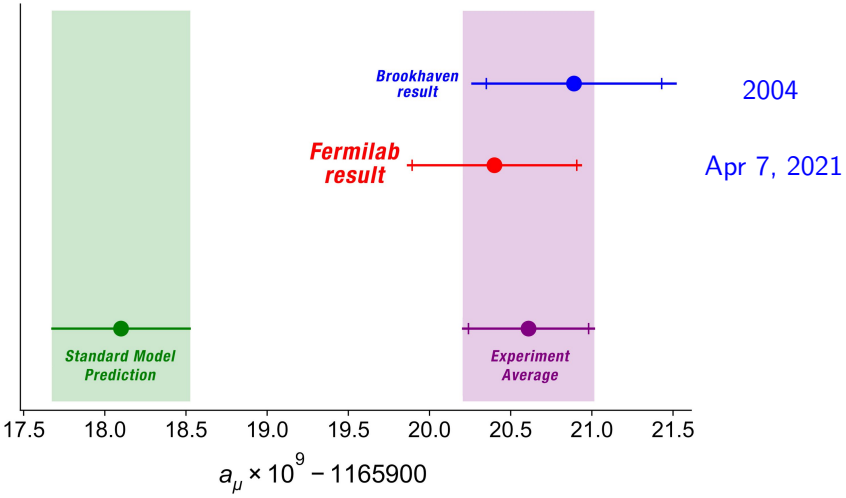
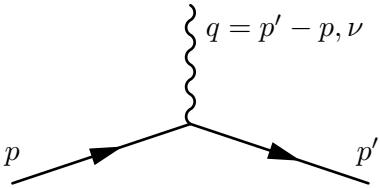
中国格点 QCD 第一届年会
线上 2021

Outline



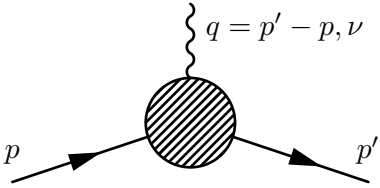
- "So far we have analyzed less than 6% of the data that the experiment will eventually collect. Although these first results are telling us that there is an intriguing difference with the Standard Model, we will learn much more in the next couple of years." – Chris Polly, Fermilab scientist, co-spokesperson for the Fermilab muon $g-2$ experiment.



Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$



$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

(Euclidean space time)

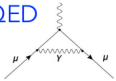
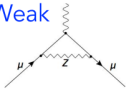
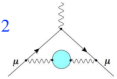
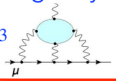
$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

- The quantity a is called the anomalous magnetic moments.
- Its value comes from quantum correction.

Muon $g-2$ Theory Initiative White paper posted 10 June 2020.

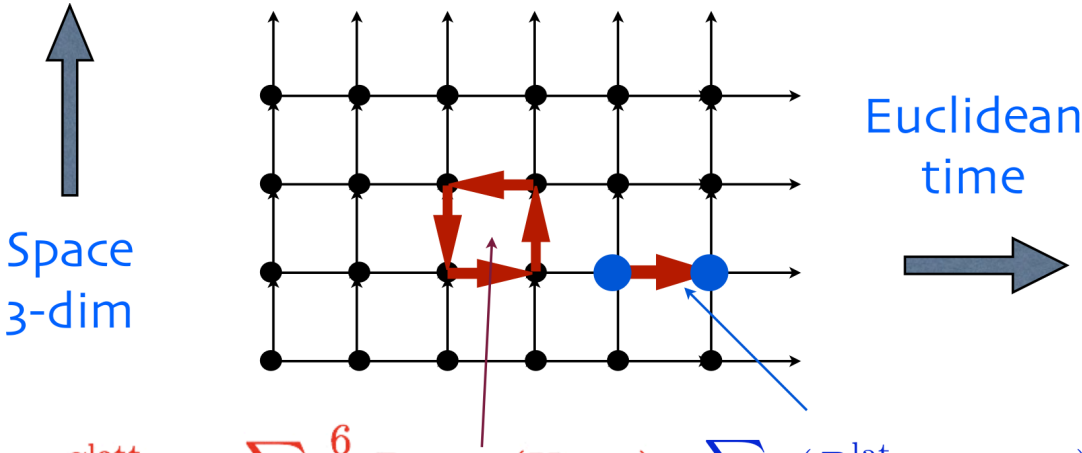
132 authors from worldwide theory + experiment community. [Phys. Rept. 887 (2020) 1-166]

$$a_\mu = a_\mu(\text{QED}) + a_\mu(\text{Weak}) + a_\mu(\text{Hadronic})$$

QED		$116\,584\,718.9\,(1) \times 10^{-11}$	0.001 ppm
Weak		$153.6\,(1.0) \times 10^{-11}$	0.01 ppm
Hadronic…			
…Vacuum Polarization (HVP)	α^2 	$6845\,(40) \times 10^{-11}$ [0.6%]	0.34 ppm
…Light-by-Light (HLbL)	α^3 	$92\,(18) \times 10^{-11}$ [20%]	0.15 ppm

- Two methods: dispersive + data ↔ lattice QCD

From Aida El-Khadra's theory talk during the Fermilab $g-2$ result announcement.



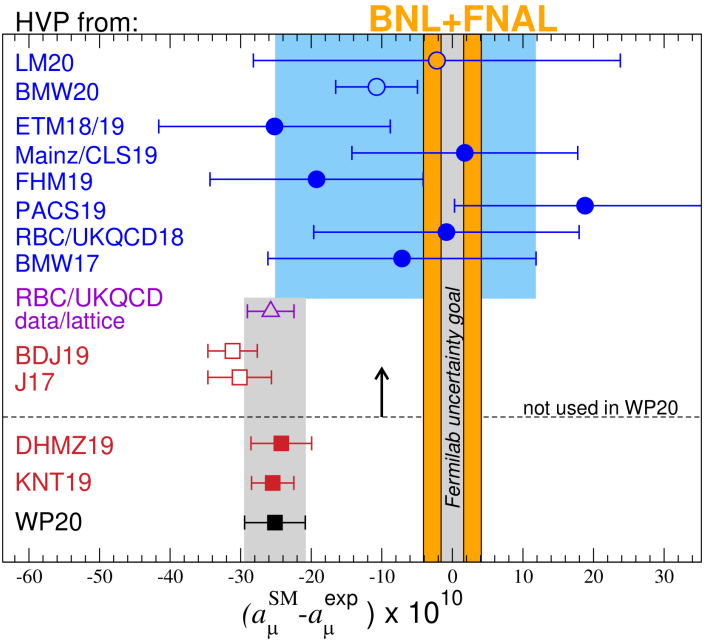
$$S_E^{\text{latt}} = -\sum_{\square} \frac{6}{g^2} \text{Re tr}_N(U_{\square,\mu\nu}) - \sum_q \overline{q}(D_\mu^{\text{lat}}\gamma_\mu + am_q)q$$

Wilson gauge action Lattice fermion action

Figure credit: Stephen R. Sharpe.

Outline

- Dispersive method (red points) is mature and reproducible.
- Lattice (blue points) errors are limited by statistics.
  Except for BMW, which beats down the statistical error, result is limited by systematic error:
  BMW 20: $707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}$
- Lattice-QCD calculations of comparable precision needed.
- Consistency is needed to claim new physics.



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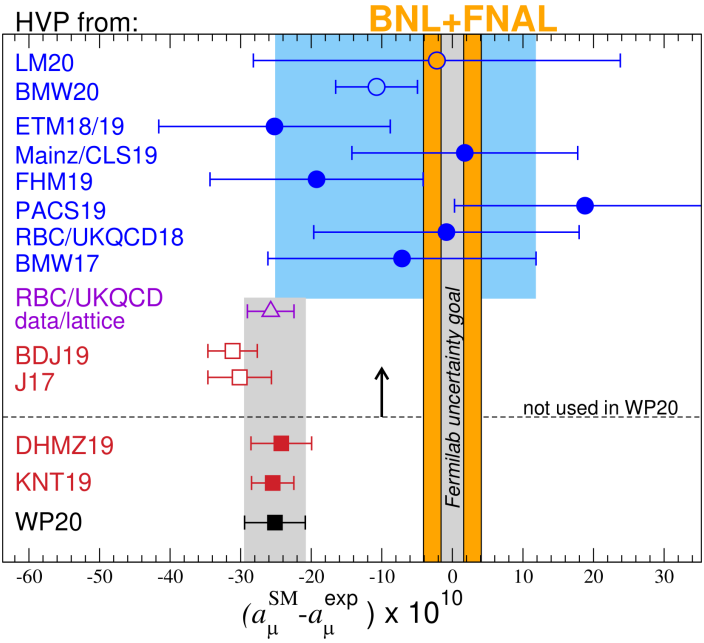

The 2011 KWLA panel is proud to award

The 2011 Ken Wilson Lattice Award

To: Xu Feng, Marcus Petschlies, Karl Jansen, and Dru B. Renner

In recognition of their paper titled

Two-flavor QCD Correction to Lepton Magnetic Moments at Leading-Order in the Electromagnetic Coupling

The 2011 KWLA Panel Members

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Two-Flavor QCD Correction to Lepton Magnetic Moments at Leading Order in the Electromagnetic Coupling

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(Received 28 March 2011; published 17 August 2011)

We present a reliable nonperturbative calculation of the QCD correction, at leading order in the electromagnetic coupling, to the anomalous magnetic moment of the electron, muon, and tau leptons using two-flavor lattice QCD. We use multiple lattice spacings, multiple volumes, and a broad range of quark masses to control the continuum, infinite-volume, and chiral limits. We examine the impact of the commonly ignored disconnected diagrams and introduce a modification to the previously used method that results in a well-controlled lattice calculation. We obtain $1.513(43) \times 10^{-12}$, $5.72(16) \times 10^{-8}$, and $2.650(54) \times 10^{-6}$ for the leading-order two-flavor QCD correction to the anomalous magnetic moment of the electron, muon, and tau, respectively, each accurate to better than 3%.

DOI: 10.1103/PhysRevLett.107.081802 PACS numbers: 13.40.Em, 12.38.Gc, 14.60.Ef

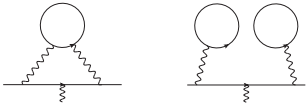
Introduction.—The experimental [1] and theoretical [2] determinations of the anomalous magnetic moment of the muon a_μ have both reached an accuracy that is better than six parts per million. This high precision reveals a discrepancy of over 3 standard deviations (3σ), which raises the possibility of physics beyond the standard model. However, the dominant error in the theory computation is due to hadronic effects that are currently not calculated but are instead either separately measured or simply modeled. This obscures the significance of the 3σ effect and makes it

perturbative expansion in the electromagnetic coupling α. Contributions from QCD first occur at the order α^2 and can be written as [4]

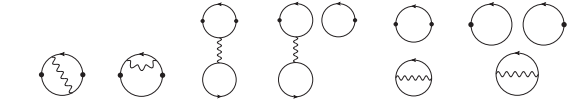
$$a_l^{\rm hvp} = \alpha^2 \int_0^\infty dQ^2 \frac{1}{Q^2} w(Q^2/m_l^2) \Pi_R(Q^2), \qquad (1)$$

where m_l is the mass of the lepton, Q is the Euclidean momentum, and $w(Q^2/m_l^2)$ is a known function. The combination $\Pi_R(Q^2) = \Pi(Q^2) - \Pi(0)$ is the renormalized hadronic vacuum polarization function $\Pi(Q^2)$, which is

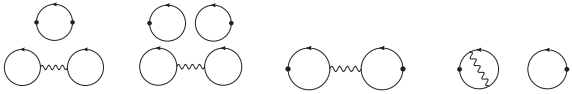
Isospin limit



QED corrections

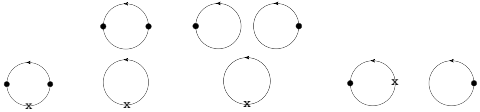


(a) V (b) S (c) T (d) T_d (e) D1 (f) $D1_d$



(g) D2 (h) $D2_d$ (i) F (j) D3

Strong isospin breaking



(a) M (b) R (c) R_d (d) O

- Need to calculate and cross check all the contributions.

T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j^{em}(\vec{x},t)J_j^{em}(0)\rangle_{\rm QCD}$$

$$a_\mu^{\rm HVP\ LO} = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2) = \sum_{t=0}^{+\infty} w(t)C(t)$$

QED and strong isospin breaking

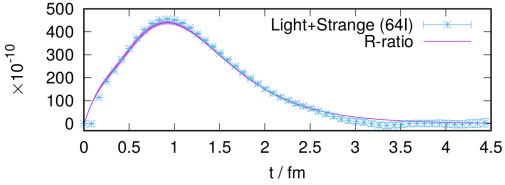
$a_\mu^{\rm HVP,\,LO}(ud)$	$a_\mu^{\rm HVP,\,LO}(s)$	$a_\mu^{\rm HVP,\,LO}(c)$	$a_{\mu,\rm disc}^{\rm HVP,\,LO}$	$\delta a_\mu^{\rm HVP}$
650.2(11.6)	53.2(0.3)	14.6(0.1)	−13.7(2.9)	7.2(3.4)

- From muon $g-2$ theory initiative white paper (2020). Value in unit of 10^{-10}
- Light quark connected diagram has the largest contribution and largest uncertainty.

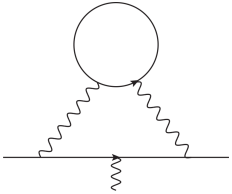
T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j^{em}(\vec{x},t)J_j^{em}(0)\rangle_{\mathrm{QCD}}$$

$$a_\mu^{\mathrm{HVP\ LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$



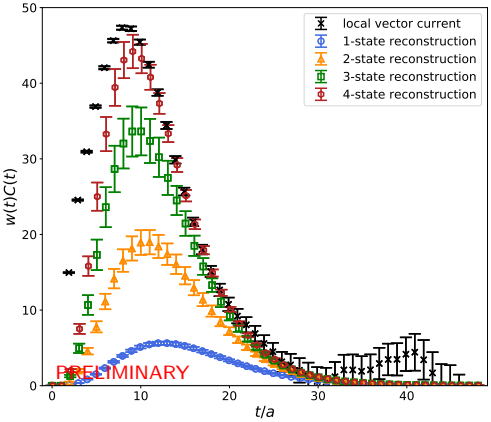
- Statistical error is mostly from:
  Light quark connected diagram at $t \gtrsim 1.5$ fm
  - More configurations (BMW 20 used $\sim 20,000$).
  - Use low modes averaging to gain full volume average. ✓
  - Bounding method on the long distance tail. ✓
  - Study the $\pi\pi$ system spectrum to calculate $C(t)$ large t.
    - Not used in any published work yet!
    - On-going efforts with promising initial results.
- Systematic error is mostly from the **continuum extrapolation**.

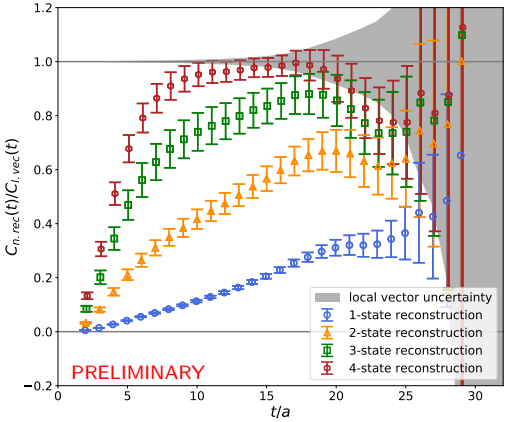


- Main idea is that: one does not have to calculate the long distance part of the correlation function directly.

$$\begin{aligned} C(t) &= \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle \\ &= \sum_n \frac{V}{3}\sum_{j=0,1,2}\langle 0|J_j(0)|n\rangle\langle n|J_j(0)|0\rangle e^{-E_n t} \end{aligned}$$

- The summation over n is limited to zero momentum states and states are normalized to "1".
- At large t, only lowest few states contribute. We only need the matrix elements $\langle n|J_j(0)|0\rangle$ and the corresponding energy E_n.
- Need to study the spectrum of the $\pi\pi$ system!
- Can reduce the statistical error beyond the gauge noise limit!





GEVP results to reconstruct long-distance behavior of local vector correlation function needed to compute connected HVP

Explicit reconstruction good estimate of correlation function at long-distance, missing excited states at short-distance

More states $\implies$ better reconstruction, can replace $C(t)$ at shorter distances

RBC-UKQCD by Aaron Meyer and Christoph Lehner

Preliminary

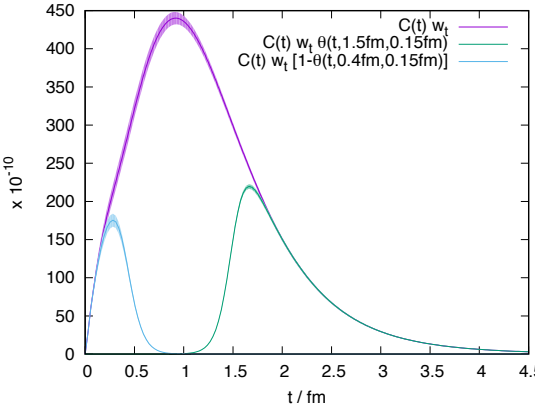
RBC-UKQCD PRL 121, 022003 (2018)

Window contribution allows a high precision study of the continuum extrapolation.

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t) \qquad w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

- Splitting sum into three parts allows crosschecks:
  - short distance $\Leftarrow$ discretization effects
  - long distance $\Leftarrow$ noisy $\pi\pi$ tail
  - intermediate (Window): sweet spot
- Can form windows from $R(e^+e^-)$ dispersive analysis too.

  Combine "window" from lattice with dispersive analysis.
- Compare "window" among lattice-QCD calculations



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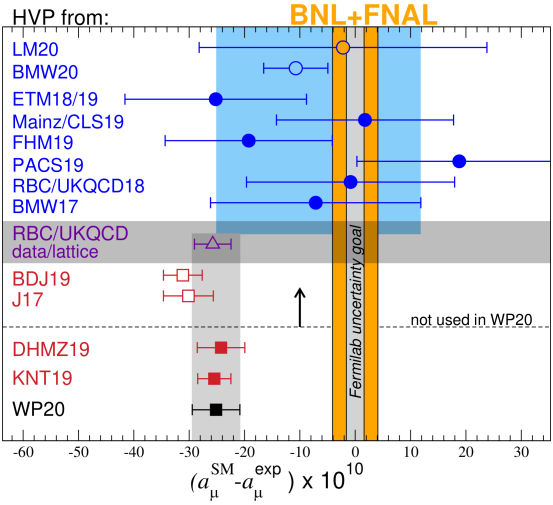

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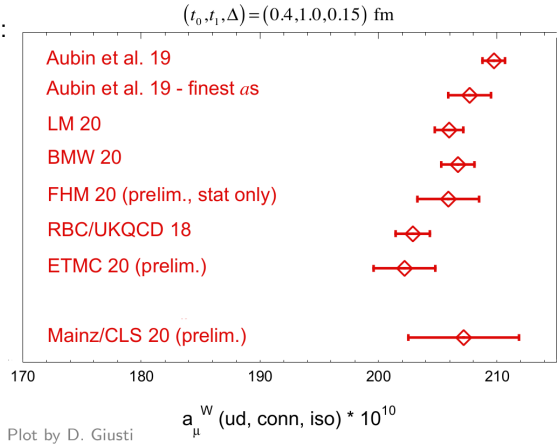
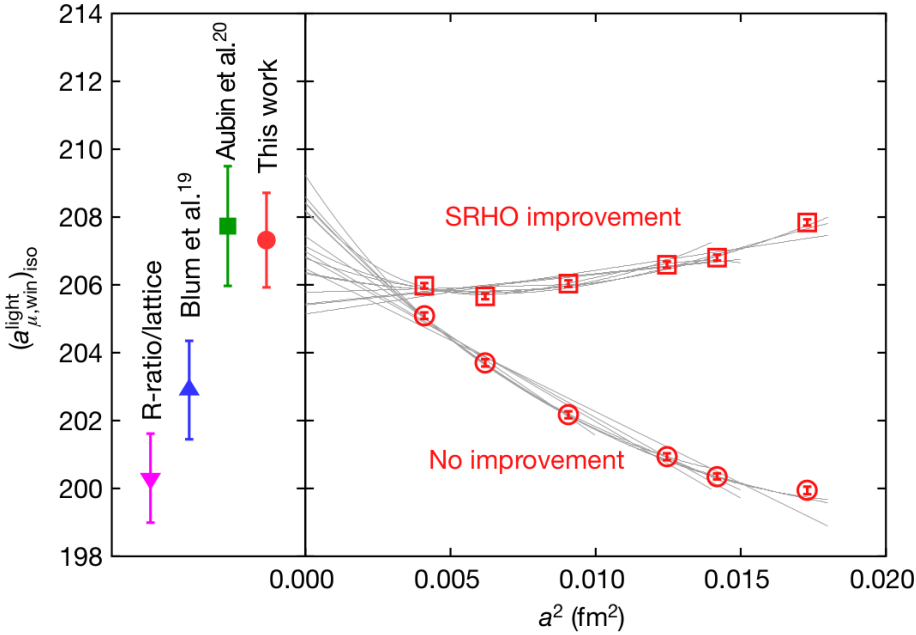


Plot by D. Giusti



- Staggered fermion has a special lattice artifacts: taste breaking effects.
- Curves show different treatments of correcting the taste breaking effects.

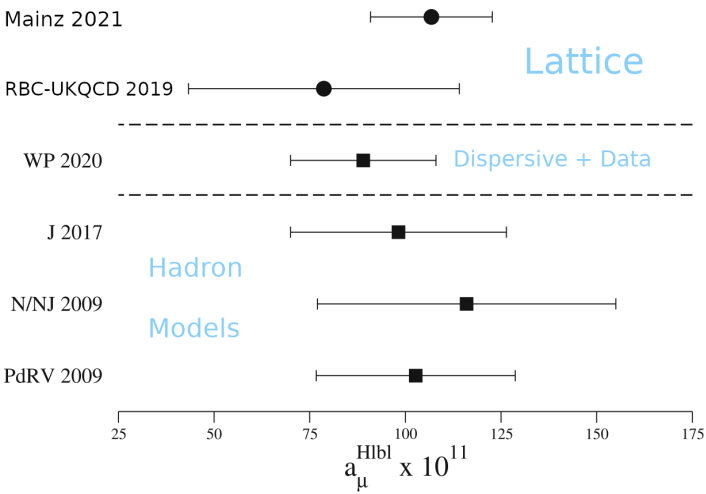
Muon $g-2$ HVP: Some comments

- The white paper result for the HVP is a community-vetted method average for the data-driven approach. It accounts for spreads in sub-contributions between individual results (KNT/DHMZ) that may not be visible in the agreement of looking at the final results for the HVP. It should be noted that its error estimate also accounts for the tension between BaBar and KLOE experimental inputs.
- We are now in the fortunate situation that we have a first lattice result with sub-percent precision (BMW). It is clear that to safely assess systematic uncertainties, most notably the one related to the choice of the lattice regulator, calculations by other lattice groups with a similar precision will be essential. The importance of having more than one lattice calculations of the same quantity and obtained with different lattice discretizations is well understood inside the lattice community.
- On the way to a method average for lattice QCD, it is prudent to also look at individual sub-contributions and their agreement, similar to what was done for the data driven approach. Here the tension for the light-quark isospin-symmetric window results between RBC/UKQCD18 and BMW20 or between the same papers for the QED disconnected results need to be addressed. This is possible since these sub-contributions are already available by different collaborations at adequate precision for such a comparison.
- Finally, it should also not be ignored that for the standard window result, there is a 3.7σ tension between the data-driven approach and the BMW20 paper.

Outline

Muon $g-2$ HLbL: overview

- Mainz 2021 is the most recent lattice result. It uses heavier pion mass with infinite volume QED kernel and extrapolate to the physical pion mass.
- RBC-UKQCD 2019 is the first lattice result. It uses physical pion mass in the finite volume QED_L scheme and extrapolate to the infinite volume.

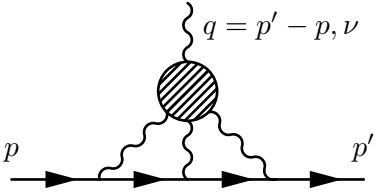


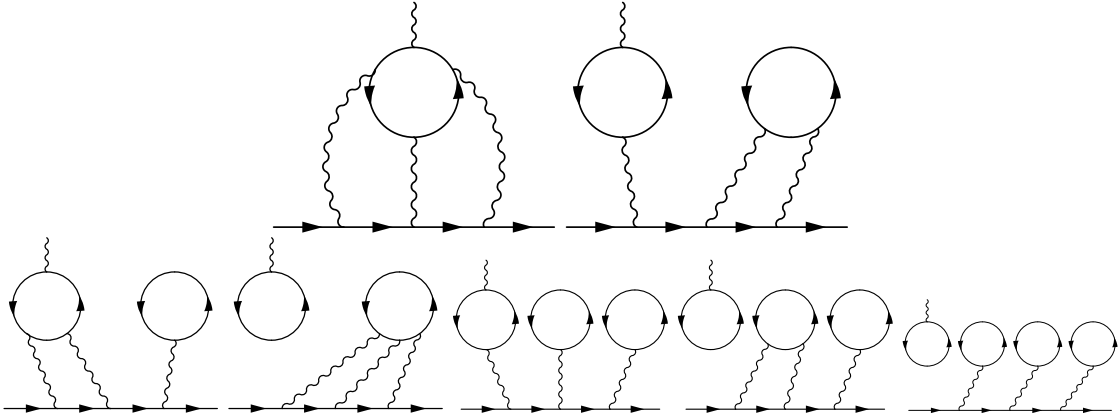
- WP 2020 result uses dispersive relations and data. It is the sum of the contributions from different cuts and poles. High energy contributions are the major source of uncertainties.
- These three results have different systematics and agree well with each other. Uncorrelated average gives: $a_\mu^{\text{HLbL}} = 9.77(1.16) \times 10^{-10}$.
- Hadronic light-by-light contribution cannot be the source of the muon $g-2$ puzzle.

Contribution	PdRV(09) [471]	N/JN(09) [472, 573]	J(17) [27]	Our estimate
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	} − 1(3)
tensors	–	–	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

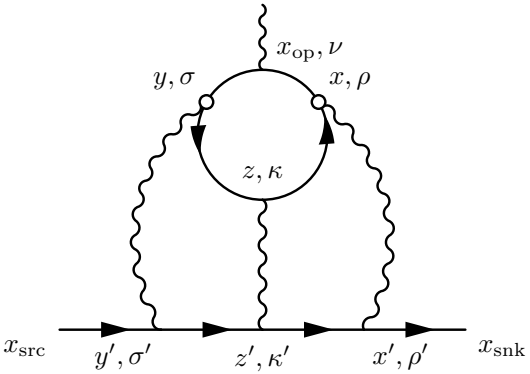
Table 15: Comparison of two frequently used compilations for HLbL in units of 10^{-11} from 2009 and a recent update with our estimate. Legend: PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus"); N/JN = Nyffeler / Jegerlehner, Nyffeler; J = Jegerlehner.

- Values in the table is in unit of 10^{-11}.
- Uncertainty of the analytically approach mostly come from the short distance part.





- There are additional different permutations of photons not shown.
- The second row diagrams are suppressed by flavor SU(3) symmetry (and small charge factors, $1/N_c$, etc). The contributions are numerically very small.



- Two point sources at x, y: randomly sample x and y.
- Importance sampling: focus on small $|x-y|$.
- Complete sampling for $|x-y| \leq 5a$ upto discrete symmetry.

$$\frac{a_\mu}{m_\mu}\bar{u}_{s'}(\vec{0})\frac{\Sigma}{2}u_s(\vec{0}) = \sum_{r=x-y}\sum_{z}\sum_{x_{\rm op}}\frac{1}{2}(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\times\bar{u}_{s'}(\vec{0})i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\rm op})u_s(\vec{0})$$

$$\vec{\mu} = \sum_{\vec{x}_{\rm op}}\frac{1}{2}(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\times\vec{J}(\vec{x}_{\rm op})$$

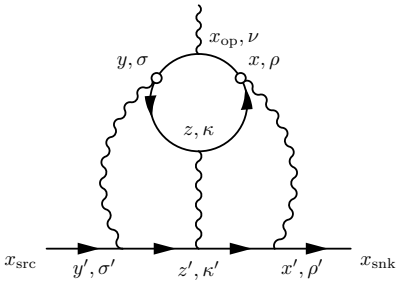
Reorder summation

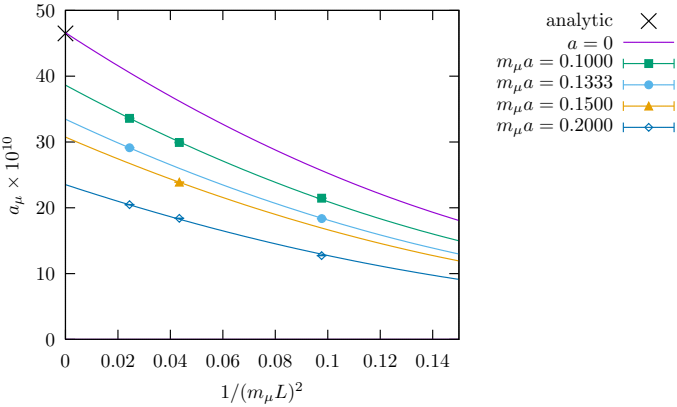
$|x-y| \leq \min(|y-z|, |x-z|)$

- Muon is plane wave, $x_{\rm ref} = (x+y)/2$.
- Sum over time component for $x_{\rm op}$.
- Only sum over $r = x-y$.

T. Blum et al 2016. (PRD 93, 1, 014503)

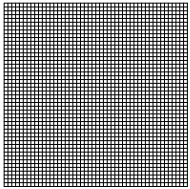
- We test our setup by computing **muon leptonic light by light** contribution to muon $g-2$.



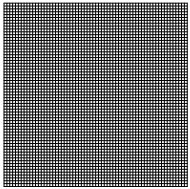


$$F_2(a,L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4}\right)(1 - c_2\, a^2 + c_2'\, a^4) \;\rightarrow\; F_2 = 46.6(2)\times 10^{-10} \qquad (19)$$

- **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.
- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.

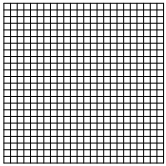


48I: $48^3 \times 96$, $5.5\,\mathrm{fm}$ box

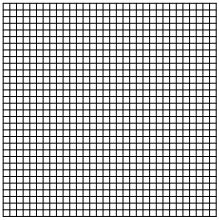


64I: $64^3 \times 128$, $5.5\,\mathrm{fm}$ box

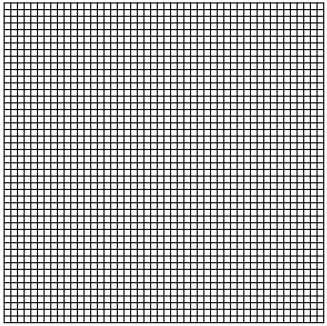
Phys. Rev. D 93, 074505 (2016)



24D: $24^3 \times 64$, $4.8\,\mathrm{fm}$ box



32D: $32^3 \times 64$, $6.4\,\mathrm{fm}$ box



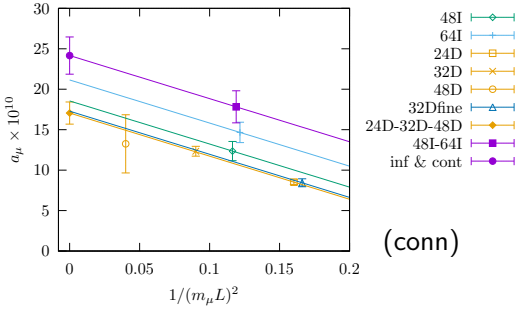
48D: $48^3 \times 64$, $9.6\,\mathrm{fm}$ box

32Dfine: $32^3 \times 64$, $4.8\,\mathrm{fm}$ box

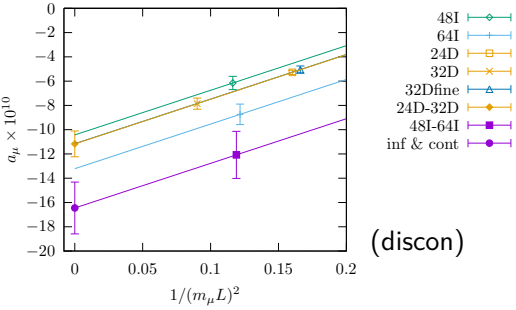
All at physical pion mass.

T. Blum et al 2020. (PRL 124, 13, 132002)

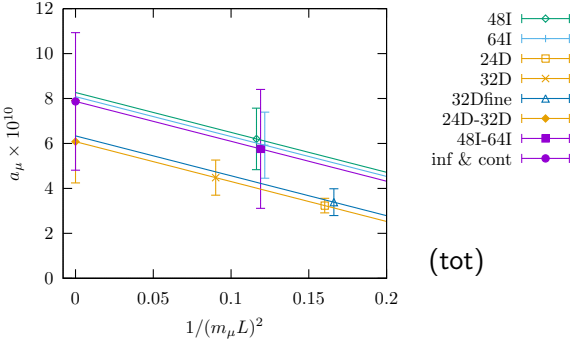
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\right)$$



(conn)



(discon)



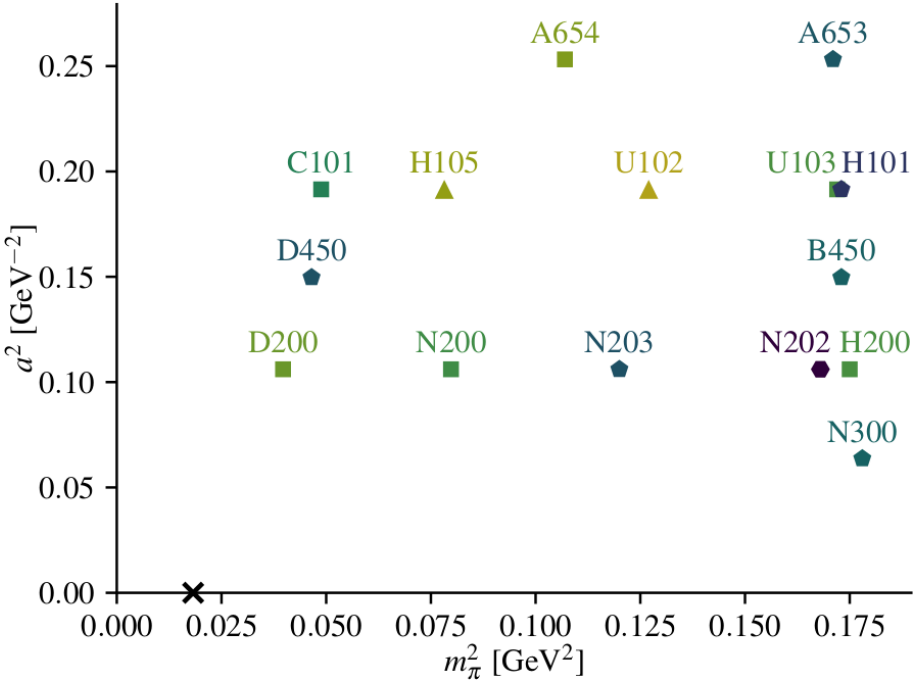
(tot)

T. Blum et al 2020. (PRL 124, 13, 132002)

	con	discon	tot
a_μ	24.16(2.30)	-16.45(2.13)	7.87(3.06)
sys hybrid $\mathcal{O}(a^2)$	0.20(0.45)	0	0.20(0.45)
sys $\mathcal{O}(1/L^3)$	2.34(0.41)	1.72(0.32)	0.83(0.56)
sys $\mathcal{O}(a^4)$	0.88(0.31)	0.71(0.28)	0.95(0.92)
sys $\mathcal{O}(a^2 \log(a^2))$	0.23(0.08)	0.25(0.09)	0.02(0.11)
sys $\mathcal{O}(a^2/L)$	4.43(1.38)	3.49(1.37)	1.08(1.57)
sys strange con	0.30	0	0.30
sys sub-discon	0	0.50	0.50
sys all	5.11(1.32)	3.99(1.29)	1.77(1.13)

- Systematic errors are estimated by alter the formula of the extrapolation.
- Same method is used for estimating the systematic error of individual and total contribution.
- Systematic error has some cancellation between the connected and disconnected diagrams.

T. Blum et al 2020. (PRL 124, 13, 132002)



E.H. Chao et al 2021. (EPJC 81 7, 651)

- Pion masses are heavier than physical value and Chiral extrapolation is used.

- For the connected and disconnected diagrams' contributions individually:

$$a_\mu(m_\pi^2, m_\pi L, a^2) = A\, e^{-m_\pi L/2} + B\, a^2 + C\; S(m_\pi^2) + D + E\, m_\pi^2\ , \tag{21}$$

$$\begin{aligned} \text{Pole} &:: \frac{1}{m_\pi^2} \\ \text{Log} &:: \log m_\pi^2 \\ \text{Log2} &:: \log^2\left(m_\pi^2\right) \\ \text{m2Log} &:: m_\pi^2 \log\left(m_\pi^2\right)\ . \end{aligned} \tag{22}$$

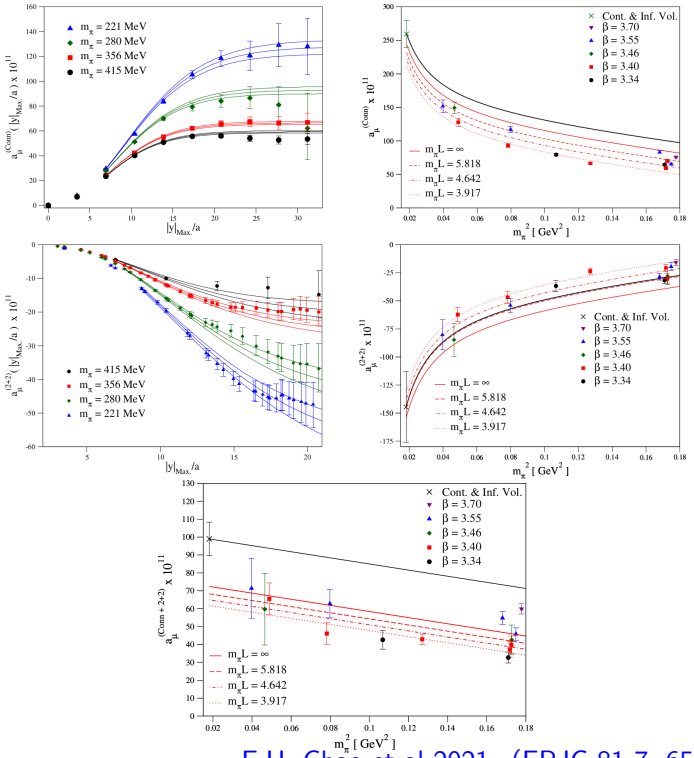
- For the total contribution:

$$a_\mu(m_\pi^2, m_\pi L, a^2) = a_\mu(0, \infty, 0)(1 + A m_\pi^2 + B e^{-m_\pi L/2} + C a^2), \tag{23}$$

  - Note: more stringent Chiral fitting form for the total contribution due to weaker pion mass dependence for the total contribution.

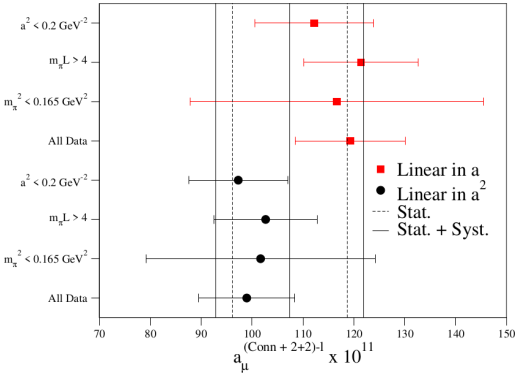
- Long distance (separation of the two vertices locations) contribution obtained by fitting an ansatz: $f(|y|) = |y|^3 A e^{-B|y|}$.
  Results only weakly depend on the form of the ansatz.

E.H. Chao et al 2021. (EPJC 81 7, 651)



E.H. Chao et al 2021. (EPJC 81 7, 651)

- Systematic uncertainty of the continuum extrapolation



Root-mean-squared deviation (9.2×10^{-11}) is estimated to be the uncertainty.

- Systematic uncertainty from Chiral extrapolation:

$$a_\mu(m_\pi^2, m_\pi L, a^2) = a_\mu(0, \infty, 0)(1 + Am_\pi^2 + Be^{-m_\pi L/2} + Ca^2), \tag{23}$$

With $Am_\pi^2 \to A_l \log(m_\pi^2/\mathrm{GeV}^2)$, half difference ($6.0 \times 10^{-11}$) is used as the estimate of the uncertainty. E.H. Chao et al 2021. (EPJC 81 7, 651)

Outline

Summary

- The errors of lattice QCD calculations comes from:
  1. finite statistics $\rightarrow$ statistical error
  2. non-zero lattice spacing $\rightarrow$ discretization error
  3. finite lattice size $\rightarrow$ finite volume error
  4. non-physical pion mass $\rightarrow$ Chiral extrapolation
     Many lattice calculations are now performed with physical pion mass, eliminating this source of the systematic errors.
- Lattice QCD calculation is playing important role in determining the hadronic contribution to muon $g-2$ and many other physical observables.
- More accurate lattice results are expected when Fermilab releases their final result.

Thank You!